Log-periodic asymptotic expansion of the spectral partition function for self-similar sets

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Dedicated to my father on the occasion of his 70th birthday

Abstract: Let $\mathcal{Z}(t)$ be the partition function (the trace of the heat semigroup) of the canonical Laplacian on a post-critically finite self-similar set (with uniform resistance scaling factor and good geometric symmetry) or on a generalized Sierpiński carpet. It is proved that $\mathcal{Z}(t) = \sum_{k=0}^{n} t^{-d_k/d_{\mathrm{w}}} G_k(-\log t) + O\bigl(\exp(-ct^{-\frac{1}{d_{\mathrm{w}}-1}})\bigr)$ as $t \downarrow 0$ for some continuous periodic functions $G_k : \mathbb{R} \to \mathbb{R}$ and $c \in (0,\infty)$. Here $d_{\mathrm{w}} \in (1,\infty)$ denotes the walk dimension, $n = 1$ for a post-critically finite self-similar set, $n = d$ for a d-dimensional generalized Sierpiński carpet, $\{d_k\}_{k=0}^{n} \subset [0,\infty)$ is strictly decreasing with $d_n = 0$, G_0 is strictly positive and G_1 is either strictly positive or strictly negative depending on the (Neumann or Dirichlet) boundary condition.

Key words. Self-similar Dirichlet form – Laplacian eigenvalues – Partition function – Short time asymptotics – Post-critically finite self-similar sets – Generalized Sierpiński carpets
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1. Introduction

Asymptotic distribution of the eigenvalues is a central topic of analysis of Laplacians and elliptic differential operators and has been studied by a huge number of people since Weyl's initiating work [52,53]. There he proved that for the eigenvalues $\{\lambda_n^U\}_{n\in\mathbb{N}}$ of the Dirichlet Laplacian $-\Delta_U$ on a (sufficiently regular) bounded non-empty open subset U of $\mathbb{R}^d$, the associated *eigenvalue counting function* $\mathcal{N}_U(\lambda) := \#\{n \in \mathbb{N} \mid \lambda_n^U \le \lambda\}$ satisfies

$$\mathcal{N}_U(\lambda) = c_d \operatorname{vol}_d(U)\lambda^{d/2} + o(\lambda^{d/2}) \qquad \text{as } \lambda \to \infty, \tag{1.1}$$

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with vol_d the Lebesgue measure on $\mathbb{R}^d$ and $c_d := (2\pi)^{-d}\,\mathrm{vol}_d\bigl(\{x \in \mathbb{R}^d \mid |x| < 1\}\bigr)$. By Karamata's Tauberian theorem [19, Section XIII.5, Theorem 2], (1.1) is equivalent to the asymptotics of the corresponding *(spectral) partition function* $\mathcal{Z}_U(t) := \mathrm{tr}\, e^{t\Delta_U}$,

$$\mathcal{Z}_U(t) = \sum_{n\in\mathbb{N}} e^{-\lambda_n^U t} = (4\pi)^{-d/2}\,\mathrm{vol}_d(U)t^{-d/2} + o(t^{-d/2}) \qquad \text{as } t \downarrow 0, \tag{1.2}$$

which in fact easily follows for any non-empty open subset U of $\mathbb{R}^d$ with $\mathrm{vol}_d(U) < \infty$ from some estimates of the Dirichlet heat kernel $p_t^U(x, y)$ on U and the expression

$$\mathcal{Z}_U(t) = \int_U p_t^U(x,x)dx, \qquad t \in (0,\infty). \tag{1.3}$$

Later in [32] Kac posed his famous question *"Can one hear the shape of a drum?"*, meaning whether the knowledge of the eigenvalues $\{\lambda_n^U\}_{n\in\mathbb{N}}$ of $-\Delta_U$ determines the geometry of the open set U. Although the answer to this original question of Kac is negative as shown by Urakawa [51] for $d \geq 4$ and by Gordon, Webb and Wolpert [25] for $d = 2$, his question motivated numerous works on further detailed asymptotics of $\mathcal{N}_U(\lambda) - c_d\,\mathrm{vol}_d(U)\lambda^{d/2}$ and $\mathcal{Z}_U(t) - (4\pi)^{-d/2}\,\mathrm{vol}_d(U)t^{-d/2}$. A key observation in this direction due to [12] is that, if the boundary of U is fractal, then the box-counting (Minkowski) dimension of the boundary of U, *not* its Hausdorff dimension, should be involved in the remainder estimate for the asymptotics (1.1) and (1.2). See e.g. [12,13, 43–47] and references therein for further details in regard to possible refinements of (1.1) and (1.2) in the settings of Euclidean domains and Riemannian manifolds.

The purpose of this paper is to establish similar detailed asymptotic behavior, *beyond the principal order term*, of the partition function of the Laplacian on self-similar sets. Our main results are stated and proved for two large classes of self-similar sets, known as *post-critically finite self-similar sets* (with additional assumption of some geometric symmetry), which are *finitely ramified*, i.e. can be made disconnected by removing a finite subset, and *generalized Sierpiński carpets*, which are *infinitely ramified*, i.e. not finitely ramified. In this introduction we illustrate our main results by treating the particular cases of the canonical Laplacians on the Sierpiński gasket and the Sierpiński carpet (see Fig. 1 below), which are among the simplest self-similar fractals and have been intensively studied. Below we will mainly focus on the case of the Sierpiński carpet, which is infinitely ramified, hence more difficult, interesting and essential.

We first recall some basics of the canonical Laplacian on the Sierpiński carpet. Let K_{SC} denote the Sierpiński carpet and let μ be the d_{f}-dimensional Hausdorff measure on K_{SC} with respect to the Euclidean metric $\rho(x, y) := |x - y|$, where $d_{\mathrm{f}} := \log_3 8$ is the Hausdorff dimension of K_{SC} with respect to ρ. A natural non-degenerate μ-symmetric diffusion on K_{SC} was constructed for the first time by Barlow and Bass in [1], and later Kusuoka and Zhou [42] also obtained one by constructing a self-similar symmetric regular Dirichlet form on $L^2(K_{\mathrm{SC}}, \mu)$. In fact, it was only very recently that these two diffusions were proved to be the same, as a consequence of the uniqueness result by Barlow, Bass, Kumagai and Teplyaev [6]. By the results of [6], together with slight additional arguments in [31, Proof of Proposition 5.1] and [36, Proposition 5.9], now it is known that there exists a unique non-zero conservative symmetric regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K_{\mathrm{SC}}, \mu)$ that is self-similar and invariant under the isometries of the enclosing unit square. Thus the canonical Laplacian on K_{SC} is obtained as the non-positive self-adjoint operator on $L^2(K_{\mathrm{SC}}, \mu)$ associated with $(\mathcal{E}, \mathcal{F})$. Furthermore by [4,

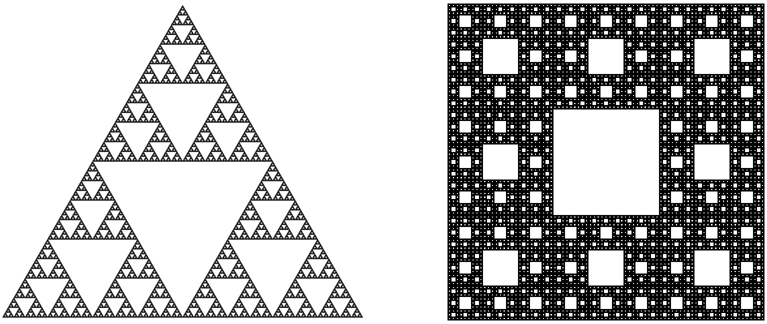

Fig. 1. Sierpiński gasket and Sierpiński carpet

6], the associated heat semigroup has a continuous heat kernel $p_t(x, y)$ which satisfies the following *sub-Gaussian estimate*

$$\frac{c_{1.1}}{t^{d_{\mathrm{f}}/d_{\mathrm{w}}}}\exp\biggl(-\Bigl(\frac{\rho(x,y)^{d_{\mathrm{w}}}}{c_{1.1}t}\Bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\biggr) \leq p_t(x,y) \leq \frac{c_{1.2}}{t^{d_{\mathrm{f}}/d_{\mathrm{w}}}}\exp\biggl(-\Bigl(\frac{\rho(x,y)^{d_{\mathrm{w}}}}{c_{1.2}t}\Bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\biggr) \tag{1.4}$$

for any $(t, x, y) \in (0, 1] \times K_{\mathrm{SC}} \times K_{\mathrm{SC}}$ for some $c_{1.1}, c_{1.2} \in (0, \infty)$ and certain specific $d_{\mathrm{w}} \in (2, \infty)$. (1.4) is called *sub*-Gaussian since the exponent d_{w}, the so-called *walk dimension of* $(K_{\mathrm{SC}}, \mu, \mathcal{E}, \mathcal{F})$, is strictly greater than 2, which is known to hold in quite some generality for Dirichlet forms on fractals. Unfortunately, the exact value of d_{w} is unknown for the Sierpiński carpet, whereas the canonical self-similar Dirichlet form on the Sierpiński gasket, constructed essentially by [24,41,8], satisfies (1.4) with $d_{\mathrm{w}} = \log_2 5$ by the famous result of Barlow and Perkins [8, Theorem 1.5].

Recall that the usual boundary V_0^{SC} of the unit square enclosing K_{SC} is considered as the *boundary of* K_{SC} in the sense that any two distinct copies of K_{SC} can intersect only on the copies of V_0^{SC}. Now let $\mathcal{Z}_{\mathrm{N}}^{\mathrm{SC}}$ be the partition function of the canonical Laplacian on K_{SC}, which satisfies the Neumann boundary condition on V_0^{SC} in a certain natural sense, and let $\mathcal{Z}_{\mathrm{D}}^{\mathrm{SC}}$ be the partition function of the Laplacian with Dirichlet boundary condition on V_0^{SC}. Then our main result for generalized Sierpiński carpets (Theorem 4.10) implies the following asymptotics of $\mathcal{Z}_{\mathrm{B}}^{\mathrm{SC}}$, $\mathrm{B} \in \{\mathrm{N}, \mathrm{D}\}$.

Theorem 1.1. *Set* $d_{\mathrm{f}} := \log_3 8$*, let* $d_{\mathrm{w}} \in (2, \infty)$ *be as in* (1.4) *for* K_{SC} *and set* $\tau := 3^{d_{\mathrm{w}}}$*. Then there exist* $c_{1.3} \in (0, \infty)$ *and continuous* $\log \tau$*-periodic functions* $G_{\mathrm{B},k} : \mathbb{R} \to \mathbb{R}$, $\mathrm{B} \in \{\mathrm{N}, \mathrm{D}\}$, $k \in \{0, 1, 2\}$*, such that for any* $\mathrm{B} \in \{\mathrm{N}, \mathrm{D}\}$*, as* $t \downarrow 0$*,*

$$\mathcal{Z}_{\mathrm{B}}^{\mathrm{SC}}(t) = t^{-d_{\mathrm{f}}/d_{\mathrm{w}}} G_{\mathrm{B},0}(-\log t) + t^{-1/d_{\mathrm{w}}} G_{\mathrm{B},1}(-\log t) + G_{\mathrm{B},2}(-\log t) + O\Bigl(\exp\bigl(-c_{1.3} t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)\Bigr). \tag{1.5}$$

Moreover, $G_{\mathrm{N},0}, G_{\mathrm{N},1}$ *are* $(0, \infty)$*-valued,* $G_{\mathrm{N},0} = G_{\mathrm{D},0}$ *and* $\frac{5}{8} G_{\mathrm{N},1} = -\frac{5}{12} G_{\mathrm{D},1}$.

Note that the factor τ appearing in the period of $G_{\mathrm{B},k}$ is the scaling factor for the time variable of the heat kernels; see Lemma 3.12, Proposition 5.3 and its proof below.

The principal order term in (1.5) was already obtained by Hambly [29, Theorem 1.1] for the present case and by the author in a more general setting in [33, Theorem 5.2], on the basis of the upper inequality in (1.4) and an argument on the associated diffusion. The new results here are the existence of the other *only finitely many* periodic functions and the *exponentially decaying* remainder estimate, and the method of the proof is essentially a refinement of Hambly's argument in [29]. Roughly speaking, the periodic functions $G_{B,k}$, $k \in \{1,2\}$, derive from the self-similarity of each "piece" of the boundary V_0^{SC}, and the order estimates t^{-0/d_w} and t^{-1/d_w} for these terms result from their Euclidean box-counting (or equivalently, Hausdorff) dimensions. More precisely, the self-similarity of the four edges constituting V_0^{SC} gives rise to $G_{B,1}$ of the second term in (1.5), while another term $G_{B,2}(-\log t)$ of constant order arises on account of the facts that the four edges intersect with each other at vertices of the unit square and that the three copies of each edge of one third length also intersect with each other. The same kind of asymptotic expansion will be proved for *any* d-dimensional generalized Sierpiński carpet, where $d+1$ log-periodic terms appear with the order estimate of the k-th term given by the Euclidean box-counting dimension of the intersection of the fractal with the $(d-k+1)$-dimensional faces of the enclosing unit hypercube.

We will prove the same result also for the canonical self-similar Dirichlet form on post-critically finite self-similar sets with good geometric symmetry, where the boundary set is finite and thereby the asymptotic expansion of the partition function involves only two log-periodic terms. *Taking advantage of the simplicity of the situation, we give a detailed proof first for this case as a good illustration of the ideas and methods*, since the complete proof for generalized Sierpiński carpets is quite involved, though still based on exactly the same ideas and methods. In this case, we can also handle the Laplacian with Dirichlet boundary condition on self-similar subsets, and then three log-periodic terms appear in the asymptotic expansion with the second term strictly negative, similarly to (1.5). For example, this result applies to the canonical Laplacian on the Sierpiński gasket with Dirichlet boundary condition on the line segment at the bottom, whose eigenvalue counting function has been recently studied in detail by [48].

Remark 1.2. Here are a few remarks on the limitations of the methods of this paper.
(1) It is very difficult to obtain any further information on the periodic functions in the asymptotic expansions. It is known that non-constant periodic functions appear in the principal order term for the canonical Laplacian on post-critically finite self-similar fractals with good geometric symmetry (see Remark 3.10 below), but this is the *only* known case and ***no*** *other periodic functions in this paper are known to be non-constant*.
(2) Unlike the case of post-critically finite self-similar sets studied e.g. in [40, 7] and [37, Chapter 4], only very little is known for the corresponding eigenvalue counting functions $\mathcal{N}_B^{SC}$ of the canonical Laplacian on the Sierpiński carpet, and in fact *even a log-periodic principal order term for the asymptotics of* $\mathcal{N}_B^{SC}$ *is* ***not*** *known.*
(3) Unfortunately, *we have to assume the scaling factors for the self-similar measure and the self-similar Dirichlet form to be uniform among all the cells* (see Definition 3.4 and (SSDF2) below), for the sake of the validity of a refined version of the renewal theorem (Theorem 2.13). In fact, Fleckinger, Levitin and Vassiliev [21, 22] and van den Berg [9, 10] obtained short time asymptotics very similar to Theorem 1.1 of the integral of the solution to the heat equation on certain von Koch snowflake domains in $\mathbb{R}^2$ with initial value 0 and boundary value 1. Since van den Berg [9, 10] allowed the domains to have different scaling factors for different pieces of the Koch curves constituting the boundary, his method could enable us to relax our assumption of the uniformity of the scaling factors, which we leave to future studies.

We close the introduction by mentioning a possible physical application of the main results of this paper. Recently there have been attempts to study mathematical physics on fractals by analyzing the poles of the spectral zeta function of the Laplacian on the basis of detailed information on the eigenvalues and the heat kernel. Obviously, the first step of any such analysis should be to show that the spectral zeta function admits a meromorphic extension to the whole complex plane $\mathbb{C}$. In fact, this step can be achieved as an application of our main results by using the expression of the spectral zeta function as the Mellin transform of the partition function; see [14,18,50] and references therein for details, and see [44,45] for basic theory of spectral zeta functions.

The rest of this paper is organized as follows. In Section 2, we collect preliminary facts and lemmas for the proofs of our main results. The key fact is Proposition 2.7, which along with Lemma 2.12 makes it possible to extract lower order terms by formal calculations of the heat kernels with Dirichlet boundary conditions on different subsets. In Section 3, after recalling basics of self-similar Dirichlet forms on post-critically finite self-similar sets, we state and prove the result (Theorem 3.9) for the partition function of the Laplacian on post-critically finite self-similar sets with Neumann and Dirichlet boundary conditions on the canonical boundary V_0. We also give a natural extension (Theorem 3.19) to the case with Dirichlet boundary condition on general self-similar subsets at the end of Section 3 without proof. In Section 4, we first collect important facts concerning generalized Sierpiński carpets and their canonical self-similar Dirichlet form and then state the main result for them (Theorem 4.10). In fact, Theorem 4.10 is essentially a special case of Theorem 4.14 on more detailed information on the lower order terms. Finally, Section 5 is devoted to the proof of Theorems 4.10 and 4.14.

Notation. In this paper, we adopt the following notation and conventions.
(1) $\mathbb{N} = \{1, 2, 3, \dots\}$, i.e. $0 \notin \mathbb{N}$.
(2) The cardinality (the number of elements) of a set A is denoted by $\#A$.
(3) We set $\sup\emptyset := \max\emptyset := 0$, $\inf\emptyset := \min\emptyset := \infty$ and set $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$ for $a, b \in [-\infty, \infty]$. All functions in this paper are assumed to be $[-\infty, \infty]$-valued.
(4) For $d \in \mathbb{N}$, $\mathbb{R}^d$ is always equipped with the Euclidean norm $|\cdot|$.
(5) Let E be a topological space. The Borel σ-field of E is denoted by $\mathcal{B}(E)$. We set $C(E) := \{u \mid u : E \to \mathbb{R}, u \text{ is continuous}\}$ and $\mathrm{supp}_E[u] := \overline{\{x \in E \mid u(x) \neq 0\}}$ for $u \in C(E)$. For $A \subset E$, $\mathrm{int}_E A$ denotes its interior in E.
(6) Let E be a set, $\rho : E \times E \to [0, \infty)$ and $x \in E$. We set $\rho(x, A) := \inf_{y \in A} \rho(x, y)$ for $A \subset E$ and $B_r(x, \rho) := \{y \in E \mid \rho(x, y) < r\}$ for $r \in (0, \infty)$.

2. Preliminaries

In this section, we prepare preliminary facts concerning the heat kernel and the eigenvalues of the Laplacian in a general framework. At the end of this section, we also state and prove a version of the renewal theorem which involves log-periodic reminder terms.

Throughout this section, we fix a compact metrizable topological space K, a finite Borel measure μ on K satisfying $\mu(U) > 0$ for any non-empty open subset U of K, and a symmetric regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$; see [23, Section 1.1] for basic notions concerning symmetric regular Dirichlet forms.

Definition 2.1. Let U be a non-empty open subset of K. We define $\mu|_U := \mu|_{\mathcal{B}(U)}$,

$$\mathcal{F}_U := \overline{\{u \in \mathcal{F} \cap C(K) \mid \mathrm{supp}_K[u] \subset U\}} \quad \text{and} \quad \mathcal{E}^U := \mathcal{E}|_{\mathcal{F}_U \times \mathcal{F}_U}, \tag{2.1}$$

where the closure is taken in the Hilbert space $\mathcal{F}$ with inner product $\mathcal{E}_1(u, v) := \mathcal{E}(u, v) + \int_K uv\,d\mu$. $(\mathcal{E}^U, \mathcal{F}_U)$ is called the *part of the Dirichlet form* $(\mathcal{E}, \mathcal{F})$ *on* U.

Since $u = 0$ μ-a.e. on $K \setminus U$ for any $u \in \mathcal{F}_U$, we can regard $\mathcal{F}_U$ as a linear subspace of $L^2(U, \mu|_U)$ in the natural manner. Under this identification, $(\mathcal{E}^U, \mathcal{F}_U)$ is a regular Dirichlet form on $L^2(U, \mu|_U)$ by [23, Theorem 1.4.2-(v) and Lemma 1.4.2-(ii)].

