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Time changes of local Dirichlet spaces by energy measures of harmonic functions

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Abstract

Given a (symmetric) recurrent local regular Dirichlet form with state space E and an associated symmetric diffusion $\{X_t\}_{t\in[0,\infty)}$ on E, we consider a function h which belongs to the extended Dirichlet space, is harmonic outside $F_1 \cup F_2$ and equal to a on F_1 and to b on F_2, where $F_1, F_2 \subset E$ are ($\mathscr{E}$-quasi-)closed sets and $a, b \in \mathbb{R}$, $a < b$. We prove that the time change of the real-valued process $\{h(X_t)\}_{t\in[0,\infty)}$ by the energy measure $\mu_{\langle h\rangle}$ of h is a reflecting Brownian motion on $[a,b]$. As an application, we also discuss asymptotic analysis of the heat kernel on the harmonic Sierpinski gasket.

Keywords. strong local Dirichlet spaces, time changes, harmonic functions, energy measures.

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1 Introduction

As presented in the celebrated work of Itô and McKean [10], any one-dimensional diffusion can be viewed as a suitable reparametrization of one-dimensional Brownian motion. Such a method of reparametrizations of stochastic processes is known as *(random) time changes*. The purpose of this paper is to present a natural extension of this fact for a symmetric diffusion on a general state space subject to certain time changes involving harmonic functions.

We illustrate our main results by treating the scale function of a one-dimensional diffusion as a particular example. For simplicity we concentrate on the case with reflecting boundaries. Then the state space has to be a compact interval and therefore without loss of generality we may assume that the state space is $[0,1]$. Let $s : [0,1] \to \mathbb{R}$ be strictly increasing and continuous, and let m be a finite Borel measure on $[0,1]$ with full support. Set

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$a := s(0)$ and $b := s(1)$. Following [4, Subsection 2.2.3], we define

$$\mathscr{F}^s := \left\{ u \in C([0,1]) \;\middle|\; u - u(0) = \int_0^{(\cdot)} \frac{du}{ds} ds \text{ for some } \frac{du}{ds} \in L^2([0,1],ds) \right\}, \tag{1.1}$$

$$\mathscr{E}^s(u,v) := \frac{1}{2} \int_0^1 \frac{du}{ds}\frac{dv}{ds} ds, \qquad u,v \in \mathscr{F}^s \tag{1.2}$$

(note that such $du/ds \in L^2([0,1],ds)$ as in (1.1) is unique for each $u \in \mathscr{F}^s$). Then $(\mathscr{E}^s, \mathscr{F}^s)$ is an irreducible recurrent strong local regular Dirichlet form on $L^2([0,1],m)$ by [4, Proposition 2.2.8], and the associated m-symmetric diffusion $X^{s,m} = (\{X_t^{s,m}\}_{t\in[0,\infty)}, \{\mathbf{P}_x\}_{x\in[0,1]})$ has the scale function s and the speed measure m. Clearly, $s \in \mathscr{F}^s$ and

$$\mathscr{E}^s(s,v) = \frac{1}{2}(v(1) - v(0)), \qquad v \in \mathscr{F}^s. \tag{1.3}$$

In particular, s is harmonic outside the boundary set $\{0,1\}$; $\mathscr{E}(s,v) = 0$ for any $v \in \mathscr{F}^s$ with $v(0) = v(1) = 0$. Moreover, we see that the $\mathscr{E}^s$-energy measure $\mu_{\langle s\rangle}$ of $s \in \mathscr{F}^s$ is equal to ds and that for any $\varphi \in C^2(\mathbb{R})$ and any $u \in \mathscr{F}^s$,

$$\mathscr{E}^s(u,\varphi(s)) = \frac{1}{2}\big(u(1)\varphi'(b) - u(0)\varphi'(a)\big) - \frac{1}{2}\int_0^1 u\varphi''(s) ds. \tag{1.4}$$

Let $x \in [0,1]$. By the theory of one-dimensional diffusions (see e.g. [19, V.46–47]), under $\mathbf{P}_x$ we can construct a continuous local martingale $M = \{M_t\}_{t\in[0,\infty)}$ and a one-dimensional Brownian motion $B = \{B_t\}_{t\in[0,\infty)}$ on the same sample space as that of $X^{s,m}$, so that $M_0 = B_0 = 0$ $\mathbf{P}_x$-a.s. and

$$s(X_t^{s,m}) = B^{[a,b]}_{\langle M\rangle_t} = s(x) + B_{\langle M\rangle_t} + L^a_{\langle M\rangle_t} - L^b_{\langle M\rangle_t} \;\text{ and }\; M_t = B_{\langle M\rangle_t}, \;\; t \in [0,\infty), \;\mathbf{P}_x\text{-a.s.}; \tag{1.5}$$

here $B^{[a,b]} = \{B_t^{[a,b]}\}_{t\in[0,\infty)}$ *is the reflecting Brownian motion started at* $s(x)$ *driven by the Brownian motion* B *with local times* $L^a = \{L_t^a\}_{t\in[0,\infty)}$ *at* a *and* $L^b = \{L_t^b\}_{t\in[0,\infty)}$ *at* b, i.e. $(B^{[a,b]}, L^a, L^b)$ is the pathwisely unique triple of $\mathbb{R}$-valued continuous processes with $B^{[a,b]}$ $[a,b]$-valued and started at $s(x)$, L^a, L^b non-decreasing and started at 0 and such that, $\mathbf{P}_x$-a.s.,

$$B_t^{[a,b]} = s(x) + B_t + L_t^a - L_t^b, \;\; t \in [0,\infty), \quad \int_0^\infty \mathbf{1}_{(a,b]}(B_t^{[a,b]}) dL_t^a = \int_0^\infty \mathbf{1}_{[a,b)}(B_t^{[a,b]}) dL_t^b = 0. \tag{1.6}$$

(1.6) is called the *Skorohod equation for the reflecting Brownian motion on* $[a,b]$ *started at* $s(x)$ *driven by* B. In particular, letting $\tau_t := \inf\{u \in [0,\infty) \mid \langle M\rangle_u > t\}$ we see that

$$s(X_{\tau_t}^{s,m}) = B_t^{[a,b]} = s(x) + B_t + L_t^a - L_t^b, \qquad t \in [0,\infty), \;\mathbf{P}_x\text{-a.s.} \tag{1.7}$$

On the other hand, since the process $N := \{N_t := L^a_{\langle M\rangle_t} - L^b_{\langle M\rangle_t}\}_{t\in[0,\infty)}$ is continuous and of bounded variation, we see that (1.5) actually gives the *Fukushima decomposition for* s:

$$s(X_t^{s,m}) - s(X_0^{s,m}) = M_t + N_t \qquad \text{for any } t \in [0,\infty), \;\mathbf{P}_x\text{-a.s.} \tag{1.8}$$

It follows that the equality (1.7) is obtained as the time change of the Fukushima decomposition (1.8) for s by the right-continuous inverse $\tau_{(\cdot)}$ of $\langle M\rangle_{(\cdot)}$.

In this paper, we extend these facts to the case of certain harmonic functions on a *general* recurrent strong local regular Dirichlet space. To state our main results, let E be a locally compact separable metrizable space with one-point compactification $E_\Delta = E \cup \{\Delta\}$, let m be a Radon measure on E with full support and let $X = (\{X_t\}_{t\in[0,\infty]}, \{\mathbf{P}_x\}_{x\in E_\Delta})$ be an m-symmetric Hunt process on E whose Dirichlet form $(\mathscr{E},\mathscr{F})$ on $L^2(E,m)$ is regular. Let $\mathscr{F}_e$ denote the associated extended Dirichlet space and let $\widetilde{u}$ denote any $\mathscr{E}$-quasi-continuous m-version of $u \in \mathscr{F}_e$, which is unique up to $\mathscr{E}$-q.e. Let $a,b \in \mathbb{R}$, $a < b$ and suppose that $F_1, F_2 \subset E$ are ($\mathscr{E}$-quasi-)closed sets admitting $\mathbf{u} \in \mathscr{F}_e$ such that $\widetilde{\mathbf{u}} = a$ $\mathscr{E}$-q.e. on F_1 and $\widetilde{\mathbf{u}} = b$ $\mathscr{E}$-q.e. on F_2. Then let $h \in \mathscr{F}_e$ be a $(F_1 \cup F_2)$-harmonic function satisfying $\widetilde{h} = \widetilde{\mathbf{u}}$ $\mathscr{E}$-q.e. on $F_1 \cup F_2$, which does exist by [9, Theorem 4.6.5]. Let

$$\widetilde{h}(X_t) - \widetilde{h}(X_0) = M_t^{[h]} + N_t^{[h]} \tag{1.9}$$

be the Fukushima decomposition for h, where $M^{[h]} = \{M_t^{[h]}\}_{t\in[0,\infty)}$ is a martingale additive functional and $N^{[h]} = \{N_t^{[h]}\}_{t\in[0,\infty)}$ is a continuous additive functional of zero energy. Note that if $(\mathscr{E},\mathscr{F})$ is strong local then $M^{[h]}$ is continuous by [9, Lemma 5.5.1 (ii)]. Let $\langle M^{[h]}\rangle = \{\langle M^{[h]}\rangle_t\}_{t\in[0,\infty)}$ be the quadratic variation of $M^{[h]}$, which is a positive continuous additive functional with Revuz measure equal to the $\mathscr{E}$-energy measure $\mu_{\langle h\rangle}$ of h. Define $\sigma_B := \inf\{t \in (0,\infty) \mid X_t \in B\}$ for $B \subset E_\Delta$. The following is a summary of the main results of this paper (Theorems 2.12 and 3.6).

Theorem 1.1. *Assume that $(\mathscr{E},\mathscr{F})$ is recurrent and strong local.*
(1) Let $\varphi \in C^2(\mathbb{R})$ satisfy $\varphi'(a) = \varphi'(b) = 0$. Then $\varphi(h) \in \mathscr{F}_e$ and

$$\mathscr{E}(u,\varphi(h)) = -\frac{1}{2}\int_E \widetilde{u}\varphi''(\widetilde{h})d\mu_{\langle h\rangle}, \quad u \in \mathscr{F}_e \cap L^\infty(E,m). \tag{1.10}$$

(2) Suppose additionally that $\sigma_{F_1} \vee \sigma_{F_2} < \infty$ $\mathbf{P}_m$-a.s., where $\mathbf{P}_m[(\cdot)] := \int_E \mathbf{P}_x[(\cdot)]dm(x)$. Let $\tau_t := \inf\{s \in [0,\infty) \mid \langle M^{[h]}\rangle_s > t\}$ for $t \in [0,\infty)$. Then for $\mathscr{E}$-q.e. $x \in E$, under $\mathbf{P}_x$, $B^h := \{M_{\tau_t}^{[h]}\}_{t\in[0,\infty)}$ is a one-dimensional Brownian motion started at 0, and $\{\widetilde{h}(X_{\tau_t})\}_{t\in[0,\infty)}$ is the reflecting Brownian motion on $[a,b]$ started at $\widetilde{h}(x)$ driven by B^h, with local times L^a at a and L^b at b equal respectively to the positive variation and the negative variation of $\{N_{\tau_t}^{[h]}\}_{t\in[0,\infty)}$.

Remark 1.2. In Theorem 1.1 it is sufficient to assume that $(\mathscr{E},\mathscr{F})$ is recurrent and *local*, since the strong locality of $(\mathscr{E},\mathscr{F})$ easily follows from its recurrence and locality.

Since the positive continuous additive functional $\langle M^{[h]}\rangle$ has the Revuz measure $\mu_{\langle h\rangle}$, $\{\widetilde{h}(X_{\tau_t})\}_{t\in[0,\infty)}$, $B^h = \{M_{\tau_t}^{[h]}\}_{t\in[0,\infty)}$ and $\{N_{\tau_t}^{[h]}\}_{t\in[0,\infty)}$ are the *time change* of the original processes *with respect to the $\mathscr{E}$-energy measure $\mu_{\langle h\rangle}$ of the harmonic function h*. By (1.10), $\varphi(\widetilde{h}) \in \mathrm{Dom}(\mathscr{L}_{\mu_{\langle h\rangle}})$ and $\mathscr{L}_{\mu_{\langle h\rangle}}(\varphi(\widetilde{h})) = \varphi''(\widetilde{h})/2$ for the generator $\mathscr{L}_{\mu_{\langle h\rangle}}$ of the *time change of $(E,m,\mathscr{E},\mathscr{F})$ by $\mu_{\langle h\rangle}$*, which is the Dirichlet space associated with $\{X_{\tau_t}\}_{t\in[0,\infty]}$ and is analytically obtained by replacing the reference measure m of the form $(\mathscr{E},\mathscr{F})$ by $\mu_{\langle h\rangle}$; see [4, Chapter 5] and [9, Section 6.2] for general theory of time changes of Dirichlet spaces.

The original motivation for this research is asymptotic analysis of the heat kernel on a fractal called the *harmonic Sierpinski gasket* (see Figure 4.2 below), which is the image of an injective harmonic map from the usual Sierpinski gasket (Figure 4.1) into $\mathbb{R}^2$ and whose heat kernel has proved to be subject to the two-sided *Gaussian* bound by [15, Theorem 6.3]. In the end of this article, we briefly describe how we can determine an on-diagonal short time asymptotic behavior of this heat kernel as an application of Theorem 1.1.

