Laplace transform as Hilbert-Schmidt operators on reproducing kernel Hilbert spaces

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Abstract

We propose a new functional-analytic framework in which the Laplace transform can be regarded as a Hilbert-Schmidt operator, by using a class of reproducing kernel Hilbert spaces. We can thus treat the Laplace transform as a compact operator, and the proposed framework and results are applicable to mathematical and numerical analysis of the Laplace transform. Results of several numerical experiments of the real inversion for the Laplace transform are also shown.

Key words. Laplace transform, reproducing kernel Hilbert spaces, real inversion

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1 Introduction

In the present research, we shall propose a new approach to analysis of the Laplace transform

$$\mathcal{L} f(p) = \int_0^\infty e^{-pt} f(t)\, dt,$$

that plays a key role in various fields of mathematics, science and engineering. In particular, finding $f = \mathcal{L}^{-1} F$ for a given function $F(p), p > 0$ appears in some areas. This is called the real inversion of the Laplace transform. Elementarily we look up the Laplace transform table for this purpose, and in applications, numerical computation of the real inversion is required. Here we should keep in mind that $\mathcal{L}^{-1}$ is not stable in standard function spaces and consequently numerical methods for the real inversion have not been established [4, 1].

In [7] Saitoh has considered an operator L from a certain reproducing kernel Hilbert space to $L^2((0,\infty), dp)$ given by

$$L f(p) := p \cdot \mathcal{L} f(p),$$

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and Fujiwara [2] has succeeded in numerical computation of the real inversion. One of their key ideas is the compactness of L in their settings, which yields traditional inverse analysis such as Tikhonov regularization [5] and singular value decomposition [3]. Another remarkable point is that they obtain concrete representation of the adjoint operator in the form of an integral transform based on their reproducing kernel Hilbert space for the sake of effective numerical computation [2].

Their numerical methods are simple and remarkable though it has a failure that it is applicable to only the case where image functions are square integrable (with respect to the Lebesgue measure). For example, their method does not work for the Laplace transforms of polynomials, which are not square integrable. To overcome the limitation, in this paper we treat the operator L on weighted Hilbert spaces and prove that L is a Hilbert-Schmidt operator, which gives fruitful information for not only L but also $\mathcal{L}$. Our method is applicable to functions in a wider class by choosing weights of both its domain and range.

This article is organized as follows. In the next section we introduce the framework and state the main theorem (Theorem 2.7), and its proof is provided in Section 3. The proof requires a variant of the well-known *Mercer's theorem* on uniform convergence of the eigenfunction expansion for continuous integral kernels, which we prove in Appendix for completeness. In Section 4, we discuss the main theorem from the viewpoint of real inversion of the Laplace transform and its numerical realization on digital computers through examples.

2 Framework and the main theorem

We first introduce a certain class of reproducing kernel Hilbert spaces, which serve as the domain of our Laplace transform operator. Throughout this and next sections, all functions are assumed to be $\mathbb{R}$-valued.

Notation. (1) We follow the convention that $\mathbb{N} = \{1,2,3,\dots\}$, i.e. $0 \notin \mathbb{N}$.
(2) We write $x \wedge y := \min\{x,y\}$ for $x,y \in \mathbb{R}$. We also set $0^{-1} := \infty$.

Definition 2.1. Let $\rho : (0,\infty) \to [0,\infty)$ be Borel measurable and suppose that $\int_0^T \rho(t)dt < \infty$ for any $T \in (0,\infty)$. We define

$$K_\rho(x,y) := \int_0^{x\wedge y} \rho(t)dt \qquad \text{for } x,y \in [0,\infty), \tag{2.1}$$

$$H_{K_\rho} := \left\{ f \;\middle|\; f : [0,\infty) \to \mathbb{R},\ f = \int_0^{(\cdot)} h(t)dt \text{ for some } h \in L^2((0,\infty),\rho(t)^{-1}dt) \right\}. \tag{2.2}$$

Note that $\int_0^T |h(t)|dt < \infty$ for any $h \in L^2((0,\infty),\rho(t)^{-1}dt)$ and any $T \in (0,\infty)$, since $h = 0$ a.e. on $\rho^{-1}(0)$ and $h/\sqrt{\rho}, \sqrt{\rho} \in L^2((0,T),dt)$. Therefore every $h \in L^2((0,\infty),\rho(t)^{-1}dt)$ determines a continuous function $f \in H_{K_\rho}$ by $f(x) := \int_0^x h(t)dt$, $x \in [0,\infty)$. Conversely for each $f \in H_{K_\rho}$, such $h \in L^2((0,\infty),\rho(t)^{-1}dt)$ as in (2.2) is unique because $h = f'$ a.e. on $(0,\infty)$ by Lebesgue's differentiation theorem.

The following proposition is easily verified.

Proposition 2.2. *Let ρ be the same as in Definition 2.1. For $f,g \in H_{K_\rho}$, we define*

$$\langle f,g\rangle_{H_{K_\rho}} := \int_0^\infty f'(t)g'(t)\frac{dt}{\rho(t)}. \tag{2.3}$$

Then H_{K_ρ} is a Hibert space with inner product $\langle\cdot,\cdot\rangle_{H_{K_\rho}}$. Moreover, K_ρ is the reproducing kernel of H_{K_ρ}, that is, K_ρ is the unique $\mathbb{R}$-valued function on $[0,\infty)^2$ possessing the following two properties:
(RP1) $K_\rho(\cdot,x)(=: K_\rho^x) \in H_{K_\rho}$ *for any $x \in [0,\infty)$.*
(RP2) $\langle f, K_\rho^x\rangle_{H_{K_\rho}} = f(x)$ *for any $f \in H_{K_\rho}$ and any $x \in [0,\infty)$.*

Next we formulate the Laplace transform operator.

