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Remarks on non-diagonality conditions for Sierpinski carpets

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Abstract.

We prove the equivalence of three different formulations of non-diagonality for Sierpinski carpets given by Barlow, Bass, Kumagai and Teplyaev [5], by Hino [10] and by Kigami [12]. We also derive some geometric property of Sierpinski carpets from the non-diagonality. As an application, we give a simpler treatment of a criterion stated in Kigami [12, Section 3.4] for the volume doubling property of self-similar measures on Sierpinski carpets.

§1. Introduction

The notion of fractal sets goes back to B. B. Mandelbrot [16]. He has pointed out there that lots of objects in nature can be regarded as consisting of *'completely non-smooth'* parts like unrectifiable curves and surfaces with infinite area, and he has called such non-smooth sets *'fractals'*. Since then fractal-like structure of objects in nature has been discovered in various fields of science such as physics, chemistry and biology, and study of fractals has grown to be an important research area of science.

Mathematical analysis on fractals has originated from the need for rigorous methods of analysis of physical phenomena, like propagation of heat and wave, on fractals. Mathematically, propagation of heat and wave is described by solutions of heat and wave equations

$$\partial_t u(t,x) = \Delta_x u(t,x), \qquad \partial_t^2 u(t,x) = \Delta_x u(t,x), \tag{1.1}$$

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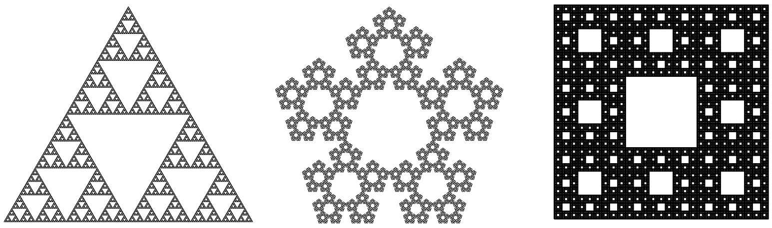

(a) *The Sierpinski gasket* (b) *The pentagasket* (c) *The Sierpinski carpet*

Figure 1. Examples of self-similar fractals

where Δ denotes the *'Laplacian'* on the space, which is typically defined as a differential operator. Definition of the Laplacian on smooth spaces like Riemannian manifolds is more or less trivial, although it gets complicated when we consider more singular spaces such as Alexandrov spaces. On the contrary, the usual notion of differentiation does not work on fractal spaces, and therefore even existence of a (canonical) Laplacian on a given fractal space is highly non-trivial.

To establish theory of Laplacians on fractals, we need to choose an appropriate class of fractal spaces to work on. One handy choice is the class of *self-similar fractals* like the Sierpinski gasket (Figure 1(a)), the pentagasket (Figure 1(b)) and the Sierpinski carpet (Figure 1(c)). A self-similar fractal is defined as the unique non-empty compact set that is invariant with respect to a given finite collection of injective contraction maps. Study of analysis on fractals so far has concentrated mainly on self-similar fractals due to their mathematical accessibility, but even on them the analysis is difficult and lots of fundamental or interesting questions are still open.

The first significant step for achievement of theory of analysis on fractals is the result of Goldstein [9] and Kusuoka [13]. They have constructed a canonical 'Brownian motion on the Sierpinski gasket' by taking the scaling limit of a sequence of random walks on graphs, and hence a canonical Laplacian has been obtained as the infinitesimal generator of this Brownian motion. Barlow and Perkins [7] have studied the properties of this Brownian motion in detail, and have proved that its transition density (the heat kernel corresponding to the Laplacian) $p_t(x,y)$ is subject to the following two-sided *sub-Gaussian estimate*:

$$(1.2) \qquad p_t(x,y) \asymp c_1 t^{-d_f/d_w} \exp\biggl(-c_2\Bigl(\frac{|x-y|^{d_w}}{t}\Bigr)^{\frac{1}{d_w-1}}\biggr), \qquad t \in (0,1],$$

where $\asymp$ means $\leq$ and $\geq$ with different $c_1, c_2 \in (0, \infty)$, $d_f := \log_2 3$ is the Hausdorff dimension of the Sierpinski gasket with respect to the Euclidean metric and $d_w := \log_2 5$ is called the *walk dimension* of the Sierpinski gasket. Note that the Sierpinski gasket is *finitely ramified*, i.e. we can disconnect it by removing finitely many points. This property of the gasket plays essential roles in the above results. The above results have been generalized to a class of finitely ramified self-similar fractals called *(affine) nested fractals*; see [15, 8]. The Sierpinski gasket and the pentagasket are typical examples of nested fractals. A lot of other interesting results are also known in the case of affine nested fractals; See [1, 11, 17] and the references therein.