Definition 2.2 (CHK)**.** We say that $(K, \mu, \mathcal{E}, \mathcal{F})$ *satisfies* (CHK), or simply (CHK) *holds*, if and only if the Markovian semigroup $\{T_t\}_{t\in(0,\infty)}$ on $L^2(K, \mu)$ associated with $(\mathcal{E}, \mathcal{F})$ admits a *continuous integral kernel* p, i.e. a continuous function $p = p_t(x, y) : (0, \infty) \times K \times K \to \mathbb{R}$ such that for any $u \in L^2(K, \mu)$ and any $t \in (0, \infty)$,

$$T_t u = \int_K p_t(\cdot, y)u(y)d\mu(y) \qquad \mu\text{-a.e. on } K. \tag{2.2}$$

Clearly, such p, if it exists, is unique and satisfies $p_t(x, y) = p_t(y, x) \geq 0$ for any $(t, x, y) \in (0, \infty) \times K \times K$. p is called the *(continuous) heat kernel of* $(K, \mu, \mathcal{E}, \mathcal{F})$.

The next lemma introduces a Hunt process X associated with $(K, \mu, \mathcal{E}, \mathcal{F})$ and its part X^U on U, which will be used in some of the proofs in this section; see [23, Section A.2] and [15, Section A.1] for details on Hunt processes. For each non-empty open subset U of K, let $U_\partial := U \cup \{\partial_U\}$ denote its one-point compactification.

Lemma 2.3. *Suppose that* (CHK) *holds and that*

$$\lim_{t\downarrow 0} \int_U p_t(x, y)d\mu(y) = 1 \qquad \text{for any open subset } U \text{ of } K \text{ and any } x \in U. \tag{2.3}$$

(1) *There exists a Hunt process* $X = \bigl(\Omega, \mathcal{M}, \{X_t\}_{t\in[0,\infty]}, \{\mathbf{P}_x\}_{x\in K_\partial}\bigr)$ *on* K *such that* $\mathbf{P}_x[X_t \in A] = \int_A p_t(x, y)d\mu(y)$ *for any* $(t, x) \in (0, \infty) \times K$ *and any* $A \in \mathcal{B}(K)$.
(2) *For* $A \in \mathcal{B}(K_\partial)$*, set* $\dot{\sigma}_A(\omega) := \inf\{t \in [0, \infty) \mid X_t(\omega) \in A\}$*,* $\omega \in \Omega$*, and* $\tau_A := \dot{\sigma}_{K_\partial \setminus A}$*. Let* U *be a non-empty open subset of* K*. For* $t \in [0, \infty]$ *and* $\omega \in \Omega$*, define* $X^U_t(\omega) := X_t(\omega)$ *if* $t < \tau_U(\omega)$ *and* $X^U_t(\omega) := \partial_U$ *if* $t \geq \tau_U(\omega)$*. Also set* $\mathbf{P}_{\partial_U} := \mathbf{P}_{\partial_K}$*. Then* $X^U := \bigl(\Omega, \mathcal{M}, \{X^U_t\}_{t\in[0,\infty]}, \{\mathbf{P}_x\}_{x\in U_\partial}\bigr)$ *is a Hunt process on* U*. Moreover, the Markovian semigroup* $\{T^U_t\}_{t\in(0,\infty)}$ *on* $L^2(U, \mu|_U)$ *associated with* $(\mathcal{E}^U, \mathcal{F}_U)$ *admits a unique continuous integral kernel* $p^U = p^U_t(x, y) : (0, \infty) \times U \times U \to \mathbb{R}$*, and* $\mathbf{P}_x[X^U_t \in A] = \int_A p^U_t(x, y)d\mu(y)$ *for any* $(t, x) \in (0, \infty) \times U$ *and any* $A \in \mathcal{B}(U)$.

Proof. Set $\mathcal{P}_t(x, A) := \int_A p_t(x, y)d\mu(y)$ for $(t, x) \in (0, \infty) \times K$ and $A \in \mathcal{B}(K)$, so that $\{\mathcal{P}_t\}_{t\in(0,\infty)}$ is a sub-Markovian transition function on $(K, \mathcal{B}(K))$. Then $\mathcal{P}_t u := \int_K u(y)\mathcal{P}_t(\cdot, dy) \in C(K)$ for any $u \in L^1(K, \mu)$ by the compactness of K and (CHK), and (2.3) means that for any $u \in C(K)$, $\lim_{t\downarrow 0} \mathcal{P}_t u(x) = u(x)$ for each $x \in K$, where the convergence is in fact uniform in $x \in K$ by [11, Chapter I, Exercise 9.13]. Now the assertions follow in exactly the same way as [33, Proof of Lemma 7.11]. □

Throughout the rest of this section, we assume that (CHK) and (2.3) hold.

Definition 2.4. Let U be a non-empty open subset of K. The integral kernel p^U of $\{T^U_t\}_{t\in(0,\infty)}$ as in Lemma 2.3-(2) is called the *Dirichlet heat kernel on* U. We extend p^U to $(0, \infty) \times K \times K$ by setting $p^U|_{(0,\infty)\times(K\times K\setminus U\times U)} := 0$.
We also set $p^{\emptyset}_t(x, y) := 0$ for any $(t, x, y) \in (0, \infty) \times K \times K$.

Clearly $0 \leq p^U_t(x, y) = p^U_t(y, x) \leq p_t(x, y)$ for any $(t, x, y) \in (0, \infty) \times K \times K$. Note that for $t \in (0, \infty)$, $p^U_t : K \times K \to [0, \infty)$ may *not* be continuous on $K \times K$.

Lemma 2.5. *Let U be a non-empty open subset of K. If U is arcwise connected, then $p_t^U(x, y) > 0$ for any $(t, x, y) \in (0, \infty) \times U \times U$.*

Proof. This follows from Lemma 2.3-(2), $p^U \leq p$ and [33, Proposition A.3-(2)]. □

Recall that $(\mathcal{E}, \mathcal{F})$ is called *local* if and only if $\mathcal{E}(u, v) = 0$ for any $u, v \in \mathcal{F} \cap C(K)$ with $\mathrm{supp}_K[u] \cap \mathrm{supp}_K[v] = \emptyset$; see [23, Section 1.1 and Theorem 3.1.2]. The following lemma can be easily verified by using [23, Theorem 1.4.2-(ii) and Exercise 1.4.1].

Lemma 2.6. *Assume that $(\mathcal{E}, \mathcal{F})$ is local. Let U, V be open subsets of K with $U \cap V = \emptyset$. Then $p_t^{U \cup V}(x, y) = p_t^U(x, y) + p_t^V(x, y)$ for any $(t, x, y) \in (0, \infty) \times K \times K$.*

The following proposition plays essential roles in the proofs of our main theorems. For $A \subset K$, the closure of A in K is denoted by $\overline{A}$.

Proposition 2.7. *Let U be an open subset of K, let J_0 be a finite set and let U_j be an open subset of K for each $j \in J_0$. Set $U_J := \bigcap_{j \in J} U_j$ for $J \subset J_0$ ($U_\emptyset := K$) and define*

$$p_t^U(x, y | \{U_j\}_{j \in J_0}) := \sum_{J \subset J_0} (-1)^{\# J} p_t^{U \cap U_J}(x, y) \tag{2.4}$$

for $(t, x, y) \in (0, \infty) \times K \times K$. Then we have the following statements.
(1) $0 \leq p_t^U(x, y | \{U_j\}_{j \in J_0}) \leq p_t^U(x, y)$ *for any $(t, x, y) \in (0, \infty) \times K \times K$.*
(2) *Assume that $(\mathcal{E}, \mathcal{F})$ is local. Let $k \in J_0$. Then for any $(t, x, y) \in (0, \infty) \times \overline{U_k} \times \overline{U_k}$,*

$$p_t^U(x, y | \{U_j\}_{j \in J_0}) \leq \sup_{s \in [t/2, t]} \sup_{z \in \overline{U_k} \setminus U_k} p_s(x, z) + \sup_{s \in [t/2, t]} \sup_{z \in \overline{U_k} \setminus U_k} p_s(z, y). \tag{2.5}$$

Proof. Let $(t, x, y) \in (0, \infty) \times K \times K$. If $x \notin U$ or $y \notin U$ then $p_t^U(x, y | \{U_j\}_{j \in J_0}) = 0$. Suppose $x, y \in U$ and set $J_{x,y} := \{j \in J_0 \mid x, y \in U_j\}$. Then $p_t^U(x, y | \{U_j\}_{j \in J_0}) = p_t^U(x, y | \{U_j\}_{j \in J_{x,y}})$ since $p_t^{U \cap U_J}(x, y) = 0$ for $J \subset J_0$ with $J \not\subset J_{x,y}$, and for each $J \subset J_{x,y}$, $x, y \in U \cap U_J$ and hence $p_t^{U \cap U_J}(x, \cdot)$ is continuous at y. Therefore choosing a metric ρ on K compatible with the original topology of K, following the notation in Lemma 2.3 and noting $\tau_{U \cap U_J} = \tau_U \wedge \min_{j \in J} \tau_{U_j}$ for $J \subset J_{x,y}$, we obtain

$$\begin{aligned}
& p_t^U(x, y | \{U_j\}_{j \in J_0}) = p_t^U(x, y | \{U_j\}_{j \in J_{x,y}}) \\
&= \lim_{s \downarrow 0} \frac{1}{\mu(B_s(y, \rho))} \sum_{J \subset J_{x,y}} (-1)^{\# J} \int_{B_s(y,\rho)} p_t^{U \cap U_J}(x, z) d\mu(z) \\
&= \lim_{s \downarrow 0} \frac{1}{\mu(B_s(y, \rho))} \sum_{J \subset J_{x,y}} (-1)^{\# J} \mathbf{P}_x[X_t \in B_s(y, \rho),\, t < \tau_U \wedge \min_{j \in J} \tau_{U_j}] \\
&= \lim_{s \downarrow 0} \frac{\mathbf{P}_x[X_t \in B_s(y, \rho),\, t < \tau_U] - \mathbf{P}_x[X_t \in B_s(y, \rho),\, t < \tau_U \wedge \max_{j \in J_{x,y}} \tau_{U_j}]}{\mu(B_s(y, \rho))} \\
&= \lim_{s \downarrow 0} \frac{\mathbf{P}_x[X_t \in B_s(y, \rho),\, \max_{j \in J_{x,y}} \tau_{U_j} \leq t < \tau_U]}{\mu(B_s(y, \rho))} \qquad (2.6) \\
&\leq \lim_{s \downarrow 0} \frac{\mathbf{P}_x[X_t \in B_s(y, \rho),\, t < \tau_U]}{\mu(B_s(y, \rho))} = \lim_{s \downarrow 0} \frac{\int_{B_s(y,\rho)} p_t^U(x, z) d\mu(z)}{\mu(B_s(y, \rho))} = p_t^U(x, y),
\end{aligned}$$

where we used the inclusion-exclusion formula for the equality in the fourth line. The expression (2.6) also shows that $p_t^U(x, y | \{U_j\}_{j \in J_0}) \geq 0$, proving (1).

For (2), let $k \in J_0$. Since $p_t^U(x, y|\{U_j\}_{j\in J_0}) = 0$ if $x \notin U$ or $y \notin U$, we may assume that $x, y \in U$. If $x \in \overline{U_k} \setminus U_k$ or $y \in \overline{U_k} \setminus U_k$, then $p_t^U(x, y|\{U_j\}_{j\in J_0}) \leq p_t^U(x, y) \leq p_t(x, y)$, which is obviously bounded by the right-hand side of (2.5). Now suppose $x, y \in U_k$. Then $k \in J_{x,y}$ and hence by (2.6),

$$p_t^U(x, y|\{U_j\}_{j\in J_0}) \leq \lim_{s\downarrow 0} \frac{\mathbf{P}_x[X_t \in B_s(y,\rho),\ \tau_{U_k} \leq t]}{\mu(B_s(y,\rho))} = p_t(x, y) - p_t^{U_k}(x, y),$$

which is bounded by the right-hand side of (2.5) by [27, Theorem 5.1] (or [26, Theorem 10.4]) together with the continuity of p, that of p^{U_k} and the compactness of $\overline{U_k}$. □

Definition 2.8. For a non-empty open subset U of K, the non-positive self-adjoint operator on $L^2(U, \mu|_U)$ associated with $(\mathcal{E}^U, \mathcal{F}_U)$ (the generator of $\{T_t^U\}_{t\in(0,\infty)}$; see [23, Section 1.3]) is denoted by Δ_U, and its domain is written as $\mathcal{D}[\Delta_U]$.

Recall that $\mathcal{D}[\Delta_U] \subset \mathcal{F}_U$ and that for $u \in \mathcal{F}_U$ and $f \in L^2(U, \mu|_U)$,

$$u \in \mathcal{D}[\Delta_U] \text{ and } -\Delta_U u = f \quad \text{if and only if} \quad \mathcal{E}(u, v) = \int_U f v \, d\mu \text{ for any } v \in \mathcal{F}_U. \tag{2.7}$$

Our main interest is in short time asymptotic behavior of the partition function, which is defined as follows.

Definition 2.9. Let U be a non-empty open subset of K. Noting that Δ_U has discrete spectrum and that $\operatorname{tr} T_t^U < \infty$ for $t \in (0,\infty)$ by [16, Theorem 2.1.4], let $\{\lambda_n^U\}_{n=1}^{n_U}$ be the non-decreasing enumeration of all the eigenvalues of $-\Delta_U$, where each eigenvalue is repeated according to its multiplicity and $n_U := \dim L^2(U, \mu|_U) \in \mathbb{N} \cup \{\infty\}$. The *eigenvalue counting function* $\mathcal{N}_U$ and the *partition function* $\mathcal{Z}_U$ *on* U (or *of the Dirichlet space* $(U, \mu|_U, \mathcal{E}^U, \mathcal{F}_U)$) are defined respectively by, for $\lambda \in \mathbb{R}$ and $t \in (0,\infty)$,

$$\mathcal{N}_U(\lambda) := \#\{n \in \mathbb{N} \mid n \leq n_U,\ \lambda_n^U \leq \lambda\}, \tag{2.8}$$

$$\mathcal{Z}_U(t) := \operatorname{tr} T_t^U = \sum_{n=1}^{n_U} e^{-\lambda_n^U t} = \int_{\mathbb{R}} e^{-\lambda t} d\mathcal{N}_U(\lambda) = \int_U p_t^U(x, x) d\mu(x). \tag{2.9}$$

We also set $\mathcal{N}_\emptyset(\lambda) := 0$ for $\lambda \in \mathbb{R}$ and $\mathcal{Z}_\emptyset(t) := 0$ for $t \in (0,\infty)$.

In the situation of Definition 2.9, $\mathcal{N}_U(\lambda) < \infty$ for any $\lambda \in \mathbb{R}$ since $\lim_{n\to\infty} \lambda_n^U = \infty$ when $n_U = \infty$, and $\mathcal{Z}_U$ is $(0,\infty)$-valued and continuous. Moreover, we have the following basic facts for $\{\lambda_n^U\}_{n=1}^{n_U}$ and $\mathcal{Z}_U$. Recall that by [17, Theorems 4.5.1 and 4.5.3], the smallest eigenvalue λ_1^U of $-\Delta_U$ is given by $\lambda_1^U = \inf_{u\in\mathcal{F}_U\setminus\{0\}} \mathcal{E}(u, u)/\int_U u^2 d\mu$, which easily implies that λ_1^U is non-increasing in U and that for $u \in \mathcal{F}_U$,

$$u \in \mathcal{D}[\Delta_U] \text{ and } -\Delta_U u = \lambda_1^U u \quad \text{if and only if} \quad \mathcal{E}(u, u) \leq \lambda_1^U \int_U u^2 d\mu. \tag{2.10}$$

Lemma 2.10. *Let U be an arcwise connected non-empty open subset of K and let φ_1^U be an eigenfunction of $-\Delta_U$ with eigenvalue λ_1^U. Then $\varphi_1^U|_U \in C(U)$, $\varphi_1^U|_U$ is either $(0,\infty)$-valued or $(-\infty,0)$-valued, and $\{u \in \mathcal{D}[\Delta_U] \mid -\Delta_U u = \lambda_1^U u\} = \mathbb{R}\varphi_1^U$.*

Proof. Since $e^{-\lambda_1^U}\varphi_1^U = T_1^U \varphi_1^U = \int_U p_1^U(\cdot, y)\varphi_1^U(y) d\mu(y)$ μ-a.e., $\varphi_1^U|_U \in C(U)$ by $0 \leq p_1^U \leq p_1$ and the continuity of $p_1^U|_{U\times U}$. By virtue of Lemma 2.5, $\varphi_1^U|_U$ is $(0,\infty)$-valued if $\varphi_1^U \geq 0$ μ-a.e., and the claims follow from [49, Theorem XIII.44]. □

Lemma 2.11. *Let U be a non-empty open subset of K. Then*

$$e^{-\lambda_1^U t} \le \mathcal{Z}_U(t) \le \mathcal{Z}_U(1) e^{\lambda_1^U} e^{-\lambda_1^U t}, \qquad t \in [1,\infty). \tag{2.11}$$

Proof. Let $\{\lambda_n^U\}_{n=1}^{n_U}$ be as in Definition 2.9. Then for $t \in [1,\infty)$, $1 \le e^{\lambda_1^U t}\mathcal{Z}_U(t) = \sum_{n=1}^{n_U} e^{(\lambda_1^U - \lambda_n^U)t} \le \sum_{n=1}^{n_U} e^{\lambda_1^U - \lambda_n^U} = \mathcal{Z}_U(1)e^{\lambda_1^U}$ since $\lambda_1^U - \lambda_n^U \le 0$ for each n. □

At the last of this section, we present two fundamental tools for the proofs of our main results. The first lemma is used to relate differences of heat kernels with different boundary conditions to alternating sums of the form (2.4). The second tool is a version of the renewal theorem which yields log-periodic asymptotic behavior of functions.

Lemma 2.12. *Let J_0 be a finite set and let $a_J \in \mathbb{R}$ for each $J \subset J_0$. Then*

$$a_{J_0} = \sum_{J \subset J_0} \sum_{A \subset J} (-1)^{\# A} a_{J \setminus A}. \tag{2.12}$$

Proof. For each $J \subset J_0$, the coefficient of the term a_J in the summation in (2.12) is $\sum_{A \subset J_0 \setminus J} (-1)^{\#(J \cup A)}$, which is 1 for $J = J_0$ and 0 for $J \subsetneq J_0$, proving the lemma. □

Theorem 2.13. *Let $\alpha_0 \in \mathbb{R}$, $\gamma \in (1,\infty)$ and $n \in \mathbb{N}$. For each $k \in \{1,\dots,n\}$, let $\alpha_k \in (-\infty, \alpha_0)$ and let $G_k : \mathbb{R} \to \mathbb{R}$ be bounded and $\log\gamma$-periodic. Assume that $\mathcal{Z} : (0,\infty) \to \mathbb{R}$ and $\mathcal{R} : (0,1] \to [0,\infty)$ satisfy*

$$\left| \mathcal{Z}(t) - \gamma^{\alpha_0}\mathcal{Z}(\gamma t) - \sum_{k=1}^{n} t^{-\alpha_k} G_k(-\log t) \right| \le \mathcal{R}(t) \le c t^{-\alpha_0+\varepsilon}, \qquad t \in (0,1] \tag{2.13}$$

and $|\mathcal{Z}(t)| \le c t^{-\alpha_0-\varepsilon}$ for any $t \in [1,\infty)$ for some $c, \varepsilon \in (0,\infty)$. Then there exists a unique $\log\gamma$-periodic function $G_0 : \mathbb{R} \to \mathbb{R}$ such that for any $t \in (0,\gamma]$,

$$\left| \mathcal{Z}(t) - t^{-\alpha_0} G_0(-\log t) + \sum_{k=1}^{n} \frac{t^{-\alpha_k} G_k(-\log t)}{\gamma^{\alpha_0-\alpha_k} - 1} \right| \le \sum_{j \in \mathbb{N}} \frac{\mathcal{R}(\gamma^{-j} t)}{\gamma^{\alpha_0 j}} =: \mathcal{R}_{\alpha_0,\gamma}(t). \tag{2.14}$$

Moreover, G_0 is bounded, and if $\mathcal{Z}$ is continuous then so is G_0.

