We introduce the notion of conformal walk dimension, which serves as a bridge between elliptic and parabolic Harnack inequalities. The importance of this notion is due to the fact that, for a given strongly local, regular symmetric Dirichlet space equipped with a complete doubling metric (MMD space), the finiteness of the conformal walk dimension characterizes the elliptic Harnack inequality. Roughly speaking, the conformal walk dimension of an MMD space is defined as the infimum over all possible values of the walk dimension with which the parabolic Harnack inequality can be made to hold by a time change of the associated diffusion and by a quasisymmetric change of the metric. We show that the conformal walk dimension of any MMD space satisfying the elliptic Harnack inequality is two, and provide a necessary condition for a pair of such changes to attain the infimum defining the conformal walk dimension when it is attained for at least one such pair. We also prove a necessary condition for the existence of such a pair attaining the infimum in the setting of a self-similar Dirichlet form on a self-similar set, and apply it to show that the infimum fails to be attained for the Vicsek set and the \( N \)-dimensional Sierpiński gasket with \( N \geq 3 \), in contrast to the attainment for the two-dimensional Sierpiński gasket due to Kigami [Math. Ann. 340 (2008), no. 4, 781–804].

2020 Mathematics Subject Classification: Primary 30L10, 31C25, 31E05, 35K08; Secondary 28A80, 60G30, 60J46, 60J60.

Keywords: Conformal walk dimension, symmetric diffusion, time change, quasisymmetry, elliptic Harnack inequality, parabolic Harnack inequality, sub-Gaussian heat kernel estimate, walk dimension, self-similar fractal

Contents

1 Introduction 2

---

*This work was supported by the Research Institute for Mathematical Sciences, an International Joint Usage/Research Center located in Kyoto University.

†Research partially supported by JSPS KAKENHI Grant Number JP17H02849.

‡Research partially supported by NSERC and the Canada research chairs program.
1 Introduction

What is the “best” way to parametrize a space? This vaguely stated question is the motivation for our work and several earlier works. By a parametrization, we mean a bijection $f : X \to M$ between the given space $X$ and another “model space” $M$ with more desirable properties. For example, the Riemann mapping theorem (or more generally, the uniformization theorem for Riemann surfaces) and geometric flows like the Ricci flow...
can be viewed as an attempt to answer the above question. In the Riemann mapping theorem example, \( X \) is a proper simply connected domain in \( \mathbb{C} \), \( M \) is the unit disk, and \( f \) is a conformal map. In the Ricci flow example, \( X \) is a Riemannian manifold, \( M \) is a Riemannian manifold with constant Ricci curvature, and \( f \) is a diffeomorphism. This work aims to formulate and answer this question for spaces satisfying Harnack inequalities. In this work, \( X \) is a space equipped with a symmetric diffusion that satisfies the elliptic Harnack inequality, \( M \) satisfies the stronger parabolic Harnack inequality and \( f \) is a quasisymmetry along with a time change of the diffusion on \( X \) (quasisymmetry is an analogue of conformal maps for metric spaces).

This paper uses quasiconformal geometry and time change of diffusion processes to understand the relationship between elliptic and parabolic Harnack inequalities. The analysis using quasiconformal geometry also leads to a natural uniformization problem for spaces satisfying the elliptic Harnack inequality. Our results can be viewed as a bridge between analysis in smooth and fractal spaces and also as a bridge between elliptic and parabolic Harnack inequalities.