On the other hand, the Sierpinski carpet is one of the simplest examples of *infinitely ramified* self-similar fractals: The Sierpinski carpet consists of eight smaller copies of itself and each pair of adjacent smaller copies intersects on a one-dimensional interval, which is an infinite set. This fact makes every stage of analysis much more difficult, and it seems natural to try to extend the results in the finitely ramified case to infinitely ramified self-similar sets. Study of the Sierpinski carpet and its analogues may be considered as the first step of this trial.

In [2], Barlow and Bass have constructed a 'Brownian motion on the Sierpinski carpet' as the scaling limit of a sequence of reflected Brownian motions on Euclidean domains approximating the carpet. Later in [3] they have established the two-sided sub-Gaussian bound (1.2) of the corresponding heat kernel. On the other hand, Kusuoka and Zhou [14] used graph approximations of the carpet and Dirichlet form techniques to provide an alternative construction of a Brownian motion. Uniqueness of such a Brownian motion had been an open problem in this field for about two decades, but has recently been proved by Barlow, Bass, Kumagai and Teplyaev [5]. In particular, two Brownian motions given by [2] and [14] are the same.

By [4, 5], it is known that these results are valid for *generalized Sierpinski carpets* (GSCs for short), a class of infinitely ramified self-similar fractals constructed from the d-dimensional unit cube $[0, 1]^d$, $d \geq 2$, in a similar way to the Sierpinski carpet. Barlow and Bass [4] have assumed that GSCs satisfy a geometric condition called *non-diagonality*. This condition is not so essential, but without it the details of the argument in [4] would get significantly complicated and the Brownian motion constructed on the GSC would exhibit strange behaviors. Therefore the non-diagonality condition is indispensable in keeping the class of GSCs reasonable for our analysis. Barlow, Bass, Kumagai and Teplyaev [5] have recently realized, however, that the non-diagonality condition given in [4] is too weak for some part of the arguments of [4], and in [5] they have given a stronger formulation of non-diagonality. On the other hand, Hino [10] has also introduced a simpler reformulation of non-diagonality to discuss some properties of the domain of the Dirichlet form on GSCs.

In fact, before [5] and [10], Kigami [12, Section 3.4] has introduced another stronger formulation of non-diagonality to give a criterion for the *volume doubling property* of self-similar measures on generalized Sierpinski carpets. The volume doubling property is known to be closely related with the validity of the sub-Gaussian heat kernel bounds; see [6] and the references therein. By [12, Theorem 3.2.3], in the case of reasonable self-similar fractals like GSCs and affine nested fractals, the volume doubling property with respect to a certain distance d is equivalent to the sub-Gaussian heat kernel upper bound with respect to the same d, where d needs to be *'adapted'* to the measure and the Dirichlet form in a suitable sense.

Consequently, there are at least three different reformulations of non-diagonality for GSCs. The purpose of this article is to guarantee the equivalence of the three formulations and derive some consequence of the non-diagonality.

The organization of this paper is as follows. In the beginning of Section 2 we introduce our framework and define generalized Sierpinski carpets (GSCs) as well as the three formulations of non-diagonality for GSCs. Then in the rest of Section 2, we first prove the equivalence of the three reformulations of non-diagonality, and secondly, we derive some geometric property of GSCs as an easy consequence of the non-diagonality. As an application, in Section 3 we provide a simpler description of the results of Kigami [12, Section 3.4] on the volume doubling property of self-similar measures on Sierpinski carpets.

Convention. In this article, we follow the convention that $0 \notin \mathbb{N}$. We abbreviate *'if and only if'* to *'iff'*.

§2. Non-diagonality for Sierpinski carpets and its consequence

The following framework will be fixed throughout this article.

Framework 2.1. Let $d \in \mathbb{N}$ and set $Q_0 := [0,1]^d$. Let $L \in \mathbb{N}$, $L \geq 2$ and set $\mathcal{Q}_m := \bigl\{\prod_{i=1}^{d}\bigl[(k_i-1)L^{-m}, k_i L^{-m}\bigr] \mid k_1,\dots,k_d \in \{1,\dots,L^m\}\bigr\}$ for each $m \in \mathbb{N}$. For each $q = \prod_{i=1}^{d}\bigl[(k_i-1)L^{-m}, k_i L^{-m}\bigr] \in \mathcal{Q}_m$, set $z^q = (z^q_1,\dots,z^q_d) := \bigl((k_1-1)L^{-m},\dots,(k_d-1)L^{-m}\bigr)$.