Proof. We follow [37, Proofs of Theorems 4.1.5 and B.4.3]. Define $f, g : \mathbb{R} \to \mathbb{R}$ by

$$f(s) := e^{-\alpha_0 s}\mathcal{Z}(e^{-s}) \qquad \text{and} \qquad g(s) := e^{-\alpha_0 s}\bigl(\mathcal{Z}(e^{-s}) - \gamma^{\alpha_0}\mathcal{Z}(\gamma e^{-s})\bigr),$$

so that $f = f(\cdot - \log\gamma) + g$. Then the assumptions easily imply that the series $G_0 := \sum_{j \in \mathbb{Z}} g(\cdot - j\log\gamma)$ is uniformly absolutely convergent on any compact subset of $\mathbb{R}$ and defines a bounded $\log\gamma$-periodic function, which is continuous if $\mathcal{Z}$ is continuous. Since $\lim_{s \to -\infty} f(s) = 0$ by the latter assumption, a repeated use of $f = f(\cdot - \log\gamma) + g$ shows $f = \sum_{j \in \mathbb{N}\cup\{0\}} g(\cdot - j\log\gamma)$. Thus $G_0 - f = \sum_{j \in \mathbb{N}} g(\cdot + j\log\gamma)$, and then (2.14) easily follows by using the $\log\gamma$-periodicity of G_k for $k \in \{1,\dots,n\}$ to sum up (2.13), or more precisely, $\bigl|g(s) - \sum_{k=1}^{n} e^{-(\alpha_0-\alpha_k)s} G_k(s)\bigr| \le e^{-\alpha_0 s}\mathcal{R}(e^{-s})$, $s \in [0,\infty)$.

Finally, the uniqueness of G_0 is immediate from its $\log\gamma$-periodicity and the bound $\mathcal{R}_{\alpha_0,\gamma}(t) \le c(\gamma^{\varepsilon}-1)^{-1} t^{-\alpha_0+\varepsilon}$, $t \in (0,\gamma]$ implied by the upper inequality of (2.13). □

3. Post-critically finite self-similar sets

In this section, we first introduce our framework of a post-critically finite self-similar set equipped with a self-similar Dirichlet form, and then state and prove our main result (Theorem 3.9) on asymptotic expansion of the partition function in this framework. We also present a natural extension (Theorem 3.19) to the partition function with Dirichlet boundary condition on general self-similar subsets at the end of this section without proof. The case of generalized Sierpiński carpets is studied later in Sections 4 and 5.

Let us start with standard notions concerning self-similar sets. We refer to [37, Chapter 1] and [38, Section 1.2] for details. Throughout this section, we fix a compact metrizable topological space K with $\#K \geq 2$, a non-empty finite set S and a continuous injective map $F_i : K \to K$ for each $i \in S$. We set $\mathcal{L} := (K, S, \{F_i\}_{i\in S})$.

Definition 3.1. (1) Let $W_0 := \{\emptyset\}$, where $\emptyset$ is an element called the *empty word*, let $W_m := S^m = \{w_1 \ldots w_m \mid w_i \in S \text{ for } i \in \{1,\ldots,m\}\}$ for $m \in \mathbb{N}$ and let $W_* := \bigcup_{m\in\mathbb{N}\cup\{0\}} W_m$. For $w \in W_*$, the unique $m \in \mathbb{N}\cup\{0\}$ satisfying $w \in W_m$ is denoted by $|w|$ and called the *length of* w.
(2) We set $\Sigma := S^{\mathbb{N}} = \{\omega_1\omega_2\omega_3\ldots \mid \omega_i \in S \text{ for } i \in \mathbb{N}\}$, which is always equipped with the product topology, and define the *shift map* $\sigma : \Sigma \to \Sigma$ by $\sigma(\omega_1\omega_2\omega_3\ldots) := \omega_2\omega_3\omega_4\ldots$. For $i \in S$ we define $\sigma_i : \Sigma \to \Sigma$ by $\sigma_i(\omega_1\omega_2\omega_3\ldots) := i\omega_1\omega_2\omega_3\ldots$. For $\omega = \omega_1\omega_2\omega_3\ldots \in \Sigma$ and $m \in \mathbb{N}\cup\{0\}$, we write $[\omega]_m := \omega_1\ldots\omega_m \in W_m$.
(3) For $w = w_1\ldots w_m \in W_*$, we set $F_w := F_{w_1}\circ\cdots\circ F_{w_m}$ ($F_\emptyset := \mathrm{id}_K$), $K_w := F_w(K)$, $\sigma_w := \sigma_{w_1}\circ\cdots\circ\sigma_{w_m}$ ($\sigma_\emptyset := \mathrm{id}_\Sigma$) and $\Sigma_w := \sigma_w(\Sigma)$.

Definition 3.2. $\mathcal{L}$ is called a *self-similar structure* if and only if there exists a continuous surjective map $\pi : \Sigma \to K$ such that $F_i \circ \pi = \pi \circ \sigma_i$ for any $i \in S$. Note that such a map π, if it exists, is unique and satisfies $\{\pi(\omega)\} = \bigcap_{m\in\mathbb{N}} K_{[\omega]_m}$ for any $\omega \in \Sigma$.

In what follows we always assume that $\mathcal{L}$ is a self-similar structure, so that $\#S \geq 2$.

Definition 3.3. (1) We define the *critical set* $\mathcal{C}$ and the *post-critical set* $\mathcal{P}$ *of* $\mathcal{L}$ by

$$\mathcal{C} := \pi^{-1}\Bigl(\bigcup_{i,j\in S, i\neq j} K_i \cap K_j\Bigr) \qquad \text{and} \qquad \mathcal{P} := \bigcup_{n\in\mathbb{N}} \sigma^n(\mathcal{C}). \tag{3.1}$$

$\mathcal{L}$ is called *post-critically finite*, or *p.-c. f.* for short, if and only if $\mathcal{P}$ is a finite set.
(2) We set $V_0 := \pi(\mathcal{P})$, $V_m := \bigcup_{w\in W_m} F_w(V_0)$ for $m \in \mathbb{N}$ and $V_* := \bigcup_{m\in\mathbb{N}} V_m$.
(3) We set $K^I := K \setminus \overline{V_0}$ and $K^I_w := F_w(K^I)$ for $w \in W_*$.

V_0 should be considered as the *"boundary"* of the self-similar set K; recall that $K_w \cap K_v = F_w(V_0) \cap F_v(V_0)$ for any $w, v \in W_*$ with $\Sigma_w \cap \Sigma_v = \emptyset$ by [37, Proposition 1.3.5-(2)]. According to [37, Lemma 1.3.11], $V_{m-1} \subset V_m$ for any $m \in \mathbb{N}$, and if $V_0 \neq \emptyset$ then V_* is dense in K. Furthermore by [33, Lemma 2.11], K^I_w is open in K and $K^I_w \subset K^I$ for any $w \in W_*$.

Note that by [37, Theorem 1.6.2], K is connected if and only if any $i, j \in S$ admit $n \in \mathbb{N}$ and $\{i_k\}_{k=0}^n \subset S$ with $i_0 = i$ and $i_n = j$ such that $K_{i_{k-1}} \cap K_{i_k} \neq \emptyset$ for any $k \in \{1,\ldots,n\}$, and if K is connected then it is arcwise connected.

Definition 3.4. A Borel probability measure μ on K is called a *self-similar measure on* $\mathcal{L}$ *with uniform weight* if and only if $\mu = \frac{1}{\#S}\sum_{i\in S} \mu \circ F_i^{-1}$ (as Borel measures on K).

Such a measure μ always exists. Indeed, if ν is the Bernoulli measure on Σ with weight $\bigl(\frac{1}{\#S}\bigr)_{i\in S}$, then $\nu \circ \pi^{-1}$ is a self-similar measure on $\mathcal{L}$ with uniform weight; see [37, Section 1.4] for details. Moreover by [38, Theorem 1.2.7 and its proof], if $K \neq \overline{V_0}$

and μ is a self-similar measure on $\mathcal{L}$ with uniform weight, then $\mu(K_w) = \frac{1}{(\#S)^{|w|}}$ and $\mu(F_w(\overline{V_0})) = 0$ for any $w \in W_*$. In particular, if $K \neq \overline{V_0}$, then a self-similar measure μ on $\mathcal{L}$ with uniform weight is unique and satisfies $\mu = (\#S)^{|w|}\mu \circ F_w$ for any $w \in W_*$.

In the rest of this section, we assume that $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ is a post-critically finite self-similar structure with K connected and $\#K \geq 2$. In particular, $2 \leq \#V_0 < \infty$ and V_* is countably infinite and dense in K, so that $K \neq \overline{V_0} = V_0$. Before stating the main theorem of this section, we briefly recall the construction and basic properties of a self-similar Dirichlet form on such $\mathcal{L}$; see [37, Chapter 3] for details.

Let $D = (D_{xy})_{x,y \in V_0}$ be a real symmetric matrix of size $\#V_0$ (which we also regard as a linear operator on $\mathbb{R}^{V_0}$) such that

(D1) $\{u \in \mathbb{R}^{V_0} \mid Du = 0\} = \mathbb{R}\mathbf{1}_{V_0}$,
(D2) $D_{xy} \geq 0$ for any $x, y \in V_0$ with $x \neq y$.

We define $\mathcal{E}^{(0)}(u, v) := -\sum_{x,y \in V_0} D_{xy}u(y)v(x)$ for $u, v \in \mathbb{R}^{V_0}$, so that $(\mathcal{E}^{(0)}, \mathbb{R}^{V_0})$ is a Dirichlet form on $L^2(V_0, \#)$ with # denoting the counting measure on V_0. Furthermore let $r \in (0, \infty)$ and define

$$\mathcal{E}^{(m)}(u, v) := \frac{1}{r^m} \sum_{w \in W_m} \mathcal{E}^{(0)}(u \circ F_w|_{V_0}, v \circ F_w|_{V_0}), \qquad u, v \in \mathbb{R}^{V_m} \tag{3.2}$$

for each $m \in \mathbb{N}$. We assume that (D, r) is a *harmonic structure on $\mathcal{L}$*, i.e. (D, r) satisfies

$$\mathcal{E}^{(0)}(u, u) = \inf\{\mathcal{E}^{(1)}(v, v) \mid v \in \mathbb{R}^{V_1}, v|_{V_0} = u\} \qquad \text{for any } u \in \mathbb{R}^{V_0}.$$

Then $\mathcal{E}^{(m)}(u, u) = \min_{v \in \mathbb{R}^{V_n}, v|_{V_m} = u} \mathcal{E}^{(n)}(v, v)$ for any $m, n \in \mathbb{N} \cup \{0\}$ with $m \leq n$ and any $u \in \mathbb{R}^{V_m}$ by [37, Proposition 3.1.3], and $r \in (0, 1)$ by [37, Proposition 3.1.8]. In particular, $\{\mathcal{E}^{(m)}(u|_{V_m}, u|_{V_m})\}_{m \in \mathbb{N} \cup \{0\}}$ is non-decreasing and hence has the limit in $[0, \infty]$ for any $u \in C(K)$, and we define

$$\begin{aligned} \mathcal{F} &:= \{u \in C(K) \mid \lim_{m\to\infty} \mathcal{E}^{(m)}(u|_{V_m}, u|_{V_m}) < \infty\}, \\ \mathcal{E}(u, v) &:= \lim_{m\to\infty} \mathcal{E}^{(m)}(u|_{V_m}, v|_{V_m}) \in \mathbb{R}, \qquad u, v \in \mathcal{F}. \end{aligned} \tag{3.3}$$

$(\mathcal{E}, \mathcal{F})$ is easily seen to satisfy the following two self-similarity properties (note that $\mathcal{F} \cap C(K) = \mathcal{F}$ in the present setting):

(SSDF1) $\mathcal{F} \cap C(K) = \{u \in C(K) \mid u \circ F_i \in \mathcal{F} \text{ for any } i \in S\}$.
(SSDF2) For any $u \in \mathcal{F} \cap C(K)$,

$$\mathcal{E}(u, u) = \frac{1}{r} \sum_{i \in S} \mathcal{E}(u \circ F_i, u \circ F_i). \tag{3.4}$$

By [37, Theorem 3.3.4], $(\mathcal{E}, \mathcal{F})$ is a resistance form on K whose resistance metric $R : K \times K \to [0, \infty)$ is compatible with the original topology of K, and then [39, Corollary 6.4 and Theorem 9.4] imply that $(\mathcal{E}, \mathcal{F})$ is a non-zero symmetric regular Dirichlet form on $L^2(K, \mu)$, where μ denotes the self-similar measure on $\mathcal{L}$ with uniform weight. (See [37, Definition 2.3.1] or [39, Definition 3.1] for the definition of resistance forms and their resistance metrics.) $(\mathcal{E}, \mathcal{F})$ is local by [33, Lemma 3.4], and it easily follows from [39, (3.1)] and [23, Theorem 1.4.2-(iv)] that $\mathcal{F}_U = \{u \in \mathcal{F} \mid u|_{K \setminus U} = 0\}$ for any non-empty open subset U of K. Moreover, (CHK) and (2.3) hold by [39, Theorem 10.4] (or by [37, Section 5.1 and Proposition 5.2.6-(2)]).

The following condition $(\mathrm{UHK})_{d_{\mathrm{w}}}$ is required for the main theorem of this section. We set $\tau := \#S/r$ and $d_{\mathrm{s}} := 2\log_\tau \#S$ and call them the *time scaling factor* and the *spectral dimension of* $(\mathcal{L},\mu,\mathcal{E},\mathcal{F},r)$, respectively. τ naturally appears as the scaling factor for the time variable of the heat kernels; see Lemma 3.12 below.

Definition 3.5. Let $d_{\mathrm{w}} \in (1,\infty)$. We say that $(K,\mu,\mathcal{E},\mathcal{F})$ *satisfies* $(\mathrm{UHK})_{d_{\mathrm{w}}}$, or simply $(\mathrm{UHK})_{d_{\mathrm{w}}}$ *holds*, if and only if there exist a metric ρ on K compatible with the original topology of K and $c_{3.1}, c_{3.2} \in (0,\infty)$ such that for any $(t,x,y) \in (0,1] \times K \times K$,

$$p_t(x,y) \le c_{3.1} t^{-d_{\mathrm{s}}/2} \exp\biggl(-c_{3.2}\Bigl(\frac{\rho(x,y)^{d_{\mathrm{w}}}}{t}\Bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\biggr). \tag{3.5}$$

In fact, $(\mathrm{UHK})_{d_{\mathrm{w}}}$ is *always* satisfied in the present setting, as follows.

Proposition 3.6. *Set* $d_{\mathrm{w}}^R := \log_{1/r}\tau$. *Then* $(\mathrm{UHK})_{d_{\mathrm{w}}^R}$ *holds with the metric ρ given by the resistance metric R of* $(\mathcal{E},\mathcal{F})$.

Proof. This is a consequence of [39, Theorem 15.10], whose assumptions can be easily verified by using [37, Lemmas 3.3.5 and 4.2.4]; see [35, Lemma 2.5] for details. □

Remark 3.7. *The power* $\frac{1}{d_{\mathrm{w}}^R-1}$ *in* (3.5) *with* $d_{\mathrm{w}} = d_{\mathrm{w}}^R$, *which appears in* $(\mathrm{UHK})_{d_{\mathrm{w}}^R}$, *is not best possible in general*; see [30] for a sharp two-sided estimate of $p_t(x,y)$ in the present setting and see [28, Section 6] for a generalization of such an estimate in the framework of a metric measure space. We have introduced the condition $(\mathrm{UHK})_{d_{\mathrm{w}}}$ to state and prove our main results with the best exponent in the remainder estimates.

We also need the following definition for the main theorem of this section.

Definition 3.8. We define the *symmetry group* $\mathcal{G}$ *of* $(\mathcal{L},(D,r),\mu)$ by

$$\mathcal{G} := \left\{ g \;\middle|\; \begin{array}{l} g \text{ is a homeomorphism from } K \text{ to itself}, g(V_0)=V_0, \mu\circ g=\mu, \\ u\circ g, u\circ g^{-1} \in \mathcal{F} \text{ and } \mathcal{E}(u\circ g, u\circ g) = \mathcal{E}(u,u) \text{ for any } u\in\mathcal{F} \end{array} \right\}, \tag{3.6}$$

which clearly forms a subgroup of the group of homeomorphisms of K.

The following is the main theorem of this section. The subscripts $_{\mathrm{N}}$ and $_{\mathrm{D}}$ stand for the Neumann and Dirichlet boundary conditions on V_0, respectively.

Theorem 3.9. *Let* $q \in V_0$, $d_{\mathrm{w}} \in (1,\infty)$ *and suppose* $\{g(q) \mid g \in \mathcal{G}\} = V_0$ *and that* $(\mathrm{UHK})_{d_{\mathrm{w}}}$ *holds. Set* $\mathcal{Z}_{\mathrm{N}} := \mathcal{Z}_K$, $\mathcal{Z}_{\mathrm{D}} := \mathcal{Z}_{K^I}$, $n_{\mathrm{N}} := \frac{\#S\#V_0-\#V_1}{\#S-1}$ *and* $n_{\mathrm{D}} := -\frac{\#(V_1\setminus V_0)}{\#S-1}$ (note that $n_{\mathrm{N}} > 0 > n_{\mathrm{D}}$). *Then there exist* $c_{3.3} \in (0,\infty)$ *and continuous* $\log\tau$*-periodic functions* $G_0, G_1 : \mathbb{R} \to (0,\infty)$ *such that for any* $\mathrm{B} \in \{\mathrm{N},\mathrm{D}\}$, *as* $t \downarrow 0$,

$$\mathcal{Z}_{\mathrm{B}}(t) = t^{-d_{\mathrm{s}}/2}G_0(-\log t) + n_{\mathrm{B}}G_1(-\log t) + O\Bigl(\exp\bigl(-c_{3.3}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)\Bigr). \tag{3.7}$$

Remark 3.10. (1) G_0 in Theorem 3.9 can be easily constructed from [40, Theorem 2.4 and Corollary 2.5] or [37, Theorem 4.1.5], where the log-periodic principal term in the asymptotic behavior of $\mathcal{N}_{\mathrm{N}} := \mathcal{N}_K$ and $\mathcal{N}_{\mathrm{D}} := \mathcal{N}_{K^I}$ has been obtained. Moreover, G_0 is easily seen to be non-constant under certain mild conditions on $\mathcal{G}$ by [7, Theorem 4.4 and Section 5] or [37, Theorem 4.3.4 and Section 4.4] (see also [35, Lemma 3.5]) together with Karamata's Tauberian theorem [19, Section XIII.5, Theorem 2].
(2) The strict positivity of G_1 in Theorem 3.9 could be derived from the author's general result [33, Theorem 7.7], but we provide an alternative simpler proof of this fact; see the proof of Proposition 3.15 below.

(3) If $\mathcal{L} = (K, S, \{F_i\}_{i\in S})$ is an affine nested fractal satisfying $\#(F_i(V_0) \cap F_j(V_0)) \le 1$ for any $i, j \in S$ with $i \ne j$, then there exists a unique harmonic structure (D, r) on $\mathcal{L}$ compatible with its spatial symmetry, and $\{g(q) \mid g \in \mathcal{G}\} = V_0$ holds. Thus Theorem 3.9 applies, and if $\#V_0 \ge 3$ in addition then G_0 is non-constant; see [37, Sections 3.8 and 4.4], [35, Section 4] and references therein for details. Moreover, in this situation it has been proved in [20, Theorems 5.7 and 6.1] that there exist $d_{\mathrm{w}} \in (1, \infty)$ and a metric ρ on K compatible with the original topology of K such that $(\mathrm{UHK})_{d_{\mathrm{w}}}$ and the matching lower bound hold, so that d_{w} is the smallest for $(\mathrm{UHK})_{d_{\mathrm{w}}}$ to be valid.

In the rest of this section, we give the proof of Theorem 3.9, which requires a few intermediate steps. We start with the following easy lemmas. Recall (see Definition 2.9) that λ_1^U denotes the smallest eigenvalue of $-\Delta_U$ for a non-empty open subset U of K.