The organization of this paper is as follows. In Section 2, we first recall basics of analytic theory of regular Dirichlet forms, then study fundamental properties of harmonic functions and prove Theorem 1.1 (1) (Theorem 2.12). In Section 3, we present a few general results concerning the sample path properties of additive functionals, then give the precise statement of Theorem 1.1 (2) in Theorem 3.6 and prove it. Section 4 is devoted to an application of the main results to asymptotic analysis of the heat kernel on the harmonic Sierpinski gasket.

Notation. In this paper, we adopt the following notations and conventions.
(1) $\mathbb{N} = \{1,2,3,\dots\}$, i.e. $0 \notin \mathbb{N}$.
(2) We set $\inf\emptyset := \infty$. We write $a \vee b := \max\{a,b\}$, $a \wedge b := \min\{a,b\}$, $a^+ := a \vee 0$ and $a^- := -(a \wedge 0)$ for $a,b \in [-\infty,\infty]$. We use the same notations for (equivalence classes of) functions. All functions treated in this paper are assumed to be $\mathbb{R}$-valued or $[-\infty,\infty]$-valued.
(3) Let $(E,\mathscr{B})$ be a measurable space. For a positive measure μ on $(E,\mathscr{B})$, let $\mathscr{B}^\mu$ denote the μ-completion of $\mathscr{B}$. A *signed measure on* $(E,\mathscr{B})$ is by definition an $\mathbb{R}$-*valued* countably additive set function on $\mathscr{B}$.
(4) Let E be a topological space. The Borel σ-field of E is denoted by $\mathscr{B}(E)$. We set $C(E) := \{f \mid f : E \to \mathbb{R}, f \text{ is continuous}\}$ and $C_c(E) := \{f \in C(E) \mid \mathrm{supp}_E[f] \text{ is compact}\}$, where $\mathrm{supp}_E[f] := \overline{\{x \in E \mid f(x) \neq 0\}}$. Also set $\|f\|_\infty := \sup_{x\in E}|f(x)|$ for $f : E \to [-\infty,\infty]$.

2 Harmonic functions and their energy measures

In the first half of this section, we briefly recall basic facts from analytic theory of regular Dirichlet forms; see [4, 8, 9, 18] for details. Throughout this section, let E be a locally compact separable metrizable space, m be a Radon measure on E with *full support*, i.e. such that $m(G) > 0$ for any non-empty open subset G of E (recall that a *Radon measure on* E is by definition a positive Borel measure on E for which every compact set is of finite measure), and let $(\mathscr{E},\mathscr{F})$ be a (symmetric) regular Dirichlet form on $L^2(E,m)$.

Let $\mathscr{F}_e$ be the extended Dirichlet space associated with $(\mathscr{E},\mathscr{F})$; $u \in \mathscr{F}_e$ if and only if u is an (m-equivalence class of) Borel measurable $\mathbb{R}$-valued function admitting $\{u_n\}_{n\in\mathbb{N}} \subset \mathscr{F}$ such that $\lim_{k,\ell\to\infty}\mathscr{E}(u_k-u_\ell,u_k-u_\ell) = 0$ and $\lim_{n\to\infty}u_n = u$ m-a.e. We extend $\mathscr{E}$ to a non-negative definite symmetric bilinear form on $\mathscr{F}_e$ by setting $\mathscr{E}(u,u) := \lim_{n\to\infty}\mathscr{E}(u_n,u_n)$ with u,u_n as above, so that $\lim_{n\to\infty}\|u-u_n\|_{\mathscr{E}} = 0$, where we write $\|u\|_{\mathscr{E}} := \mathscr{E}(u,u)^{1/2}$ for $u \in \mathscr{F}_e$. We have $\mathscr{F} = \mathscr{F}_e \cap L^2(E,m)$ by [9, Theorem 1.5.2 (iii)]. By [9, Corollary 1.6.3], $\varphi(u) \in \mathscr{F}_e$ and $\mathscr{E}(\varphi(u),\varphi(u)) \le \mathscr{E}(u,u)$ for $u \in \mathscr{F}_e$ and a *normal contraction* φ, i.e. a function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(0) = 0$ and $|\varphi(s)-\varphi(t)| \le |s-t|$ for any $s,t \in \mathbb{R}$. We write $\mathscr{F}_{e,b} := \mathscr{F}_e \cap L^\infty(E,m)$, which is an algebra under pointwise sum and multiplication by [9, Corollary 1.6.3].

Definition 2.1. We define the 1-*capacity* $\mathrm{Cap}_{\mathscr{E}}$ *associated with* $(\mathscr{E},\mathscr{F})$ by

$$\mathrm{cap}_{\mathscr{E}}(U) := \inf\{\mathscr{E}_1(u,u) \mid u \in \mathscr{F}, u \ge 1 \text{ } m\text{-a.e. on } U\}, \quad U \subset E \text{ open}, \tag{2.1}$$

$$\mathrm{Cap}_{\mathscr{E}}(A) := \inf\{\mathrm{cap}_{\mathscr{E}}(U) \mid U \subset E \text{ open}, A \subset U\}, \qquad A \subset E, \tag{2.2}$$

where $\mathscr{E}_1(u,v) := \mathscr{E}(u,v) + \int_E uvdm$ for $u,v \in \mathscr{F}$. $N \subset E$ is called *$\mathscr{E}$-polar* if and only if $\mathrm{Cap}_{\mathscr{E}}(N) = 0$. Moreover, let $A \subset E$ and let $\mathscr{S}(x)$ be a statement on $x \in A$. Then we say that *$\mathscr{S}$ holds $\mathscr{E}$-q.e. on A*, or *$\mathscr{S}(x)$ for $\mathscr{E}$-q.e. $x \in A$*, if and only if the set $\{x \in A \mid \mathscr{S}(x)$ fails$\}$ is $\mathscr{E}$-polar. When $A = E$ we simply say '*$\mathscr{S}$ holds $\mathscr{E}$-q.e.*' instead.

Clearly, $\mathrm{Cap}_{\mathscr{E}}$ is an extension of $\mathrm{cap}_{\mathscr{E}}$ and $m(A) \leq \mathrm{Cap}_{\mathscr{E}}(A)$ for any $A \in \mathscr{B}(E)$. By [9, Theorem A.1.2], $\mathrm{Cap}_{\mathscr{E}}$ is countably subadditive.

Next we define $\mathscr{E}$-quasi notions by utilizing $\mathrm{Cap}_{\mathscr{E}}$, as follows.

Definition 2.2. (1) A non-decreasing sequence $\{F_k\}_{k\in\mathbb{N}}$ of closed sets in E is called an *$\mathscr{E}$-nest* if and only if $\lim_{k\to\infty} \mathrm{Cap}_{\mathscr{E}}(K \setminus F_k) = 0$ for any compact subset K of E.
(2) A function $u : E \setminus N \to [-\infty,\infty]$, defined outside an $\mathscr{E}$-polar set N, is called *$\mathscr{E}$-quasi-continuous* if and only if there exists an $\mathscr{E}$-nest $\{F_k\}_{k\in\mathbb{N}}$ such that $\bigcup_{k\in\mathbb{N}} F_k \subset E \setminus N$ and $u|_{F_k}$ is $\mathbb{R}$-valued continuous for each $k \in \mathbb{N}$.
(3) A subset E_0 of E is called *$\mathscr{E}$-quasi-open* (resp. *$\mathscr{E}$-quasi-closed*) if and only if there exists an $\mathscr{E}$-nest $\{F_k\}_{k\in\mathbb{N}}$ such that $E_0 \cap F_k$ is open (resp. closed) in F_k for each $k \in \mathbb{N}$, with F_k equipped with the relative topology inherited from E.

Given an $\mathscr{E}$-nest $\{F_k\}_{k\in\mathbb{N}}$, $E \setminus \bigcup_{k\in\mathbb{N}} F_k$ is an $\mathscr{E}$-polar set. If we set $E_1 := E_0 \cap \bigcup_{k\in\mathbb{N}} F_k$ in the situation of (3) above, then $E_1 \in \mathscr{B}(E)$, E_1 is $\mathscr{E}$-quasi-open (resp. $\mathscr{E}$-quasi-closed) and $E_0 \setminus E_1$ is $\mathscr{E}$-polar. A $[-\infty,\infty]$-valued function u defined $\mathscr{E}$-q.e. is $\mathscr{E}$-quasi-continuous if and only if u is $\mathbb{R}$-valued $\mathscr{E}$-q.e. and $u^{-1}(I)$ is $\mathscr{E}$-quasi-open for any open subset I of $\mathbb{R}$. If u is $\mathscr{E}$-quasi-continuous, then $u \geq 0$ m-a.e. if and only if $u \geq 0$ $\mathscr{E}$-q.e. by [9, Lemma 2.1.4], and u admits a Borel measurable $\mathscr{E}$-quasi-continuous function $v : E \to \mathbb{R}$ such that $u = v$ $\mathscr{E}$-q.e. By [9, Theorem 2.1.7], for any $u \in \mathscr{F}_e$ there exists an $\mathscr{E}$-quasi-continuous function v such that $u = v$ m-a.e., and such v is called an *$\mathscr{E}$-quasi-countinuous m-version of u*, which is unique up to $\mathscr{E}$-q.e. For $u \in \mathscr{F}_e$, let $\widetilde{u}$ denote any $\mathscr{E}$-quasi-countinuous m-version of u.

Definition 2.3. Let μ be a positive Borel measure on E *charging no $\mathscr{E}$-polar set*, i.e. $\mu(N) = 0$ for any $\mathscr{E}$-polar $N \in \mathscr{B}(E)$. (Note that then every $\mathscr{E}$-polar, $\mathscr{E}$-quasi-open or $\mathscr{E}$-quasi-closed set belongs to $\mathscr{B}(E)^\mu$ and that every $\mathscr{E}$-quasi-continuous function defined $\mathscr{E}$-q.e. is $\mathscr{B}(E)^\mu$-measurable.)
(1) μ is called an *$\mathscr{E}$-smooth measure* if and only if $\mu(F_k) < \infty$ for any $k \in \mathbb{N}$ for some $\mathscr{E}$-nest $\{F_k\}_{k\in\mathbb{N}}$. The collection of all $\mathscr{E}$-smooth measures is denoted by $S^{\mathscr{E}}$.
(2) $F_\mu \subset E$ is called an *$\mathscr{E}$-quasi-support of μ* if and only if it is $\mathscr{E}$-quasi-closed, $\mu(E \setminus F_\mu) = 0$ and $F_\mu \setminus F$ is $\mathscr{E}$-polar for any $\mathscr{E}$-quasi-closed set $F \subset E$ with $\mu(E \setminus F) = 0$.

If $\mu \in S^{\mathscr{E}}$ and $\{F_k\}_{k\in\mathbb{N}}$ is an $\mathscr{E}$-nest for μ as in Definition 2.3 (1), then $\mu(E \setminus \bigcup_{k\in\mathbb{N}} F_k) = 0$ and hence μ is σ-finite. Any Radon measure on E charging no $\mathscr{E}$-polar set belongs to $S^{\mathscr{E}}$; it suffices to set $F_k := \overline{G_k}$, where $\{G_k\}_{k\in\mathbb{N}}$ is a non-decreasing sequence of relatively compact open subsets of E with $\bigcup_{k\in\mathbb{N}} G_k = E$. By [9, Theorem 4.6.3], every $\mu \in S^{\mathscr{E}}$ admits an $\mathscr{E}$-quasi-support $F_\mu \in \mathscr{B}(E)$.

Associated with $u \in \mathscr{F}_{e,b}$ is the *$\mathscr{E}$-energy measure* $\mu_{\langle u \rangle}$; by [9, Theorem 5.2.3] we have

$$\int_E \widetilde{f} d\mu_{\langle u \rangle} = 2\mathscr{E}(uf,u) - \mathscr{E}(u^2,f), \qquad u,f \in \mathscr{F}_{e,b}, \tag{2.3}$$

where, for each $u \in \mathscr{F}_{e,b}$, $\mu_{\langle u \rangle}(\in S^{\mathscr{E}}$ by [9, Lemma 3.2.4]) is defined as the unique positive Borel measure on E satisfying (2.3) for any $f \in \mathscr{F} \cap C_c(E)$ with f in place of $\widetilde{f}$ in the integrand. For $u \in \mathscr{F}_{e,b}$, [2, Proposition I.4.1.1] implies that $\mu_{\langle u\rangle}(E) \le 2\mathscr{E}(u,u)$ and that

$$\mu_{\langle \varphi(u)\rangle} \le \mu_{\langle u\rangle} \quad \text{for any normal contraction } \varphi. \tag{2.4}$$

Also for $u,v \in \mathscr{F}_{e,b}$, we define a Borel signed measure $\mu_{\langle u,v\rangle}$ on E by $\mu_{\langle u,v\rangle} := (\mu_{\langle u+v\rangle} - \mu_{\langle u-v\rangle})/4$. (2.3) yields

$$\int_E \widetilde{f} d\mu_{\langle u,v\rangle} = \mathscr{E}(uf,v) + \mathscr{E}(vf,u) - \mathscr{E}(uv,f), \quad u,v,f \in \mathscr{F}_{e,b}, \tag{2.5}$$

and hence $\mathscr{F}_{e,b} \times \mathscr{F}_{e,b} \ni (u,v) \mapsto \mu_{\langle u,v\rangle}$ is bilinear and symmetric. Therefore we easily see that for any $u,v \in \mathscr{F}_{e,b}$ and any bounded Borel measurable $f : E \to [0,\infty)$,

$$\left[\int_E f d\mu_{\langle u,v\rangle}\right]^2 \le \int_E f d\mu_{\langle u\rangle} \int_E f d\mu_{\langle v\rangle}, \tag{2.6}$$

$$\left|\left[\int_E f d\mu_{\langle u\rangle}\right]^{1/2} - \left[\int_E f d\mu_{\langle v\rangle}\right]^{1/2}\right|^2 \le \int_E f d\mu_{\langle u-v\rangle} \le 2\|f\|_\infty \|u-v\|_{\mathscr{E}}^2. \tag{2.7}$$

Then by a limiting procedure using (2.4), (2.7) and [9, Corollary 1.6.3], for *any* $u,v \in \mathscr{F}_e$ we can uniquely define a finite $\mathscr{E}$-smooth measure $\mu_{\langle u\rangle}$ and $\mu_{\langle u,v\rangle} := (\mu_{\langle u+v\rangle} - \mu_{\langle u-v\rangle})/4$ so that $\mu_{\langle u\rangle}(E) \le 2\mathscr{E}(u,u)$ and $\mathscr{F}_e \times \mathscr{F}_e \ni (u,v) \mapsto \mu_{\langle u,v\rangle}$ is bilinear and symmetric. Again we get (2.6) and (2.7) in the same way, and we can also verify (2.4) by using Banach-Saks theorem (see [4, Theorem A.4.1] or [18, Theorem A.2.2]). It is immediate by (2.6) and (2.7) that $\mu_{\langle u_1,v\rangle} = \mu_{\langle u_2,v\rangle}$ for $u_1,u_2,v \in \mathscr{F}_e$ with $\|u_1 - u_2\|_{\mathscr{E}} = 0$. Moreover, if $(\mathscr{E},\mathscr{F})$ is strong local, then we have the following chain rule for $\mu_{\langle\cdot\rangle}$, which often plays essential roles in analysis of strong local Dirichlet forms.