Let $S \subset \mathcal{Q}_1$ be non-empty, and for each $q \in S$ we define $F_q : \mathbb{R}^d \to \mathbb{R}^d$ by $F_q(x) := L^{-1}x + z^q$, so that $F_q(Q_0) = q \subset Q_0$. For $q = q_1 \dots q_m \in S^m$, $m \in \mathbb{N}$, we set $F_q := F_{q_1} \circ \cdots \circ F_{q_m}$, $Q_q := F_q(Q_0)$

and $z^q = (z_1^q, \dots, z_d^q) := z^{Q_q}$. We also set $\mathcal{Q}_m^S := \{Q_q \mid q \in S^m\} (\subset \mathcal{Q}_m)$ and $Q_m^S := \bigcup_{q \in S^m} Q_q$ for $m \in \mathbb{N}$.

The non-diagonality conditions are defined in terms of S, as follows.

Definition 2.2 (Non-diagonality, [5, 10, 12])**.** We define three *non-diagonality* conditions $\mathbf{(ND)_U}$, $\mathbf{(ND)_H}$ and $\mathbf{(ND)_K}$ on S as follows.

$\mathbf{(ND)_U}$ For any $m \in \mathbb{N}$ and any d-dimensional cube $B \subset Q_0$ of side length $2L^{-m}$ which is the union of elements of $\mathcal{Q}_m$, $\mathrm{int}_{\mathbb{R}^d}(B \cap Q_1^S)$ is either empty or connected.

$\mathbf{(ND)_H}$ For any d-dimensional rectangle $B \subset Q_0$ with each side length L^{-1} or $2L^{-1}$ which is the union of elements of $\mathcal{Q}_1$, $\mathrm{int}_{\mathbb{R}^d}(B \cap Q_1^S)$ is either empty or connected.

$\mathbf{(ND)_K}$ For any $x \in Q_1^S$, there exists $\eta \in (0, \infty)$ such that for any $r \in (0, \eta)$, $\mathrm{int}_{\mathbb{R}^d}(Q_1^S \cap B_r(x))$ is non-empty and connected, where $B_r(x) := \{y \in \mathbb{R}^d \mid |y - x| \leq r\}$ with $|\cdot|$ the Euclidean norm.

$\mathbf{(ND)_U}$ is due to [5, Section 2.2] and it is a modified version of the original non-diagonality condition given in [4]. $\mathbf{(ND)_H}$ is due to Hino [10], and $\mathbf{(ND)_K}$ is provided in [12, Definition 3.4.1].

Definition 2.3 (Generalized Sierpinski carpets)**.** Let $\mathrm{GSC}(d, L, S)$ be the self-similar set associated with $\{F_q\}_{q \in S}$, i.e. $\mathrm{GSC}(d, L, S) := \bigcap_{m \in \mathbb{N}} Q_m^S$, which is the unique non-empty compact $K \subset \mathbb{R}^d$ such that $K = \bigcup_{q \in S} F_q(K)$. We call $\mathrm{GSC}(d, L, S)$ a *generalized Sierpinski carpet*, *GSC* for short, iff S satisfies $\mathbf{(ND)_H}$ and the following two conditions:

(GSC1) If $q \in S$, $k \in \{1, \dots, d\}$ and $\{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_k \in \{0, 1\}\} \cap q \neq \emptyset$ then $R_k(q) \in S$, where $R_k : \mathbb{R}^d \to \mathbb{R}^d$ denotes the reflection in the hyperplane $\{(x_1, \dots, x_d) \in \mathbb{R}^d \mid x_k = 1/2\}$.

(GSC2) (Borders included) $\displaystyle\bigcup_{j=1}^{d} \{(0, \dots, 0, x_j, 0, \dots, 0) \mid x_j \in [0, 1]\} \subset Q_1^S$.

Note that the above definition of generalized Sierpinski carpets is broader than the usual one in [4, 5]. We have omitted the conditions which are not required in the arguments below.

The following is the first main theorem of this article.

Theorem 2.4. *The three conditions* $\mathbf{(ND)_U}$, $\mathbf{(ND)_H}$ *and* $\mathbf{(ND)_K}$ *are equivalent to each other.*

Proof. We set $I := \{1, \dots, d\}$ in this proof.