Lemma 3.11. *Let U, V be non-empty open subsets of K. If V is connected and $U \subsetneqq V$, then $\lambda_1^U > \lambda_1^V$. In particular, if $U \ne K$, then $\lambda_1^U > 0$.*

Proof. The latter assertion follows from the former since K is connected and $\lambda_1^K = 0$. Suppose V is connected and $U \subsetneqq V$. Note that then V is arcwise connected since K is locally arcwise connected by [37, Proposition 1.3.6] and the connectedness of K. Suppose $\lambda_1^U \le \lambda_1^V$ and let φ_1^U be an eigenfunction of $-\Delta_U$ with eigenvalue λ_1^U. Then $\varphi_1^U \in \mathcal{F}_V$ and $\mathcal{E}(\varphi_1^U, \varphi_1^U) \le \lambda_1^V \int_V (\varphi_1^U)^2 d\mu$ by $U \subset V$, so that φ_1^U would be an eigenfunction of $-\Delta_V$ with eigenvalue λ_1^V on account of (2.10). Now Lemma 2.10 would imply that either $\varphi_1^U > 0$ on V or $\varphi_1^U < 0$ on V, which would contradict $\varphi_1^U \in \mathcal{F}_U = \{u \in \mathcal{F} \mid u|_{K\setminus U} = 0\}$ since $U \subsetneqq V$. Thus $\lambda_1^U > \lambda_1^V$. □

Lemma 3.12. *Let U be a non-empty open subset of K^I and let $w \in W_*$. Then*

$$\mathcal{Z}_U(\tau^{|w|}t) = \mathcal{Z}_{F_w(U)}(t), \qquad t \in (0, \infty). \tag{3.8}$$

Proof. This is proved in exactly the same way as [36, Lemma 3.4]. □

Definition 3.13. For $m \in \mathbb{N} \cup \{0\}$ and $x \in V_m$, we define

$$W_{m,x} := \{w \in W_m \mid x \in K_w\}, \ \ n_{x,m} := \#W_{m,x} \ \text{ and } \ U_m^x := \{x\} \cup \bigcup\nolimits_{w\in W_{m,x}} K_w^I;$$

note that U_m^x is open in K. We also set $U^q := U_0^q = K^I \cup \{q\}$ for $q \in V_0$.

Proposition 3.14. *Let $q \in V_0$ and suppose $\{g(q) \mid g \in \mathcal{G}\} = V_0$. Let $m \in \mathbb{N} \cup \{0\}$ and $x \in V_m$. Then for any $t \in (0, \infty)$,*

$$\mathcal{Z}_{U_m^x}(t) = \mathcal{Z}_{U^q}(\tau^m t) + (n_{x,m} - 1)\mathcal{Z}_{\mathrm{D}}(\tau^m t). \tag{3.9}$$

Proof. For each $U \in \{U^q, K^I\}$, let $\{\varphi_n^U\}_{n\in\mathbb{N}}$ be a complete orthonormal system of $L^2(U, \mu|_U)$ consisting of eigenfunctions of $-\Delta_U$ with eigenvalues $\{\lambda_n^U\}_{n\in\mathbb{N}}$, which we use below to write down all the eigenfunctions of $-\Delta_{U_m^x}$. Let $\{a_k\}_{k=1}^{n_{x,m}} \subset \mathbb{R}^{W_{m,x}}$, $a_k = (a_{k,w})_{w\in W_{m,x}}$, be an orthonormal basis of $\mathbb{R}^{W_{m,x}}$ with $a_1 = n_{x,m}^{-1/2}\mathbf{1}_{W_{m,x}}$. For each $w \in W_{m,x}$, $x \in K_w \cap V_m = F_w(V_0)$, and hence by $\{g(q) \mid g \in \mathcal{G}\} = V_0$ we can choose $g_w \in \mathcal{G}$ so that $x = F_w(g_w(q))$. Now for $n \in \mathbb{N}$, we define $\varphi_{n,k} : K \to \mathbb{R}$, $k \in \{1, \dots, n_{x,m}\}$, by $\varphi_{n,k}|_{K\setminus U_m^x} := 0$ and

$$\varphi_{n,k}|_{K_w^I\cup\{x\}} := \begin{cases} n_{x,m}^{-1/2}(\#S)^{m/2}\varphi_n^{U^q} \circ g_w^{-1} \circ F_w^{-1}|_{K_w^I\cup\{x\}} & \text{if } k = 1 \\ a_{k,w}(\#S)^{m/2}\varphi_n^{K^I} \circ g_w^{-1} \circ F_w^{-1}|_{K_w^I\cup\{x\}} & \text{if } k \ge 2 \end{cases} \tag{3.10}$$

for each $w \in W_{m,x}$; note that the value at x of the right-hand side of (3.10) is independent of $w \in W_{m,x}$ by $\varphi_n^{K^I}(q) = 0$. Then the expression (3.10) of $\varphi_{n,k}$ extends to K_w for each $w \in W_{m,x}$, hence $\varphi_{n,k} \in C(K)$, and it follows from $\{g_w\}_{w\in W_{m,x}} \subset \mathcal{G}$, (SSDF1) and $\varphi_{n,k}|_{K\setminus U_m^x} = 0$ that $\varphi_{n,k} \in \mathcal{F}_{U_m^x}$.

Next we prove that $\{\varphi_{n,k}\}_{n\in\mathbb{N},\, k\in\{1,\dots,n_{x,m}\}}$ is a complete orthonormal system of $L^2(U_m^x, \mu|_{U_m^x})$. Indeed, it is easily seen to be orthonormal in $L^2(U_m^x, \mu|_{U_m^x})$ by a direct calculation using $\mu = (\#S)^{|w|}\mu \circ F_w$ and $\{g_w\}_{w\in W_{m,x}} \subset \mathcal{G}$. Let $u \in L^2(U_m^x, \mu|_{U_m^x})$ and suppose that $\int_{U_m^x} u\varphi_{n,k}d\mu = 0$ for any $(n,k) \in \mathbb{N} \times \{1,\dots,n_{x,m}\}$. Then for any $n \in \mathbb{N}$, $0 = n_{x,m}^{1/2}(\#S)^{m/2}\int_{U_m^x} u\varphi_{n,1}d\mu = \int_{U^q} \varphi_n^{U^q}\bigl(\sum_{w\in W_{m,x}} u \circ F_w \circ g_w\bigr)d\mu$, and hence $\sum_{w\in W_{m,x}} u \circ F_w \circ g_w = 0$ μ-a.e. by the completeness of $\{\varphi_n^{U^q}\}_{n\in\mathbb{N}}$. On the other hand, for each $n \in \mathbb{N}$, $\int_{U_m^x} u\varphi_{n,k}d\mu = 0$, $k \in \{2,\dots,n_{x,m}\}$, implies that $(\#S)^m\int_{K_w}(\varphi_n^{K^I} \circ g_w^{-1} \circ F_w^{-1})ud\mu = \int_{K^I}(u \circ F_w \circ g_w)\varphi_n^{K^I}d\mu$ is independent of $w \in W_{m,x}$, so that $u \circ F_w \circ g_w = u \circ F_v \circ g_v$ μ-a.e. for any $w, v \in W_{m,x}$ by the completeness of $\{\varphi_n^{K^I}\}_{n\in\mathbb{N}}$. Thus $0 = n_{x,m}^{-1}\sum_{v\in W_{m,x}} u\circ F_v\circ g_v = u \circ F_w \circ g_w$ μ-a.e. for any $w \in W_{m,x}$ and hence $u = 0$ μ-a.e. on U_m^x.

Finally, we show that for $n \in \mathbb{N}$, $\varphi_{n,k}$ is an eigenfunction of $-\Delta_{U_m^x}$ with eigenvalue $\tau^m\lambda_n^{U^q}$ for $k = 1$ and $\tau^m\lambda_n^{K^I}$ for $k \in \{2,\dots,n_{x,m}\}$. Let $n \in \mathbb{N}$. By using (SSDF2), $\{g_w\}_{w\in W_{m,x}} \subset \mathcal{G}$ and the fact that $\varphi_n^{U^q} \in \mathcal{D}[\Delta_{U^q}]$ and $-\Delta_{U^q}\varphi_n^{U^q} = \lambda_n^{U^q}\varphi_n^{U^q}$ (recall (2.7)), we easily see that $\mathcal{E}(\varphi_{n,1}, u) = \tau^m\lambda_1^{U^q}\int_{U_m^x} \varphi_{n,1}ud\mu$ for any $u \in \mathcal{F}_{U_m^x}$. For the proof for $k \in \{2,\dots,n_{x,m}\}$, for each $u : U_m^x \to \mathbb{R}$ we define $Pu : U_m^x \to \mathbb{R}$ by

$$Pu|_{K_w^I\cup\{x\}} := n_{x,m}^{-1}\sum_{v\in W_{m,x}} u\circ F_v\circ g_v\circ g_w^{-1}\circ F_w^{-1}|_{K_w^I\cup\{x\}}, \quad w \in W_{m,x}, \tag{3.11}$$

so that $Pu(x) := u(x)$ regardless of choices of $w \in W_{m,x}$ in (3.11). Identifying each $u \in \mathcal{F}_{U_m^x}$ with $u|_{U_m^x} \in C(U_m^x)$, we have $P(\mathcal{F}_{U_m^x}) \subset \mathcal{F}_{U_m^x}$ by virtue of (SSDF1) and $\{g_w\}_{w\in W_{m,x}} \subset \mathcal{G}$. Now let $k \in \{2,\dots,n_{x,m}\}$ and $u \in \mathcal{F}_{U_m^x}$. (SSDF2) and $\{g_w\}_{w\in W_{m,x}} \subset \mathcal{G}$ yield $\mathcal{E}(\varphi_{n,k}, Pu) = 0 = \int_{U_m^x}\varphi_{n,k}Pud\mu$, hence $\mathcal{E}(\varphi_{n,k}, u) = \mathcal{E}(\varphi_{n,k}, u-Pu)$ and $\int_{U_m^x}\varphi_{n,k}ud\mu = \int_{U_m^x}\varphi_{n,k}(u-Pu)d\mu$, which together with $\varphi_n^{K^I} \in \mathcal{D}[\Delta_{K^I}]$, $-\Delta_{K^I}\varphi_n^{K^I} = \lambda_n^{K^I}\varphi_n^{K^I}$, easily imply $\mathcal{E}(\varphi_{n,k}, u) = \tau^m\lambda_n^{K^I}\int_{U_m^x}\varphi_{n,k}ud\mu$ since $(u-Pu)\circ F_w \in \mathcal{F}_{K^I}$ for any $w \in W_{m,x}$ by $u-Pu \in \mathcal{F}_{U_m^x}$ and $(u-Pu)(x) = 0$.

Thus it follows that $\{\lambda_{n,k}\}_{n\in\mathbb{N},\, k\in\{1,\dots,n_{x,m}\}}$, $\lambda_{n,1} := \tau^m\lambda_n^{U^q}$ and $\lambda_{n,k} := \tau^m\lambda_n^{K^I}$, $k \in \{2,\dots,n_{x,m}\}$, gives an enumeration of all the eigenvalues of $-\Delta_{U_m^x}$ with each eigenvalue repeated according to its multiplicity, and hence (3.9) follows. □

Proposition 3.15. *Let $q \in V_0$, $d_{\mathrm{w}} \in (1,\infty)$ and suppose $\{g(q) \mid g \in \mathcal{G}\} = V_0$ and that* $(\mathrm{UHK})_{d_{\mathrm{w}}}$ *holds. Then there exist $c_{3.4} \in (0,\infty)$ and a continuous $\log\tau$-periodic function $G_1 : \mathbb{R} \to (0,\infty)$ such that, as $t \downarrow 0$,*

$$\mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t) = G_1(-\log t) + O\Bigl(\exp\bigl(-c_{3.4}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)\Bigr). \tag{3.12}$$

Remark 3.16. *The periodic function G_1 in the conclusion of Theorem* 3.9 *is nothing but G_1 given by Proposition* 3.15 *as above*; see the end of the proof of Theorem 3.9 below.

Proof. Let us verify the assumptions of Theorem 2.13 with $\alpha_0 = 0$ and $\gamma = \tau$ for $\mathcal{Z}_{U^q} - \mathcal{Z}_{\mathrm{D}}$. Since $n_{q,1} = 1$ by [35, Remark 6.4], $W_{1,q} = \{i\}$ and $U_1^q = K_i^I \cup \{q\}$ for some $i \in S$, and $q \in K_i \cap V_1 = F_i(V_0)$. Let $t \in (0,\infty)$. Recalling (2.4), by Proposition 3.14, Lemma 3.12, $U_1^q \subset U^q$ and Proposition 2.7-(1) we have

$$\begin{aligned}\mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t) - \bigl(\mathcal{Z}_{U^q}(\tau t) - \mathcal{Z}_{\mathrm{D}}(\tau t)\bigr) &= \mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t) - \mathcal{Z}_{U_1^q}(t) + \mathcal{Z}_{K_i^I}(t)\\ &= \int_K p_t^{U^q}\bigl(x,x\big|\{K\setminus\{q\},U_1^q\}\bigr)d\mu(x) \geq 0. \quad (3.13)\end{aligned}$$

Let ρ be the metric on K in $(\mathrm{UHK})_{d_{\mathrm{w}}}$ and let $x \in K$. Since $\rho(x,q)\vee\rho(x,F_i(V_0)\setminus\{q\}) \geq \rho(q,F_i(V_0)\setminus\{q\})/2$ and $(K\setminus K_i)\cup(F_i(V_0)\setminus\{q\}) \subset K\setminus U_1^q$, Proposition 2.7, (3.5) and $d_{\mathrm{s}} < 2$ imply that for $t \in (0,1]$,

$$\begin{aligned}0 &\leq p_t^{U^q}\bigl(x,x\big|\{K\setminus\{q\},U_1^q\}\bigr) \quad (3.14)\\ &\leq \begin{cases} 4c_{3.1}t^{-d_{\mathrm{s}}/2}\exp\Bigl(-c_{3.2}\bigl(\frac{\rho(q,F_i(V_0)\setminus\{q\})^{d_{\mathrm{w}}}}{2^{d_{\mathrm{w}}}t}\bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\Bigr) & \text{if } x \in K_i\\ 4c_{3.1}t^{-d_{\mathrm{s}}/2}\exp\Bigl(-c_{3.2}\bigl(\frac{\rho(q,x)^{d_{\mathrm{w}}}}{t}\bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\Bigr) & \text{if } x \in K\setminus K_i\end{cases}\\ &\leq 4c_{3.1}t^{-d_{\mathrm{s}}/2}\exp\bigl(-c_{3.4}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr),\end{aligned}$$

where $c_{3.4} := c_{3.2}\bigl(\rho(q,K\setminus U_1^q)/2\bigr)^{\frac{d_{\mathrm{w}}}{d_{\mathrm{w}}-1}}$. Now (3.13) and (3.14) together imply that (2.13) holds for $\mathcal{Z}_{U^q} - \mathcal{Z}_{\mathrm{D}}$ with $\alpha_0 = 0$, $\gamma = \tau$, $n = 1$, $\alpha_1 = -1$, $G_1 = 0$ and $\mathcal{R}(t) := 4c_{3.1}t^{-d_{\mathrm{s}}/2}\exp\bigl(-c_{3.4}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)$. Moreover $\sum_{j\in\mathbb{N}}\mathcal{R}(\tau^{-j}t) \leq c_{3.5}\exp\bigl(-c_{3.4}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)$ for any $t \in (0,\tau]$ for some $c_{3.5} \in (0,\infty)$, $0 \leq \mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t) \leq \mathcal{Z}_{U^q}(1)e^{\lambda_1^{U^q}}e^{-\lambda_1^{U^q}t}$ for any $t \in [1,\infty)$ by $p^{K^I} \leq p^{U^q}$ and Lemma 2.11, and $\lambda_1^{U^q} > 0$ by Lemma 3.11. Hence Theorem 2.13 applies to $\mathcal{Z}_{U^q} - \mathcal{Z}_{\mathrm{D}}$ to yield a continuous $\log\tau$-periodic function $G_1 : \mathbb{R} \to \mathbb{R}$ satisfying (3.12).

It remains to show that G_1 is $(0,\infty)$-valued. Note that U^q is connected since K^I is connected by $\{g(q) \mid g \in \mathcal{G}\} = V_0$ and [37, Proposition 1.6.9]. Therefore $\lambda_1^{K^I} > \lambda_1^{U^q}$ by Lemma 3.11, so that $\mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t) > 0$ for any $t \in [T,\infty)$ for some $T \in [1,\infty)$ in view of Lemma 2.11. Now since $\mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t) \geq \mathcal{Z}_{U^q}(\tau t) - \mathcal{Z}_{\mathrm{D}}(\tau t)$ for any $t \in (0,\infty)$ by (3.13), $\inf_{t\in(0,\tau T]}\bigl(\mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t)\bigr) = \min_{t\in[T,\tau T]}\bigl(\mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t)\bigr) > 0$, which together with (3.12) shows that G_1 is $(0,\infty)$-valued. □

Proposition 3.17. *Let $q \in V_0$, $d_{\mathrm{w}} \in (1,\infty)$ and suppose $\{g(q) \mid g \in \mathcal{G}\} = V_0$ and that* $(\mathrm{UHK})_{d_{\mathrm{w}}}$ *holds. Then there exists $c_{3.6} \in (0,\infty)$ such that, as $t \downarrow 0$,*

$$\mathcal{Z}_{\mathrm{N}}(t) - \mathcal{Z}_{\mathrm{D}}(t) = (\#V_0)\bigl(\mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t)\bigr) + O\Bigl(\exp\bigl(-c_{3.6}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)\Bigr), \quad (3.15)$$

$$\mathcal{Z}_{\mathrm{D}}(t) - (\#S)\mathcal{Z}_{\mathrm{D}}(\tau t) = \#(V_1\setminus V_0)\bigl(\mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t)\bigr) + O\Bigl(\exp\bigl(-c_{3.6}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)\Bigr). \quad (3.16)$$

Proof. Let ρ be the metric on K in $(\mathrm{UHK})_{d_{\mathrm{w}}}$ and set $\delta := \min_{x\in V_1}\rho(x,K\setminus U_1^x)/2 > 0$, so that $\rho(z,x)\vee\rho(z,y) \geq \delta$ for any $z \in K$ and any $x,y \in V_1$ with $x \neq y$. Let $t \in (0,\infty)$. Since $\mathcal{Z}_{U^x} = \mathcal{Z}_{U^q}$ for any $x \in V_0$ by Proposition 3.14, we see from Lemma 2.12, Proposition 2.7 and (3.5) that

$$\begin{aligned}&\mathcal{Z}_{\mathrm{N}}(t) - \mathcal{Z}_{\mathrm{D}}(t) - (\#V_0)\bigl(\mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t)\bigr) \quad (3.17)\\ &= \int_K\Bigl(p_t(y,y) - p_t^{K^I}(y,y) - \sum_{x\in V_0}\bigl(p_t^{U^x}(y,y) - p_t^{K^I}(y,y)\bigr)\Bigr)d\mu(y)\end{aligned}$$

$$
\begin{aligned}
&= \int_K \Bigl(\sum_{\emptyset \neq V \subset V_0} \sum_{A \subset V} (-1)^{\#A} p_t^{K^I \cup (V\setminus A)}(y,y) - \sum_{x \in V_0} p_t^{U^x}(y, y | \{K \setminus \{x\}\}) \Bigr) d\mu(y) \\
&= \int_K \Bigl(\sum_{\emptyset \neq V \subset V_0} p_t^{K^I \cup V}(y, y | \{K \setminus \{x\}\}_{x \in V}) - \sum_{x \in V_0} p_t^{U^x}(y, y | \{K \setminus \{x\}\}) \Bigr) d\mu(y) \\
&= \sum\nolimits_{V \subset V_0, \#V \geq 2} \int_K p_t^{K^I \cup V}(y, y | \{K \setminus \{x\}\}_{x \in V}) d\mu(y) \ (\geq 0) \\
&\leq \sum\nolimits_{V \subset V_0, \#V \geq 2} \int_K 2 \min_{x \in V} \Bigl\{ \sup_{s \in [t/2, t]} p_s(x, y) \Bigr\} d\mu(y) \\
&\leq 2^{\#V_0 + 2} c_{3.1} t^{-d_{\mathrm{s}}/2} \exp\bigl(-c_{3.2} (\delta^{d_{\mathrm{w}}}/t)^{\frac{1}{d_{\mathrm{w}}-1}} \bigr)
\end{aligned}
$$

where the inequality in the last line of (3.17) is valid only for $t \in (0, 1]$.