Lemma 2.4 ([9, Theorem 3.2.2]). *Let $n \in \mathbb{N}$, $u_1,\ldots,u_n \in \mathscr{F}_e$ and let $\varphi = \varphi(x_1,\ldots,x_n) \in C^1(\mathbb{R}^n)$ satisfy $\varphi(0) = 0$. Suppose either that $u_1,\ldots,u_n \in \mathscr{F}_{e,b}$ or that $\partial\varphi/\partial x_i$ is bounded on $\mathbb{R}^n$ for any $i \in \{1,\ldots,n\}$. Then $\varphi(u_1,\ldots,u_n) \in \mathscr{F}_e$. Moreover, if in addition $(\mathscr{E},\mathscr{F})$ is strong local, then for any $v \in \mathscr{F}_e$,*

$$d\mu_{\langle \varphi(u_1,\ldots,u_n),v\rangle} = \sum_{i=1}^n \frac{\partial\varphi}{\partial x_i}(\widetilde{u}_1,\ldots,\widetilde{u}_n) d\mu_{\langle u_i,v\rangle}. \tag{2.8}$$

Remark 2.5. [2, Proposition I.4.1.1] and [9, Theorems 3.2.2 and 5.2.3] are stated mainly for functions in $\mathscr{F} \cap L^\infty(E,m)$ or $\mathscr{F}$ and not necessarily for those in $\mathscr{F}_{e,b}$ or $\mathscr{F}_e$, but we easily see that they are valid for functions in $\mathscr{F}_e$ in the following manner:

[2, Proposition I.4.1.1] can be easily extended to functions in $\mathscr{F}_{e,b}$ by using Banach-Saks theorem. For the other two theorems, choose $\eta \in L^1(E,m) \cap L^\infty(E,m)$ so that $\eta > 0$ m-a.e., and set $\mathscr{F}^\eta := \mathscr{F}_e \cap L^2(E,\eta\cdot m)$, where $(\eta\cdot m)(A) := \int_A \eta dm$ for $A \in \mathscr{B}(E)$. Then by [9, Theorem 6.2.1], $(\mathscr{E},\mathscr{F}^\eta)$ is a regular Dirichlet form on $L^2(E,\eta\cdot m)$, and by [9, Theorem 3.1.2, Problems 3.1.1 and 1.4.1] it is strong local if $(\mathscr{E},\mathscr{F})$ is. By [9, Corollary 4.6.1 and the argument before Lemma 6.2.9], the notion of $\mathscr{E}$-nest and the $\mathscr{E}$-quasi notions with respect to $(\mathscr{E},\mathscr{F}^\eta)$ (on $L^2(E,\eta\cdot m)$) coincide with those with respect to $(\mathscr{E},\mathscr{F})$ (on $L^2(E,m)$). Moreover, $\mathscr{F}^\eta \cap L^\infty(E,\eta\cdot m) = \mathscr{F}_{e,b}$, and for any $u_1,\ldots,u_n \in \mathscr{F}_e$ we can choose η as above so that $u_i \in L^2(E,\eta\cdot m)$ and hence $u_i \in \mathscr{F}^\eta$ for $i \in \{1,\ldots,n\}$. Now [9, Theorems 3.2.2 and 5.2.3] applied to functions in $\mathscr{F}^\eta \cap L^\infty(E,\eta\cdot m)$ or $\mathscr{F}^\eta$ yield the desired assertions for functions in $\mathscr{F}_{e,b}$ or $\mathscr{F}_e$.

We now start our study of harmonic functions and their $\mathscr{E}$-energy measures. First, we give the definition of harmonic functions.

Definition 2.6. Let $F \subset E$ be $\mathscr{E}$-quasi-closed and set $\mathscr{F}_F^u := \{v \in \mathscr{F}_e \mid \widetilde{v} = \widetilde{u}\ \mathscr{E}\text{-q.e. on } F\}$ for $u \in \mathscr{F}_e$. We call $h \in \mathscr{F}_e$ *F-harmonic* if and only if

$$\mathscr{E}(h,h) = \inf\{\mathscr{E}(v,v) \mid v \in \mathscr{F}_F^h\} \quad \text{or equivalently,} \quad \mathscr{E}(h,v) = 0, \quad {}^\forall v \in \mathscr{F}_F^0. \tag{2.9}$$

Let $F \subset E$ be $\mathscr{E}$-quasi-closed. The equivalence of the two conditions in (2.9) for $h \in \mathscr{F}_e$ is obvious. Let $u \in \mathscr{F}_e$. Then by [9, Theorems 4.1.3, 4.2.1 (ii), 4.6.5 and A.2.6 (i)], there exists an F-harmonic function $h \in \mathscr{F}_F^u$. (2.9) implies that, if $h_1, h_2 \in \mathscr{F}_F^u$ are F-harmonic then $\|h_1 - h_2\|_{\mathscr{E}} = 0$ and hence $\mu_{\langle h_1 \rangle} = \mu_{\langle h_2 \rangle}$. Also by (2.9), if $h \in \mathscr{F}_F^u$ is F-harmonic and φ is a normal contraction such that $\varphi(\widetilde{u}) = \widetilde{u}\ \mathscr{E}$-q.e. on F, then $\varphi(h)$ is also an F-harmonic function belonging to $\mathscr{F}_F^u$.

The following lemma will be used in the proof of Lemma 3.3.

Lemma 2.7. *Let $u \in \mathscr{F}_e$. If $F \subset E$ is an $\mathscr{E}$-quasi-support of $\mu_{\langle u \rangle}$ then u is F-harmonic.*

Proof. Let $v \in \mathscr{F}_F^0$, $\ell \in \mathbb{N}$ and set $u_\ell := (-\ell) \vee (u \wedge \ell)$ and $v_\ell := (-\ell) \vee (v \wedge \ell)$. Let $n \in \mathbb{N}$, $n \geq 2$. Then (2.5) yields $2\mathscr{E}(u_\ell, v_\ell{}^n) = \int_E \widetilde{v_\ell} d\mu_{\langle u_\ell, v_\ell{}^{n-1} \rangle} + \int_E (\widetilde{v_\ell})^{n-1} d\mu_{\langle u_\ell, v_\ell \rangle} = 0$, where the latter equality follows by $\mu_{\langle u \rangle}(E \setminus F) = 0$, (2.4) and (2.6).

Next let $\varphi \in C^2(\mathbb{R})$ satisfy $\varphi(0) = \varphi'(0) = 0$. Then $\varphi(v_\ell) \in \mathscr{F}_e$ by Lemma 2.4. By approximating φ'' uniformly on $[-\ell, \ell]$ by polynomials, we see that there exists a sequence of polynomials $\{\varphi_n\}_{n \in \mathbb{N}}$ such that $\varphi_n(0) = \varphi_n'(0) = 0$ and $\sup_{x \in [-\ell,\ell]} |\varphi_n'(x) - \varphi'(x)| \to 0$ as $n \to \infty$. The argument in the previous paragraph yields $\mathscr{E}(u_\ell, \varphi_n(v_\ell)) = 0$, and letting $n \to \infty$ results in $\mathscr{E}(u_\ell, \varphi(v_\ell)) = 0$ since $\lim_{n\to\infty} \|\varphi_n(v_\ell) - \varphi(v_\ell)\|_{\mathscr{E}} = 0$ by [9, (3.2.27)].

Finally, choose $f \in C^1(\mathbb{R})$ so that $0 \leq f \leq 1$, $f(0) = 0$ and $f(x) = 1$ for $|x| \geq 1$, and set $\psi_n(x) := \int_0^x f(ny)dy$. Then $\psi_n \in C^2(\mathbb{R})$ and $\psi_n(0) = \psi_n'(0) = 0$. Similarly to [9, Corollary 1.6.3] we have $\lim_{n\to\infty} \|v_\ell - \psi_n(v_\ell)\|_{\mathscr{E}} = 0$ and hence $\mathscr{E}(u_\ell, v_\ell) = \lim_{n\to\infty} \mathscr{E}(u_\ell, \psi_n(v_\ell)) = 0$. Now letting $\ell \to \infty$ yields $\mathscr{E}(u, v) = 0$ by [9, Corollary 1.6.3]. Thus u is F-harmonic. ∎

Given an $\mathscr{E}$-quasi-closed set $F \subset E$ and $u \in \mathscr{F}_e$, an F-harmonic function $h \in \mathscr{F}_F^u$ may *not* be unique since $(\mathscr{E}, \mathscr{F})$ is *not* assumed to be irreducible. Nevertheless we still have a kind of equivalence between F-harmonic functions belonging to $\mathscr{F}_F^u$, as follows.

Lemma 2.8. *Let $F \subset E$ be $\mathscr{E}$-quasi-closed, $u \in \mathscr{F}_e$ and $h_1, h_2 \in \mathscr{F}_F^u$ be F-harmonic. Then*
(1) $\widetilde{h_1} = \widetilde{h_2}$ $\mu_{\langle h_1 \rangle}$-a.e. (Recall that $\mu_{\langle h_1 \rangle} = \mu_{\langle h_2 \rangle}$.)
(2) Let $\varphi \in C^1(\mathbb{R})$ satisfy $\varphi(0) = 0$. Suppose either that $h_2 \in \mathscr{F}_{e,b}$ or that φ' is bounded on $\mathbb{R}$. Then $\varphi(h_1), \varphi(h_2) \in \mathscr{F}_e$ and $\|\varphi(h_1) - \varphi(h_2)\|_{\mathscr{E}} = 0$.

Proof. (1) Let $f := |h_1 - h_2| \wedge 1$. Then $f \in \mathscr{F}_F^0$ and $\|f\|_{\mathscr{E}} \leq \|h_1 - h_2\|_{\mathscr{E}} = 0$. Let $\ell \in \mathbb{N}$ and $g_\ell := (-\ell) \vee (h_1 \wedge \ell)$. (2.3) implies that $\int_E \widetilde{f} d\mu_{\langle g_\ell \rangle} = 2\mathscr{E}(g_\ell f, g_\ell) - \mathscr{E}(g_\ell^2, f) = 2\mathscr{E}(g_\ell f, g_\ell)$. [4, Exercise 1.1.10] together with $\|f\|_{\mathscr{E}} = 0$ yields $\|g_\ell f\|_{\mathscr{E}} \leq \|f\|_{L^\infty(E,m)} \|g_\ell\|_{\mathscr{E}} \leq \|h_1\|_{\mathscr{E}}$. $\mathscr{E}(g_\ell f, h_1) = 0$ by (2.9), and then $0 \leq \int_E \widetilde{f} d\mu_{\langle g_\ell \rangle} = 2\mathscr{E}(g_\ell f, g_\ell - h_1) \leq \|h_1\|_{\mathscr{E}} \|g_\ell - h_1\|_{\mathscr{E}}$. Since $\lim_{\ell\to\infty} \|g_\ell - h_1\|_{\mathscr{E}} = 0$ by [9, Corollary 1.6.3], letting $\ell \to \infty$ and (2.7) lead to $\int_E \widetilde{f} d\mu_{\langle h_1 \rangle} = 0$, which yields the assertion since $\widetilde{f} = |\widetilde{h_1} - \widetilde{h_2}| \wedge 1$ $\mathscr{E}$-q.e. and hence $\mu_{\langle h_1 \rangle}$-a.e.
(2) First suppose either that $h_1, h_2 \in \mathscr{F}_{e,b}$ or that φ' is bounded. Then for some $\psi \in C^1(\mathbb{R})$ and $c \in (0, \infty)$, $c\psi$ is a normal contraction and $\varphi(h_i) = \psi(h_i)$ m-a.e. for $i = 1, 2$. Thus

$\varphi(h_i) \in \mathscr{F}_e$ and $c^2\mu_{\langle\varphi(h_i)\rangle} \le \mu_{\langle h_1\rangle}$ for $i = 1,2$, in view of (2.4). Let F_0 be an $\mathscr{E}$-quasi-support of $\mu_{\langle h_1\rangle}$. Then (1) and [9, Theorem 4.6.2] imply that $\widetilde{h_1} = \widetilde{h_2}$ $\mathscr{E}$-q.e. on F_0, and hence $\varphi(\widetilde{h_1}) = \varphi(\widetilde{h_2})$ $\mathscr{E}$-q.e. on F_0. Since F_0 is $\mathscr{E}$-quasi-closed and $\mu_{\langle\varphi(h_i)\rangle}(E \setminus F_0) = 0$ for $i = 1,2$, Lemma 2.7 yields $\mathscr{E}(\varphi(h_1),\varphi(h_1)) = \mathscr{E}(\varphi(h_2),\varphi(h_2)) = \mathscr{E}(\varphi(h_1),\varphi(h_2))$, from which it is immediate that $\|\varphi(h_1) - \varphi(h_2)\|_{\mathscr{E}} = 0$.