$\mathbf{(ND)_U} \Rightarrow \mathbf{(ND)_H}$: Let $B \subset Q_0$ be a d-dimensional rectangle with each side length L^{-1} or $2L^{-1}$ which is the union of elements of $\mathcal{Q}_1$. Then

$$B = \prod_{i \in I_0} \left[\frac{k_i - 1}{L}, \frac{k_i + 1}{L} \right] \times \prod_{i \in I \setminus I_0} \left[\frac{k_i - 1}{L}, \frac{k_i}{L} \right], \tag{2.1}$$

for some $I_0 \subset I$ and $(k_i)_{i \in I} \in \{1, \ldots, L-1\}^{I_0} \times \{1, \ldots, L\}^{I \setminus I_0}$. Let $C := \prod_{i \in I_0} \left[k_i L^{-1} - L^{-2}, k_i L^{-1} + L^{-2} \right] \times \prod_{i \in I \setminus I_0} \left[k_i L^{-1} - 2L^{-2}, k_i L^{-1} \right]$ $(\subset B)$ and $f\bigl((x_i + k_i L^{-1})_{i \in I}\bigr) := (x_i L + k_i L^{-1})_{i \in I_0} \times (x_i L/2 + k_i L^{-1})_{i \in I \setminus I_0}$ for $(x_i)_{i \in I} \in \mathbb{R}^d$. Then $f : \mathbb{R}^d \to \mathbb{R}^d$ is a bijective affine map and $f(C) = B$. We easily see that $f(q \cap \mathrm{int}_{\mathbb{R}^d} C) = q \cap \mathrm{int}_{\mathbb{R}^d} B$ for any $q \in \mathcal{Q}_1$. Therefore

$$f\bigl(Q_1^S \cap \mathrm{int}_{\mathbb{R}^d} C\bigr) = \bigcup_{q \in S} f(q \cap \mathrm{int}_{\mathbb{R}^d} C) = \bigcup_{q \in S} (q \cap \mathrm{int}_{\mathbb{R}^d} B) = Q_1^S \cap \mathrm{int}_{\mathbb{R}^d} B.$$

Hence f maps $\mathrm{int}_{\mathbb{R}^d}(C \cap Q_1^S) = \mathrm{int}_{\mathbb{R}^d}\bigl(Q_1^S \cap \mathrm{int}_{\mathbb{R}^d} C\bigr)$, which is either empty or connected by $\mathbf{(ND)_U}$, homeomorphically onto $\mathrm{int}_{\mathbb{R}^d}\bigl(Q_1^S \cap \mathrm{int}_{\mathbb{R}^d} B\bigr) = \mathrm{int}_{\mathbb{R}^d}(B \cap Q_1^S)$. Thus $\mathbf{(ND)_H}$ follows.

$\mathbf{(ND)_H} \Rightarrow \mathbf{(ND)_U}$: Let $m \in \mathbb{N}$ and let $C \subset Q_0$ be a d-dimensional cube of side length $2L^{-m}$ which is the union of 2^d distinct elements of $\mathcal{Q}_m$. If $m = 1$ then $\mathbf{(ND)_H}$ immediately implies that $\mathrm{int}_{\mathbb{R}^d}(C \cap Q_1^S)$ is either empty or connected. Therefore we may assume that $m \geq 2$. Then

$$C = \prod_{i \in I_0} \left[\frac{k_i}{L} - \frac{1}{L^m}, \frac{k_i}{L} + \frac{1}{L^m} \right] \times \prod_{i \in I \setminus I_0} \left[\frac{k_i}{L} - \frac{\ell_i + 2}{L^m}, \frac{k_i}{L} - \frac{\ell_i}{L^m} \right]$$

for some $I_0 \subset I$, $(k_i)_{i \in I} \in \{1, \ldots, L-1\}^{I_0} \times \{0, \ldots, L-1\}^{I \setminus I_0}$ and $(\ell_i)_{i \in I \setminus I_0} \in \{0, \ldots, L^{m-1} - 2\}^{I \setminus I_0}$. Define B $(\supset C)$ by (2.1) and

$$f\left(\left(x_i + \frac{k_i}{L} \right)_{i \in I} \right) := \left(\frac{x_i}{L^{m-1}} + \frac{k_i}{L} \right)_{i \in I_0} \times \left(\frac{2x_i}{L^{m-1}} + \frac{k_i}{L} - \frac{\ell_i}{L^m} \right)_{i \in I \setminus I_0}$$

for $(x_i)_{i \in I} \in \mathbb{R}^d$. Then again we easily see that $f : \mathbb{R}^d \to \mathbb{R}^d$ is a bijective affine map satisfying $f(q \cap \mathrm{int}_{\mathbb{R}^d} B) = q \cap \mathrm{int}_{\mathbb{R}^d} C$ for any $q \in S$ and $f(B) = C$. Hence $f\bigl(Q_1^S \cap \mathrm{int}_{\mathbb{R}^d} B\bigr) = Q_1^S \cap \mathrm{int}_{\mathbb{R}^d} C$ and $f\bigl(\mathrm{int}_{\mathbb{R}^d}(B \cap Q_1^S)\bigr) = \mathrm{int}_{\mathbb{R}^d}(C \cap Q_1^S)$. Now using $\mathbf{(ND)_H}$ for B yields $\mathbf{(ND)_U}$.