We informally describe the setup and results. A more precise treatment is given in Section 2. The setup of this work is a metric space equipped with a Radon measure \( m \) with full support and an \( m \)-symmetric diffusion process. Equivalently, we consider a metric space \((X, d)\) equipped with such \( m \) and a strongly local, regular symmetric Dirichlet form \((\mathcal{E}, \mathcal{F})\) on \( L^2(X, m) \). We always assume that \( B(x, r) := B_d(x, r) := \{y \in X \mid d(x, y) < r\} \) has compact closure in \( X \) for any \((x, r) \in X \times (0, \infty)\) and that \( X \) contains at least two elements, and call \((X, d, m, \mathcal{E}, \mathcal{F})\) a metric measure Dirichlet space or an MMD space for short. Associated to an MMD space \((X, d, m, \mathcal{E}, \mathcal{F})\) is a non-negative self-adjoint operator \( L \) on \( L^2(X, m) \) such that the corresponding Markov semigroup \((P_t)_{t \geq 0}\) is given by \( P_t = e^{-tL} \). The operator \( L \) is called the generator of \((X, d, m, \mathcal{E}, \mathcal{F})\), which is an analog of the Laplace operator in the abstract setting of MMD spaces. We refer to \([\text{FOT, CF}]\) for the theory of Dirichlet forms.

We recall that this setup includes Brownian motion on a Riemannian manifold, where \( d \) is the Riemannian distance function, \( m \) is the Riemannian measure, \( \mathcal{F} \) is the Sobolev space \( W^{1,2} \), and \( \mathcal{E}(f, f) = \int |\nabla f|^2 \, dm \), where \( \nabla \) denotes the Riemannian gradient. In this case, the corresponding generator \( L \) is the Laplace–Beltrami operator with a minus sign (so that \( L \) is non-negative definite). This setup also covers non-smooth settings like diffusions on fractals including the Sierpiński gasket and the Sierpiński carpet. We refer the reader to \([\text{Bar98}]\) for an introduction to diffusions on fractals. Random walks on graphs can also be studied in this framework because the corresponding cable processes share many properties with the original random walks (see \([\text{BB04}]\) for this approach).

An MMD space has an associated sheaf of harmonic and caloric functions. Roughly speaking, harmonic and caloric functions are generalization of solutions to the “Laplace equation” \( \Delta h \equiv 0 \) and the “heat equation” \( \partial_t u - \Delta u \equiv 0 \), respectively. Let \( L \) denote the generator of an MMD space \((X, d, m, \mathcal{E}, \mathcal{F})\). Let \( h : U \to \mathbb{R} \) be a measurable function in an open set \( U \). We say that \( h \) is harmonic in \( U \), if it satisfies \( \mathcal{L}h \equiv 0 \) in \( U \) interpreted in a weak sense. Similarly, we say that a space-time function \( u : (a, b) \times U \to \mathbb{R} \) is caloric in \((a, b) \times U \) if it satisfies the “heat equation” \( \partial_t u + \mathcal{L}u \equiv 0 \) interpreted in a weak sense.
Harnack inequalities are fundamental regularity estimates that have numerous applications in partial differential equations and probability theory. We refer to [Kas] for a nice survey on Harnack inequality and its variants. We recall the (scale-invariant) elliptic and parabolic Harnack inequalities. We say that an MMD space \((X,d,m,\mathcal{E},\mathcal{F})\) satisfies the elliptic Harnack inequality (abbreviated as EHI), if there exist \(C > 1\) and \(\delta \in (0,1)\) such that for all \(x \in X, r > 0\) and for any non-negative harmonic function \(h\) on the ball \(B(x,r)\), we have
\[
\text{ess sup}_{B(x,\delta r)} h \leq C \text{ess inf}_{B(x,\delta r)} h.
\]
We say that an MMD space \((X,d,m,\mathcal{E},\mathcal{F})\) satisfies the parabolic Harnack inequality with walk dimension \(\beta > 0\) (abbreviated as PHI(\(\beta\))), if there exist \(0 < C_1 < C_2 < C_3 < C_4 < \infty, C_5 > 1\) and \(\delta \in (0,1)\) such that for all \(x \in X, r > 0\) and for any non-negative bounded caloric function \(u\) on the space-time cylinder \(Q = (a,a+C_4 r^\beta) \times B(x,r)\), we have
\[
\text{ess sup}_{Q_-} u \leq C_5 \text{ess inf}_{Q_+} u,
\]
where \(Q_- = (a+C_1 r^\beta, a+C_2 r^\beta) \times B(x,\delta r)\) and \(Q_+ = (a+C_3 r^\beta, a+C_4 r^\beta) \times B(x,\delta r)\).