Definition 2.3 (Laplace transform)**.** Let

$$\mathcal{D}[\mathcal{L}] := \bigcap_{p\in(0,\infty)} L^1((0,\infty), e^{-pt}dt). \tag{2.4}$$

For $f \in \mathcal{D}[\mathcal{L}]$, we define the *Laplace transform* $\mathcal{L}f : (0,\infty) \to \mathbb{R}$ *of* f by

$$\mathcal{L}f(p) := \int_0^\infty e^{-pt}f(t)dt, \quad p \in (0,\infty), \tag{2.5}$$

and also define $Lf : (0,\infty) \to \mathbb{R}$ by

$$Lf(p) := p\cdot\mathcal{L}f(p), \quad p \in (0,\infty). \tag{2.6}$$

$\mathcal{L}f$ and Lf are, of course, real analytic on $(0,\infty)$.

Proposition 2.4. *Let $\rho : (0,\infty) \to [0,\infty)$ be Borel measurable and suppose $\rho \in \mathcal{D}[\mathcal{L}]$. Then $L^2((0,\infty),\rho(t)^{-1}dt) \cup H_{K_\rho} \subset \mathcal{D}[\mathcal{L}]$. Moreover, $Lf = \mathcal{L}f'$ for any $f \in H_{K_\rho}$.*

Proof. Let $T, p \in (0,\infty)$. Since $\int_0^T \rho(t)dt \le \int_0^T e^{pT}e^{-pt}\rho(t)dt \le e^{pT}\int_0^\infty e^{-pt}\rho(t)dt < \infty$, K_ρ and H_{K_ρ} are defined respectively by (2.1) and (2.2). If $h \in L^2((0,\infty),\rho(t)^{-1}dt)$, then $e^{-p(\cdot)}h \in L^1((0,\infty),dt)$ by $e^{-p(\cdot)}\sqrt{\rho}, h/\sqrt{\rho} \in L^2((0,\infty),dt)$ and hence $h \in \mathcal{D}[\mathcal{L}]$.

Let $f \in H_{K_\rho}$ and define $F \in H_{K_\rho}$ by $F(x) := \int_0^x |f'(t)|dt$. Then $|f| \le F$ on $[0,\infty)$. The integration by parts formula for absolutely continuous functions yields

$$p\int_0^T e^{-pt}|f(t)|dt \le p\int_0^T e^{-pt}F(t)dt = -e^{-pT}F(T) + \int_0^T e^{-pt}|f'(t)|dt. \tag{2.7}$$

Since $|f'| \in L^2((0,\infty),\rho(t)^{-1}dt) \subset \mathcal{D}[\mathcal{L}]$, letting $T \to \infty$ in (2.7) together with $F \ge 0$ yields

$$p\int_0^\infty e^{-pt}|f(t)|dt \le p\int_0^\infty e^{-pt}F(t)dt \le \int_0^\infty e^{-pt}|f'(t)|dt < \infty.$$

Thus $f \in \mathcal{D}[\mathcal{L}]$. By the dominated convergence, $f(0) = 0$ and integration by parts again,

$$\mathcal{L}f'(p) - Lf(p) = \lim_{T\to\infty}\int_0^T e^{-pt}(f'(t) - pf(t))dt = \lim_{T\to\infty} e^{-pT}f(T),$$

which has to be equal to 0 since $\int_0^\infty e^{-pt}|f(t)|dt < \infty$. Hence $Lf = \mathcal{L}f'$. ■

To state the main theorem, we put the following assumption **(HS)**:

$\rho : (0,\infty) \to [0,\infty)$ is Borel measurable and satisfies $\rho \in \mathcal{D}[\mathcal{L}]$ and $\mathcal{L}\rho(1) > 0$, μ is a positive Borel measure on $(0,\infty)$ and $\mathcal{A}(\rho,\mu) := \int_{(0,\infty)} \mathcal{L}\rho(2p)d\mu(p) < \infty$. **(HS)**

Remark 2.5. (1) For a $[0,\infty)$-valued Borel measurable function $\rho \in \mathcal{D}[\mathcal{L}]$, the condition $\mathcal{L}\rho(1) > 0$ fails if and only if $\rho = 0$ a.e. with respect to the Lebesgue measure.
(2) **(HS)** implies that $\mu(K) < \infty$ for any compact subset of $(0,\infty)$ since $\mathcal{L}\rho(2p)$ is $(0,\infty)$-valued, continuous in $p \in (0,\infty)$ and belongs to $L^1((0,\infty),\mu)$. In particular, μ is σ-finite.

Proposition 2.6. *Assume* **(HS)**. *Then L defines a bounded linear operator $L : H_{K_\rho} \to L^2((0,\infty),\mu)$.*

Proof. Let $f \in H_{K_\rho}$ and $p \in (0,\infty)$. Then by Proposition 2.4 and Hölder's inequality,

$$|Lf(p)|^2 = |\mathcal{L} f'(p)|^2 = \left| \int_0^\infty e^{-pt} \rho(t)^{1/2} \frac{f'(t)}{\sqrt{\rho(t)}} dt \right|^2 \le \mathcal{L}\rho(2p) \|f\|_{H_{K_\rho}}^2. \tag{2.8}$$

Integrating (2.8) with respect to $d\mu(p)$ yields $\|Lf\|_{L^2(\mu)} \le \sqrt{\mathcal{A}(\rho,\mu)} \|f\|_{H_{K_\rho}}$. ■

In fact, the same assumption **(HS)** implies that L is a Hilbert-Schmidt operator and that LL^* admits an integral kernel $\mathcal{L}\rho(p+q)$, which is the main theorem of this paper.