$\mathbf{(ND)_H} \Leftrightarrow \mathbf{(ND)_K}$: This is easily proved by replacing the rectangle C in the above proof with $B_r(x)$ and constructing an appropriate homeomorphism $f : \mathbb{R}^d \to \mathbb{R}^d$ satisfying $f(B_r(x)) = B$. Q.E.D.

Remark. $\mathbf{(ND)_U} \Rightarrow \mathbf{(ND)_H}$ of the above proof immediately shows that $\mathbf{(ND)_U}, \mathbf{(ND)_H}$ and $\mathbf{(ND)_K}$ are also equivalent to the following:

$\mathbf{(ND)'_U}$ Any d-dimensional cube $B \subset Q_0$ of side length $2L^{-2}$ which is the union of elements of $\mathcal{Q}_2$, $\mathrm{int}_{\mathbb{R}^d}(B \cap Q_1^S)$ is either empty or connected.

Next, we establish an easy consequence of $\mathbf{(ND)_H}$. The following proposition is due to Hino [10].

Notation. Let $m \in \mathbb{N}$. For $w, v \in S^m$, we define $\delta_m(w,v) := L^m \sum_{i=1}^d |z_i^w - z_i^v|$.

Note that $\delta_m(w,v) \in \{0,\dots,d\}$ if $Q_w \cap Q_v \neq \emptyset$, and that $\delta_m(w,v) = 1$ iff Q_w and Q_v intersect on a $(d-1)$-dimensional face.

Proposition 2.5 (Hino [10]). $\mathbf{(ND)_H}$ *holds iff for any* $i, j \in S$ *with* $Q_i \cap Q_j \neq \emptyset$ *there exists a sequence* $\{n(\ell)\}_{\ell=0}^{\delta_1(i,j)} \subset S$ *such that* $n(0) = i$, $n(\delta_1(i,j)) = j$ *and* $\delta_1(n(\ell-1), n(\ell)) = 1$ *for any* $\ell \in \{1,\dots,\delta_1(i,j)\}$.

Remark. For $i, j, n(\ell) \in S$ as above, $Q_i \cap Q_j = \bigcap_{\ell=0}^{\delta_1(i,j)} Q_{n(\ell)}$, which is a $(d - \delta_1(i,j))$-dimensional cube of side length L^{-1}.

Proof. The 'only if' part is easily proved. To show the converse, assume $\mathbf{(ND)_H}$, let $i, j \in S$ satisfy $Q_i \cap Q_j \neq \emptyset$ and let $J := \delta_1(i,j)(\in \{0,\dots,d\})$. The proof is by induction in J. The statement is clear if $J \leq 1$, so suppose $J \geq 2$ and let $B := \bigcup\{q \in \mathcal{Q}_1 \mid q \supset Q_i \cap Q_j\}$, which is a d-dimensional rectangle with each side length L^{-1} or L^{-2}. Then since $\mathrm{int}_{\mathbb{R}^d}(B \cap Q_1^S) \supset \mathrm{int}_{\mathbb{R}^d} Q_i \cup \mathrm{int}_{\mathbb{R}^d} Q_j \neq \emptyset$, $\mathbf{(ND)_H}$ implies that $\mathrm{int}_{\mathbb{R}^d}(B \cap Q_1^S)$ is connected. Therefore we may choose $q \in S$ so that $Q_q \supset Q_i \cap Q_j$ and $\delta_1(q,j) = 1$. Then $L\left|z_k^q - z_k^j\right| = 1$ for a unique $k \in \{1,\dots,d\}$ and $z_k^i = z_k^q$. Hence $\delta_1(i,q) = J - 1$. Now the induction hypothesis implies that there exists $\{n(\ell)\}_{\ell=0}^{J-1} \subset S$ such that $n(0) = i$, $n(J-1) = q$ and $\delta_1(n(\ell-1), n(\ell)) = 1$ for any $\ell \in \{1,\dots,J-1\}$. Setting $n(J) := j$ completes the induction procedure. Q.E.D.

Furthermore, $\mathbf{(ND)_H}$ together with (GSC1) yields the following stronger statement, which is our second main result.

Theorem 2.6. *Assume* $\mathbf{(ND)_H}$ *and* (GSC1). *Let* $m \in \mathbb{N}$, *let* $w, v \in S^m$ *satisfy* $Q_w \cap Q_v \neq \emptyset$ *and set* $J := \delta_m(w,v)(\in \{0,\dots,d\})$. *Then there exists a sequence* $\{n(\ell)\}_{\ell=0}^J \subset S^m$ *such that* $n(0) = w$, $n(J) = v$ *and* $\delta_m(n(\ell-1), n(\ell)) = 1$ *for any* $\ell \in \{1,\dots,J\}$.