We briefly review some earlier works on Harnack inequalities, referring the reader to [Kas] for a more detailed survey of the literature. In a series of celebrated works, Moser showed EHI and PHI(2) for uniformly elliptic divergence form operators on \(\mathbb{R}^n\) [Mos61, Mos64]. Cheng, Li and Yau obtained gradient estimates that imply EHI and PHI(2) for Riemannian manifolds with non-negative Ricci curvature [CY, LY, Yau]. Grigor’yan and Saloff-Coste independently characterized PHI(2) using the volume doubling property and the Poincaré inequality [Gri, Sal]. This characterization was extended to PHI(\(\beta\)) by Barlow, Bass and Kumagai [BB04, BBK]. A similar characterization of the simpler elliptic Harnack inequality remained open until recently [Bas13, BM, BCM].

Note that every harmonic function lifts to a caloric function. More precisely, if \(h\) is harmonic on \(B(x,r)\), then \(u(t,x) = h(x)\) is caloric on \((a,b) \times B(x,r)\) for all \(b > a\). This lift immediately shows that
\[
\text{PHI}(\beta) \Rightarrow \text{EHI}, \quad \text{for all } \beta > 0. \tag{1.1}
\]
However, the converse of the above implication fails. Indeed, Delmotte has constructed an example of a space that satisfies EHI but fails to satisfy PHI(\(\beta\)) for any \(\beta > 0\) [Del] (see also [BCM]). Nevertheless, one can characterize the elliptic Harnack inequality in terms of the parabolic Harnack inequality [BM, BCM].

The main idea behind the characterization of EHI is to reparametrize the space and time of the associated diffusion process so that it satisfies PHI(\(\beta\)) for some \(\beta > 0\). In the theory of regular symmetric Dirichlet forms the Revuz correspondence provides a

\[\footnote{To be precise, while for our purpose we need to formulate EHI without assuming the boundedness of non-negative harmonic functions \(h\), we follow for simplicity the formulation of PHI(\(\beta\)) in [BGK, Subsection 3.1] which a priori requires the boundedness of non-negative caloric functions \(u\), and then (1.1) is obvious only under the extra assumption of the boundedness of \(h\). It turns out that this extra assumption can be dropped, but the proof of this fact is non-trivial; see Theorem 4.5 and its proof for details.}
bijection between the time changes of the process and the family of smooth measures. Roughly speaking, smooth measures are Radon measures that do not charge any set of capacity zero. If $\mu$ is a smooth measure for an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$, then it defines a “time-changed” Dirichlet space $(X, d, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ as well as a “time-changed” Markov process. We say that a measure $\mu$ is admissible, if $\mu$ is a smooth measure and has full quasi-support for the Dirichlet form $(\mathcal{E}, \mathcal{F})$, which amounts to saying that $\mu$ represents a “time change” keeping the form $\mathcal{E}$ essentially unchanged (see Definitions 2.7 and 2.8). We denote the collection of admissible measures by $\mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$. Next, we recall the definition of conformal gauge.

**Definition 1.1 (Conformal gauge).** Let $(X, d)$ be a metric space and $\theta$ be another metric on $X$. We say that $d$ is quasisymmetric to $\theta$, if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that

$$\frac{\theta(x, a)}{\theta(x, b)} \leq \eta \left(\frac{d(x, a)}{d(x, b)}\right)$$

for all triples of points $x, a, b \in X$, $x \neq b$.