Theorem 2.7. *Assume* **(HS)**. *Then $L : H_{K_\rho} \to L^2((0,\infty),\mu)$ is a Hilbert-Schmidt operator with Hilbert-Schmidt norm $\sqrt{\mathcal{A}(\rho,\mu)}$. Moreover, for any $\varphi \in L^2((0,\infty),\mu)$,*

$$LL^*\varphi(p) = \int_{(0,\infty)} \mathcal{L}\rho(p+q)\varphi(q)d\mu(q), \quad p \in (0,\infty), \tag{2.9}$$

where $LL^*\varphi$ is seen as a real analytic function on $(0,\infty)$ given by (2.6) with $f = L^*\varphi$.

Since every Hilbert-Schmidt operator is compact, we have the following corollary.

Corollary 2.8. *Assume* **(HS)**. *Then $L : H_{K_\rho} \to L^2((0,\infty),\mu)$ is a compact operator.*

The next section is devoted to the proof of Theorem 2.7.

3 Proof of Theorem 2.7

We first show the equality (2.9). Let $\varphi \in L^2((0,\infty),\mu)$ and $t \in (0,\infty)$. We may assume $\varphi \ge 0$ without loss of generality. By the reproducing property of K_ρ (Proposition 2.2), Proposition 2.4, $(K_\rho^t)' = \rho \mathbf{1}_{(0,t)}$ and Fubini's theorem,

$$\begin{aligned} L^*\varphi(t) &= \langle L^*\varphi, K_\rho^t \rangle_{H_{K_\rho}} = \langle \varphi, LK_\rho^t \rangle_{L^2(\mu)} = \int_{(0,\infty)} \varphi(q) \mathcal{L}[\rho \mathbf{1}_{(0,t)}](q) d\mu(q) \\ &= \int_{(0,\infty)} \varphi(q) \Big(\int_0^t e^{-qs}\rho(s)ds \Big) d\mu(q) = \int_0^t \Big(\int_{(0,\infty)} e^{-qs}\rho(s)\varphi(q)d\mu(q) \Big) ds. \end{aligned}$$

Therefore $(L^*\varphi)'(t) = \int_{(0,\infty)} e^{-qt}\rho(t)\varphi(q)d\mu(q)$, and then for any $p \in (0,\infty)$,

$$LL^*\varphi(p) = \mathcal{L}[(L^*\varphi)'](p) = \int_0^\infty e^{-pt}\Big(\int_{(0,\infty)} e^{-qt}\rho(t)\varphi(q)d\mu(q)\Big)dt$$
$$= \int_{(0,\infty)} \varphi(q)\Big(\int_0^\infty e^{-(p+q)t}\rho(t)dt\Big)d\mu(q) = \int_{(0,\infty)} \mathcal{L}\rho(p+q)\varphi(q)d\mu(q).$$

Thus (2.9) follows.

We next prove that LL^* is a Hilbert-Schmidt operator. Let $k(p) := \sqrt{\mathcal{L}\rho(2p)}$. Then $k \in L^2((0,\infty),\mu)$ and $\|k\|^2_{L^2(\mu)} = \mathcal{A}(\rho,\mu)$ by **(HS)**, and for any $p,q \in (0,\infty)$,

$$\mathcal{L}\rho(p+q) = \int_0^\infty e^{-(p+q)t}\rho(t)dt \le \sqrt{\int_0^\infty e^{-2pt}\rho(t)dt\int_0^\infty e^{-2qt}\rho(t)dt} = k(p)k(q).$$

Therefore,

$$\int_{(0,\infty)^2} |\mathcal{L}\rho(p+q)|^2 d(\mu\times\mu)(p,q) \le \int_{(0,\infty)^2} k(p)^2k(q)^2 d\mu(p)d\mu(q) = \|k\|^4_{L^2(\mu)} < \infty,$$

i.e. LL^* has an integral kernel belonging to $L^2((0,\infty)^2,\mu\times\mu)$. Hence LL^* is a Hilbert-Schmidt operator and in particular it is compact.

Since LL^* is compact and self-adjoint, it admits the following spectral decomposition:

$$LL^* = \sum_{n=1}^N \lambda_n \langle \cdot, \varphi_n\rangle_{L^2(\mu)}\varphi_n \quad \text{(convergent in operator norm)}, \tag{3.1}$$

where $N \in \mathbb{N}\cup\{0,\infty\}$, $\{\varphi_n\}_{n=1}^N$ is a complete orthonormal system of $(\ker LL^*)^\perp$, $\{\lambda_n\}_{n=1}^N \subset (0,\infty)$ is non-increasing, and $\lim_{n\to\infty}\lambda_n = 0$ if $N = \infty$. For each n, $\varphi_n = \lambda_n^{-1}LL^*\varphi_n$ in $L^2((0,\infty),\mu)$. Therefore φ_n is uniquely determined as a continuous function on

$$\operatorname{supp}[\mu] := \{p \in (0,\infty) \mid \mu(V) > 0 \text{ for any open neighborhood } V \text{ of } p \text{ in } (0,\infty)\},$$

which is the smallest closed subset of $(0,\infty)$ whose complement has 0 μ-measure. Since $\mu((0,\infty)\setminus\operatorname{supp}[\mu]) = 0$ we may regard μ as a positive Borel measure on $\operatorname{supp}[\mu]$ and LL^* as a compact operator on $L^2(\operatorname{supp}[\mu],\mu)$. By a version of so-called *Mercer's theorem* (see Theorem A.1 and its proof below for details), the integral kernel $\mathcal{L}\rho(p+q)$ of LL^* admits the eigenfunction expansion

$$\mathcal{L}\rho(p+q) = \sum_{n=1}^N \lambda_n\varphi_n(p)\varphi_n(q), \qquad p,q \in \operatorname{supp}[\mu], \tag{3.2}$$

where the series is uniformly absolutely convergent on every compact subset of $\operatorname{supp}[\mu]\times\operatorname{supp}[\mu]$. Then by (3.2) and the monotone convergence,

$$\mathcal{A}(\rho,\mu) = \int_{\operatorname{supp}[\mu]} (\mathcal{L}\rho)(2p)d\mu(p) = \sum_{n=1}^N \lambda_n \int_{\operatorname{supp}[\mu]} \varphi_n(p)^2 d\mu(p) = \sum_{n=1}^N \lambda_n.$$