Remark. (1) The remark after Proposition 2.5 applies: $Q_w \cap Q_v = \bigcap_{\ell=0}^J Q_{n(\ell)}$, which is a $(d-J)$-dimensional cube of side length L^{-m}.
(2) (GSC1) cannot be omitted in Theorem 2.6. Indeed, if $d = 2$, $L = 3$ and $S = \mathcal{Q}_1 \setminus \left\{\prod_{i=1}^2 \left[\frac{k_i-1}{3}, \frac{k_i}{3}\right] \mid (k_1,k_2) \in \{(1,2),(3,1),(3,3)\}\right\}$, then $\mathbf{(ND)_H}$ holds but both (GSC1) and the conclusion of Theorem 2.6 fail.

Proof. First, by (GSC1) we easily have the following $(\mathrm{GSC1})_m$:

$(\mathrm{GSC1})_m$ If $q \in S^m$, $k \in \{1,\dots,d\}$ and $Q_q \cap \{(x_1,\dots,x_d) \in \mathbb{R}^d \mid x_k \in \{0,1\}\} \neq \emptyset$ then $R_k(Q_q) \in \mathcal{Q}_m^S$, where R_k is the same as in (GSC1).

The proof of Theorem 2.6 is by induction in m. The statement for $m = 1$ follows by Proposition 2.5, so suppose $m \geq 2$. Choose $i, j \in S$ so that $Q_w \subset Q_i$ and $Q_v \subset Q_j$. Then we have $J_0 := \delta_1(i, j) \in \{0, \ldots, J\}$. By Proposition 2.5, there exists $\{\alpha(\ell)\}_{\ell=0}^{J_0} \subset S$ such that $\alpha(0) = i$, $\alpha(J_0) = j$ and $\delta_1(\alpha(\ell-1), \alpha(\ell)) = 1$ for any $\ell \in \{1, \ldots, J_0\}$. Then for any $\ell \in \{0, \ldots, J_0\}$, $Q_{\alpha(\ell)} \supset Q_i \cap Q_j \supset Q_w \cap Q_j$, which is a $(d - J_0)$-dimensional cube of side length L^{-m}. We define $\{n(\ell)\}_{\ell=0}^{J_0} \subset S^m$ inductively as follows. First let $n(0) := w$. Next, let $\ell \in \{1, \ldots, J_0\}$ and suppose we have $n(\ell - 1) \in S^m$ satisfying $Q_w \cap Q_j \subset Q_{n(\ell-1)} \subset Q_{\alpha(\ell-1)}$. Then we easily see that $Q_{n(\ell-1)} \cap (Q_{\alpha(\ell-1)} \cap Q_{\alpha(\ell)}) \neq \emptyset$. Therefore $(\mathrm{GSC1})_{m-1}$ implies that the reflection of $Q_{n(\ell-1)}$ in the $(d-1)$-dimensional face $Q_{\alpha(\ell-1)} \cap Q_{\alpha(\ell)}$ belongs to $\mathcal{Q}_m^S$, which we define as $Q_{n(\ell)}$ with $n(\ell) \in S^m$. This definition of $n(\ell)$ immediately yields $\delta_m(n(\ell-1), n(\ell)) = 1$ and $Q_w \cap Q_j \subset Q_{n(\ell)} \subset Q_{\alpha(\ell)}$. Thus we get $\{n(\ell)\}_{\ell=0}^{J_0} \subset S^m$ satisfying $n(0) = w$ and $\delta_m(n(\ell-1), n(\ell)) = 1$ for any $\ell \in \{1, \ldots, J_0\}$. We also have $Q_w \cap Q_j \subset Q_{n(J_0)} \subset Q_{\alpha(J_0)} = Q_j$, which easily yields $\delta_m(n(J_0), v) = J - J_0$ and $Q_{n(J_0)} \cap Q_v \supset Q_w \cap Q_v \neq \emptyset$. Since $Q_{n(J_0)} \cup Q_v \subset Q_j$, an application of the induction hypothesis to elements of $\mathcal{Q}_m^S$ included in Q_j yields the existence of $\{\beta(\ell)\}_{\ell=0}^{J-J_0} \subset S^m$ such that $\beta(0) = n(J_0)$, $\beta(J - J_0) = v$ and $\delta_m(\beta(\ell-1), \beta(\ell)) = 1$ for any $\ell \in \{1, \ldots, J - J_0\}$. Setting $n(\ell) := \beta(\ell - J_0)$ for $\ell \in \{J_0 + 1, \ldots, J\}$ completes the proof. Q.E.D.