Next suppose only that $h_2 \in \mathscr{F}_{e,b}$. Let $\ell \in \mathbb{N}$ satisfy $\ell \ge \|h_2\|_{L^\infty(E,m)}$ and let g_ℓ be as in (1). Then since $|\widetilde{h_1}| = |\widetilde{u}| = |\widetilde{h_2}| \le \ell$ $\mathscr{E}$-q.e. on F, g_ℓ is also an F-harmonic function belonging to $\mathscr{F}^u_F$ and hence $\varphi(g_\ell), \varphi(h_2) \in \mathscr{F}_e$ and $\|\varphi(g_\ell) - \varphi(h_2)\|_{\mathscr{E}} = 0$ by the previous paragraph. Since $\lim_{\ell\to\infty}\varphi(g_\ell) = \varphi(h_1)$ m-a.e. and $\|\varphi(g_k) - \varphi(g_\ell)\|_{\mathscr{E}} = 0$ for $k,\ell \ge \|h_2\|_{L^\infty(E,m)}$, an argument similar to [4, Proof of Lemma 1.1.12] shows $\varphi(h_1) \in \mathscr{F}_e$ and $\|\varphi(h_1) - \varphi(g_\ell)\|_{\mathscr{E}} = 0$ for $\ell \ge \|h_2\|_{L^\infty(E,m)}$. Thus we obtain $\|\varphi(h_1) - \varphi(h_2)\|_{\mathscr{E}} = 0$. ■

In the main results of this paper, we put the following assumption **(BC)**:

$(\mathscr{E},\mathscr{F})$ is recurrent, i.e. $\mathbf{1} \in \mathscr{F}_e$ and $\mathscr{E}(\mathbf{1},\mathbf{1}) = 0$. $F_1, F_2 \subset E$ are $\mathscr{E}$-quasi-closed and admit $\mathbf{u} \in \mathscr{F}_e$ such that $\widetilde{\mathbf{u}} = 0$ $\mathscr{E}$-q.e. on F_1 and $\widetilde{\mathbf{u}} = 1$ $\mathscr{E}$-q.e. on F_2. $a,b \in \mathbb{R}$, $a < b$ and $h \in \mathscr{F}^{a,b}$ is $(F_1 \cup F_2)$-harmonic, where $\mathscr{F}^{s,t} := \mathscr{F}^{s\mathbf{1}+(t-s)\mathbf{u}}_{F_1\cup F_2}$ for $s,t \in \mathbb{R}$. **(BC)**

In the situation of **(BC)**, $F_1 \cap F_2$ is $\mathscr{E}$-polar and there does exist an $(F_1 \cup F_2)$-harmonic function $h \in \mathscr{F}^{a,b}$. Such $\mathbf{u} \in \mathscr{F}_e$ as in **(BC)** exists if $F_1 \cap F_2 = \emptyset$ and if either F_1 is closed and F_2 is compact or vice versa, since $(\mathscr{E},\mathscr{F})$ is regular and $\mathbf{1} \in \mathscr{F}_e$.

The following proposition is due to Fitzsimmons [7].

Proposition 2.9 ([7, (2.7)]). *Assume* **(BC)**. *Then there exists a unique Borel signed measure λ on E charging no $\mathscr{E}$-polar set, such that*

$$\mathscr{E}(h,v) = -(b-a)\int_E \widetilde{v}\,d\lambda, \qquad v \in \mathscr{F}_{e,b}, \tag{2.10}$$

and λ is independent of particular choices of a,b and h. Moreover, let $\lambda_1(A) := \lambda(A \setminus F_2)$ and $\lambda_2(A) := -\lambda(A \setminus F_1)$ for $A \in \mathscr{B}(E)$. Then $\lambda_1, \lambda_2 \in S^{\mathscr{E}}$, $\lambda = \lambda_1 - \lambda_2$, $\lambda_1(E \setminus F_1) = \lambda_2(E \setminus F_2) = 0$ and $\lambda_1(F_1) = \lambda_2(F_2) = (b-a)^{-2}\mathscr{E}(h,h)$.

Note that, if λ is a Borel signed measure on E charging no $\mathscr{E}$-polar set, then so is its total variation $|\lambda|$ and hence $\int_E \widetilde{v}\,d\lambda$ for $v \in \mathscr{F}_{e,b}$ and $\lambda(A \setminus F_i)$ for $A \in \mathscr{B}(E)$, $i = 1,2$ are defined.

The proof of Proposition 2.9 given by Fitzsimmons [7, (2.7)] is based on its probabilistic counterpart shown in [3, Proof of Theorem 3.2]. We give an alternative analytic proof here.

Proof. Let $h_{0,1} \in \mathscr{F}^{0,1}$ be $(F_1 \cup F_2)$-harmonic. Then $a\mathbf{1} + (b-a)h_{0,1} (\in \mathscr{F}^{a,b})$ is also $(F_1 \cup F_2)$-harmonic and hence $\|(b-a)h_{0,1} - h\|_{\mathscr{E}} = 0$ by (2.9). Thus $\mathscr{E}(h,v) = (b-a)\mathscr{E}(h_{0,1},v)$ for $v \in \mathscr{F}_e$, and therefore it suffices to show the assertions for $h_{0,1}$ instead of h. Since $\|h_{0,1} - (0 \vee h_{0,1}) \wedge 1\|_{\mathscr{E}} = 0$ by (2.9), we may assume $0 \le h_{0,1} \le 1$ m-a.e. Let $h_{1,0} := \mathbf{1} - h_{0,1}$ and $v \in \mathscr{F}_{e,b}$. Choose $(F_1 \cup F_2)$-harmonic functions $u_{0,1} \in \mathscr{F}^{vh_{0,1}}_{F_1\cup F_2}$ and $u_{1,0} \in \mathscr{F}^{vh_{1,0}}_{F_1\cup F_2}$ so that $|u_{0,1}| \vee |u_{1,0}| \le \|v\|_{L^\infty(E,m)}$ m-a.e. (2.9) yields $\mathscr{E}(h_{0,1}, u_{1,0}h_{0,1}) = \mathscr{E}(h_{1,0}, u_{0,1}h_{1,0}) = 0$ and therefore by (2.3) and (2.9),

$$\begin{aligned}\mathscr{E}(h_{0,1},v) &= \mathscr{E}(h_{0,1},vh_{1,0})+\mathscr{E}(\mathbf{1}-h_{1,0},vh_{0,1}) = \mathscr{E}(h_{0,1},u_{1,0})-\mathscr{E}(h_{1,0},u_{0,1})\\ &= -\big(2\mathscr{E}(h_{0,1},u_{1,0}h_{0,1})-\mathscr{E}((h_{0,1})^2,u_{1,0})\big)+\big(2\mathscr{E}(h_{1,0},u_{0,1}h_{1,0})-\mathscr{E}((h_{1,0})^2,u_{0,1})\big)\\ &= -\int_E \widetilde{u_{1,0}}d\mu_{\langle h_{0,1}\rangle}+\int_E \widetilde{u_{0,1}}d\mu_{\langle h_{1,0}\rangle}. \qquad (2.11)\end{aligned}$$

It follows from (2.11) that $|\mathscr{E}(h_{0,1},v)|\le 4\|v\|_{L^\infty(E,m)}\mathscr{E}(h_{0,1},h_{0,1})$ for any $v\in\mathscr{F}_{e,b}$. Then [8, Theorem 4.2] and a time change argument as in Remark 2.5 imply the existence of a Borel signed measure λ on E charging no $\mathscr{E}$-polar set and satisfying (2.10).

Let $G\subset E$ be $\mathscr{E}$-quasi-open. By [9, Lemma 4.6.1] we can choose $u_G\in\mathscr{F}^0_{E\setminus G}$ so that $\widetilde{u_G}>0$ $\mathscr{E}$-q.e. on G. Let $u_k:=(0\vee ku_G)\wedge 1$ for $k\in\mathbb{N}$. Then $u_k\in\mathscr{F}^0_{E\setminus G}$ and $\lim_{k\to\infty}\widetilde{u_k}=\mathbf{1}_G$ $\mathscr{E}$-q.e. Now (2.10) yields $\lambda(G)=\lim_{k\to\infty}\int_E\widetilde{u_k}d\lambda=-\lim_{k\to\infty}\mathscr{E}(h_{0,1},u_k)$. Therefore the values of λ for $\mathscr{E}$-quasi-open sets are uniquely determined by the property (2.10), and the Dynkin class theorem [12, Theorem 2.1.3] implies the uniqueness of λ.

Next we prove that $\lambda_1\in S^{\mathscr{E}}$. Let $\lambda_1=\lambda_1^+-\lambda_1^-$ be the Hahn decomposition of λ_1. It suffices to show that λ_1 is a positive measure, i.e. $\lambda_1^-=0$. Since $\lambda_1|_{\mathscr{B}(F_2)}=0$ we can choose $L\in\mathscr{B}(E\setminus F_2)$ so that $\lambda_1^+(L)=\lambda_1^-(E\setminus L)=0$. Let $K\subset L$ be a closed subset of E. Since $E\setminus F_2$ and $E\setminus(F_2\cup K)$ are $\mathscr{E}$-quasi-open, by the previous paragraph there exist $\{u_k\}_{k\in\mathbb{N}},\{v_k\}_{k\in\mathbb{N}}\subset\mathscr{F}^0_{F_2}$ such that $|u_k|\vee|v_k|\le 1$ m-a.e. for $k\in\mathbb{N}$, $\lim_{k\to\infty}\widetilde{u_k}=\mathbf{1}_{E\setminus F_2}$ $\mathscr{E}$-q.e. and $\lim_{k\to\infty}\widetilde{v_k}=\mathbf{1}_{E\setminus(F_2\cup K)}$ $\mathscr{E}$-q.e. Then we easily see from (2.10) and (2.11) that $\int_E(\widetilde{u_k}-\widetilde{v_k})^+d\lambda=-\mathscr{E}(h_{0,1},(u_k-v_k)^+)\ge 0$, and letting $k\to\infty$ yields $0\le\lambda(K)=\lambda_1(K)=-\lambda_1^-(K)$, i.e. $\lambda_1^-(K)=0$. Now $\lambda_1^-(L)=\sup\{\lambda_1^-(K)\mid K\subset L, K \text{ is closed in } E\}=0$ by [5, Theorem 7.1.3] and hence $\lambda_1^-=0$. In exactly the same way we have $\lambda_2\in S^{\mathscr{E}}$, and in particular $\lambda|_{\mathscr{B}(E\setminus(F_1\cup F_2))}=0$. Therefore $\lambda=\lambda_1-\lambda_2$ and $\lambda_1(E\setminus F_1)=\lambda_2(E\setminus F_2)=0$. Finally, letting $v:=\mathbf{1}$ and $v:=h_{0,1}$ in (2.10) yields $\lambda_1(F_1)=\lambda_2(F_2)=\mathscr{E}(h_{0,1},h_{0,1})$. ■

Remark 2.10. The boundary value $\widetilde{h}=\left\{\begin{smallmatrix} a \text{ on } F_1\\ b \text{ on } F_2\end{smallmatrix}\right.$ $\mathscr{E}$-q.e. is essential in Proposition 2.9. In fact, for general $u\in\mathscr{F}_e$ and an $\mathscr{E}$-quasi-closed set $F\subset E$, there may not exist such a Borel signed measure λ on E as in (2.10) *even if $h\in\mathscr{F}^u_F$ is F-harmonic.*

A simple application of Lemma 2.4 and Proposition 2.9 yields the following fact due to Fitzsimmons [7], which is used in Section 4. See [7, Proposition 2.9] for a proof. Note that, if $u\in\mathscr{F}_e$ then by $\mu_{\langle u\rangle}\in S^{\mathscr{E}}$ we can regard $\widetilde{u}$ as a measurable map $\widetilde{u}:\big(E,\mathscr{B}(E)^{\mu_{\langle u\rangle}},\mu_{\langle u\rangle}\big)\to(\mathbb{R},\mathscr{B}(\mathbb{R}))$, and therefore the image measure $\mu_{\langle u\rangle}\circ\widetilde{u}^{-1}$ on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$ is defined.