On the other hand, $(\#S)\mathcal{Z}_{\mathrm{D}}(\tau t) = \sum_{i \in S} \mathcal{Z}_{K_i^I}(t) = \mathcal{Z}_{K \setminus V_1}(t)$ and $n_{x,1}\mathcal{Z}_{\mathrm{D}}(\tau t) = \sum_{i \in W_{1,x}} \mathcal{Z}_{K_i^I}(t) = \mathcal{Z}_{U_1^x \setminus \{x\}}(t)$ for any $x \in V_1$ by Lemmas 3.12 and 2.6, and hence $\mathcal{Z}_{U^q}(\tau t) - \mathcal{Z}_{\mathrm{D}}(\tau t) = \mathcal{Z}_{U_1^x}(t) - \mathcal{Z}_{U_1^x \setminus \{x\}}(t)$ for any $x \in V_1$ by Proposition 3.14. Therefore similarly to (3.17), setting $K_1^V := (K \setminus V_1) \cup V$ for each $V \subset V_1 \setminus V_0$, from Lemmas 2.12, 2.6, Proposition 2.7 and (3.5) we obtain

$$
\begin{aligned}
&\mathcal{Z}_{\mathrm{D}}(t) - (\#S)\mathcal{Z}_{\mathrm{D}}(\tau t) - \#(V_1 \setminus V_0)\bigl(\mathcal{Z}_{U^q}(\tau t) - \mathcal{Z}_{\mathrm{D}}(\tau t) \bigr) \qquad (3.18) \\
&= \int_K \Bigl(p_t^{K^I}(y,y) - p_t^{K \setminus V_1}(y,y) - \sum_{x \in V_1 \setminus V_0} \bigl(p_t^{U_1^x}(y,y) - p_t^{U_1^x \setminus \{x\}}(y,y) \bigr) \Bigr) d\mu(y) \\
&= \int_K \Bigl(\sum_{\emptyset \neq V \subset V_1 \setminus V_0} \sum_{A \subset V} (-1)^{\#A} p_t^{K_1^{V \setminus A}}(y,y) - \sum_{x \in V_1 \setminus V_0} p_t^{U_1^x}(y, y | K \setminus \{x\}) \Bigr) d\mu(y) \\
&= \int_K \Bigl(\sum_{\emptyset \neq V \subset V_1 \setminus V_0} p_t^{K_1^V}(y, y | \{K \setminus \{x\}\}_{x \in V}) - \sum_{x \in V_1 \setminus V_0} p_t^{U_1^x}(y, y | K \setminus \{x\}) \Bigr) d\mu(y) \\
&= \sum\nolimits_{V \subset V_1 \setminus V_0, \#V \geq 2} \int_K p_t^{K_1^V}(y, y | \{K \setminus \{x\}\}_{x \in V}) d\mu(y) \ (\geq 0) \\
&\leq \sum\nolimits_{V \subset V_1 \setminus V_0, \#V \geq 2} \int_K 2 \min_{x \in V} \Bigl\{ \sup_{s \in [t/2, t]} p_s(x, y) \Bigr\} d\mu(y) \\
&\leq 2^{\#(V_1 \setminus V_0) + 2} c_{3.1} t^{-d_{\mathrm{s}}/2} \exp\bigl(-c_{3.2} (\delta^{d_{\mathrm{w}}}/t)^{\frac{1}{d_{\mathrm{w}}-1}} \bigr)
\end{aligned}
$$

where again the inequality in the last line of (3.18) follows only for $t \in (0, 1]$.

Now with $c_{3.6} := c_{3.2}\delta^{\frac{d_{\mathrm{w}}}{d_{\mathrm{w}}-1}}/2$, (3.15) is immediate from (3.17), whereas (3.16) follows from (3.18), (3.13) and (3.14) since $c_{3.4} \geq c_{3.2}\delta^{\frac{d_{\mathrm{w}}}{d_{\mathrm{w}}-1}}$ for $c_{3.4}$ as in (3.14). $\square$

Proof of Theorem 3.9. (3.16) and Proposition 3.15 together yields (2.13) for $\mathcal{Z}_{\mathrm{D}}$ with $\alpha_0 = d_{\mathrm{s}}/2$, $\gamma = \tau$, $n = 1$, $\alpha_1 = 0$, $-n_{\mathrm{D}}(\#S - 1)G_1$ in place of G_1, and $\mathcal{R}(t) := c_{3.7} \exp\bigl(-(c_{3.4} \wedge c_{3.6}) t^{-\frac{1}{d_{\mathrm{w}}-1}} \bigr)$ for some $c_{3.7} \in (0, \infty)$. Moreover, $\lambda_1^{K^I} > 0$ by Lemma 3.11 and $0 \leq \mathcal{Z}_D(t) \leq \mathcal{Z}_D(1) e^{\lambda_1^{K^I}} e^{-\lambda_1^{K^I} t}$ for any $t \in [1, \infty)$ by Lemma 2.11. Hence we can apply Theorem 2.13 to $\mathcal{Z}_{\mathrm{D}}$ to conclude that there exists a continuous $\log \tau$-periodic function $G_0 : \mathbb{R} \to \mathbb{R}$ such that (3.7) holds for B $=$ D with G_1 as in Proposition 3.15 and $c_{3.3} := c_{3.4} \wedge c_{3.6}$. Then (3.7) for B $=$ N follows from that for

$\mathrm{B} = \mathrm{D}$, (3.15) and Proposition 3.15. Finally, since $\mathcal{Z}_{U^q} - \mathcal{Z}_{\mathrm{D}} \geq 0$ by Proposition 2.7-(1), $\mathcal{Z}_{\mathrm{D}}(t) \geq (\#S)\mathcal{Z}_{\mathrm{D}}(\tau t)$ and hence $t^{d_{\mathrm{s}}/2}\mathcal{Z}_{\mathrm{D}}(t) \geq (\tau t)^{d_{\mathrm{s}}/2}\mathcal{Z}_{\mathrm{D}}(\tau t)$ for any $t \in (0,\infty)$ by (3.18), and it follows that $\inf_{t\in(0,1]} t^{d_{\mathrm{s}}/2}\mathcal{Z}_{\mathrm{D}}(t) = \min_{t\in[\tau^{-1},1]} t^{d_{\mathrm{s}}/2}\mathcal{Z}_{\mathrm{D}}(t) > 0$, which together with (3.7) shows that G_0 is $(0,\infty)$-valued. □

In fact, by a very similar argument we can also prove the following theorem.

Definition 3.18. Let $m \in \mathbb{N}$ and let $X \subset W_m$ be non-empty. We set $d_X := \frac{2}{m}\log_\tau \#X$,

$$\Sigma[X] := \bigl\{\omega \in \Sigma \bigm| [\sigma^{mn}(\omega)]_m \in X \text{ for any } n \in \mathbb{N}\cup\{0\}\bigr\} \quad \text{and} \quad K[X] := \pi(\Sigma[X]).$$

Theorem 3.19. *Let $q \in V_0$, $d_{\mathrm{w}} \in (1,\infty)$, suppose $\{g(q) \mid g \in \mathcal{G}\} = V_0$ and that $(\mathrm{UHK})_{d_{\mathrm{w}}}$ holds, and let $n_{\mathrm{N}}, n_{\mathrm{D}}, G_0, G_1$ be as in Theorem 3.9. Let $m \in \mathbb{N}$, let $X \subsetneqq W_m$ be such that $\#X \geq 2$ and set $\mathcal{Z}_{X,\mathrm{N}} := \mathcal{Z}_{K\setminus K[X]}$, $\mathcal{Z}_{X,\mathrm{D}} := \mathcal{Z}_{K^I\setminus K[X]}$, $n_{X,\mathrm{N}} := \frac{\#X\#(K[X]\cap V_0)-\#(K[X]\cap V_m)}{\#X-1}$ and $n_{X,\mathrm{D}} := -\frac{\#(K[X]\cap(V_m\setminus V_0))}{\#X-1}$* (note that $n_{X,\mathrm{N}} \geq 0 \geq n_{X,\mathrm{D}}$). *Then there exist $c_{3.8} \in (0,\infty)$ which is independent of m and X and a continuous $m\log\tau$-periodic function $G_X : \mathbb{R} \to (0,\infty)$ such that for any* $\mathrm{B} \in \{\mathrm{N},\mathrm{D}\}$, *as $t \downarrow 0$,*

$$\mathcal{Z}_{X,\mathrm{B}}(t) = t^{-d_{\mathrm{s}}/2}G_0(-\log t) - t^{-d_X/2}G_X(-\log t) + (n_{\mathrm{B}} - n_{X,\mathrm{B}})G_1(-\log t) + O\Bigl(\exp\bigl(-c_{3.8}(c_{m,X}/t)^{\frac{1}{d_{\mathrm{w}}-1}}\bigr)\Bigr), \quad (3.19)$$

where $c_{m,X} := \tau^{-m} \wedge \bigl(\min_{x\in V_m\setminus K[X]}\rho(x,K[X]) \wedge \min_{x\in V_m}\rho(x,V_m\setminus\{x\})\bigr)^{d_{\mathrm{w}}}$ with ρ the metric on K given in $(\mathrm{UHK})_{d_{\mathrm{w}}}$.

The following proposition is the core of the proof of Theorem 3.19.

Proposition 3.20. *Let $q \in V_0$ and suppose $\{g(q) \mid g \in \mathcal{G}\} = V_0$. Let $m \in \mathbb{N}$ and let $X \subsetneqq W_m$ be non-empty. Then for any $t \in (0,\infty)$,*

$$\mathcal{Z}_{\mathrm{N}}(t) - \mathcal{Z}_{X,\mathrm{N}}(t) - \bigl(\mathcal{Z}_{\mathrm{D}}(t) - \mathcal{Z}_{X,\mathrm{D}}(t)\bigr) - \#(K[X]\cap V_0)\bigl(\mathcal{Z}_{U^q}(t) - \mathcal{Z}_{\mathrm{D}}(t)\bigr)$$
$$= \sum_{\substack{V\subset V_0 \\ \#V\geq 2 \text{ or } V\not\subset K[X]}} \int_K p_t^{K^I\cup V}(y,y|\{K\setminus K[X]\}\cup\{K\setminus\{x\}\}_{x\in V})\,d\mu(y), \quad (3.20)$$

$$\mathcal{Z}_{\mathrm{D}}(t) - \mathcal{Z}_{X,\mathrm{D}}(t) - (\#X)\bigl(\mathcal{Z}_{\mathrm{D}}(\tau^m t) - \mathcal{Z}_{X,\mathrm{D}}(\tau^m t)\bigr) - \#\bigl(K[X]\cap(V_m\setminus V_0)\bigr)\bigl(\mathcal{Z}_{U^q}(\tau^m t) - \mathcal{Z}_{\mathrm{D}}(\tau^m t)\bigr) \quad (3.21)$$
$$= \sum_{\substack{V\subset V_m\setminus V_0 \\ \#V\geq 2 \text{ or } V\not\subset K[X]}} \int_K p_t^{(K\setminus V_m)\cup V}(y,y|\{K\setminus K[X]\}\cup\{K\setminus\{x\}\}_{x\in V})\,d\mu(y).$$

Proposition 3.20 follows from Lemmas 2.6, 2.12, 3.12 and Proposition 3.14 very similarly to (3.17) and (3.18). Theorem 3.19 can be proved by using Propositions 2.7, 3.15, 3.20 and $(\mathrm{UHK})_{d_{\mathrm{w}}}$ to apply Theorem 2.13 to $\mathcal{Z}_{\mathrm{D}} - \mathcal{Z}_{X,\mathrm{D}}$. We omit the details.

4. Sierpiński carpets

Our main concern in the rest of this paper is the case of generalized Sierpiński carpets, which are among the most typical examples of *infinitely ramified* self-similar fractals and have been intensively studied e.g. in [1–6,42,38,29,33,31].

In this section, we first collect fundamental facts concerning generalized Sierpiński carpets and their canonical self-similar Dirichlet form and then state our main theorems (Theorems 4.10 and 4.14) of asymptotic expansion of the partition function for them. The proof of Theorems 4.10 and 4.14 is given in the next section.

We fix the following setting throughout this and the next sections.

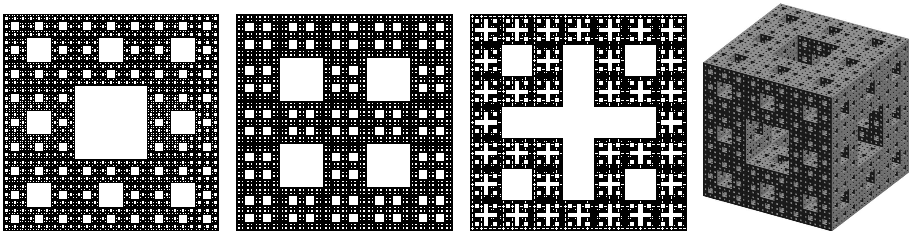

Fig. 2. Sierpiński carpet, some other generalized Sierpiński carpets with $d = 2$ and Menger sponge

Framework 4.1. Let $d, l \in \mathbb{N}$, $d \geq 2$, $l \geq 2$ and set $Q_0 := [0,1]^d$. Let $S \subset \{0,1,\dots,l-1\}^d$ be non-empty, and for each $i \in S$ define $f_i : \mathbb{R}^d \to \mathbb{R}^d$ by $f_i(x) := l^{-1}i + l^{-1}x$. Set $Q_1 := \bigcup_{i\in S} f_i(Q_0)$, which satisfies $Q_1 \subset Q_0$. Let K be the *self-similar set associated with* $\{f_i\}_{i\in S}$, i.e. the unique non-empty compact subset of $\mathbb{R}^d$ such that $K = \bigcup_{i\in S} f_i(K)$, and set $F_i := f_i|_K$ for $i \in S$, so that $\mathrm{GSC}(d,l,S) := (K, S, \{F_i\}_{i\in S})$ is a self-similar structure. Also let ρ be the Euclidean metric on K given by $\rho(x,y) := |x-y|$, set $d_{\mathrm{f}} := \log_l \#S$ and let μ be the self-similar measure on $\mathcal{L}$ with uniform weight.

Recall that d_{f} is the Hausdorff dimension of (K,ρ) and that μ is a constant multiple of the d_{f}-dimensional Hausdorff measure on (K,ρ); see e.g. [37, Theorem 1.5.7 and Proposition 1.5.8].

The following definition is essentially due to M. T. Barlow and R. F. Bass [5].

Definition 4.2 (Generalized Sierpiński carpets, [6, Subsection 2.2])**.** $\mathrm{GSC}(d,l,S)$ is called a *generalized Sierpiński carpet* if and only if S satisfies the following conditions:

(GSC1) (Symmetry) $f(Q_1) = Q_1$ for any isometry f of $\mathbb{R}^d$ with $f(Q_0) = Q_0$.
(GSC2) (Connectedness) Q_1 is connected.
(GSC3) (Non-diagonality) $\mathrm{int}_{\mathbb{R}^d}\bigl(Q_1 \cap \prod_{k=1}^d [(i_k-\varepsilon_k)l^{-1}, (i_k+1)l^{-1}]\bigr)$ is either empty or connected for any $(i_k)_{k=1}^d \in \mathbb{Z}^d$ and any $(\varepsilon_k)_{k=1}^d \in \{0,1\}^d$.
(GSC4) (Borders included) $\{(x_1, 0, \dots, 0) \in \mathbb{R}^d \mid x_1 \in [0,1]\} \subset Q_1$.

As special cases of Definition 4.2, $\mathrm{GSC}(2,3,S_{\mathrm{SC}})$ and $\mathrm{GSC}(3,3,S_{\mathrm{MS}})$ are called the *Sierpiński carpet* and the *Menger sponge*, respectively, where $S_{\mathrm{SC}} := \{0,1,2\}^2 \setminus \{(1,1)\}$ and $S_{\mathrm{MS}} := \bigl\{(i_1,i_2,i_3) \in \{0,1,2\}^3 \bigm| \sum_{k=1}^3 \mathbf{1}_{\{1\}}(i_k) \leq 1\bigr\}$ (see Fig. 2 above).

We remark that there are several equivalent ways of stating the non-diagonality condition, as in the following proposition.

Proposition 4.3 ([34, §2])**.** *Set* $|x|_1 := \sum_{k=1}^d |x_k|$ *for* $x = (x_k)_{k=1}^d \in \mathbb{R}^d$. *Then* (GSC3) *is equivalent to any one of the following three conditions:*

$(\mathrm{ND})_{\mathbb{N}}$ $\mathrm{int}_{\mathbb{R}^d}\bigl(Q_1 \cap \prod_{k=1}^d [(i_k-1)l^{-m}, (i_k+1)l^{-m}]\bigr)$ *is either empty or connected for any* $m \in \mathbb{N}$ *and any* $(i_k)_{k=1}^d \in \{1,\dots,l^m-1\}^d$.
$(\mathrm{ND})_2$ *The case of* $m = 2$ *of* $(\mathrm{ND})_{\mathbb{N}}$ *holds.*
(NDF) *For any* $i, j \in S$ *with* $f_i(Q_0) \cap f_j(Q_0) \neq \emptyset$ *there exists* $\{n(k)\}_{k=0}^{|i-j|_1} \subset S$ *such that* $n(0) = i$, $n(|i-j|_1) = j$ *and* $|n(k) - n(k+1)|_1 = 1$ *for any* $k \in \{0,\dots,|i-j|_1 - 1\}$.

Remark 4.4. Only the case of $m = 1$ of $(\mathrm{ND})_{\mathbb{N}}$ was assumed in the original definition of generalized Sierpiński carpets in [5, Section 2], but Barlow, Bass, Kumagai and

Teplyaev [6] have recently realized that it is too weak for [5, Proof of Theorem 3.19] and has to be replaced by $(\mathrm{ND})_{\mathbb{N}}$ (or equivalently, by (GSC3)).

In the rest of this section, we assume that $\mathcal{L} := \mathrm{GSC}(d, l, S) = (K, S, \{F_i\}_{i\in S})$ is a generalized Sierpiński carpet. Then we easily see the following proposition.

Proposition 4.5. *Set* $S_{k,\varepsilon} := \{(i_n)_{n=1}^d \in S \mid i_k = (l-1)\varepsilon\}$ *for* $k \in \{1, 2, \dots, d\}$ *and* $\varepsilon \in \{0, 1\}$. *Then* $\mathcal{P} = \bigcup_{k=1}^d (S_{k,0}^{\mathbb{N}} \cup S_{k,1}^{\mathbb{N}})$ *and* $\overline{V_0} = V_0 = K \setminus (0,1)^d \neq K$.

There are two established ways of constructing a non-degenerate μ-symmetric diffusion on K, or equivalently, a non-zero conservative local regular Dirichlet form on $L^2(K, \mu)$, one by Barlow and Bass [1,5] using the reflecting Brownian motions on the domains approximating K, and the other by Kusuoka and Zhou [42] based on graph approximations. It had been a long-standing problem to prove that the constructions in [1,5] and in [42] give rise to the same diffusion on K, which Barlow, Bass, Kumagai and Teplyaev [6] have finally solved by proving the uniqueness of a non-zero conservative symmetric regular Dirichlet form on $L^2(K, \mu)$ possessing certain local symmetry. As a consequence of the results in [6], after some additional arguments in [31,36] we have the following unique existence of a canonical self-similar Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$. Recall the discussion following (3.3) for (SSDF1) and (SSDF2).

Definition 4.6. We define $\mathcal{G}_0 := \{f|_K \mid f \text{ is an isometry of } \mathbb{R}^d \text{ with } f(Q_0) = Q_0\}$, which forms a subgroup of the group of homeomorphisms of K by virtue of (GSC1).

Theorem 4.7 ([6, Theorems 1.2 and 4.32], cf. [31, Proposition 5.1], [36, Proposition 5.9])**.** *There exists a unique (up to constant multiples of $\mathcal{E}$) non-zero conservative symmetric regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ satisfying* (SSDF1), (SSDF2) *for some* $r \in (0, \infty)$ *and the following condition:*

(GSCDF) *If* $u \in \mathcal{F} \cap C(K)$ *and* $g \in \mathcal{G}_0$ *then* $u \circ g \in \mathcal{F}$ *and* $\mathcal{E}(u \circ g, u \circ g) = \mathcal{E}(u, u)$.