§3. Application: A criterion for the volume doubling property

Now we use Theorem 2.6 to provide a simpler description of the result of Kigami [12, Theorem 3.4.3]. Throughout this section, we put the following framework and notations in addition to Framework 2.1. See [11, Chapter 1] and [12, Section 1.2] for basics on self-similar sets.

Framework 3.1. Set $W_\#(S) := \bigcup_{m \in \mathbb{N}} S^m$. Let $K := \mathrm{GSC}(d, L, S)$ be a GSC and let $\mathcal{L} := (K, S, \{F_q\}_{q \in S})$ denote the self-similar structure associated with $\{F_q\}_{q \in S}$. For $q = q_1 \ldots q_m \in S^m$, $m \in \mathbb{N}$, set $K_q := F_q(K)$, $|q| := m$ and $q_{[-1]} := q_1 \ldots q_{m-1}$ ($:= \emptyset$ if $m = 1$, where $\emptyset$ is an element called the *empty word*).

For $s \in [0, 1]$ and $k \in \{1, \ldots, d\}$, set $H_{k,s} := \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_k = s\}$ and $S_{k,s} := \{q \in S \mid q \cap H_{k,s} \neq \emptyset\}$. Note that, by (GSC1), the reflection $R_k : \mathbb{R}^d \to \mathbb{R}^d$ in the hyperplane $H_{k,1/2}$ induces natural bijections $S_{k,0} \to S_{k,1}$ and $S_{k,1} \to S_{k,0}$ given by $q \mapsto R_k(q)$.

Definition 3.2 (Self-similar scales and related notions)**.** Let $\mathbf{a} = (a_q)_{q \in S} \in (0, 1)^S$. We define the following notions:
(0) $\mathbf{a}$ is called *weakly symmetric* iff $a_q = a_{R_k(q)}$ for any $k \in \{1, \ldots, d\}$ and any $q \in S_{k,0}$.

(1) $a_\emptyset := 1$ and $a_{q_1\dots q_m} := a_{q_1}\dots a_{q_m}$ for $q_1\dots q_m \in S^m$, $m \in \mathbb{N}$. $\Lambda_s(\mathbf{a}) := \{q \in W_\#(S) \mid a_{q_{[-1]}} > s \geq a_q\}$ for $s \in (0,1)$ and $\mathcal{S}(\mathbf{a}) := \{\Lambda_s(\mathbf{a})\}_{s\in(0,1)}$. We call $\mathcal{S}(\mathbf{a})$ the *self-similar scale on $S^{\mathbb{N}}$ with weight* $\mathbf{a}$.
(2) $\Lambda_{s,q}(\mathbf{a}) := \{w \in \Lambda_s(\mathbf{a}) \mid K_w \cap K_q \neq \emptyset\}$ for $s \in (0,1)$, $q \in W_\#(S)$. We call $(\mathcal{L}, \mathcal{S}(\mathbf{a}))$ *locally finite* iff $\sup\{\#\Lambda_{s,q}(\mathbf{a}) \mid s \in (0,1), q \in \Lambda_s(\mathbf{a})\} < \infty$.
(3) $\mathcal{IP}(\mathcal{L}, \mathbf{a}) := \{(w,v) \mid w, v \in \Lambda_s(\mathbf{a})\ {}^\exists s \in (0,1), w \neq v, K_w \cap K_v \neq \emptyset\}$.
(4) $U_s(x,\mathbf{a}) := \bigcup_{q\in\Lambda_s(\mathbf{a}), x\in K_q}\bigl(\bigcup_{w\in\Lambda_{s,q}(\mathbf{a})} K_w\bigr)$ for $(s,x) \in (0,1)\times K$.

Clearly, $\{U_s(x,\mathbf{a})\}_{s\in(0,1)}$ is decreasing as $s \downarrow 0$ and is a fundamental system of neighborhoods of x in K. Now as an application of Theorem 2.6, we have the following propositions.

Proposition 3.3 ([12, Theorem 3.4.4 and Proof of Lemma 3.5.16])**.** *Let $\mathbf{a} = (a_i)_{i\in S} \in (0,1)^S$. Then $(\mathcal{L}, \mathcal{S}(\mathbf{a}))$ is locally finite iff $\mathbf{a}$ is weakly symmetric, which also implies that the sets $\{|w|-|v| \mid (w,v) \in \mathcal{IP}(\mathcal{L},\mathbf{a})\}$ and $\{a_w/a_v \mid (w,v) \in \mathcal{IP}(\mathcal{L},\mathbf{a})\}$ are both finite.*