Corollary 2.11 ([7, Proposition 2.9]). *Assume* **(BC)** *and that $(\mathscr{E},\mathscr{F})$ is strong local. Let dy denote the Lebesgue measure on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$. Then*

$$\mu_{\langle h\rangle}\circ\widetilde{h}^{-1}=\frac{2\mathscr{E}(h,h)}{b-a}\mathbf{1}_{[a,b]}dy. \qquad (2.12)$$

Now we can state and prove the main theorem of this section, which is in fact an easy consequence of the strong locality, Lemmas 2.4, 2.8 and Proposition 2.9. Note that $\mu_{\langle u,v\rangle}(E)=2\mathscr{E}(u,v)$ for $u,v\in\mathscr{F}_e$ if $(\mathscr{E},\mathscr{F})$ is recurrent, which follows by (2.3). Recall that $\varphi(h)\in\mathscr{F}_e$ for any $\varphi\in C^1(\mathbb{R})$ in the situation of **(BC)** by Lemma 2.8 and $\mathbf{1}\in\mathscr{F}_e$.

Theorem 2.12. *Assume* **(BC)** *and that* $(\mathscr{E},\mathscr{F})$ *is strong local.*
(1) Set $\kappa := 2(b-a)^{-1}\mathscr{E}(h,h)$. *Then for any* $\varphi,\psi\in C^1(\mathbb{R})$,

$$\mathscr{E}(\varphi(h),\psi(h)) = \frac{\kappa}{2}\int_a^b \varphi'(y)\psi'(y)dy. \tag{2.13}$$

(2) Let $\varphi\in C^2(\mathbb{R})$ *and* $u\in\mathscr{F}_{e,b}$. *Then*

$$\mathscr{E}(u,\varphi(h)) = (b-a)\Big(\varphi'(b)\int_{F_2}\widetilde{u}d\lambda_2 - \varphi'(a)\int_{F_1}\widetilde{u}d\lambda_1\Big) - \frac{1}{2}\int_E \widetilde{u}\varphi''(\widetilde{h})d\mu_{\langle h\rangle}. \tag{2.14}$$

Proof. (1) Since we may assume that $h\in\mathscr{F}_{e,b}$ by Lemma 2.8, it follows from Lemma 2.4 and (2.12) that for $\varphi,\psi\in C^1(\mathbb{R})$,

$$2\mathscr{E}(\varphi(h),\psi(h)) = \mu_{\langle\varphi(h),\psi(h)\rangle}(E) = \int_E \varphi'(\widetilde{h})\psi'(\widetilde{h})d\mu_{\langle h\rangle} = \kappa\int_a^b \varphi'(y)\psi'(y)dy.$$

(2) Using (2.5), Lemma 2.4 and Proposition 2.9, we have

$$\begin{aligned}
2\mathscr{E}(u,\varphi(h)) &= \mu_{\langle u,\varphi(h)\rangle}(E) = \int_E \varphi'(\widetilde{h})d\mu_{\langle u,h\rangle} \\
&= \mathscr{E}(u\varphi'(h),h) + \mathscr{E}(h\varphi'(h),u) - \mathscr{E}(uh,\varphi'(h)) \\
&= 2\mathscr{E}(\varphi'(h)u,h) - \big(\mathscr{E}(hu,\varphi'(h)) + \mathscr{E}(\varphi'(h)u,h) - \mathscr{E}(h\varphi'(h),u)\big) \\
&= -2(b-a)\int_E \varphi'(\widetilde{h})\widetilde{u}d\lambda - \int_E \widetilde{u}d\mu_{\langle h,\varphi'(h)\rangle} \\
&= 2(b-a)\Big(\varphi'(b)\int_{F_2}\widetilde{u}d\lambda_2 - \varphi'(a)\int_{F_1}\widetilde{u}d\lambda_1\Big) - \int_E \widetilde{u}\varphi''(\widetilde{h})d\mu_{\langle h\rangle},
\end{aligned}$$

proving (2.14). ∎

3 Reflecting Brownian motion arising from time change by $\mu_{\langle h\rangle}$

Throughout this section, we follow the notations introduced in the previous section. The main purpose of this section is to give the precise statement of Theorem 1.1 (2) in Theorem 3.6 and to prove it. In the first part of this section, we recall basics on the m-symmetric Hunt process corresponding to $(\mathscr{E},\mathscr{F})$ and its additive functionals. See [4, 9] for details. Let $E_\Delta := E\cup\{\Delta\}$ denote the one-point compactification of E. In what follows, the measure m is extended to $\mathscr{B}(E_\Delta)$ by setting $m(\{\Delta\}) := 0$, and a $[-\infty,\infty]$-valued function f defined ($\mathscr{E}$-q.e.) on E is always set to be 0 at Δ when needed; $f(\Delta) := 0$.

We fix an m-symmetric Hunt process $X = \big(\Omega,\mathcal{M},\{X_t\}_{t\in[0,\infty]},\{\mathbf{P}_x\}_{x\in E_\Delta}\big)$ on E with life time ζ and shift operators $\{\theta_t\}_{t\in[0,\infty]}$ whose Dirichlet form on $L^2(E,m)$ is $(\mathscr{E},\mathscr{F})$. Such X does exist by [9, Theorem 7.2.1]. Let $\mathcal{F}_* = \{\mathcal{F}_t\}_{t\in[0,\infty]}$ be the minimum completed admissible filtration as in [9, p.311], which is right-continuous by [9, Theorem A.2.1]. For each σ-finite positive Borel measure μ on E_Δ and $A\in\mathcal{F}_\infty$, the function $E_\Delta\ni x\mapsto \mathbf{P}_x[A]$ is $\mathscr{B}(E_\Delta)^\mu$-measurable, and associated with μ is a measure $\mathbf{P}_\mu$ on $(\Omega,\mathcal{F}_\infty)$ given by $\mathbf{P}_\mu[A] := \int_{E_\Delta} \mathbf{P}_x[A]d\mu(x)$. Clearly $\mathbf{P}_\mu[\Omega] = \mu(E_\Delta)$, and $\mathbf{P}_\mu$ is σ-finite since $\mathbf{P}_\mu[X_0 \in B] = \int_{E_\Delta} \mathbf{1}_B d\mu = \mu(B)$ for $B \in \mathscr{B}(E_\Delta)$. Let $\mathbf{E}_x[(\cdot)]$ and $\mathbf{E}_\mu[(\cdot)]$ denote expectations (that is, integrals on Ω) under the measures $\mathbf{P}_x$ and $\mathbf{P}_\mu$, respectively. For $(t,\omega) \in (0,\infty)\times\Omega$ let $X_{t-}(\omega) := \lim_{s\to t, s<t} X_s(\omega) \in E_\Delta$, which exists by the definition [9, p.314] of X being a Hunt process. For $B \subset E_\Delta$ and $\omega \in \Omega$, we set $\sigma_B(\omega) := \inf\{t \in (0,\infty) \mid X_t(\omega) \in B\}$, $\dot{\sigma}_B(\omega) := \inf\{t \in [0,\infty) \mid X_t(\omega) \in B\}$ and $\hat{\sigma}_B(\omega) := \inf\{t \in (0,\infty) \mid X_{t-}(\omega) \in B\}$, which are $\mathcal{F}_*$-stopping times if $B \in \mathscr{B}(E_\Delta)$ by [9, Theorem A.2.3]. We call $N \in \mathscr{B}(E)$ a *properly exceptional set for* X if and only if $m(N) = 0$ and $\mathbf{P}_x[\dot{\sigma}_N \wedge \hat{\sigma}_N = \infty] = 1$ for any $x \in E \setminus N$. By [9, Theorem 4.2.1 (ii)], every properly exceptional set for X is $\mathscr{E}$-polar, and conversely any $\mathscr{E}$-polar set is included in a Borel properly exceptional set for X by [9, Theorem 4.1.1].

Definition 3.1. (1) A family $A = \{A_t\}_{t\in[0,\infty)}$ of $[-\infty,\infty]$-valued function on Ω is called an *additive functional of* X if and only if A_t is $\mathcal{F}_t$-measurable for each $t \in [0,\infty)$ and there exist a set $\Lambda \in \mathcal{F}_\infty$ and a properly exceptional set $N \in \mathscr{B}(E)$ for X such that the following conditions **(AF1)** and **(AF2)** hold:

$$\mathbf{P}_x[\Lambda] = 1 \text{ for any } x \in E \setminus N \text{ and } \theta_t(\Lambda) \subset \Lambda \text{ for any } t \in [0,\infty). \tag{AF1}$$

For each $\omega \in \Lambda$, $t \mapsto A_t(\omega)$ is right-continuous on $[0,\infty)$, $\mathbb{R}$-valued on $[0,\zeta(\omega))$ and has finite left limits on $(0,\zeta(\omega))$, $A_0(\omega) = 0$, $A_{s+t}(\omega) = A_t(\omega) + A_s \circ \theta_t(\omega)$ for any $s,t \in [0,\infty)$, and if $\zeta(\omega) < \infty$ then $A_t(\omega) = A_{\zeta(\omega)}(\omega)$ for any $t \in [\zeta(\omega),\infty)$. **(AF2)**

We call such sets Λ and N as above a *defining set* and an *exceptional set*, respectively, *of* A.
(2) An additive functional $A = \{A_t\}_{t\in[0,\infty)}$ of X is called *positive continuous*, *finite continuous* or *finite cadlag*, respectively, if and only if we can choose a defining set Λ of A so that for each $\omega \in \Lambda$, the function $t \mapsto A_t(\omega)$ on $[0,\infty)$ is $[0,\infty]$-valued continuous, $\mathbb{R}$-valued continuous, or $\mathbb{R}$-valued right-continuous with finite left limits on $(0,\infty)$, respectively. The collection of all positive continuous additive functionals of X is denoted by $\mathbf{A}_c^+$.
(3) We call two additive functionals $A = \{A_t\}_{t\in[0,\infty)}$ and $B = \{B_t\}_{t\in[0,\infty)}$ of X *equivalent*, and write $A \sim_{\mathbf{A}} B$, if and only if $\mathbf{P}_m[A_t \neq B_t] = 0$ for any $t \in [0,\infty)$.

By [4, Lemma A.3.2 and Theorem 3.1.5], $A \sim_{\mathbf{A}} B$ if and only if there exist a defining set $\Lambda \in \mathcal{F}_\infty$ and an exceptional set $N \in \mathscr{B}(E)$, respectively, of both A and B such that $A_t(\omega) = B_t(\omega)$ for any $(t,\omega) \in [0,\infty)\times\Lambda$. Equivalent additive functionals of X are always identified henceforth, and any equality among additive functionals of X will always mean the equivalence $\sim_{\mathbf{A}}$. Let $A = \{A_t\}_{t\in[0,\infty)}$ be an additive functional of X with defining set $\Lambda \in \mathcal{F}_\infty$ and exceptional set $N \in \mathscr{B}(E)$. Let $\Lambda_0 := \Lambda \cap \{\dot{\sigma}_N \wedge \hat{\sigma}_N = \infty\}$. Then we easily see that Λ_0 is also a defining set of A and belongs to $\mathcal{F}_0$. Hence by setting $A_t|_{\Omega\setminus\Lambda_0} := 0$, we may, and *always do*, assume that *every additive functional* $A = \{A_t\}_{t\in[0,\infty)}$ *of* X *with defining set* Λ *and exceptional set* N *satisfies* $\Lambda \subset \{\dot{\sigma}_N \wedge \hat{\sigma}_N = \infty\}$, $\Lambda \in \mathcal{F}_0$ *and* $A_t|_{\Omega\setminus\Lambda} = 0$ *for* $t \in [0,\infty)$.

By [9, Theorems 5.1.3 and 5.1.4], there is a natural bijection $\mathbf{A}_c^+/\sim_{\mathbf{A}} \to S^{\mathscr{E}}$, $A \mapsto \mu_A$, called the *Revuz correspondence*; for $A = \{A_t\}_{t\in[0,\infty)} \in \mathbf{A}_c^+$, μ_A is the unique $\mathscr{E}$-smooth measure such that for any $t \in (0,\infty)$ and any $f,\eta : E \to [0,\infty]$ Borel measurable,

$$\int_E \mathbf{E}_x\Big[\int_0^t f(X_s)dA_s\Big]\eta(x)dm(x) = \int_0^t \int_E \mathbf{E}_x[\eta(X_s)]f(x)d\mu_A(x)ds. \tag{3.1}$$

μ_A is called the *Revuz measure of* $A \in \mathbf{A}_c^+$. For an additive functional $A = \{A_t\}_{t\in[0,\infty)}$ of X, its *energy* $\mathbf{e}_{\mathbf{A}}(A)$ is defined as $\mathbf{e}_{\mathbf{A}}(A) := \lim_{t\downarrow 0}(2t)^{-1}\mathbf{E}_m[A_t^2]$ whenever the limit exists.