Moreover, $r \in (0, \infty)$ *for which* (SSDF2) *holds is unique and satisfies* $r \leq l^{-2}\#S$.

$(\mathcal{E}, \mathcal{F})$ is local by [33, Lemma 3.4], and $(K, \mu, \mathcal{E}, \mathcal{F})$ also satisfies (CHK) and (2.3) by the following theorem. As before we define the *time scaling factor* τ and the *spectral dimension* d_{s} *of* $(\mathcal{L}, \mu, \mathcal{E}, \mathcal{F}, r)$ by $\tau := \#S/r$ and $d_{\mathrm{s}} := 2\log_\tau \#S$, respectively.

Theorem 4.8 ([5, Theorem 1.3], [6, Theorem 4.30 and Remark 4.33])**.** $(K, \mu, \mathcal{E}, \mathcal{F})$ *satisfies* (CHK) *and there exist* $c_{4.1}, c_{4.2} \in (0, \infty)$ *such that, with* $d_{\mathrm{w}} := \log_l \tau$ (note that $d_{\mathrm{w}} \geq 2$ and that $d_{\mathrm{f}}/d_{\mathrm{w}} = d_{\mathrm{s}}/2$)*, for any* $(t, x, y) \in (0, 1] \times K \times K$,

$$\frac{c_{4.1}}{t^{d_{\mathrm{f}}/d_{\mathrm{w}}}} \exp\biggl(-\Bigl(\frac{\rho(x,y)^{d_{\mathrm{w}}}}{c_{4.1}t}\Bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\biggr) \leq p_t(x,y) \leq \frac{c_{4.2}}{t^{d_{\mathrm{f}}/d_{\mathrm{w}}}} \exp\biggl(-\Bigl(\frac{\rho(x,y)^{d_{\mathrm{w}}}}{c_{4.2}t}\Bigr)^{\frac{1}{d_{\mathrm{w}}-1}}\biggr). \tag{4.1}$$

Remark 4.9. *The strict inequality* $d_{\mathrm{w}} > 2$ *holds if* $\#S < l^d$. In the case of $d = 2$, this estimate follows from [3, Proof of Proposition 5.2] (see also [4, (2.5)]), whereas for $d \geq 3$ this fact is stated in [5, Remarks 5.4-1.] just without proof.

Now we are in the stage of stating our main theorems of asymptotic expansion of the partition function on generalized Sierpiński carpets. Recall that we set $K^I := K \setminus V_0$.

Theorem 4.10. *Set* $d_k := \log_l \#\bigl(S \cap (\mathbb{Z}^{d-k} \times \{0\}^k)\bigr)$ *for* $k \in \{0, 1, \dots, d\}$ $(d_0 = d_{\mathrm{f}})$*,* $\mathcal{Z}_{\mathrm{N}} := \mathcal{Z}_K$*,* $\mathcal{Z}_{\mathrm{D}} := \mathcal{Z}_{K^I}$*,* $n_{\mathrm{N}} := \frac{\#\{(i,j)\in S\times S \mid |i-j|=1\}}{2\#(S\setminus(\mathbb{Z}^{d-1}\times\{0\}))}$ *and* $n_{\mathrm{D}} := n_{\mathrm{N}} - 2d$ (note that

$n_{\mathrm{N}} > 0 > n_{\mathrm{D}}$). *Then there exist* $c_{4.3} \in (0,\infty)$ *and continuous* $\log\tau$*-periodic functions* $G_{\mathrm{B},k} : \mathbb{R} \to \mathbb{R}$, $\mathrm{B} \in \{\mathrm{N},\mathrm{D}\}$, $k \in \{0,\ldots,d\}$, *such that for any* $\mathrm{B} \in \{\mathrm{N},\mathrm{D}\}$, *as* $t \downarrow 0$,

$$\mathcal{Z}_{\mathrm{B}}(t) = \sum_{k=0}^{d} t^{-d_k/d_{\mathrm{w}}} G_{\mathrm{B},k}(-\log t) + O\Bigl(\exp\bigl(-c_{4.3}t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr)\Bigr). \tag{4.2}$$

Moreover, $G_{\mathrm{N},0}, G_{\mathrm{N},1}$ *are* $(0,\infty)$*-valued,* $G_{\mathrm{N},0} = G_{\mathrm{D},0}$ *and* $n_{\mathrm{N}}^{-1} G_{\mathrm{N},1} = n_{\mathrm{D}}^{-1} G_{\mathrm{D},1}$.

Note that $d_{d-1} = 1$ and $d_d = 0$ by (GSC4) and that for each $k \in \{0,1,\ldots,d\}$, d_k is the box-counting and Hausdorff dimensions of $K \cap (\mathbb{R}^{d-k} \times \{0\}^k)$ with respect to ρ by [33, Propositions 2.24 and 6.7] and [37, Corollary 1.5.9] in view of $K \cap (\mathbb{R}^{d-k} \times \{0\}^k) = K[S \cap (\mathbb{Z}^{d-k} \times \{0\}^k)]$ with the notation of Definition 3.18. Note also that $d_k > d_{k+1}$ for any $k \in \{0,\ldots,d-1\}$ since $\bigl((l-1)\mathbf{1}_{\{1,\ldots,d-k\}}(j)\bigr)_{j=1}^{d} \in S \cap (\mathbb{Z}^{d-k} \times \{0\}^k) \setminus (\mathbb{Z}^{d-k-1} \times \{0\}^{k+1})$ by (GSC4) and (GSC1).

Remark 4.11. (1) As remarked after Theorem 1.1 in the introduction, $G_{\mathrm{D},0}$ in Theorem 4.10 was obtained by Hambly [29, Theorem 4.1] where it was proved that

$$\mathcal{Z}_{\mathrm{D}}(t) = t^{-d_{\mathrm{f}}/d_{\mathrm{w}}} G_{\mathrm{D},0}(-\log t) + O(t^{-d_1/d_{\mathrm{w}}}) \qquad \text{as } t \downarrow 0, \tag{4.3}$$

and similarly for $\mathcal{Z}_{\mathrm{N}}$ by a slight modification of [29, Proof of Theorem 4.1] (or by [33, Theorem 5.11]). Unfortunately, however, *a log-periodic principal order term similar to* (4.3) *for the asymptotics of* $\mathcal{N}_{\mathrm{N}} := \mathcal{N}_K$ *and* $\mathcal{N}_{\mathrm{D}} := \mathcal{N}_{K^I}$ *is* ***not*** *known when* $\#S < l^d$.
(2) The strict positivity of $G_{\mathrm{N},1}$ (the strict negativity of $G_{\mathrm{D},1}$) in Theorem 4.10 follows from [33, Theorem 7.7] (see [33, Theorem 8.5]), but we give an alternative simpler proof of this fact in the next section as a special case of Theorem 4.14 below.

Remark 4.12. It should be possible to prove asymptotics analogous to (3.19) of Theorem 3.19 for generalized Sierpiński carpets, but the statement and the proof would get much more complicated than for Theorem 3.19 because of the possible complexity of the boundary set $K[X]$ which does not arise in the setting of the previous section. Since the simplest case of the Neumann and Dirichlet boundary conditions on V_0 treated in Theorem 4.10 is already quite involved, we content ourselves with just this case.

In fact, the assertion of Theorem 4.10 for $\mathcal{Z}_{\mathrm{D}}$ is a special case of the following more general theorem, which requires the following definition.

Definition 4.13. Let $\varepsilon = (\varepsilon_k)_{k=1}^d \in \{0,1\}^d$, $m \in \mathbb{N} \cup \{0\}$ and $i = (i_k)_{k=1}^d \in \mathbb{R}^d$.
(1) We set $|\varepsilon| := \sum_{k=1}^d \varepsilon_k$, $\varepsilon^{-1}(j) := \{k \in \{1,\ldots,d\} \mid \varepsilon_k = j\}$ for $j \in \{0,1\}$, $S_\varepsilon := S \cap (\mathbb{Z}^{\varepsilon^{-1}(0)} \times \{0\}^{\varepsilon^{-1}(1)})$, $\mathcal{I}_m^{i,\varepsilon} := \{(k, i_k + jl^{-m-1}) \mid k \in \varepsilon^{-1}(0), j \in \{1,\ldots,l-1\}\}$, $\mathcal{I}_i(J) := \{(k,i_k) \mid k \in J\}$ for $J \subset \{1,\ldots,d\}$, and $\mathcal{I}_{i,\varepsilon} := \mathcal{I}_i(\varepsilon^{-1}(1))$.
(2) We define $R_m^{i,\varepsilon} := \prod_{k=1}^d [i_k - \varepsilon_k l^{-m}, i_k + l^{-m}]$, $U_m^{i,\varepsilon} := K \cap \operatorname{int}_{\mathbb{R}^d} R_m^{i,\varepsilon}$, $W_m^{i,\varepsilon} := \{w \in W_m \mid K_w \subset R_m^{i,\varepsilon}\}$, $\hat{R}_m^{i,\varepsilon} := \mathbb{R}^{\varepsilon^{-1}(0)} \times \prod_{k\in\varepsilon^{-1}(1)} [i_k - l^{-m-1}, i_k + l^{-m-1}]$ and $\hat{U}_m^{i,\varepsilon} := K \cap \operatorname{int}_{\mathbb{R}^d}(R_m^{i,\varepsilon} \cap \hat{R}_m^{i,\varepsilon})$. Note that $U_m^{i,\varepsilon} \neq \emptyset$ if and only if $W_m^{i,\varepsilon} \neq \emptyset$.
(3) We set $H_{k,s} := \{(x_1,\ldots,x_d) \in \mathbb{R}^d \mid x_k = s\}$ for $k \in \{1,\ldots,d\}$ and $s \in \mathbb{R}$ and $H_{\mathcal{I}} := \bigcup_{(k,s)\in\mathcal{I}} H_{k,s}$ for $\mathcal{I} \subset \{1,\ldots,d\} \times \mathbb{R}$. Recalling (2.4), for $t \in (0,\infty)$ we define

$$\mathcal{Z}_m^{i,\varepsilon}(t) := \sum_{\mathcal{I} \subset \mathcal{I}_{i,\varepsilon}} (-1)^{\#\mathcal{I}} \mathcal{Z}_{U_m^{i,\varepsilon} \setminus H_{\mathcal{I}}}(t) = \int_{U_m^{i,\varepsilon}} p_t^{U_m^{i,\varepsilon}}(x, x | \{K \setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}}) d\mu(x). \tag{4.4}$$

Theorem 4.14. *Let $\varepsilon \in \{0,1\}^d$, $m \in \mathbb{N} \cup \{0\}$, $i \in l^{-m}\mathbb{Z}^d$ and assume $W_m^{i,\varepsilon} \neq \emptyset$. Then there exist $c_{4.4}, c_{4.5} \in (0,\infty)$ which are independent of ε, m, i, and continuous* $\log\tau$*-periodic functions $G_m^{i,\varepsilon,k} : \mathbb{R} \to \mathbb{R}$, $k \in \{|\varepsilon|, \dots, d\}$, such that for any $t \in (0,\tau]$,*

$$\left| \mathcal{Z}_m^{i,\varepsilon}(\tau^{-m}t) - \sum_{k=|\varepsilon|}^{d} t^{-d_k/d_{\mathrm{w}}} G_m^{i,\varepsilon,k}(-\log t) \right| \leq c_{4.4} \exp\bigl(-c_{4.5} t^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr). \tag{4.5}$$

Moreover, $G_m^{i,\varepsilon,|\varepsilon|}$ is $(0,\infty)$-valued, and $G_m^{i,\varepsilon,|\varepsilon|+1}$ is $(-\infty,0)$-valued if $|\varepsilon| < d$.

Remark 4.15. (1) *Only finitely many periodic functions $G_m^{i,\varepsilon,k}$ arise in Theorem* 4.14. Indeed, according to Lemma 5.2 and Proposition 5.3 below, for each $\varepsilon \in \{0,1\}^d$ there exists $\Xi_\varepsilon \subset \bigcup_{m \in \mathbb{N}\cup\{0\}} (l^{-m}\mathbb{Z}^d \times \{m\})$ with $\#\Xi_\varepsilon \leq 2^{2^{|\varepsilon|}}$ such that for any $m \in \mathbb{N} \cup \{0\}$ and any $i \in l^{-m}\mathbb{Z}^d$ we can choose $(j,n) \in \Xi_\varepsilon$ so that $\mathcal{Z}_m^{i,\varepsilon}(\tau^{-m}t) = \mathcal{Z}_n^{j,\varepsilon}(\tau^{-n}t)$ for any $t \in (0,\infty)$. In particular, the set of *all* continuous $\log\tau$-periodic functions appearing in Theorem 4.14 is given by $\bigl\{G_m^{i,\varepsilon,k} \bigm| \varepsilon \in \{0,1\}^d, (i,m) \in \Xi_\varepsilon, k \in \{|\varepsilon|, \dots, d\}\bigr\}$, *which has at most $(d+1)2^{d+2^d}$ elements.*
(2) $\mathcal{Z}_m^{i,\varepsilon}(\tau^{-m}t)$ *is independent of $\varepsilon \in \{0,1\}^d$, $m \in \mathbb{N} \cup \{0\}$ and $i \in l^{-m}\mathbb{Z}^d$ as long as $|\varepsilon| = 1$ and $W_m^{i,\varepsilon} \neq \emptyset$, and $G_{\mathrm{N},1} = n_{\mathrm{N}} G_m^{i,\varepsilon,1}$ and $G_{\mathrm{D},1} = n_{\mathrm{D}} G_m^{i,\varepsilon,1}$ for any such ε, m, i;* see Proposition 5.7 and (5.13) below.

The proof of Theorems 4.10 and 4.14 is given in the next section.

5. Proof of Theorems 4.10 and 4.14

Throughout this section, we fix the setting of Framework 4.1 and assume that $\mathcal{L} := \mathrm{GSC}(d,l,S) = (K, S, \{F_i\}_{i \in S})$ is a generalized Sierpiński carpet and that $(\mathcal{E}, \mathcal{F})$ is the Dirichlet form on $L^2(K,\mu)$ given by Theorem 4.7. As in the previous section, we set $\tau := \#S/r$, $d_{\mathrm{w}} := \log_l \tau$ and $d_k := \log_l \#\bigl(S \cap (\mathbb{Z}^{d-k} \times \{0\}^k)\bigr)$ for $k \in \{0,1,\dots,d\}$ ($d_0 = d_{\mathrm{f}}$), and we also follow the notation introduced in Definitions 4.6 and 4.13.

Similarly to the proofs of Theorems 3.9 and 3.19, we need several intermediate steps to prove Theorems 4.10 and 4.14. We start with some discussion on the scaling property between open subsets of K of the form $U_m^{i,\varepsilon}$.

Definition 5.1. Let $\varepsilon \in \{0,1\}^d$, $m, n \in \mathbb{N} \cup \{0\}$, $i \in l^{-m}\mathbb{Z}^d$ and $j \in l^{-n}\mathbb{Z}^d$. Define $F_{m,n}^{i,j} : \mathbb{R}^d \to \mathbb{R}^d$ by $F_{m,n}^{i,j}(x) := j + l^{m-n}(x - i)$, so that $F_{m,n}^{i,j}(R_m^{i,\varepsilon}) = R_n^{j,\varepsilon}$. We say that (i,m) is *ε-equivalent* to (j,n), and write $(i,m) \overset{\varepsilon}{\sim} (j,n)$, if and only if there exists a bijection $\varphi : W_m^{i,\varepsilon} \to W_n^{j,\varepsilon}$ such that $F_{m,n}^{i,j}|_{K_w} = F_{\varphi(w)} \circ F_w^{-1}$ for any $w \in W_m^{i,\varepsilon}$.

Clearly, $\overset{\varepsilon}{\sim}$ is an equivalence relation on $\bigcup_{m \in \mathbb{N}\cup\{0\}} (l^{-m}\mathbb{Z}^d \times \{m\})$.

Lemma 5.2. *Let $\varepsilon \in \{0,1\}^d$. Then there exists $\Xi_\varepsilon \subset \bigcup_{m \in \mathbb{N}\cup\{0\}} (l^{-m}\mathbb{Z}^d \times \{m\})$ with $\#\Xi_\varepsilon \leq 2^{2^{|\varepsilon|}}$ such that whenever $m \in \mathbb{N} \cup \{0\}$ and $i \in l^{-m}\mathbb{Z}^d$, $(i,m) \overset{\varepsilon}{\sim} (j,n)$ for some $(j,n) \in \Xi_\varepsilon$.*

Proof. For $m \in \mathbb{N} \cup \{0\}$ and $i = (i_k)_{k=1}^d \in l^{-m}\mathbb{Z}^d$, set

$$A_m^{i,\varepsilon} := \left\{ (\delta_k)_{k=1}^d \in \{0\}^{\varepsilon^{-1}(0)} \times \{0,1\}^{\varepsilon^{-1}(1)} \;\middle|\; \prod_{k=1}^{d} \left[i_k - \frac{\delta_k}{l^m}, i_k + \frac{1-\delta_k}{l^m} \right] \subset Q_m \right\}, \tag{5.1}$$

where $Q_m := \bigcup_{w_1\dots w_m\in W_m} f_{w_1}\circ\cdots\circ f_{w_m}(Q_0)$. Then it is easy to see that for $m,n\in\mathbb{N}\cup\{0\}$, $i\in l^{-m}\mathbb{Z}^d$ and $j\in l^{-n}\mathbb{Z}^d$, $(i,m)\overset{\varepsilon}{\sim}(j,n)$ if and only if $A^{i,\varepsilon}_m = A^{j,\varepsilon}_n$. Thus the set $\bigl(\bigcup_{m\in\mathbb{N}\cup\{0\}}(l^{-m}\mathbb{Z}^d\times\{m\})\bigr)/\overset{\varepsilon}{\sim}$ has at most $2^{2^{|\varepsilon|}}$ elements and hence the assertion follows by choosing a representative from each of the equivalence classes under $\overset{\varepsilon}{\sim}$. □

Proposition 5.3. *Let $\varepsilon\in\{0,1\}^d$, $m,n\in\mathbb{N}\cup\{0\}$, $i\in l^{-m}\mathbb{Z}^d$ and $j\in l^{-n}\mathbb{Z}^d$. If $(i,m)\overset{\varepsilon}{\sim}(j,n)$, then $\mathcal{Z}_U(\tau^{n-m}t)=\mathcal{Z}_{F^{i,j}_{m,n}(U)}(t)$ for any $t\in(0,\infty)$ and any non-empty open subset U of $U^{i,\varepsilon}_m$, and in particular $\mathcal{Z}^{i,\varepsilon}_m(\tau^{n-m}t)=\mathcal{Z}^{j,\varepsilon}_n(t)$ for any $t\in(0,\infty)$.*