On the other hand, let $\{\psi_m\}_{m=1}^M$ ($M \in \mathbb{N}\cup\{0,\infty\}$) be a complete orthonormal system of $\ker LL^*$ (note that $L^2((0,\infty),\mu)$ is separable since μ is σ-finite and the Borel σ-field of

$(0,\infty)$ is countably generated). Then $\{\varphi_n\}_{n=1}^N \cup \{\psi\}_{m=1}^M$ is a complete orthonormal system of $L^2((0,\infty),\mu)$. Hence

$$\|L\|_{\mathrm{HS}}^2 = \|L^*\|_{\mathrm{HS}}^2 = \sum_{n=1}^N \|L^*\varphi_n\|_{L^2(\mu)}^2 + \sum_{m=1}^M \|L^*\psi_n\|_{L^2(\mu)}^2$$
$$= \sum_{n=1}^N \langle LL^*\varphi_n, \varphi_n\rangle_{L^2(\mu)} + \sum_{m=1}^M \langle LL^*\psi_n, \psi_n\rangle_{L^2(\mu)} = \sum_{n=1}^N \lambda_n = \mathcal{A}(\rho,\mu) < \infty,$$

where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm. Thus the proof is complete. $\blacksquare_{\textbf{Theorem 2.7}}$

4 Examples

Now we present some results of numerical real inversion of the Laplace transform, that is, numerical computation of $\mathcal{L}^{-1}F$ for a given function F. Our numerical schemes rely heavily on the compactness of L as well as the explicit expression (2.9) of LL^*. Let us first exhibit some examples, under which we have performed real inversion.

Example 1 ([7, 2]) If $\rho(t) = te^{-t}$ and μ is the Lebesgue measure dp on $(0,\infty)$, then the assumption **(HS)** is satisfied with $\mathcal{A}(\rho,\mu) = 1/2$. This gives

$$L^*\varphi(t) = \int_0^\infty \frac{1}{(p+1)^2}\left\{1 - e^{-t(p+1)}\big(t(p+1)+1\big)\right\}\varphi(p)dp$$

and

$$LL^*\varphi(p) = \int_0^\infty \frac{\varphi(q)}{(1+p+q)^2}dq.$$

Example 2 Set $\rho(t) = (t+1)^{2d}$ for a fixed $d \in \mathbb{N}$ and $d\mu(p) = \exp\left(-p - \frac{1}{p}\right)dp$. Then this couple satisfies **(HS)** and gives

$$L^*\varphi(t) = \int_0^\infty \varphi(p)\frac{(2d)!}{p^{2d+1}}\left\{e_{2d}(p)\,e^{-p} - e_{2d}\big(p(1+t)\big)\,e^{-p(1+t)}\right\}e^{-\frac{1}{p}}\,dp$$

and

$$LL^*\varphi(p) = \int_0^\infty \varphi(q)\frac{(2d)!}{(p+q)^{2d+1}}\,e_{2d}(p+q)\,e^{-p-\frac{1}{p}}\,dp,$$

where

$$e_k(p) := 1 + p + \frac{p^2}{2!} + \cdots + \frac{p^k}{k!}.$$

We have $\mathcal{A}(\rho,\mu) \approx 0.38\,(d=1)$, $6.39\,(d=2)$, $828.95\,(d=3)$, $458661.07\,(d=4)$. Note that any polynomial vanishing at 0 with degree less than or equal to d belongs to H_{K_ρ} in this case. Therefore this example is applicable to the case where the original function is a polynomial with degree less than or equal to d.

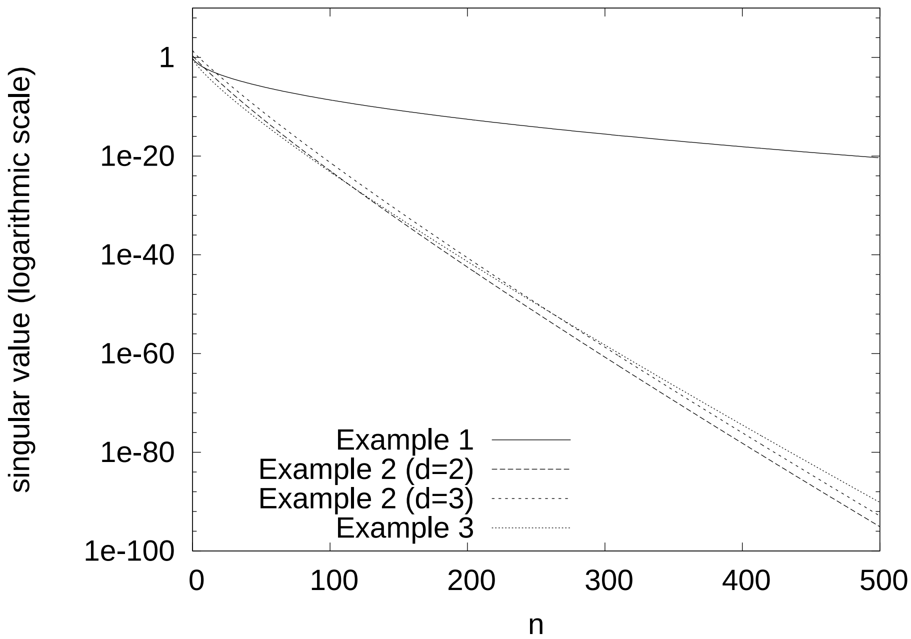


Figure 1: Decay of singular values of L in examples

Example 3 Let $\rho(t) = e^{\sqrt{t}}$ and $d\mu(p) = \exp\left(-p - \frac{1}{p}\right) dp$. This couple again satisfies **(HS)** with $\mathcal{A}(\rho, \mu) \approx 0.2463$. We have