The conformal gauge of a metric space $(X, d)$ is defined as

$$\mathcal{J}(X, d) := \{ \theta : X \times X \to [0, \infty) \mid \theta \text{ is a metric on } X, \ d \text{ is quasisymmetric to } \theta \}. \quad (1.2)$$

By [Hei, Proposition 10.6], being quasisymmetric is an equivalence relation among metrics. That is,

$$d \in \mathcal{J}(X, d) \quad \text{and} \quad \mathcal{J}(X, \theta) = \mathcal{J}(X, d) \quad \text{for all } \theta \in \mathcal{J}(X, d). \quad (1.3)$$

The notion of quasisymmetry is an extension of conformal map to the context of metric spaces. Quasisymmetric maps on the real line were introduced by Beurling and Ahlfors, and were studied as boundary values of quasiconformal self-maps of the upper half-plane [BA]. The above definition on general metric spaces is due to Tukia and Väisälä [TV]. This is the reason behind the terminology “conformal gauge”. We refer to [Hei, HK] for expositions to the theory of quasisymmetric maps and quasiconformal geometry on metric spaces.

To characterize the elliptic Harnack inequality, we reparametrize the space by choosing a new metric in the conformal gauge of $(X, d)$ and we reparametrize time by choosing a new symmetric measure that is admissible. More precisely, given an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfying EHI, we seek to find a metric $\theta \in \mathcal{J}(X, d)$ and a measure $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$ such that the corresponding time-changed MMD space $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ equipped with the new metric $\theta$ satisfies PHI($\beta$) for some $\beta > 0$. In other words, we seek to upgrade EHI to PHI($\beta$) by reparametrizing space and time. This motivates the notion of conformal walk dimension.

**Definition 1.2 (Conformal walk dimension).** The conformal walk dimension $d_{cw}$ of an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ is defined as

$$d_{cw} = \inf \left\{ \beta > 0 \mid \text{there exist } \mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F}) \text{ and } \theta \in \mathcal{J}(X, d) \text{ such that } (X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu) \text{ satisfies PHI(}\beta\text{)} \right\}, \quad (1.4)$$

where $\inf \emptyset = +\infty$ and $(\mathcal{E}^\mu, \mathcal{F}^\mu)$ denotes the time-changed Dirichlet form on $L^2(X, \mu)$. 

5
We remark that, if \((X, d, m, \mathcal{E}, \mathcal{F})\) satisfies \(\text{PHI}(\beta)\), then it is easy to see that for any \(\alpha \in (0, 1]\) the MMD space \((X, d^\alpha, m, \mathcal{E}, \mathcal{F})\) satisfies \(\text{PHI}(\beta/\alpha)\) and \(d^\alpha \in \mathcal{J}(X, d)\). This shows that it is easy to increase the walk dimension by changing the metric to a different one in the conformal gauge, but it is non-trivial to decrease the walk dimension. This explains the “infimum” in (1.4). Another remark, which is based on an observation in [Hin02, Section 1], is that the lower bound
\[
d_{cw} \geq 2
\]
is essentially known to experts; indeed, (1.5) can be obtained from the so-called Varadhan-type Gaussian off-diagonal asymptotics of the associated Markov semigroup due to [AH, Theorem 2.7] combined with the characterization of \(\text{PHI}(\beta)\) for \(\beta > 1\) by the heat kernel estimates \(\text{HKE}(\beta)\) with walk dimension \(\beta\) (see Definition 4.1 and Theorem 4.5).

Two natural questions arise. What is the value of \(d_{cw}\)? When is the infimum in (1.4) attained? The answer to the first question is given below. We assume that our metric space \((X, d)\) satisfies the metric doubling property, i.e., admits \(N \in \mathbb{N}\) such that for all \(x \in X\) and \(r > 0\), the ball \(B(x, r)\) can be covered by \(N\) balls of radii \(r/2\).

Our first main result (Theorem 2.9) is that the value of the conformal walk dimension is always two, i.e., we always have the equality in (1.5), for any MMD space satisfying the metric doubling property and the elliptic Harnack inequality. In other words, we have the equivalence among the following three conditions, sharpening the existing characterization of the elliptic Harnack inequality:

(a) \((X, d, m, \mathcal{E}, \mathcal{F})\) satisfies the elliptic Harnack inequality.

(b) \(d_{cw} < \infty\).

(c) \(d_{cw} = 2\).