Proof. Note that $F^{i,j}_{m,n}(H_{k,i_k})=H_{k,j_k}$ for any $k\in\{1,\dots,d\}$, where $i=(i_k)_{k=1}^d$ and $j=(j_k)_{k=1}^d$. By $(i,m)\overset{\varepsilon}{\sim}(j,n)$ there exists a bijection $\varphi: W^{i,\varepsilon}_m\to W^{j,\varepsilon}_n$ such that $F^{i,j}_{m,n}|_{K_w}=F_{\varphi(w)}\circ F_w^{-1}$ for any $w\in W^{i,\varepsilon}_m$, then $F^{i,j}_{m,n}\bigl(\bigcup_{w\in W^{i,\varepsilon}_m}K_w\bigr)=\bigcup_{v\in W^{j,\varepsilon}_n}K_v$ and hence $F^{i,j}_{m,n}(U^{i,\varepsilon}_m)=U^{j,\varepsilon}_n$. If $W^{i,\varepsilon}_m=\emptyset$, then $U^{i,\varepsilon}_m=U^{j,\varepsilon}_n=\emptyset$ and thus $\mathcal{Z}^{i,\varepsilon}_m(\tau^{n-m}t)=\mathcal{Z}^{j,\varepsilon}_n(t)=0$ for any $t\in(0,\infty)$. Assume $W^{i,\varepsilon}_m\neq\emptyset$, let U be a non-empty open subset of $U^{i,\varepsilon}_m$ and set $(F^{i,j}_{m,n})^U_*u := u\circ(F^{i,j}_{m,n})^{-1}|_{F^{i,j}_{m,n}(U)}$ for $u:U\to[-\infty,\infty]$, so that $\int_U|u|d\mu=(\#S)^{n-m}\int_{F^{i,j}_{m,n}(U)}\bigl|(F^{i,j}_{m,n})^U_*u\bigr|d\mu$ if u is Borel measurable. Then $(F^{i,j}_{m,n})^U_*$ defines a bijective linear map $(F^{i,j}_{m,n})^U_*: L^2(U,\mu|_U)\to L^2\bigl(F^{i,j}_{m,n}(U),\mu|_{F^{i,j}_{m,n}(U)}\bigr)$. Moreover, regarding $\{u\in\mathcal{F}\cap C(K)\mid \operatorname{supp}_K[u]\subset V\}=:\mathcal{C}_V$ and $\mathcal{F}_V$ as subsets of $L^2(V,\mu|_V)$ for each non-empty open subset V of K, we have $(F^{i,j}_{m,n})^U_*(\mathcal{F}_U)=\mathcal{F}_{F^{i,j}_{m,n}(U)}$ and $\mathcal{E}(u,u)=r^{n-m}\mathcal{E}\bigl((F^{i,j}_{m,n})^U_*u,(F^{i,j}_{m,n})^U_*u\bigr)$ for any $u\in\mathcal{F}_U$, since the same are valid with $\mathcal{C}_U,\mathcal{C}_{F^{i,j}_{m,n}(U)}$ in place of $\mathcal{F}_U,\mathcal{F}_{F^{i,j}_{m,n}(U)}$ by (SSDF1) and (SSDF2). Now it follows from the above facts and [23, Lemma 1.3.4-(i)] that $(F^{i,j}_{m,n})^U_*T^U_{\tau^{n-m}t}=T^{F^{i,j}_{m,n}(U)}_t(F^{i,j}_{m,n})^U_*$ for any $t\in(0,\infty)$, which together with the uniqueness of p^U implies that $p^U_{\tau^{n-m}t}(x,y)=(\#S)^{m-n}p^{F^{i,j}_{m,n}(U)}_t\bigl(F^{i,j}_{m,n}(x),F^{i,j}_{m,n}(y)\bigr)$ for any $(t,x,y)\in(0,\infty)\times U\times U$. Thus for $t\in(0,\infty)$, $\mathcal{Z}_U(\tau^{n-m}t)=\mathcal{Z}_{F^{i,j}_{m,n}(U)}(t)$, which with $U=U^{i,\varepsilon}_m\setminus H_{\mathcal{I}_i(J)}$, $J\subset\varepsilon^{-1}(1)$, yields $\mathcal{Z}^{i,\varepsilon}_m(\tau^{n-m}t)=\mathcal{Z}^{j,\varepsilon}_n(t)$ by virtue of $F^{i,j}_{m,n}(H_{k,i_k})=H_{k,j_k}$, $k\in\varepsilon^{-1}(1)$. □

Lemma 5.4. *Let $\varepsilon\in\{0,1\}^d$, $m\in\mathbb{N}\cup\{0\}$ and $i\in l^{-m}\mathbb{Z}^d$. Then for any $j\in S_\varepsilon$, $(i,m)\overset{\varepsilon}{\sim}(i+l^{-m-1}j,m+1)$. Moreover,*

$$\hat{U}^{i,\varepsilon}_m\setminus H_{\mathcal{I}^{i,\varepsilon}_m}=\bigcup_{j\in S_\varepsilon}U^{i+l^{-m-1}j,\varepsilon}_{m+1}\qquad\textit{(disjoint)}.\tag{5.2}$$

Proof. These assertions are immediate from Definition 4.13 and (GSC1). □

Recall (see Definition 2.9) that λ^U_1 denotes the smallest eigenvalue of $-\Delta_U$ for a non-empty open subset U of K.

Lemma 5.5. *Let $\varepsilon\in\{0,1\}^d$, $m\in\mathbb{N}\cup\{0\}$, $i\in l^{-m}\mathbb{Z}^d$ and assume $W^{i,\varepsilon}_m\neq\emptyset$. Then $\lambda^{U^{i,\varepsilon}_m}_1>0$, and $0\le\mathcal{Z}^{i,\varepsilon}_m(t)\le\mathcal{Z}_{U^{i,\varepsilon}_m}(t)\le 2^dc_{4.2}$ for any $t\in[\tau^{-m},\infty)$. Moreover, there exists $T^{i,\varepsilon}_m\in[1,\infty)$ such that $\mathcal{Z}^{i,\varepsilon}_m(t)>0$ for any $t\in[T^{i,\varepsilon}_m,\infty)$.*

Proof. Since $0_{\mathbb{R}^d} \in S_\varepsilon$ by (GSC4), we have $(i,m) \overset{\varepsilon}{\sim} (i,m+2)$ by Lemma 5.4, then $\mathcal{Z}_{U_m^{i,\varepsilon}}(\tau^2 t) = \mathcal{Z}_{U_{m+2}^{i,\varepsilon}}(t)$ for $t \in (0,\infty)$ by Proposition 5.3 and hence $\tau^2 \lambda_1^{U_m^{i,\varepsilon}} = \lambda_1^{U_{m+2}^{i,\varepsilon}}$ by Lemma 2.11. On the other hand, Theorem 4.8 easily shows that $(\mathcal{E},\mathcal{F})$ is irreducible, so that $\{u \in \mathcal{F} \mid \mathcal{E}(u,u) = 0\} = \mathbb{R}\mathbf{1}_K$ by [15, Theorem 2.1.11], and $\mathbf{1}_K \notin \mathcal{F}_{U_{m+2}^{i,\varepsilon}}$ since $\mu(K \setminus U_{m+2}^{i,\varepsilon}) > 0$ by (GSC4). Thus $\lambda_1^{U_{m+2}^{i,\varepsilon}} > 0$ and $\lambda_1^{U_m^{i,\varepsilon}} = \tau^{-2}\lambda_1^{U_{m+2}^{i,\varepsilon}} > 0$.

Next let $t \in [\tau^{-m},\infty)$. Since $p_{(\cdot)}(x,x)$ is non-increasing for any $x \in K$ by [16, (2.1.4)], it follows from Proposition 2.7-(1), Theorem 4.8 and $\#W_m^{i,\varepsilon} \le 2^d$ that

$$0 \le \mathcal{Z}_m^{i,\varepsilon}(t) \le \mathcal{Z}_{U_m^{i,\varepsilon}}(t) \le \int_{U_m^{i,\varepsilon}} p_t(x,x)d\mu(x) \le \int_{U_m^{i,\varepsilon}} p_{\tau^{-m}}(x,x)d\mu(x)$$
$$\le c_{4.2}(\tau^{-m})^{-d_{\mathrm{f}}/d_{\mathrm{w}}}\mu(U_m^{i,\varepsilon}) = (\#W_m^{i,\varepsilon})c_{4.2} \le 2^d c_{4.2}.$$

For the last assertion, by $\mathcal{Z}_m^{i,\varepsilon} = \sum_{\mathcal{I}\subset\mathcal{I}_{i,\varepsilon}}(-1)^{\#\mathcal{I}}\mathcal{Z}_{U_m^{i,\varepsilon}\setminus H_{\mathcal{I}}}$ and Lemma 2.11 it suffices to show that $\lambda_1^{U_m^{i,\varepsilon}\setminus H_{\mathcal{I}}} > \lambda_1^{U_m^{i,\varepsilon}}$ for $\mathcal{I} \subset \mathcal{I}_{i,\varepsilon}$ with $\mathcal{I} \ne \emptyset$. Suppose $\lambda_1^{U_m^{i,\varepsilon}\setminus H_{\mathcal{I}}} \le \lambda_1^{U_m^{i,\varepsilon}}$ and let $\varphi_1^{\mathcal{I}}$ be an eigenfunction of $-\Delta_{U_m^{i,\varepsilon}\setminus H_{\mathcal{I}}}$ with eigenvalue $\lambda_1^{U_m^{i,\varepsilon}\setminus H_{\mathcal{I}}}$. Then $\varphi_1^{\mathcal{I}} \in \mathcal{F}_{U_m^{i,\varepsilon}\setminus H_{\mathcal{I}}} \subset \mathcal{F}_{U_m^{i,\varepsilon}}$ and $\mathcal{E}(\varphi_1^{\mathcal{I}},\varphi_1^{\mathcal{I}}) \le \lambda_1^{U_m^{i,\varepsilon}} \int_{U_m^{i,\varepsilon}}(\varphi_1^{\mathcal{I}})^2 d\mu$, so that $\varphi_1^{\mathcal{I}}$ would be an eigenfunction of $-\Delta_{U_m^{i,\varepsilon}}$ with eigenvalue $\lambda_1^{U_m^{i,\varepsilon}}$ in view of (2.10). Furthermore from (GSC1), (GSC2) and (GSC4) we can easily verify that $U_m^{i,\varepsilon}$ is arcwise connected, and hence Lemma 2.10 would imply that there would exist a μ-version of $\varphi_1^{\mathcal{I}}$ satisfying $\varphi_1^{\mathcal{I}}|_{U_m^{i,\varepsilon}} \in C(U_m^{i,\varepsilon})$ and that then either $\varphi_1^{\mathcal{I}} > 0$ on $U_m^{i,\varepsilon}$ or $\varphi_1^{\mathcal{I}} < 0$ on $U_m^{i,\varepsilon}$. On the other hand, let $\mathrm{Cap}_{\mathcal{E}}$ denote the 1-capacity associated with $(K,\mu,\mathcal{E},\mathcal{F})$ defined by $\mathrm{Cap}_{\mathcal{E}}(A) := \inf_{U\subset K \text{ open in } K, A\subset U} \inf_{u\in\mathcal{F}, u\ge 1\ \mu\text{-a.e. on } U} \mathcal{E}_1(u,u)$ for each $A \subset K$ (see [23, Section 2.1]). Then $\mathrm{Cap}_{\mathcal{E}}(U_m^{i,\varepsilon} \cap H_{\mathcal{I}}) > 0$ by [33, Lemma 7.14 and Theorem 7.18], but $\mathrm{Cap}_{\mathcal{E}}(U_m^{i,\varepsilon} \cap H_{\mathcal{I}}) = \mathrm{Cap}_{\mathcal{E}}\bigl(\{x \in U_m^{i,\varepsilon} \cap H_{\mathcal{I}} \mid \varphi_1^{\mathcal{I}}(x) \ne 0\}\bigr) = 0$ by $\varphi_1^{\mathcal{I}} \ne 0$ on $U_m^{i,\varepsilon}$, $\varphi_1^{\mathcal{I}} \in \mathcal{F}_{U_m^{i,\varepsilon}\setminus H_{\mathcal{I}}}$ and [23, Corollary 2.3.1], a contradiction. Thus $\lambda_1^{U_m^{i,\varepsilon}\setminus H_{\mathcal{I}}} > \lambda_1^{U_m^{i,\varepsilon}}$. □

The key step for the proof of Theorems 4.10 and 4.14 is the following proposition, which together with Lemma 5.5 allows us to apply Theorem 2.13 repeatedly to conclude (4.5) and then to verify the strict positivity of $G_m^{i,\varepsilon,|\varepsilon|}$ and $-G_m^{i,\varepsilon,|\varepsilon|+1}$.

Proposition 5.6. *Let $\varepsilon \in \{0,1\}^d$, $m \in \mathbb{N} \cup \{0\}$ and $i \in l^{-m}\mathbb{Z}^d$. Then there exist $c_{5.1}, c_{5.2} \in (0,\infty)$ which are independent of ε, m, i, such that for any $t \in (0,\infty)$,*

$$0 \le \mathcal{Z}_m^{i,\varepsilon}(t) - \tau^{d|\varepsilon|/d_{\mathrm{w}}}\mathcal{Z}_m^{i,\varepsilon}(\tau t) - \sum_{\delta\in\{0,1\}^d,\, \varepsilon^{-1}(1)\subsetneq\delta^{-1}(1)} \sum_{j\in J_{\varepsilon,\delta}} \mathcal{Z}_{m+1}^{i+l^{-m-1}j,\delta}(t)$$
$$\le c_{5.1}\exp\bigl(-c_{5.2}(\tau^m t)^{-\frac{1}{d_{\mathrm{w}}-1}}\bigr), \tag{5.3}$$

where $J_{\varepsilon,\delta} := \{0,\ldots,l-1\}^{\delta^{-1}(0)} \times \{1,\ldots,l-1\}^{\varepsilon^{-1}(0)\cap\delta^{-1}(1)} \times \{0\}^{\varepsilon^{-1}(1)}$.

Proof. Set $\mathcal{I}_0 := \mathcal{I}_m^{i,\varepsilon}$ and let $t \in (0,\infty)$. Noting that $\tau^{d|\varepsilon|/d_{\mathrm{w}}} = \#S_\varepsilon$ by (GSC1), we see from Proposition 5.3, Lemmas 5.4 and 2.6 that

$$\mathcal{Z}_m^{i,\varepsilon}(t) - \tau^{d|\varepsilon|/d_{\mathrm{w}}}\mathcal{Z}_m^{i,\varepsilon}(\tau t) = \mathcal{Z}_m^{i,\varepsilon}(t) - \sum_{j\in S_\varepsilon} \mathcal{Z}_{m+1}^{i+l^{-m-1}j,\varepsilon}(t) \tag{5.4}$$

$$
\begin{aligned}
&= \mathcal{Z}^{i,\varepsilon}_m(t) - \sum_{j\in S_\varepsilon} \int_K p_t^{U^{i+l^{-m-1}j,\varepsilon}_{m+1}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}})d\mu(x)\\
&= \mathcal{Z}^{i,\varepsilon}_m(t) - \int_K p_t^{\hat{U}^{i,\varepsilon}_m\setminus H_{\mathcal{I}_0}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}})d\mu(x)\\
&= \int_{U^{i,\varepsilon}_m}\Bigl(p_t^{U^{i,\varepsilon}_m}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}}) - p_t^{\hat{U}^{i,\varepsilon}_m}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}})\\
&\quad + p_t^{\hat{U}^{i,\varepsilon}_m}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}}) - p_t^{\hat{U}^{i,\varepsilon}_m\setminus H_{\mathcal{I}_0}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}})\Bigr)d\mu(x).
\end{aligned}
$$

Let $x\in K$. Since $\hat{U}^{i,\varepsilon}_m = U^{i,\varepsilon}_m \cap \mathrm{int}_{\mathbb{R}^d}\,\hat{R}^{i,\varepsilon}_m$, Proposition 2.7 and Theorem 4.8 yield

$$
\begin{aligned}
&p_t^{U^{i,\varepsilon}_m}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}}) - p_t^{\hat{U}^{i,\varepsilon}_m}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}}) \qquad (5.5)\\
&= p_t^{U^{i,\varepsilon}_m}(x,x|\{K\cap \mathrm{int}_{\mathbb{R}^d}\,\hat{R}^{i,\varepsilon}_m\}\cup\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}})\ (\geq 0)\\
&\leq \frac{2c_{4.2}}{(t/2)^{d_\mathrm{f}/d_\mathrm{w}}}\exp\biggl(-\Bigl(\frac{\rho(x,K\setminus \mathrm{int}_{\mathbb{R}^d}\,\hat{R}^{i,\varepsilon}_m)\vee\max_{(k,s)\in\mathcal{I}_{i,\varepsilon}}\rho(x,K\cap H_{k,s})}{(c_{4.2}t)^{1/d_\mathrm{w}}}\Bigr)^{\frac{d_\mathrm{w}}{d_\mathrm{w}-1}}\biggr)\\
&\leq 2^{1+d_\mathrm{f}/d_\mathrm{w}}c_{4.2}t^{-d_\mathrm{f}/d_\mathrm{w}}\exp\bigl(-(2^{d_\mathrm{w}}c_{4.2}\tau^{m+1}t)^{-\frac{1}{d_\mathrm{w}-1}}\bigr)
\end{aligned}
$$

where we have the inequality in the third line of (5.5) only for $t\in(0,1]$.

On the other hand, setting $J^0_\delta := \{0,\dots,l-1\}^{\delta^{-1}(0)}\times\{0\}^{\delta^{-1}(1)}$ and $J^1_{\varepsilon,\delta} := \{0\}^{\delta^{-1}(0)}\times\{1,\dots,l-1\}^{\varepsilon^{-1}(0)\cap\delta^{-1}(1)}\times\{0\}^{\varepsilon^{-1}(1)}$ for $\delta\in\{0,1\}^d$ with $\varepsilon^{-1}(1)\subsetneqq\delta^{-1}(1)$, by using Lemmas 2.12 and 2.6 we get

$$
\begin{aligned}
&p_t^{\hat{U}^{i,\varepsilon}_m}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}}) - p_t^{\hat{U}^{i,\varepsilon}_m\setminus H_{\mathcal{I}_0}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}}) \qquad (5.6)\\
&= \sum_{J\subset\varepsilon^{-1}(1)}(-1)^{\#J}\Bigl(p_t^{\hat{U}^{i,\varepsilon}_m\setminus H_{\mathcal{I}_i(J)}}(x,x) - p_t^{\hat{U}^{i,\varepsilon}_m\setminus H_{\mathcal{I}_0\cup\mathcal{I}_i(J)}}(x,x)\Bigr)\\
&= \sum_{J\subset\varepsilon^{-1}(1)}(-1)^{\#J}\sum_{\emptyset\neq\mathcal{I}\subset\mathcal{I}_0}\sum_{A\subset\mathcal{I}}(-1)^{\#A}p_t^{\hat{U}^{i,\varepsilon}_m\setminus H_{(\mathcal{I}_0\setminus\mathcal{I})\cup A\cup\mathcal{I}_i(J)}}(x,x)\\
&= \sum_{\emptyset\neq\mathcal{I}\subset\mathcal{I}_0}\sum_{A\subset\mathcal{I}\cup\mathcal{I}_{i,\varepsilon}}(-1)^{\#A}p_t^{\hat{U}^{i,\varepsilon}_m\setminus H_{(\mathcal{I}_0\setminus\mathcal{I})\cup A}}(x,x)\\
&= \sum_{\emptyset\neq\mathcal{I}\subset\mathcal{I}_0}p_t^{\hat{U}^{i,\varepsilon}_m\setminus H_{\mathcal{I}_0\setminus\mathcal{I}}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}\cup\mathcal{I}_{i,\varepsilon}})\\
&= \sum\nolimits_{\mathcal{I}\subset\mathcal{I}_0,\,\#(\mathcal{I}\cap(\{k\}\times\mathbb{R}))\geq 2\text{ for some }k\in\varepsilon^{-1}(0)}p_t^{\hat{U}^{i,\varepsilon}_m\setminus H_{\mathcal{I}_0\setminus\mathcal{I}}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}\cup\mathcal{I}_{i,\varepsilon}})\\
&\quad + \sum\nolimits_{\substack{\delta\in\{0,1\}^d\\ \varepsilon^{-1}(1)\subsetneqq\delta^{-1}(1)}}\sum\nolimits_{j\in l^{-m-1}J^1_{\varepsilon,\delta}}p_t^{\hat{U}^{i,\varepsilon}_m\setminus H_{\mathcal{I}_0\setminus\mathcal{I}_{i+j,\delta}}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i+j,\delta}})\\
&= \sum\nolimits_{\mathcal{I}\subset\mathcal{I}_0,\,\#(\mathcal{I}\cap(\{k\}\times\mathbb{R}))\geq 2\text{ for some }k\in\varepsilon^{-1}(0)}p_t^{\hat{U}^{i,\varepsilon}_m\setminus H_{\mathcal{I}_0\setminus\mathcal{I}}}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}\cup\mathcal{I}_{i,\varepsilon}})\\
&\quad + \sum_{\substack{\delta\in\{0,1\}^d\\ \varepsilon^{-1}(1)\subsetneqq\delta^{-1}(1)}}\sum_{j\in l^{-m-1}J^1_{\varepsilon,\delta}}\Bigl(p_t^{(\hat{U}^{i,\varepsilon}_m\setminus H_{\mathcal{I}_0\setminus\mathcal{I}_{i+j,\delta}})\setminus\hat{R}^{i+j,\delta}_m}(x,x|\{K\setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i+j,\delta}})
\end{aligned}
$$

$$+ \sum_{j_0 \in l^{-m-1} J^0_\delta} p_t^{U^{i+j+j_0,\delta}_{m+1}}(x,x|\{K \setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i+j+j_0,\delta}})\Bigg).$$