$$L^*\varphi(t) = \int_0^\infty \frac{\varphi(p)}{p} \left[1 - e^{-tp+\sqrt{t}} + \frac{e^{\frac{1}{4p}}}{\sqrt{p}} \left\{ \mathrm{Erf}\left(\frac{1}{2\sqrt{p}}\right) + \mathrm{Erf}\left(\sqrt{tp} - \frac{1}{2\sqrt{p}}\right) \right\} \right] e^{-p-\frac{1}{p}}\, dp$$

and

$$LL^*\varphi(p) = \int_0^\infty \varphi(q) \frac{1}{p+q} \left[1 + \frac{1}{\sqrt{p+q}} e^{\frac{1}{4(p+q)}}\, \mathrm{Erfc}\left(-\frac{1}{2\sqrt{p+q}}\right)\right] e^{-p-\frac{1}{p}}\, dp,$$

where

$$\mathrm{Erf}(x) = \int_0^x e^{-t^2}\, dt, \quad \mathrm{Erfc}(x) = \int_x^\infty e^{-t^2}\, dt.$$

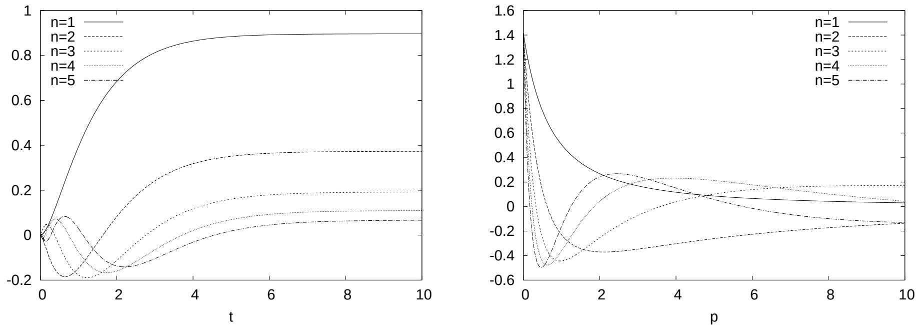
This example is interesting in that H_{K_ρ} contains all polynomials vanishing at 0.

For each setting, the singular values $\{\mu_n\}_{n\in\mathbb{N}}$ of L decay as shown in Figure 1 and Table 1. Figure 2 (a) and (b) show the singular functions $\{g_n\}_{n\in\mathbb{N}}$ and $\{\varphi_n\}_{n\in\mathbb{N}}$ for Example 1 respectively. Numerical computation is done by multiple-precision arithmetic with 250 decimal digits precision.

We can consider a real inversion for $Lf = F \in L(H_{K_\rho})$ through the spectral cut-off (i.e. the cut-off of the singular value decomposition) of the operator L, as described in [3]; note

n	Example 1	Example 2 ($d=2$)	Example 2 ($d=3$)	Example 3
1	0.633851	2.20972	25.6401	0.461443
2	0.264109	1.05542	11.5005	0.166329
3	0.135979	0.535993	5.42578	0.0679611
4	0.0777156	0.278896	2.67786	0.0297729
5	0.0474348	0.145443	1.37678	0.0136621
10	0.00676128	0.00582987	0.0699739	0.00041063
100	2.72037×10^{-9}	1.80121×10^{-23}	8.35375×10^{-22}	1.05223×10^{-23}
200	3.2122×10^{-13}	4.9574×10^{-43}	3.79503×10^{-41}	6.54885×10^{-42}
500	4.6898×10^{-21}	1.21975×10^{-95}	1.90012×10^{-93}	9.83059×10^{-91}

Table 1: Singular values of L in examples



(a) $g_n \in H_{K_\rho}$ (b) $\varphi_n \in L^2((0,\infty),dp)$

Figure 2: Singular functions of Example 1

that we can only treat cases with $f(0)=0$ (recall the definition of H_{K_ρ} in (2.2)). Figure 3 shows numerical spectral cut-off inversion for $f(t)=t^3/3-3t$ in Example 2 with $d=2$ or $d=3$. Polynomials of degree 3 are not included in H_{K_ρ} for Example 2 with $d=2$ thus oscillation appears in the setting of $d=2$ (the solid curve in Figure 3). On the other hand, H_{K_ρ} for Example 2 with $d=3$ includes polynomials of degree 3 and oscillation is reduced in the dotted curve in Figure 3. Similarily, Figure 4 shows a spectral cut-off in Example 2 with $d=3$ for the polynomial $f(t)=t^5/120$, which is not included in H_{K_ρ} in this setting. In the case of Example 3, H_{K_ρ} includes any polynomials vanishing at 0, thus we can get inversions shown in Figure 5 for both $f(t)=t^3/3-3t$ and $f(t)=t^5/120$. In the numerical computation, the truncation number M of the spectral cut-off is chosen as the maximum number satisfying $\mu_M>10^{-60}$ (recall that μ_n denotes the n-th largest singular value of L).