Definition 3.2. We define the space $\mathscr{M}$ of *martingale additive functionals* and the space $\mathscr{N}_c$ of *continuous additive functionals of zero energy* by

$$\mathscr{M} := \left\{ M \;\middle|\; \begin{array}{l} M = \{M_t\}_{t\in[0,\infty)} \text{ is a finite cadlag additive functional of } X \text{ such that} \\ \mathbf{E}_x[M_t^2] < \infty \text{ and } \mathbf{E}_x[M_t] = 0 \ \mathscr{E}\text{-q.e. } x \in E \text{ for each } t \in (0,\infty) \end{array} \right\},$$

$$\mathscr{N}_c := \left\{ N \;\middle|\; \begin{array}{l} N = \{N_t\}_{t\in[0,\infty)} \text{ is a finite continuous additive functional of } X, \\ \mathbf{E}_x[|N_t|] < \infty \ \mathscr{E}\text{-q.e. } x \in E \text{ for each } t \in (0,\infty) \text{ and } \mathbf{e}_{\mathbf{A}}(N) = 0 \end{array} \right\}.$$

Let $M = \{M_t\}_{t\in[0,\infty)} \in \mathscr{M}$. Then for $\mathscr{E}$-q.e. $x \in E$, M is an $(\mathcal{F}_*, \mathbf{P}_x)$-martingale with $M_0 = 0$ $\mathbf{P}_x$-a.s. and $\mathbf{E}_x[M_t^2] < \infty$ for any $t \in [0,\infty)$. As stated in [9, p.200], there exists $\langle M\rangle = \{\langle M\rangle_t\}_{t\in[0,\infty)} \in \mathbf{A}_c^+$, unique up to the equivalence $\sim_{\mathbf{A}}$ and called the *quadratic variation of* M, such that $\mathbf{E}_x[M_t^2] = \mathbf{E}_x[\langle M\rangle_t]$, $t \in [0,\infty)$ for $\mathscr{E}$-q.e. $x \in E$. We easily see that $\langle M\rangle$ is a finite continuous additive functional of X and that $\{M_t^2 - \langle M\rangle_t\}_{t\in[0,\infty)}$ is a $(\mathcal{F}_*, \mathbf{P}_x)$-martingale for $\mathscr{E}$-q.e. $x \in E$. Letting $f = \eta = \mathbf{1}$ and $A = \langle M\rangle$ in (3.1) yields $\mathbf{e}_{\mathbf{A}}(M) = \mu_{\langle M\rangle}(E)/2$. We set $\mathscr{M}^\circ := \{M \in \mathscr{M} \mid \mathbf{e}_{\mathbf{A}}(M) < \infty\}$.

Let $u \in \mathscr{F}_e$. Then associated to u is a finite cadlag additive functional $u(X) - u(X_0)$ of X given by $u(X) - u(X_0) := \{\widetilde{u}(X_t) - \widetilde{u}(X_0)\}_{t\in[0,\infty)}$ (on a defining set), whose equivalence class under $\sim_{\mathbf{A}}$ is independent of choices of an $\mathscr{E}$-quasi-continuous m-version $\widetilde{u}$ of u. The Fukushima decomposition theorem [9, Theorem 5.2.2] asserts that there exist $M^{[u]} = \{M_t^{[u]}\}_{t\in[0,\infty)} \in \mathscr{M}^\circ$ and $N^{[u]} = \{N_t^{[u]}\}_{t\in[0,\infty)} \in \mathscr{N}_c$, unique up to the equivalence $\sim_{\mathbf{A}}$, such that $u(X) - u(X_0) = M^{[u]} + N^{[u]}$, or equivalently, for $\mathscr{E}$-q.e. $x \in E$,

$$\widetilde{u}(X_t) - \widetilde{u}(X_0) = M_t^{[u]} + N_t^{[u]} \qquad \text{for any } t \in [0,\infty), \ \mathbf{P}_x\text{-a.s.} \tag{3.2}$$

Moreover, we have $\mu_{\langle M^{[u]}\rangle} = \mu_{\langle u\rangle}$ by [9, Theorem 5.2.3] and hence $\mathbf{e}_{\mathbf{A}}(u(X) - u(X_0)) = \mathbf{e}_{\mathbf{A}}(M^{[u]}) = \mu_{\langle u\rangle}(E)/2 \le \mathscr{E}(u,u)$ by [9, (5.2.3)].

Before presenting the main theorem of this section (Theorem 3.6), we provide a lemma and a proposition which concern sample path properties of additive functionals of X and will be of independent interest. They assert that $N^{[u]}$ can change its value only on an $\mathscr{E}$-quasi-support of $\mu_{\langle u\rangle}$, or in other words only when $\langle M^{[u]}\rangle$ increases, complementing [9, Theorem 5.4.1 (i) and Lemma 5.4.2 (i)].

Lemma 3.3. *Let $u \in \mathscr{F}_e$ and $F \subset E$ be an $\mathscr{E}$-quasi-support of $\mu_{\langle u\rangle}$. Then for $\mathscr{E}$-q.e. $x \in E$,*

$$N_t^{[u]}(\omega) = 0 \quad \text{for any } t \in [0, \sigma_F(\omega)), \ \ \mathbf{P}_x\text{-a.e. } \omega \in \Omega. \tag{3.3}$$

Proof. Since F is $\mathscr{E}$-quasi-closed, by [4, Proof of Theorem 3.3.3 (i)] we can choose a properly exceptional set $N \in \mathscr{B}(E)$ for X so that $B := F \cup N \in \mathscr{B}(E)$ and B is finely closed with respect to X (see [4, 9] for details concerning fine topology). Then B is also an $\mathscr{E}$-quasi-support of $\mu_{\langle u\rangle}$. Let $u_B(x) := \mathbf{E}_x[\widetilde{u}(X_{\sigma_B})]$, $x \in E$. By [9, Theorems 4.1.3, 4.2.1 (ii), 4.6.5 and A.2.6 (i)], u_B is an B-harmonic function belonging to $\mathscr{F}_B^u$, and so is u by Lemma 2.7. Thus $\|u - u_B\|_{\mathscr{E}} = 0$ and hence [9, Theorem 5.2.4] yields $N^{[u]} = N^{[u_B]}$. Now by [9, Lemma 5.4.2 (i)], for $\mathscr{E}$-q.e. $x \in E$ we have (3.3) with B in place of F, and the result follows since $\sigma_B = \sigma_F$ $\mathbf{P}_x$-a.s. for $x \in E \setminus N$ and hence for $\mathscr{E}$-q.e. $x \in E$. ■

Proposition 3.4. *Let $u \in \mathscr{F}_e$. Then for $\mathscr{E}$-q.e. $x \in E$,*

$$N_s^{[u]}(\omega) = N_t^{[u]}(\omega) \text{ for any } s,t \in [0,\infty) \text{ with } \langle M^{[u]}\rangle_s(\omega) = \langle M^{[u]}\rangle_t(\omega), \ \mathbf{P}_x\text{-a.e. } \omega \in \Omega. \tag{3.4}$$

Proof. Choose $\Lambda \in \mathcal{F}_0$ and $N \in \mathscr{B}(E)$ so that they are respectively a defining set and an exceptional set of both $N^{[u]}$ and $\langle M^{[u]} \rangle$. Then under our convention, $\Lambda_0 := \Lambda \cup \{\zeta = 0\}$ is also a defining set of them (note that either $\{\zeta = 0\} \subset A$ or $\{\zeta = 0\} \cap A = \emptyset$ for each $A \in \mathcal{F}_\infty$). Let $R(\omega) := \inf\{t \in [0,\infty) \mid \langle M^{[u]} \rangle_t(\omega) > 0\}$ for $\omega \in \Omega$ and set $B := \{x \in E \mid \mathbf{P}_x[R = 0] = 1\}$. B is an $\mathscr{E}$-quasi-support of the Revuz measure $\mu_{\langle u \rangle}$ of $\langle M^{[u]} \rangle$ by [9, Theorem 5.1.5]. Clearly, we can choose $F \in \mathscr{B}(E)$ so that $F \subset B$ and $B \setminus F$ is $\mathscr{E}$-polar, and then F is also an $\mathscr{E}$-quasi-support of $\mu_{\langle u \rangle}$. Since $\sigma_F = \sigma_B$ $\mathbf{P}_x$-a.s. for $\mathscr{E}$-q.e. $x \in E$, [9, Lemma 5.1.11] and Lemma 3.3 imply that $\mathbf{P}_x[R = \sigma_F] = 1$ and (3.3) hold for any $x \in E \setminus N_0$ for some properly exceptional set $N_0 \in \mathscr{B}(E)$ for X with $N \subset N_0$. Let $\Omega_{(3.3)}$ be the event in (3.3) and $\Omega_0 := \{R = \sigma_F\} \cap \Omega_{(3.3)} \cap \Lambda_0$, which belongs to $\mathcal{F}_\infty$ by virtue of [6, Chapter III, 13 and 33] and satisfies $\mathbf{P}_x[\Omega_0] = 1$ for any $x \in E \setminus N_0$. For $\omega \in \Omega_0$, by (3.3), the definition of $R(\omega)$ $(= \sigma_F(\omega))$ and the continuity of $N^{[u]}_{(\cdot)}(\omega) : [0,\infty) \to \mathbb{R}$ we see that $N^{[u]}_t(\omega) = 0$ for any $t \in [0,\infty)$ with $\langle M^{[u]} \rangle_t(\omega) = 0$. Now we set $\Omega^{[u]} := \bigcap_{t \in \mathbb{Q} \cap [0,\infty)} \theta_t^{-1}(\Omega_0)$. Since $\{\zeta = 0\} \subset \Lambda_0$ and $N^{[u]} = \langle M^{[u]} \rangle = 0$ on $\{\zeta = 0\}$, we have $\{\zeta = 0\} \subset \Omega_0$ and hence $\mathbf{P}_\Delta[\Omega_0] = 1$. Therefore for $x \in E \setminus N_0$, the Markov property of X ([4, Theorem A.1.21], [18, Exercise IV.1.9 (v)]) yields $\mathbf{P}_x[\theta_t^{-1}(\Omega_0)] = \mathbf{E}_x[\mathbf{P}_{X_t}[\Omega_0]] = 1$ for any $t \in [0,\infty)$ and hence $\mathbf{P}_x[\Omega^{[u]}] = 1$. We can verify (3.4) for $\omega \in \Omega^{[u]}$ by using **(AF2)** for $N^{[u]}$ and $\langle M^{[u]} \rangle$, and the proof is complete. ∎

Turning to the situation of **(BC)**, Propositions 2.9 and 3.4 result in the next proposition.

Proposition 3.5. *Assume* **(BC)**, *and for $i = 1,2$ let $A^i = \{A^i_t\}_{t \in [0,\infty)} \in \mathbf{A}_c^+$ be the positive continuous additive functional of X with Revuz measure λ_i, where $\lambda_i \in S^{\mathscr{E}}$ is as in Proposition 2.9. Suppose that $(\mathscr{E},\mathscr{F})$ is strong local and that $\sigma_{F_1} \vee \sigma_{F_2} < \infty$ $\mathbf{P}_m$-a.s. Then there exist $\Lambda \in \mathcal{F}_0$ and $N \in \mathscr{B}(E)$ which are respectively a defining set and an exceptional set of the five additive functionals $h(X) - h(X_0)$, $M^{[h]}$, A^1, A^2 and $\langle M^{[h]} \rangle$, such that the following conditions are valid:*
(i) *Let $x \in E \setminus N$. Then $\mathbf{E}_x[(M^{[h]}_t)^2] = \mathbf{E}_x[\langle M^{[h]} \rangle_t] < \infty$ and $\mathbf{E}_x[M^{[h]}_t] = 0$ for any $t \in [0,\infty)$.*
(ii) *Let $\omega \in \Lambda$. Then $\zeta(\omega) = \lim_{t \to \infty} \langle M^{[h]} \rangle_t(\omega) = \infty$, $\widetilde{h}(X_{(\cdot)}(\omega))$ is $[a,b]$-valued and $M^{[h]}_{(\cdot)}(\omega)$ is continuous. If $s,t \in [0,\infty)$ and $\langle M^{[h]} \rangle_s(\omega) = \langle M^{[h]} \rangle_t(\omega)$ then $M^{[h]}_s(\omega) = M^{[h]}_t(\omega)$ and $A^i_s(\omega) = A^i_t(\omega)$, $i = 1,2$. Also, $\widetilde{h}(X_t(\omega)) - \widetilde{h}(X_0(\omega)) = M^{[h]}_t(\omega) + (b-a)(A^1_t(\omega) - A^2_t(\omega))$ for any $t \in [0,\infty)$ and $\int_0^\infty \mathbf{1}_{(a,b]}(\widetilde{h}(X_s(\omega))dA^1_s(\omega) = \int_0^\infty \mathbf{1}_{[a,b)}(\widetilde{h}(X_s(\omega))dA^2_s(\omega) = 0$.*

Note that, on account of [9, Theorem 4.6.6 (ii)], the condition that $\sigma_{F_1} \vee \sigma_{F_2} < \infty$ $\mathbf{P}_m$-a.s. is satisfied if $(\mathscr{E},\mathscr{F})$ is irreducible and $\mathrm{Cap}_{\mathscr{E}}(F_1)\mathrm{Cap}_{\mathscr{E}}(F_2) > 0$ in addition to **(BC)**.

The proof of Proposition 3.5 is postponed until the end of this section. Using Proposition 3.5, now we can state and prove the main theorem of this section. Recall that, under our convention, $\langle M^{[h]} \rangle$ is set to be 0 on $\Omega \setminus \Lambda$ in the situation of Proposition 3.5.