Then from (5.6), Proposition 2.7-(2) and Theorem 4.8 we conclude that

$$\begin{aligned}
&p_t^{\hat{U}^{i,\varepsilon}_m}(x,x|\{K \setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}}) - p_t^{\hat{U}^{i,\varepsilon}_m \setminus H_{\mathcal{I}_0}}(x,x|\{K \setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i,\varepsilon}}) \qquad (5.7)\\
&\quad - \sum_{\substack{\delta\in\{0,1\}^d\\ \varepsilon^{-1}(1)\subsetneqq\delta^{-1}(1)}} \sum_{j\in J_{\varepsilon,\delta}} p_t^{U^{i+l^{-m-1}j,\delta}_{m+1}}\big(x,x\big|\{K \setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i+l^{-m-1}j,\delta}}\big)\\
&= \sum_{\mathcal{I}\subset\mathcal{I}_0,\,\#(\mathcal{I}\cap(\{k\}\times\mathbb{R}))\geq 2 \text{ for some } k\in\varepsilon^{-1}(0)} p_t^{\hat{U}^{i,\varepsilon}_m \setminus H_{\mathcal{I}_0\setminus\mathcal{I}}}(x,x|\{K \setminus H_{k,s}\}_{(k,s)\in\mathcal{I}\cup\mathcal{I}_{i,\varepsilon}})\\
&\quad + \sum_{\substack{\delta\in\{0,1\}^d\\ \varepsilon^{-1}(1)\subsetneqq\delta^{-1}(1)}} \sum_{j\in l^{-m-1}J^1_{\varepsilon,\delta}} p_t^{(\hat{U}^{i,\varepsilon}_m \setminus H_{\mathcal{I}_0\setminus\mathcal{I}_{i+j,\delta}})\setminus \hat{R}^{i+j,\delta}_m}(x,x|\{K \setminus H_{k,s}\}_{(k,s)\in\mathcal{I}_{i+j,\delta}})\\
&\leq 2\sum_{\mathcal{I}\subset\mathcal{I}_0,\,\#(\mathcal{I}\cap(\{k\}\times\mathbb{R}))\geq 2 \text{ for some } k\in\varepsilon^{-1}(0)} \min_{(k,u)\in\mathcal{I}\cup\mathcal{I}_{i,\varepsilon}} \Big\{ \sup_{s\in[t/2,t]} \sup_{z\in K\cap H_{k,u}} p_s(x,z)\Big\}\\
&\quad + 2 \sum_{\substack{\delta\in\{0,1\}^d\\ \varepsilon^{-1}(1)\subsetneqq\delta^{-1}(1)}} \sum_{j\in l^{-m-1}J^1_{\varepsilon,\delta}} \mathbf{1}_{K\setminus\hat{R}^{i+j,\delta}_m}(x) \min_{(k,u)\in\mathcal{I}_{i+j,\delta}} \Big\{ \sup_{s\in[t/2,t]} \sup_{z\in K\cap H_{k,u}} p_s(x,z)\Big\}\\
&\leq 2^{d(l-1)+1+d_{\mathrm{f}}/d_{\mathrm{w}}} c_{4.2} t^{-d_{\mathrm{f}}/d_{\mathrm{w}}} \exp\big(-(2^{d_{\mathrm{w}}} c_{4.2} \tau^{m+1} t)^{-\frac{1}{d_{\mathrm{w}}-1}}\big)
\end{aligned}$$

where the inequality in the last line of (5.7) is valid only for $t \in (0,1]$.

Now (5.4), (5.5), (5.7) and Proposition 2.7-(1) immediately show the lower inequality in (5.3) and also the upper inequality in (5.3) with $c_{5.2} := 4^{-1}(2c_{4.2}\tau)^{-\frac{1}{d_{\mathrm{w}}-1}}$ for $t \in (0,1]$ by virtue of $\mu(U^{i,\varepsilon}_m) = (\#S)^{-m}\#W^{i,\varepsilon}_m \leq 2^d \tau^{-md_{\mathrm{f}}/d_{\mathrm{w}}}$. Finally, the second assertion of Lemma 5.5 yields the upper inequality in (5.3) for $t \in [1,\infty)$. □

Proof of Theorem 4.14. The proof is by induction in $d - |\varepsilon|$. Suppose first $|\varepsilon| = d$. Then $0 \leq \mathcal{Z}^{i,\varepsilon}_m(\tau^{-m}t) - \mathcal{Z}^{i,\varepsilon}_m(\tau^{-m}\tau t) \leq c_{5.1}\exp\big(-c_{5.2}t^{-\frac{1}{d_{\mathrm{w}}-1}}\big)$ for any $t \in (0,\infty)$ by Proposition 5.6, which together with Lemmas 5.5 and 2.11 implies that Theorem 2.13 applies to $\mathcal{Z}^{i,\varepsilon}_m(\tau^{-m}(\cdot))$ with $\alpha_0 = 0$, $\gamma = \tau$, $n = 1$, $\alpha_1 = -1$, $G_1 = 0$ and $\mathcal{R}(t) := c_{5.1}\exp\big(-c_{5.2}t^{-\frac{1}{d_{\mathrm{w}}-1}}\big)$. Therefore there exists a continuous $\log\tau$-periodic function $G^{i,\varepsilon,d}_m : \mathbb{R} \to \mathbb{R}$ such that $\big|\mathcal{Z}^{i,\varepsilon}_m(\tau^{-m}t) - G^{i,\varepsilon,d}_m(-\log t)\big| \leq \sum_{j\in\mathbb{N}}\mathcal{R}(\tau^{-j}t) \leq c_{5.1}c_{5.3}\exp\big(-c_{5.2}(\tau^{-1}t)^{-\frac{1}{d_{\mathrm{w}}-1}}\big)$ for any $t \in (0,\tau]$, where $c_{5.3} \in (1,\infty)$ is explicit in terms of $\tau, d_{\mathrm{w}}, c_{5.2}$, proving (4.5) with $c_{4.4} = c_d := c_{5.1}c_{5.3}$ and $c_{4.5} := c_{5.2}\tau^{\frac{1}{d_{\mathrm{w}}-1}}$.

Next assume $|\varepsilon| < d$ and suppose that (4.5) for $t \in (0,\tau]$ holds with δ, n, j in place of ε, m, i and with $c_{4.4} = c_{|\varepsilon|+1}$ and $c_{4.5} = c_{5.2}\tau^{\frac{1}{d_{\mathrm{w}}-1}}$ for some $c_{|\varepsilon|+1} \in (0,\infty)$ whenever $\delta \in \{0,1\}^d$, $n \in \mathbb{N}\cup\{0\}$ and $j \in l^{-n}\mathbb{Z}^d$ satisfy $|\delta| \geq |\varepsilon|+1$ and $W^{j,\delta}_n \neq \emptyset$. For $\delta \in \{0,1\}^d$, $n \in \mathbb{N}\cup\{0\}$ and $j \in l^{-n}\mathbb{Z}^d$ with $|\delta| \geq |\varepsilon|+1$ and $W^{j,\delta}_n = \emptyset$, we set $G^{j,\delta,k}_n := 0$, $k \in \{|\delta|,\ldots,d\}$, so that (4.5) trivially holds. Then Proposition 5.6 implies that (2.13) holds for $\mathcal{Z}^{i,\varepsilon}_m(\tau^{-m}(\cdot))$ with $\alpha_0 = d_{|\varepsilon|}/d_{\mathrm{w}}$, $\gamma = \tau$, $n = d - |\varepsilon|$, $\alpha_{k-|\varepsilon|} = d_k/d_{\mathrm{w}}$ and

$$G_{k-|\varepsilon|} := \tau^{-d_k/d_{\mathrm{w}}} \sum_{\delta\in\{0,1\}^d,\,\varepsilon^{-1}(1)\subsetneqq\delta^{-1}(1),\,|\delta|\leq k} \sum_{j\in J_{\varepsilon,\delta}} G^{i+l^{-m-1}j,\delta,k}_{m+1} \qquad (5.8)$$

for $k \in \{|\varepsilon|+1,\dots,d\}$, and $\mathcal{R}(t) := (c_{5.1} + (2l)^d c_{|\varepsilon|+1}) \exp\bigl(-c_{5.2} t^{-\frac{1}{d_\mathrm{w}-1}}\bigr)$. Now from this fact and Theorem 2.13 together with Lemmas 5.5 and 2.11, we conclude that there exists a continuous $\log\tau$-periodic function $G_m^{i,\varepsilon,|\varepsilon|} : \mathbb{R} \to \mathbb{R}$ such that (4.5) for $t \in (0,\tau]$ holds with $G_m^{i,\varepsilon,k} := -\bigl(\tau^{(d_{|\varepsilon|}-d_k)/d_\mathrm{w}} - 1\bigr)^{-1} G_{k-|\varepsilon|}$ for $k \in \{|\varepsilon|+1,\dots,d\}$, $c_{4.4} = c_{|\varepsilon|} := c_{5.3}(c_{5.1} + (2l)^d c_{|\varepsilon|+1})$ and $c_{4.5} = c_{5.2}\tau^{\frac{1}{d_\mathrm{w}-1}}$, where $G_{k-|\varepsilon|}$ is as in (5.8). Thus the induction procedure in $d - |\varepsilon|$ is completed and (4.5) is proved.

It remains to prove the strict positivity of $G_m^{i,\varepsilon,|\varepsilon|}$ and $-G_m^{i,\varepsilon,|\varepsilon|+1}$. By Proposition 5.6 and Proposition 2.7-(1), $t^{d_{|\varepsilon|}/d_\mathrm{w}} \mathcal{Z}_m^{i,\varepsilon}(t) \geq (\tau t)^{d_{|\varepsilon|}/d_\mathrm{w}} \mathcal{Z}_m^{i,\varepsilon}(\tau t)$ for any $t \in (0,\infty)$, and hence

$$\inf_{t\in(0,\tau T_m^{i,\varepsilon}]} t^{d_{|\varepsilon|}/d_\mathrm{w}} \mathcal{Z}_m^{i,\varepsilon}(t) = \min_{t\in[T_m^{i,\varepsilon},\tau T_m^{i,\varepsilon}]} t^{d_{|\varepsilon|}/d_\mathrm{w}} \mathcal{Z}_m^{i,\varepsilon}(t) > 0 \tag{5.9}$$

with $T_m^{i,\varepsilon} \in [1,\infty)$ as in Lemma 5.5. Now (5.9) and (4.5) together imply that $G_m^{i,\varepsilon,|\varepsilon|}$ is $(0,\infty)$-valued. Moreover, suppose $|\varepsilon| < d$, choose $\delta \in \{0,1\}^d$ so that $\varepsilon^{-1}(1) \subsetneqq \delta^{-1}(1)$ and $|\delta| = |\varepsilon|+1$, and let $j := \bigl(\mathbf{1}_{\varepsilon^{-1}(0)\cap\delta^{-1}(1)}(k)\bigr)_{k=1}^d$. Then $j \in J_{\varepsilon,\delta}$ with $J_{\varepsilon,\delta}$ as in Proposition 5.6, and we easily see from $W_m^{i,\varepsilon} \neq \emptyset$, (GSC1) and (GSC4) that $W_{m+1}^{i+l^{-m-1}j,\delta} \neq \emptyset$. Hence by Proposition 5.6, Proposition 2.7-(1) and (5.9),

$$\inf_{t\in(0,1]} t^{d_{|\varepsilon|+1}/d_\mathrm{w}} \bigl(\mathcal{Z}_m^{i,\varepsilon}(t) - \tau^{d_{|\varepsilon|}/d_\mathrm{w}} \mathcal{Z}_m^{i,\varepsilon}(\tau t)\bigr) \geq \inf_{t\in(0,1]} t^{d_{|\delta|}/d_\mathrm{w}} \mathcal{Z}_{m+1}^{i+l^{-m-1}j,\delta}(t) > 0,$$

which together with (4.5) immediately shows that $G_m^{i,\varepsilon,|\varepsilon|+1}$ is $(-\infty,0)$-valued. □

Next we prove Theorem 4.10, for which we need the following two propositions.

Proposition 5.7. *Let $\delta,\varepsilon \in \{0,1\}^d$ satisfy $|\delta| = |\varepsilon| = 1$, let $j \in \{0,1\}$, $m \in \mathbb{N}\cup\{0\}$, $i \in l^{-m}\mathbb{Z}^d$ and suppose $W_m^{i,\varepsilon} \neq \emptyset$. Then for any $t \in (0,\infty)$,*

$$\mathcal{Z}_{U_m^{i,\varepsilon}}(t) = \mathcal{Z}_{U_0^{j\delta,\delta}}(\tau^m t) + (\#W_m^{i,\varepsilon} - 1)\mathcal{Z}_\mathrm{D}(\tau^m t) \quad \textit{and} \quad \mathcal{Z}_m^{i,\varepsilon}(t) = \mathcal{Z}_0^{j\delta,\delta}(\tau^m t). \tag{5.10}$$

Proof. Note that $\#W_m^{i,\varepsilon} \in \{1,2\}$. Since Lemma 3.12 is valid with the same proof also in the present setting, $\mathcal{Z}_m^{i,\varepsilon}(t) = \mathcal{Z}_{U_m^{i,\varepsilon}}(t) - (\#W_m^{i,\varepsilon})\mathcal{Z}_\mathrm{D}(\tau^m t)$ for any $t \in (0,\infty)$ by Lemmas 2.6 and 3.12, so that the two equalities in (5.10) are equivalent. Thus it suffices to show the former equality, which can be proved in exactly the same way as Proposition 3.14 on the basis of (SSDF1), (SSDF2) and (GSCDF). □

Proposition 5.8. *There exist $c_{5.4}, c_{5.5} \in (0,\infty)$ such that for any $t \in (0,\infty)$,*

$$0 \leq \mathcal{Z}_\mathrm{N}(t) - \mathcal{Z}_\mathrm{D}(t) - \sum_{\substack{\delta,j\in\{0,1\}^d,\,|\delta|\geq 1\\ \delta^{-1}(0)\subset j^{-1}(0)}} \mathcal{Z}_0^{j,\delta}(t) \leq c_{5.4}\exp\bigl(-c_{5.5}t^{-\frac{1}{d_\mathrm{w}-1}}\bigr). \tag{5.11}$$

Proof. Set $\mathcal{I}_0 := \{1,\dots,d\}\times\{0,1\}$ and let $t \in (0,\infty)$. We see from Lemma 2.12 that

$$\mathcal{Z}_\mathrm{N}(t) - \mathcal{Z}_\mathrm{D}(t) = \int_K \Bigl(p_t(x,x) - p_t^{K\setminus H_{\mathcal{I}_0}}(x,x)\Bigr) d\mu(x) \tag{5.12}$$
$$= \int_K \sum_{\emptyset\neq\mathcal{I}\subset\mathcal{I}_0} \sum_{A\subset\mathcal{I}} (-1)^{\#A} p_t^{K\setminus H_{(\mathcal{I}_0\setminus\mathcal{I})\cup A}}(x,x)\,d\mu(x)$$

$$= \sum_{\emptyset \neq \mathcal{I} \subset \mathcal{I}_0} \int_K p_t^{K \setminus H_{\mathcal{I}_0 \setminus \mathcal{I}}}(x, x | \{K \setminus H_{k,s}\}_{(k,s) \in \mathcal{I}}) d\mu(x)$$

$$= \sum\nolimits_{\mathcal{I} \subset \mathcal{I}_0,\, (k,0),(k,1) \in \mathcal{I} \text{ for some } k \in \{1,\dots,d\}} \int_K p_t^{K \setminus H_{\mathcal{I}_0 \setminus \mathcal{I}}}(x, x | \{K \setminus H_{k,s}\}_{(k,s) \in \mathcal{I}}) d\mu(x)$$
$$+ \sum\nolimits_{\delta, j \in \{0,1\}^d,\, |\delta| \geq 1,\, \delta^{-1}(0) \subset j^{-1}(0)} \int_K p_t^{U_0^{j,\delta}}(x, x | \{K \setminus H_{k,s}\}_{(k,s) \in \mathcal{I}_{j,\delta}}) d\mu(x)$$

$$= \sum\nolimits_{\mathcal{I} \subset \mathcal{I}_0,\, (k,0),(k,1) \in \mathcal{I} \text{ for some } k \in \{1,\dots,d\}} \int_K p_t^{K \setminus H_{\mathcal{I}_0 \setminus \mathcal{I}}}(x, x | \{K \setminus H_{k,s}\}_{(k,s) \in \mathcal{I}}) d\mu(x)$$
$$+ \sum\nolimits_{\delta, j \in \{0,1\}^d,\, |\delta| \geq 1,\, \delta^{-1}(0) \subset j^{-1}(0)} \mathcal{Z}_0^{j,\delta}(t),$$

which together with Proposition 2.7 and Theorem 4.8 easily implies (5.11), similarly to the proof of Proposition 5.6. □

Proof of Theorem 4.10. (4.2) for $\mathcal{Z}_\mathrm{D}$ with strictly positive $G_{\mathrm{D},0}, -G_{\mathrm{D},1}$ is just a special case of Theorem 4.14 with $i = \varepsilon = 0_{\mathbb{R}^d}$ and $m = 0$. Theorem 4.14 and Proposition 5.8 together imply (4.2) for $\mathcal{Z}_\mathrm{N}$, $\lim_{t \downarrow 0} t^{d_0/d_\mathrm{w}} \bigl(\mathcal{Z}_\mathrm{N}(t) - \mathcal{Z}_\mathrm{D}(t)\bigr) = 0$ and hence $G_{\mathrm{N},0} = G_{\mathrm{D},0}$.

To prove $n_\mathrm{N}^{-1} G_{\mathrm{N},1} = n_\mathrm{D}^{-1} G_{\mathrm{D},1}$, let $\delta \in \{0,1\}^d$ satisfy $|\delta| = 1$ and let $j \in \{0,1\}$. Then in view of (4.2) (with $G_{\mathrm{N},0} = G_{\mathrm{D},0}$) and Theorem 4.14, we easily see from Propositions 5.7 and 5.8 that $G_{\mathrm{N},1} = G_{\mathrm{D},1} + 2d G_{\mathbf{0}}^{j\delta,\delta,1}$, and from Proposition 5.6 with $i = \varepsilon = 0_{\mathbb{R}^d}$ and $m = 0$ and Proposition 5.7 that $\mathcal{Z}_\mathrm{D}(t) - \tau^{d_0/d_\mathrm{w}} \mathcal{Z}_\mathrm{D}(\tau t) - n_{1,0} \mathcal{Z}_{\mathbf{0}}^{j\delta,\delta}(\tau t) = o(t^{-d_1/d_\mathrm{w}})$ as $t \downarrow 0$, whence $(\tau^{d_1/d_\mathrm{w}} - \tau^{d_0/d_\mathrm{w}}) G_{\mathrm{D},1} = n_{1,0} G_{\mathbf{0}}^{j\delta,\delta,1}$; here $n_{1,0} := \#\{F_i(K \cap H_{k,s}) \mid i \in S, k \in \{1,\dots,d\}, s \in \{0,1\}, F_i(K \cap H_{k,s}) \not\subset V_0\}$ which is easily seen to be equal to $-n_\mathrm{D} \#\bigl(S \setminus (\mathbb{Z}^{d-1} \times \{0\})\bigr)$. Thus we obtain

$$G_{\mathrm{D},1} = n_\mathrm{D} G_{\mathbf{0}}^{j\delta,\delta,1} \quad \text{and} \quad G_{\mathrm{N},1} = G_{\mathrm{D},1} + 2d G_{\mathbf{0}}^{j\delta,\delta,1} = n_\mathrm{N} G_{\mathbf{0}}^{j\delta,\delta,1}, \tag{5.13}$$

so that $n_\mathrm{N}^{-1} G_{\mathrm{N},1} = G_{\mathbf{0}}^{j\delta,\delta,1} = n_\mathrm{D}^{-1} G_{\mathrm{D},1}$, completing the proof of Theorem 4.10. □

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