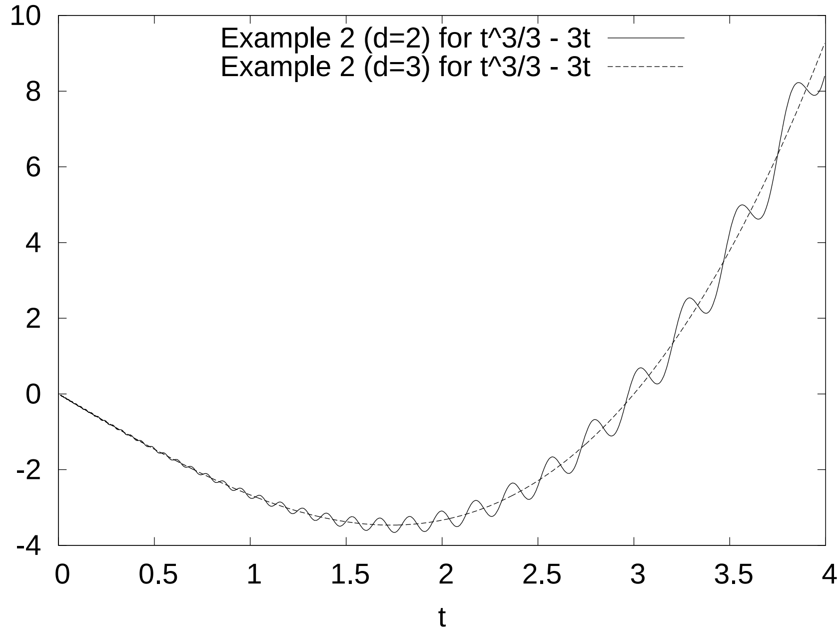


Figure 3: Real inversion for a polynomial of degree 3 in Example 2

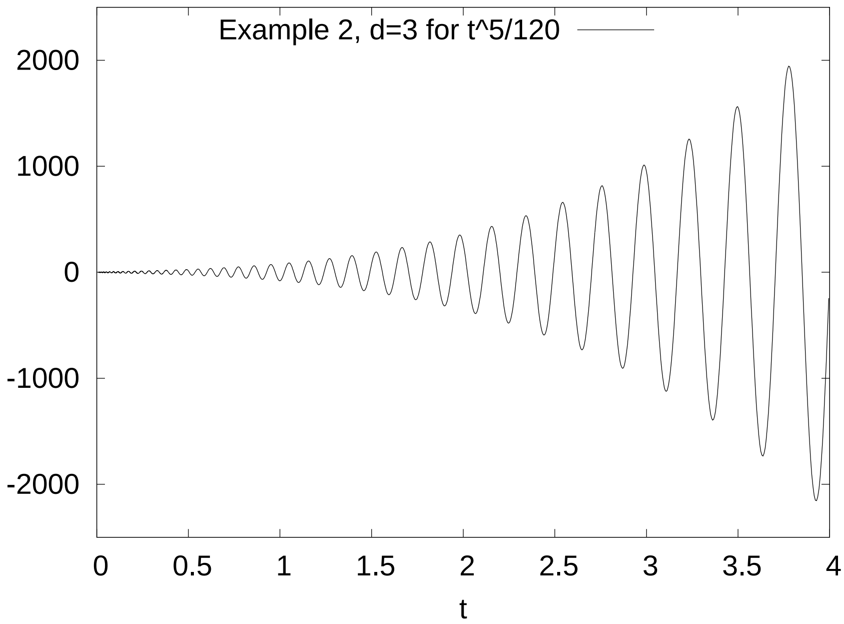


Figure 4: Real inversion for a polynomial of degree 5 in Example 2

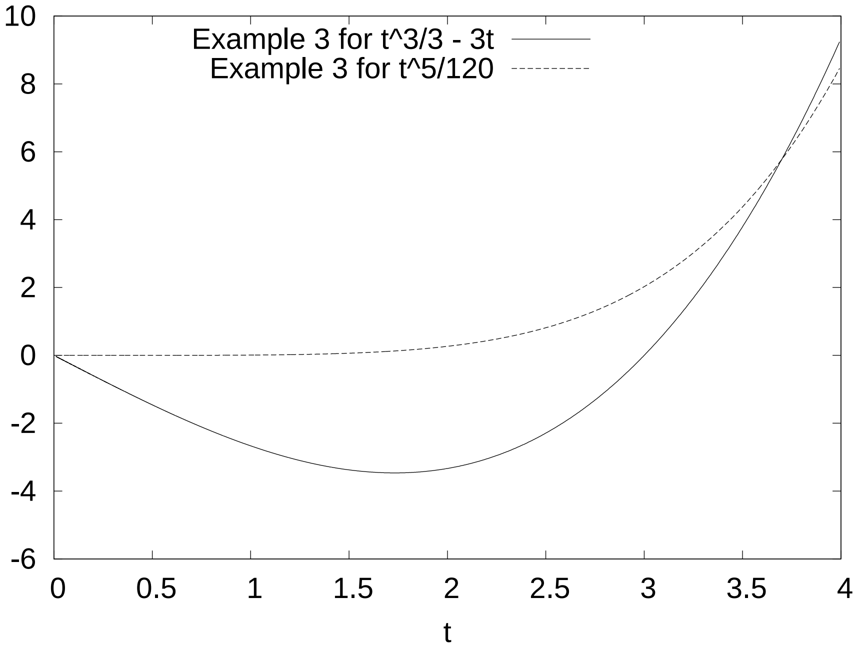


Figure 5: Real inversion for polynomials in Example 3

A Appendix — Uniform convergence of the eigenfunction expansion of integral kernels

In the proof of Theorem 2.7, we have used the fact that the eigenfunction expansion of an integral kernel is uniformly absolutely convergent on every compact subset of the domain of the kernel. If the underlying topological space is compact Hausdorff, this assertion is well-known as *Mercer's theorem* (see [6, §98] for example). A version of Mercer's theorem applicable to non-compact spaces is found in Sun [8]. We could have applied his result to our case in the proof of Theorem 2.7, but instead we provide a complete proof of such a version of the theorem in this appendix for easiness of the reading.