Theorem 3.6. *Assume* **(BC)**. *Suppose that $(\mathscr{E},\mathscr{F})$ is strong local and that $\sigma_{F_1} \vee \sigma_{F_2} < \infty$ $\mathbf{P}_m$-a.s. Let $A^1, A^2 \in \mathbf{A}_c^+$, $\Lambda \in \mathcal{F}_0$ and $N \in \mathscr{B}(E)$ be as in Proposition 3.5. Define $\{\tau_t\}_{t \in [0,\infty)}$, $\mathcal{F}^h_* = \{\mathcal{F}^h_t\}_{t \in [0,\infty)}$, $X^h = \{X^h_t\}_{t \in [0,\infty)}$, $B^h = \{B^h_t\}_{t \in [0,\infty)}$, $L^a = \{L^a_t\}_{t \in [0,\infty)}$ and $L^b = \{L^b_t\}_{t \in [0,\infty)}$ by $\tau_t := \inf\{s \in [0,\infty) \mid \langle M^{[h]} \rangle_s > t\}$ on Ω* (note that τ_t is an $\mathcal{F}_*$-stopping time), *$\mathcal{F}^h_t := \mathcal{F}_{\tau_t}$, $X^h_t := a$ on $\Omega \setminus \Lambda$, $B^h_t = L^a_t = L^b_t := 0$ on $\Omega \setminus \Lambda$ and*

$$X^h_t := \widetilde{h}(X_{\tau_t}), \quad B^h_t := M^{[h]}_{\tau_t}, \quad L^a_t := (b-a)A^1_{\tau_t}, \quad L^b_t := (b-a)A^2_{\tau_t} \quad \text{on } \Lambda. \tag{3.5}$$

Then X^h, B^h, L^a, L^b are $\mathbb{R}$-valued $\mathcal{F}^h_*$-adapted continuous processes with X^h $[a,b]$-valued, L^a, L^b non-decreasing and $B^h_0 = L^a_0 = L^b_0 = 0$ on Ω. Moreover, $(B^h, \mathcal{F}^h_*)$ is a one-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}_\infty, \mathbf{P}_x)$ for each $x \in E \setminus N$, and for any $\omega \in \Lambda$,

$$X^h_t(\omega) = \widetilde{h}(X_0(\omega)) + B^h_t(\omega) + L^a_t(\omega) - L^b_t(\omega), \quad t \in [0,\infty), \tag{3.6}$$

$$\int_0^\infty \mathbf{1}_{(a,b]}(X^h_s(\omega))dL^a_s(\omega) = \int_0^\infty \mathbf{1}_{[a,b)}(X^h_s(\omega))dL^b_s(\omega) = 0. \tag{3.7}$$

In particular, for $x \in E \setminus N$, X^h is the reflecting Brownian motion on $[a,b]$ started at $\widetilde{h}(x)$ with local times L^a at a and L^b at b, driven by the Brownian motion $(B^h, \mathcal{F}^h_*)$ on $(\Omega, \mathcal{F}_\infty, \mathbf{P}_x)$.

Proof. Note that $\tau_{(\cdot)}(\omega) : [0,\infty) \to [0,\infty]$ is right-continuous non-decreasing for any $\omega \in \Omega$ and hence that $\mathcal{F}^h_*$ is right-continuous by [12, Problem 1.2.23]. For $\omega \in \Lambda$, since $\langle M^{[h]}\rangle_s(\omega)$ is finite, continuous in s and tends to ∞ as $s \to \infty$, we have $\tau_t(\omega) < \infty$ and $\langle M^{[h]}\rangle_{\tau_t(\omega)}(\omega) = t$ for $t \in [0,\infty)$ and $\lim_{t\to\infty} \tau_t(\omega) = \infty$. Therefore X^h, B^h, L^a, L^b can be defined by (3.5) and are $\mathbb{R}$-valued $\mathcal{F}^h_*$-adapted right-continuous processes with X^h $[a,b]$-valued and L^a, L^b non-decreasing. Now the other assertions are immediate from Proposition 3.5 **(ii)** and [12, Theorem 3.4.6]; (3.7) follows since we have $\int_0^\infty \varphi(\tau_s(\omega))dL^a_s(\omega) = \int_0^\infty \varphi(s)dA^1_s(\omega)$ and $\int_0^\infty \varphi(\tau_s(\omega))dL^b_s(\omega) = \int_0^\infty \varphi(s)dA^2_s(\omega)$ for $\omega \in \Lambda$ and any Borel measurable $\varphi : [0,\infty) \to [0,\infty]$ by Proposition 3.5 **(ii)** and the Dynkin class theorem [12, Theorem 2.1.3]. ■

We prove Proposition 3.5 in the rest of this section. The following lemma is required.

Lemma 3.7. *(1) Let $F \subset E$ be $\mathscr{E}$-quasi-closed and satsify $\sigma_F < \infty$ $\mathbf{P}_m$-a.s. If $u \in \mathscr{F}^0_F$ and $\|u\|_{\mathscr{E}} = 0$ then $u = 0$.*
(2) Assume **(BC)**. *Suppose that $\sigma_{F_1 \cup F_2} < \infty$ $\mathbf{P}_m$-a.s. Then $a \leq \widetilde{h} \leq b$ $\mathscr{E}$-q.e.*

Proof. (1) $N^{[u]} = 0$ by [9, Theorem 5.2.4], and $\mu_{\langle u\rangle} = 0$, which is the Revuz measure of $\langle M^{[u]}\rangle$. Thus $\langle M^{[u]}\rangle = 0$, hence $M^{[u]} = 0$ and it follows that $\widetilde{u}(X_t) = \widetilde{u}(X_0)$ for any $t \in [0,\infty)$, $\mathbf{P}_m$-a.s. Let $B := F \cap \widetilde{u}^{-1}(0)$. Then since $u \in \mathscr{F}^0_F$, $F \setminus B$ is $\mathscr{E}$-polar and hence $\sigma_B = \sigma_F < \infty$ $\mathbf{P}_m$-a.s. The right-continuity of $\widetilde{u}(X_{(\cdot)})$ on a defining set of $u(X) - u(X_0)$ yields $\widetilde{u}(X_{\sigma_B}) = 0$ $\mathbf{P}_m$-a.s. and therefore $\widetilde{u}(X_0) = \widetilde{u}(X_{\sigma_B}) = 0$ $\mathbf{P}_m$-a.s. Hence $m(\widetilde{u}^{-1}(0)) = \mathbf{P}_m[\widetilde{u}(X_0) = 0] = 0$.
(2) $h - (a \vee h) \wedge b \in \mathscr{F}^0_{F_1 \cup F_2}$ since $\mathbf{1} \in \mathscr{F}_e$ by the recurrence, and $\|h - (a \vee h) \wedge b\|_{\mathscr{E}} = 0$ by (2.9). Therefore (1) yields $h = (a \vee h) \wedge b$, i.e. $a \leq h \leq b$ m-a.e. Thus the result follows. ■

Proof of Proposition 3.5. For $\mathscr{E}$-q.e. $x \in E$, $\mathbf{P}_x[\zeta = \infty] = 1$ by [9, Lemma 1.6.5 and Problem 4.5.1] and $\int_0^\infty \mathbf{1}_{\mathbb{R}\setminus\{a\}}(\widetilde{h}(X_s))dA^1_s = \int_0^\infty \mathbf{1}_{\mathbb{R}\setminus\{b\}}(\widetilde{h}(X_s))dA^2_s = 0$ $\mathbf{P}_x$-a.s. by $\lambda_i(E \setminus F_i) = 0$, $i = 1,2$, $\widetilde{h}|_{F_1 \cup F_2} = a\mathbf{1}_{F_1} + b\mathbf{1}_{F_2}$ $\mathscr{E}$-q.e. and [9, Lemma 5.1.11 and Theorem 5.1.5]. We also have $N^{[h]} = (b-a)(A^1 - A^2)$ by virtue of Proposition 2.9 and [9, Theorem 5.4.2].

Note that $M^{[h]}$ is a finite continuous additive functional of X by [9, Lemma 5.5.1 (ii)] and the strong locality of $(\mathscr{E}, \mathscr{F})$. Noting also Lemma 3.7 (2), we can choose a defining set $\Lambda_0 \in \mathcal{F}_0$ and an exceptional set $N_0 \in \mathscr{B}(E)$ of the five additive functionals $h(X) - h(X_0)$, $M^{[h]}$, A^1, A^2 and $\langle M^{[h]}\rangle$ so that $M^{[h]}_{(\cdot)}(\omega)$ is $\mathbb{R}$-valued continuous for each $\omega \in \Lambda_0$, $\widetilde{h}|_{E\setminus N_0}$ is $[a,b]$-valued and **(i)** of the statement holds with N_0 in place of N. Let Λ_1 be the set of $\omega \in \Lambda_0$ possessing all the properties in **(ii)** of the statement except $\lim_{t\to\infty}\langle M^{[h]}\rangle_t = \infty$. Then clearly $\theta_t(\Lambda_1) \subset \Lambda_1$ for any $t \in [0,\infty)$, and [6, Chapter III, 13 and 33] yields $\Lambda_1 \in \mathcal{F}_\infty$.

Moreover, $\mathbf{P}_x[\Lambda_1] = 1$ for $\mathscr{E}$-q.e. $x \in E \setminus N_0$ by the previous paragraph, Proposition 3.4 and [12, p.175, (4.18)]; note that $M^{[h]}$ is a continuous $(\mathcal{F}_*, \mathbf{P}_x)$-martingale with quadratic variation $\langle M^{[h]} \rangle$ for each $x \in E \setminus N_0$ by **(i)**.

We prove that $\mathbf{P}_x[\lim_{t\to\infty} \langle M^{[h]} \rangle_t = \infty] = 1$ for $\mathscr{E}$-q.e. $x \in E \setminus N_0$. The following short proof of this fact is due to the referee. Let $Z := \lim_{t\to\infty} \langle M^{[h]} \rangle_t$ and $f(x) := 1 - \mathbf{E}_x[e^{-Z}]$, $x \in E \setminus N_0$. **(AF2)** of $\langle M^{[h]} \rangle$ implies that $Z = \langle M^{[h]} \rangle_t + Z \circ \theta_t$ on Λ_0 for $t \in [0,\infty)$ and that f is excessive for the restriction $X|_{E\setminus N_0}$ of X to $E \setminus N_0$, which is defined by [9, (A.2.23)] and is an m-symmetric Hunt process on $E \setminus N_0$ by [9, Theorem A.2.8]. By the recurrence of $(\mathscr{E}, \mathscr{F})$ and [4, Lemma 3.5.5 (i)] (note also [9, Theorem 4.2.1 (ii)]), for $\mathscr{E}$-q.e. $x \in E \setminus N_0$, $\mathbf{E}_x[f(X_t)] = f(x)$ i.e. $\mathbf{E}_x[(e^{\langle M^{[h]} \rangle_t} - 1)e^{-Z}] = 0$ for $t \in [0,\infty)$ and hence $\mathbf{P}_x[Z \in \{0,\infty\}] = 1$. Set $\sigma := \sigma_{\widetilde{h}^{-1}(a)} \vee \sigma_{\widetilde{h}^{-1}(b)}$. On $\Lambda_1 \cap \{Z = 0\}$, $M_t^{[h]} = A_t^1 = A_t^2 = 0$, hence $\widetilde{h}(X_t) = \widetilde{h}(X_0)$ for $t \in [0,\infty)$ and therefore $\sigma = \infty$. Now since $\sigma < \infty$, $\mathbf{P}_m$-a.s. and hence $\mathbf{P}_x$-a.s. for $\mathscr{E}$-q.e. $x \in E$ by [4, Lemma A.2.4 (ii)] and [9, Theorem A.2.7, Theorem 4.6.1 (ii) and Lemma 2.1.4], it follows for $\mathscr{E}$-q.e. $x \in E$ that $\mathbf{P}_x[Z = 0] = \mathbf{P}_x[\Lambda_1 \cap \{Z = 0\}] = 0$ and hence $\mathbf{P}_x[Z = \infty] = 1$.

The proof is completed by choosing a properly exceptional set $N \in \mathscr{B}(E)$ for X with $N_0 \subset N$ so that $\mathbf{P}_x[\Lambda_1 \cap \{Z = \infty\}] = 1$ for any $x \in E \setminus N$ and setting $\Lambda := \Lambda_1 \cap \{Z = \infty\} \cap \{\dot{\sigma}_N \wedge \hat{\sigma}_N = \infty\}$. ■

Remark 3.8. Instead of appealing to a general result (Proposition 3.4), we could adopt the following more direct proof of Theorem 3.6, which is due to the referee: Since $h(X) - h(X_0) = M^{[h]} + (b-a)(A^1 - A^2)$ and $\int_0^\infty \mathbf{1}_{(a,b]}(\widetilde{h}(X_s))dA_s^1 = \int_0^\infty \mathbf{1}_{[a,b)}(\widetilde{h}(X_s))dA_s^2 = 0$, the triple $(\widetilde{h}(X), (b-a)A^1, (b-a)A^2)$ pathwisely solves the Skorohod equation for $(M^{[h]}, \widetilde{h}(X_0))$ on $[a,b]$. From this fact, similarly to the above proof we can prove $\mathbf{P}_x[\lim_{t\to\infty} \langle M^{[h]} \rangle_t = \infty] = 1$ for $\mathscr{E}$-q.e. $x \in E$. Moreover, since solutions for the Skorohod equation are invariant under continuous reparametrization, from $M_t^{[h]} = B^h_{\langle M^{[h]} \rangle_t}$ we can conclude that (X^h, L^a, L^b) is pathwisely the unique solution to the Skorohod equation for $(B^h, \widetilde{h}(X_0))$ on $[a,b]$.