Proposition 3.4 ([12, Theorems 1.6.6 and 3.4.3])**.** *Let $\mathbf{a} = (a_i)_{i\in S}$, $\mathbf{b} = (b_i)_{i\in S} \in (0,1)^S$. Then the following conditions are equivalent.*
(1) $\mathbf{b}$ is gentle with respect to $\mathbf{a}$, i.e. $\sup\{b_w/b_v \mid (w,v) \in \mathcal{IP}(\mathcal{L},\mathbf{a})\} < \infty$.
(2) For each $k \in \{1,\dots,d\}$, either $a_q = a_{R_k(q)}$ and $b_q = b_{R_k(q)}$ for any $q \in S_{k,0}$, or $b_q = a_q^{\eta_k}$ for any $q \in S_{k,0} \cup S_{k,1}$ for some $\eta_k \in (0,\infty)$.

These two propositions together with [12, Theorem 1.3.5] immediately yield the following theorem.

Theorem 3.5 (Volume doubling property, [12, Theorem 3.4.5])**.** *Let $\mathbf{a} = (a_i)_{i\in S} \in (0,1)^S$. Let $(\mu_i)_{i\in S} \in (0,1)^S$ satisfy $\sum_{i\in S}\mu_i = 1$ and let μ be the self-similar measure on K with weight $(\mu_i)_{i\in S}$, i.e. μ be the unique Borel probability measure on K satisfying $\mu(K_q) = \mu_q$ for any $q \in W_\#(S)$. Then the following conditions are equivalent.*
(1) $\mathbf{a}$ and $(\mu_i)_{i\in S}$ are both weakly symmetric.
(2) $(\mathcal{L}, \mathcal{S}(\mathbf{a}), \mu)$ is volume doubling (VD)*, that is, there exist $\alpha \in (0,1)$ and c_V such that $\mu(U_s(x,\mathbf{a})) \leq c_V\mu(U_{\alpha s}(x,\mathbf{a}))$ for any $(s,x) \in (0,1)$.*

In the rest of this section, we briefly present the idea of the proof of Propositions 3.3 and 3.4. The key of the proof is the following lemma.

Notation. For $w, v \in W_\#(S)$ with $Q_w \cap Q_v \neq \emptyset$, define $\delta_\#(w,v) \in \{0,\dots,d\}$ so that $Q_w \cap Q_v$ is a $(d - \delta_\#(w,v))$-dimensional cube of side length $L^{-\max\{|w|,|v|\}}$.

Note that $\delta_\#(w,v) = 0$ iff $Q_w \subset Q_v$ or $Q_w \supset Q_v$ in the above situation.

Lemma 3.6. *Let $\mathbf{a} = (a_i)_{i\in S} \in (0,1)^S$. Let $s \in (0,1)$ and $w, v \in \Lambda_s(\mathbf{a})$ and assume $Q_w \cap Q_v \neq \emptyset$. Then there exist $J \in \{0,\dots,\delta_\#(w,v)\}$*

and $\{n(\ell)\}_{\ell=0}^{J} \subset \Lambda_s(\mathbf{a})$ *such that* $n(0) = w$, $n(J) = v$ *and for any* $\ell \in \{1, \ldots, J\}$, $Q_{n(\ell-1)} \cap Q_{n(\ell)} \neq \emptyset$ *and* $\delta_{\#}(n(\ell-1), n(\ell)) = 1$.

Remark. (GSC2) is **not** needed in Lemma 3.6 and its proof below.

Proof. Noting that $\mathrm{int}_{\mathbb{R}^d}(Q_\alpha \cap Q_\beta) = \emptyset$ for any distinct $\alpha, \beta \in \Lambda_s(\mathbf{a})$ and using $(\mathrm{GSC1})_m$ for $m \in \mathbb{N}$, this lemma follows by an induction in $k := \delta_{\#}(w, v)$, similarly to Theorem 2.6. Note that $|\alpha|$ and $|\beta|$ may not equal for $\alpha, \beta \in \Lambda_s(\mathbf{a})$, which is why $J \neq \delta_{\#}(w, v)$ in general. Q.E.D.

Proof of Propositions 3.3 and 3.4. By Lemma 3.6, we easily see that $\mathcal{IP}(\mathcal{L}, \mathbf{a})$ may be replaced with $\{(w, v) \in \mathcal{IP}(\mathcal{L}, \mathbf{a}) \mid \delta_{\#}(w, v) = 1\}$ in the statements of Propositions 3.3 and 3.4. Then some direct calculations as in [12, Section 1.6] show the assertions. (The arguments in [12, Section 1.6] are quite general and use lots of notations and lemmas, whereas for the case of self-similar scales on a GSC, Lemma 3.6 suffices.) Q.E.D.

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