Notation. (1) Throughout this section, $\mathbb{F}$ denotes any one of the two fields $\mathbb{C}$ and $\mathbb{R}$ and all functions are assumed to be $\mathbb{F}$-valued. If $a \in \mathbb{F}$ or a is an $\mathbb{F}$-valued function, its complex conjugate is denoted by $\overline{a}$.
(2) Given a topological space X, $\mathscr{B}(X)$ denotes its Borel σ-field. Also for two measurable spaces $(X,\mathscr{F})$ and $(Y,\mathscr{G})$, the product σ-field of $\mathscr{F}$ and $\mathscr{G}$ is denoted by $\mathscr{F}\otimes\mathscr{G}$.

Theorem A.1. *Let X be a topological space which is locally compact (i.e. whose every point admits a compact neighborhood), and let μ be a σ-finite positive Borel measure on X satisfying $\mu(U) > 0$ for any non-empty open subset U of X. Suppose that a continuous function $K : X \times X \to \mathbb{F}$ possesses the following five properties:*
$(\mathrm{K1})$ $K(x,y) = \overline{K(y,x)}$ *for any* $x,y \in X$.

(K2) *K is $\mathscr{B}(X)\otimes\mathscr{B}(X)$-measurable and $K \in L^2(X\times X,\mu\times\mu)$.*
(K3) *$K(x,\cdot)\in L^2(X,\mu)$ for any $x\in X$.*
(K4) *$A_K f := \int_X K(\cdot,y)f(y)d\mu(y)$ defines a continuous function on X for any $f\in L^2(X,\mu)$.*
(K5) *$\langle A_K f, f\rangle_{L^2(X,\mu)} \geq 0$ for any $f\in L^2(X,\mu)$.*

Noting that A_K is a non-negative self-adjoint Hilbert-Schmidt operator on $L^2(X,\mu)$ by (K1)*,* (K2) *and* (K5)*, let*

$$A_K = \sum_{n=1}^{N} \lambda_n \langle\cdot,\varphi_n\rangle_{L^2(\mu)}\varphi_n \tag{A.1}$$

be its spectral decomposition, where $N\in\mathbb{N}\cup\{0,\infty\}$ and $\lambda_n>0$ for any $n\in\mathbb{N}$ with $n\leq N$. Then for any $x,y\in X$,

$$K(x,y) = \sum_{n=1}^{N}\lambda_n\varphi_n(x)\overline{\varphi_n(y)}, \tag{A.2}$$

where the series is uniformly absolutely convergent on every compact subset of $X\times X$.

Remark A.2. Under the assumption of Theorem A.1, we have the following statements.
(1) $\|A_K\|_{\mathrm{HS}}^2 = \|K\|_{L^2(\mu\times\mu)}^2 = \sum_{n=1}^{N}\lambda_n^2$, where $\|\cdot\|_{\mathrm{HS}}$ denotes the Hilbert-Schmidt norm.
(2) Each $\varphi_n\in L^2(X,\mu)$ is represented by a continuous function since, for μ-a.e. $x\in X$,

$$\varphi_n(x) = \lambda_n^{-1}\int_X K(x,y)\varphi_n(y)d\mu(y) \tag{A.3}$$

where the right-hand side is continuous by (K4). Such a representation of $\varphi_n\in L^2(X,\mu)$ by a continuous function is unique since every non-empty open set in X has strictly positive μ-measure. Therefore we may assume that *each φ_n is continuous* and that (A.3) *is valid for any $x\in X$*. The convergence of the series in (A.2) is on the basis of this assumption.

Proof of Theorem A.1. We follow the argument in [6, §98]. Let $n\in\mathbb{N}$, $n\leq N$ and set $K_n(x,y) := \sum_{i=1}^{n}\lambda_i\varphi_i(x)\overline{\varphi_i(y)}$. An easy calculation yields

$$\|K-K_n\|_{L^2(\mu\times\mu)}^2 = \|K\|_{L^2(\mu\times\mu)}^2 - \sum_{i=1}^{n}\lambda_i^2 \begin{cases} =0 & \text{if} \quad N<\infty,\ n=N, \\ \xrightarrow{n\to\infty} 0 & \text{if} \quad N=\infty. \end{cases} \tag{A.4}$$

If $N<\infty$, then (A.4) implies that $K=K_N$ $\mu\times\mu$-a.e., from which (A.2) immediately follows since K and K_N are continuous on $X\times X$.

Therefore we may now assume that $N=\infty$. Then (A.4) means that in $L^2(X\times X,\mu\times\mu)$ we have $K(x,y)=\sum_{i=1}^{\infty}\lambda_i\varphi_i(x)\overline{\varphi_i(y)}$ and $K(x,y)-K_n(x,y)=\sum_{i=n+1}^{\infty}\lambda_i\varphi_i(x)\overline{\varphi_i(y)}$. Hence for any $f\in L^2(X,\mu)$,

$$\begin{aligned} &\int_{X\times X}(K(x,y)-K_n(x,y))\overline{f(x)}f(y)d(\mu\times\mu)(x,y) \\ &= \sum_{i=n+1}^{\infty}\lambda_i\int_X\int_X\varphi_i(x)\overline{\varphi_i(y)}\,\overline{f(x)}f(y)d\mu(x)d\mu(y) = \sum_{i=n+1}^{\infty}\lambda_i|\langle f,\varphi_i\rangle_{L^2(\mu)}|^2 \geq 0. \end{aligned} \tag{A.5}$$