4 Example: the harmonic Sierpinski gasket

In this section, we briefly discuss an application of Theorems 2.12 and 3.6 to short time asymptotic analysis of the heat kernel on a fractal called the harmonic Sierpinski gasket. Let $V_0 = \{q_1, q_2, q_3\} \subset \mathbb{R}^2$ be the set of the three vertices of an equilateral triangle, and for $i \in \{1,2,3\} =: S$ define $F_i : \mathbb{R}^2 \to \mathbb{R}^2$ by $F_i(x) := (x + q_i)/2$. The *Sierpinski gasket* (Figure 4.1) is defined as the unique non-empty compact set $K \subset \mathbb{R}^2$ that satisfies $K = \bigcup_{i\in S} F_i(K)$. As studied in [1, 14, 20], a standard resistance form $(\mathscr{E}, \mathscr{F})$ is defined on the Sierpinski gasket K and its resistance metric is compatible with the original (Euclidean) topology of K. Therefore $\mathscr{F} \subset C(K)$, and for any finite positive Borel measure m on K with full support, $(\mathscr{E}, \mathscr{F})$ is an irreducible recurrent strong local regular Dirichlet form on $L^2(K,m)$ with jointly continuous heat kernel $p_m = p_m(t,x,y) : (0,\infty) \times K \times K \to (0,\infty)$. (See [14, Chapter 2] and [16, Part I] for basic theory of resistance forms.) Moreover, its extended Dirichlet space is equal to $\mathscr{F}$, the empty set $\emptyset$ is the only $\mathscr{E}$-polar set and every $\mathscr{E}$-quasi-open set is open, independently of the reference measure m. Therefore harmonic functions with respect to $(\mathscr{E}, \mathscr{F})$ are defined as in Definition 2.6, also independently of m.

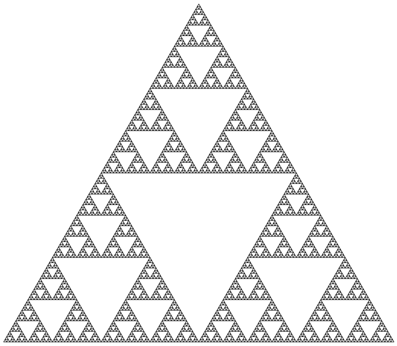

Figure 4.1: *The Sierpinski gasket*

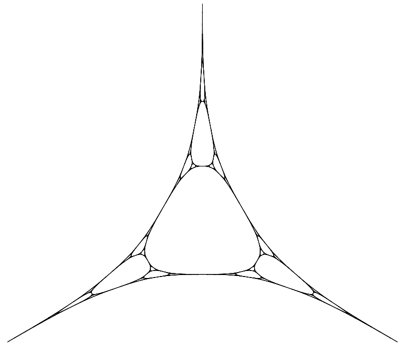

Figure 4.2: *The harmonic Sierpinski gasket*

Let $h_1, h_2 \in \mathscr{F}$ be V_0-harmonic functions with $h_1(q_1) = h_2(q_1) = 0$, $h_1(q_2) = h_1(q_3) = 1$ and $h_2(q_2) = -h_2(q_3) = 1/\sqrt{3}$, so that $\mathscr{E}(h_1,h_1) = \mathscr{E}(h_2,h_2) > 0$ and $\mathscr{E}(h_1,h_2) = 0$. By multiplying $\mathscr{E}$ by a constant, we assume that $2\mathscr{E}(h_1,h_1) = 2\mathscr{E}(h_2,h_2) = 1$. By [13, Theorem 3.6], the continuous map $\Phi : K \to \mathbb{R}^2$ defined by $\Phi(x) := (h_1(x), h_2(x))$ is injective and hence a homeomorphism from K onto its image $K_H := \Phi(K)$, which is called the *harmonic Sierpinski gasket* (Figure 4.2). Note that then we may regard $\mathscr{F}$ as a linear subspace of $C(K_H)$ and hence $(\mathscr{E},\mathscr{F})$ as a resistance form on K_H.

Kusuoka [17] and Kigami [13, 15] have studied the properties of the Dirichlet form $(\mathscr{E},\mathscr{F})$ using the *Kusuoka measure* $\mu := \mu_{\langle h_1 \rangle} + \mu_{\langle h_2 \rangle}$ as the reference measure of the form $(\mathscr{E},\mathscr{F})$; note that the $\mathscr{E}$-energy measures $\mu_{\langle h_1 \rangle}$ and $\mu_{\langle h_2 \rangle}$ have full supports since h_1 and h_2 are non-constant on any non-empty open subset of K_H by [14, Example 3.2.6]. Their results suggest that analytic properties of the Dirichlet space $(K_H, \mu, \mathscr{E}, \mathscr{F})$ are closely related with the geometry of K_H. For example, for $x, y \in K_H$, define

$$d_H(x,y) := \inf\{|\gamma|_{\mathrm{Euc}} \mid \gamma : [0,1] \to K_H,\ \gamma \text{ is continuous},\ \gamma(0) = x,\ \gamma(1) = y\}, \tag{4.1}$$

where $|\gamma|_{\mathrm{Euc}}$ denotes the length of γ with respect to the Euclidean metric. Then d_H is a metric on K_H compatible with the original topology of K_H, and Kigami [15, Theorem 6.3] has shown that the jointly continuous heat kernel p_μ associated with the Dirichlet space $(K_H, \mu, \mathscr{E}, \mathscr{F})$ is subject to the two-sided *Gaussian* bound: for $(t,x,y) \in (0,\infty) \times K_H \times K_H$,

$$\frac{c_1}{\mu\big(B_{\sqrt{t}}(x,d_H)\big)} \exp\Big(-c_2 \frac{d_H(x,y)^2}{t}\Big) \le p_\mu(t,x,y) \le \frac{c_3}{\mu\big(B_{\sqrt{t}}(x,d_H)\big)} \exp\Big(-c_4 \frac{d_H(x,y)^2}{t}\Big), \tag{4.2}$$

where $B_r(x,d_H) := \{y \in K_H \mid d_H(x,y) < r\}$ and $c_1, c_2, c_3, c_4 \in (0,\infty)$.

We are interested in asymptotic behaviors of this heat kernel $p_\mu(t,x,y)$ as $t \downarrow 0$. Let $o := (0,0) = \Phi(q_1) \in \Phi(V_0)$. In view of the picture of K_H (Figure 4.2), around the point o, K_H looks very much like a one-dimensional interval. From this observation, it is natural to expect that the behavior of $p_\mu(t,o,o)$ as $t \downarrow 0$ will be similar to that of the transition density of Brownian motion. This is in fact the case, as we shall see below.

First, we can prove the following theorem based on our main results and [7, Proposition 2.9] (Corollary 2.11). Recall that $p_{\mu_{\langle h_1 \rangle}}$ is the jointly continuous heat kernel associated with the Dirichlet space $(K_H, \mu_{\langle h_1 \rangle}, \mathscr{E}, \mathscr{F})$ (note that here the reference measure is $\mu_{\langle h_1 \rangle}$, not μ).

Theorem 4.1. *Let* $p_{[0,1]} : (0,\infty) \times [0,1] \times [0,1] \to (0,\infty)$ *denote the transition density of the reflecting Brownian motion on* $[0,1]$. *Then*

$$p_{\mu_{\langle h_1 \rangle}}(t,o,x) = p_{[0,1]}(t,0,h_1(x)), \qquad (t,x) \in (0,\infty) \times K_H. \tag{4.3}$$

Note that $h_1 : K_H \to [0,1]$ is the projection on the real axis and that $h_1(o) = 0$.

The proof of Theorem 4.1 is given in the end of this section. On the other hand, essentially as a consequence of the following estimate on the decay of the measure $\mu_{\langle h_2 \rangle}$

$$\frac{1}{15}\varepsilon^{\beta} \le \frac{\mu_{\langle h_2 \rangle}\bigl(h_1^{-1}([0,\varepsilon))\bigr)}{\mu_{\langle h_1 \rangle}\bigl(h_1^{-1}([0,\varepsilon))\bigr)} \le 15\varepsilon^{\beta}, \qquad \varepsilon \in (0,1], \tag{4.4}$$

where $\beta := 2\log_{5/3} 3 = 4.30132\ldots$, we can verify

$$p_{\mu}(t,o,o) - p_{\mu_{\langle h_1 \rangle}}(t,o,o) = O(t^{(\beta-1)/2}) \qquad \text{as } t \downarrow 0; \tag{4.5}$$

note that (4.4) reflects our observation that the shape of K_H around o resembles that of a one-dimensional interval.

Since $p_{[0,1]}(t,0,0) = 2/\sqrt{2\pi t} + O(\exp(-ct^{-1}))$ as $t \downarrow 0$ for some $c \in (0,\infty)$, from Theorem 4.1 and (4.5) we conclude that

$$p_{\mu}(t,o,o) = \frac{1}{\sqrt{2\pi t}}\bigl(2 + O(t^{\beta/2})\bigr) \qquad \text{as } t \downarrow 0. \tag{4.6}$$

Furthermore, we remark that the same result is true at any $x \in \Phi(V_*)$, where inductively $V_n := \bigcup_{i \in S} F_i(V_{n-1})$ for $n \in \mathbb{N}$ and $V_* := \bigcup_{n \in \mathbb{N}} V_n$; there exists $\xi_x \in (0,\infty)$ such that

$$p_{\mu}(t,x,x) = \frac{1}{\xi_x\sqrt{2\pi t}}\bigl(1 + O(t^{\beta/2})\bigr) \qquad \text{as } t \downarrow 0. \tag{4.7}$$

We do not go into the details of (4.5), (4.6) and (4.7) here. The proofs of these results, along with much more detailed information on the asymptotics of p_μ, will be treated in a forthcoming paper [11] by the author.

Now we close this paper with the proof of Theorem 4.1.

Proof of Theorem 4.1. Let $X^1 = \bigl(\Omega, \mathcal{M}, \{X_t^1\}_{t\in[0,\infty]}, \{\mathbf{P}_x\}_{x \in (K_H)_\Delta}\bigr)$ be a $\mu_{\langle h_1 \rangle}$-symmetric diffusion on K_H whose Dirichlet form on $L^2(K_H, \mu_{\langle h_1 \rangle})$ is $(\mathscr{E}, \mathscr{F})$. Then since the empty set $\emptyset$ is the only $\mathscr{E}$-polar set and $\langle M^{[h_1]} \rangle = \{t\}_{t \in [0,\infty)}$ as additive functionals of X^1 by (3.1), Theorem 3.6 implies that $\{h_1(X_t^1)\}_{t\in[0,\infty)}$ is a reflecting Brownian motion started at $h_1(x)$ under $\mathbf{P}_x$ for any $x \in K_H$. Let $t \in (0,\infty)$ and $x \in K_H$. Then $\mathbf{P}_x[X_t^1 \in dy] = p_{\mu_{\langle h_1 \rangle}}(t,x,y)d\mu_{\langle h_1 \rangle}(y)$ by [16, Theorem 10.4], and therefore by the symmetry of $p_{\mu_{\langle h_1 \rangle}}$ and $p_{[0,1]}$ we have

$$\int_{K_H} p_{\mu_{\langle h_1 \rangle}}(t,y,x) f(h_1(y))d\mu_{\langle h_1 \rangle}(y) = \mathbf{E}_x[f(h_1(X_t^1))] = \int_0^1 p_{[0,1]}(t,y,h_1(x))f(y)dy \tag{4.8}$$

for any $f \in L^2([0,1],dy)$. Note that $\mu_{\langle h_1 \rangle}\bigl(h_1^{-1}([0,\varepsilon))\bigr) = \varepsilon$ for $\varepsilon \in (0,1]$ by Corollary 2.11 and that $\{h_1^{-1}([0,\varepsilon))\}_{\varepsilon \in (0,1]}$ is a fundamental system of neighborhoods of $o \in K_H$ by $h_1^{-1}(0) = \{o\}$ and [14, Theorem 3.2.14]. Therefore letting $f := \varepsilon^{-1}\mathbf{1}_{[0,\varepsilon)}$ in (4.8) and then $\varepsilon \downarrow 0$ result in (4.3) by the continuity of $p_{\mu_{\langle h_1 \rangle}}$ and $p_{[0,1]}$. ■

Remark 4.2. We can prove (4.8) also in an analytic way by using Theorem 2.12 (2), as follows: Let $\mathscr{L}_1$ be the non-positive self-adjoint operator on $L^2(K_H, \mu_{\langle h_1 \rangle})$ associated with $(K_H, \mu_{\langle h_1 \rangle}, \mathscr{E}, \mathscr{F})$. By Corollary 2.11 and Theorem 2.12 (2), $\{\mathbf{1}\} \cup \{\sqrt{2}\cos(n\pi h_1)\}_{n \in \mathbb{N}}$ is an orthonormal system in $L^2(K_H, \mu_{\langle h_1 \rangle})$ consisting of eigenfunctions of $-\mathscr{L}_1$ with eigenvalues 0 and $n^2\pi^2/2$ respectively. Then by using this fact, we can calculate the integral $\int_{K_H} p_{\mu_{\langle h_1 \rangle}}(t,y,x) f(h_1(y)) d\mu_{\langle h_1 \rangle}(y)$ for $f \in L^2([0,1],dy)$ to verify (4.8); see [11, Proof of Proposition 4.9] for details.

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Note added in proof. After this paper was accepted for publication, Professor Zoran Vondraček informed the author of his paper [21], for which the author would like to thank him. In fact, one of his main theorems, [21, Theorem 2.2], is very similar to our Theorem 3.6. (Note that [21, Theorem 2.2] is valid without the condition that *"$u^{-1}(\{a\})$ has empty fine interior for any $a \in \mathbb{R}$"*, in view of Remark 3.8 above.) Theorem 3.6 can be considered as a natural adaptation of [21, Theorem 2.2] in the framework of a recurrent local Dirichlet space, and Theorem 2.12 provides the analytic counterpart of those results in this situation.

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