Let $x\in X$ and suppose $K(x,x)<K_n(x,x)$. By the continuity of K and K_n, there exists an open neighborhood U of x in X such that $K<K_n$ on $U\times U$. Then $\mu(V)>0$, and since μ is σ-finite we can choose $V \in \mathscr{B}(X)$ so that $V \subset U$ and $0 < \mu(V) < \infty$. Now for $f := \mathbf{1}_V (\in L^2(X,\mu))$, the integral in the first line of (A.5) is strictly negative, which contradicts (A.5). Thus $K(x,x) \geq K_n(x,x)$ and letting $n \to \infty$ leads to

$$\sum_{i=1}^{\infty} \lambda_i |\varphi_i(x)|^2 \leq K(x,x) < \infty \quad \text{for any } x \in X. \tag{A.6}$$

Then for any compact subset W of X, by the Cauchy-Schwarz inequality and (A.6),

$$\begin{aligned}\sup_{y\in W}\left|\sum_{i=m+1}^{n} |\lambda_i \varphi_i(x)\overline{\varphi_i(y)}|\right|^2 &\leq \sup_{y\in W}\left(\sum_{i=m+1}^{n} \lambda_i |\varphi_i(x)|^2 \sum_{i=m+1}^{n} \lambda_i |\varphi_i(y)|^2\right) \\ &\leq \sup_{y\in W} K(y,y) \sum_{i=m+1}^{n} \lambda_i |\varphi_i(x)|^2 \xrightarrow{m,n\to\infty} 0.\end{aligned}$$

Hence for each $x \in X$ the series $\sum_{i=1}^{\infty} \lambda_i \varphi_i(x)\overline{\varphi_i}$ of continuous functions on X is uniformly absolutely convergent on every compact subset of X, and the limit, which we call H_x, is again a continuous function on X since X is locally compact.

On the other hand, if $\psi \in \ker A_K$ then $\int_X K(x,y)\psi(y)d\mu(y) = 0$ for μ-a.e. $x \in X$, and the same is ture for *any* $x \in X$ since $\int_X K(\cdot,y)\psi(y)d\mu(y)$ is continuous. Now let $x \in X$. Then

$$\langle \varphi_n, \overline{K(x,\cdot)}\rangle_{L^2(\mu)} = \int_X K(x,y)\varphi_n(y)d\mu(y) = \lambda_n \varphi_n(x), \quad n \in \mathbb{N}, \tag{A.7}$$

$$\langle \psi, \overline{K(x,\cdot)}\rangle_{L^2(\mu)} = \int_X K(x,y)\psi(y)d\mu(y) = 0, \qquad \psi \in \ker A_K. \tag{A.8}$$

$\overline{K(x,\cdot)} \in (\ker A_K)^{\perp}$ by (A.8), and then (A.7) yields the series expansion

$$K(x,\cdot) = \sum_{i=1}^{\infty} \lambda_i \varphi_i(x)\overline{\varphi_i} \tag{A.9}$$

in $L^2(X,\mu)$ since $\{\varphi_n\}_{n\in\mathbb{N}}$ is a complete orthonormal system of $(\ker A_K)^{\perp}$. We may choose a subsequence $\{n_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ so that $\sum_{i=1}^{n_k} \lambda_i \varphi_i(x)\overline{\varphi_i(y)}$ converges to $K(x,y)$ as $k \to \infty$ for μ-a.e. $y \in X$, and it converges to $H_x(y)$ for any $y \in X$ by the previous paragraph. Thus $K(x,\cdot) = H_x$ μ-a.e. and hence everywhere on X by the continuity of $K(x,\cdot)$ and H_x. In other words, for each $x \in X$, the expansion (A.9) is valid both in $L^2(X,\mu)$ and in the sense of uniform absolute convergence on every compact subset of X. In particular,

$$K(x,x) = \sum_{i=1}^{\infty} \lambda_i |\varphi_i(x)|^2 \quad \text{for any } x \in X, \tag{A.10}$$

where the convergence is monotonically non-decreasing. Since the limit $K(x,x)$ and each term $\lambda_i|\varphi_i(x)|^2$ in the series of (A.10) are continuous in $x \in X$, Dini's theorem implies the uniform convergence of the expansion (A.10) on every compact subset of X.

Now let Γ be a compact subset of $X \times X$ and set $\Gamma_1 := \{x \in X \mid (x,y) \in \Gamma \text{ for some } y \in X\}$ and $\Gamma_2 := \{y \in X \mid (x,y) \in \Gamma \text{ for some } x \in X\}$. Then Γ_1 and Γ_2 are compact subsets of X and $\Gamma \subset \Gamma_1 \times \Gamma_2$. Therefore

$$\sup_{(x,y)\in\Gamma}\left|\sum_{i=n}^{\infty} |\lambda_i \varphi_i(x)\overline{\varphi_i(y)}|\right|^2 \leq \sup_{(x,y)\in\Gamma}\left(\sum_{i=n}^{\infty} \lambda_i |\varphi_i(x)|^2 \sum_{i=n}^{\infty} \lambda_i |\varphi_i(y)|^2\right)$$

$$\leq \sup_{x\in\Gamma_1} \sum_{i=n}^{\infty} \lambda_i |\varphi_i(x)|^2 \cdot \sup_{y\in\Gamma_2} \sum_{i=n}^{\infty} \lambda_i |\varphi_i(y)|^2 \xrightarrow{n\to\infty} 0.$$

This completes the proof since $K(x,y) = \sum_{i=1}^{\infty} \lambda_i \varphi_i(x)\overline{\varphi_i(y)}$ for any $x,y \in X$ by (A.9). ■

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