The equivalence between (a) and (b) is contained in [BM, BCM]. That (c) implies (b) is obvious. Our contribution to the above equivalence is the proof that (a) implies (c). Therefore our result sharpens the characterization of EHI in [BM, BCM]. The result that (a) implies (c) is particularly interesting on fractals as we explain below. Diffusions on many regular fractals are known to satisfy \(\text{PHI}(\beta)\) with \(\beta > 2\). These are often called anomalous diffusions to distinguish them from the classical smooth settings like the Euclidean space where one often has Gaussian space-time scaling and \(\text{PHI}(2)\).

However, by the above equivalence one can “improve” from \(\text{PHI}(\beta)\) to \(\text{PHI}(2 + \varepsilon)\) for any \(\varepsilon > 0\) even on fractals. So this result serves as a bridge between anomalous space-time scaling in fractals and Gaussian space-time scaling seen in smooth settings.

2Since \((\mathcal{E}, \mathcal{F})\) satisfies the locality assumption in [AH] by its strong locality and [CF, Theorem 2.4.3], we can apply [AH, Theorem 2.7], which \(\text{HKE}(\beta)\) with \(\beta \in (1, 2)\) would contradict in view of the finiteness of the limit in [AH, Theorem 2.7] implied by [AH, Proposition 5.1] and \(\text{HKE}(\beta)\). Therefore \(\text{HKE}(\beta)\) for \(\beta \in (1, 2)\) cannot hold, hence neither can \(\text{PHI}(\beta/\alpha)\) for any \(\beta \in (0, 2)\) and any \(\alpha \in (\beta/2, \beta) \cap (0, 1]\) by Theorem 4.5 and thus neither can \(\text{PHI}(\beta)\) for any \(\beta \in (0, 2)\) by the previous remark, proving (1.5).

In Lemma 4.7, we will give an alternative proof of (1.5) based on a relatively simple result from [Mur20].
It is worth mentioning that the proof that (a) implies (b) in [BM, BCM] do not give a universal upper bound for $d_{cw}$. The bound on $d_{cw}$ obtained there depends on the constants in EHI and could be arbitrarily large. To improve the previous (a)-implies-(b) result to the (a)-implies-(c) result, we need a new construction of metrics and measures.

We briefly discuss this new construction in the proof that (a) implies (c), which we will achieve in Section 4. The inspiration behind our argument is the uniformization theorem for Riemann surfaces. In the proof of the uniformization theorem, the Green’s function of a Riemann surface (or a subset of the surface) plays an essential role in constructing the uniformizing map [Mar19, Chapter 15]. We use certain cutoff functions across annuli with small Dirichlet energy at different scales and locations as a substitute for the Green’s function. It is helpful to think of these cutoff functions as equilibrium potentials across annuli. Roughly speaking, the diameter of a ball under the new metric $\theta \in J(X, d)$ for our construction is proportional to the average gradient of the equilibrium potential chosen at a suitable location and scale.

On a technical level, our proof relies heavily on the theory of Gromov hyperbolic spaces. We view $X$ as the boundary of a Gromov hyperbolic space called the hyperbolic filling. The conformal gauge of $X$ is essentially in a bijective correspondence to the bi-Lipschitz changes of the metric on the hyperbolic filling. A desired bi-Lipschitz change of the metric on the hyperbolic filling is constructed using equilibrium potentials as described above. A major ingredient in the proof is a combinatorial description of the conformal gauge due to Carrasco Piaggio [Car13], which we will adapt for our purpose in Section 3.

Our first main result described above (Theorem 2.9) is a partial converse to the trivial implication $\text{PHI}(\beta) \Rightarrow \text{EHI}$ in (1.1). The equivalence between (a) and (c) clarifies the extent to which the converse of this trivial implication holds. Although the value of $d_{cw}$ has a simple description, the following questions remain open in general.

**Problem 1.3.** (1) (Attainment problem) Given an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ that satisfies the metric doubling property and the elliptic Harnack inequality, is the infimum in (1.4) attained?

(2) (Gaussian uniformization problem) Given an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ that satisfies the metric doubling property and the elliptic Harnack inequality, describe all the pairs $(\theta, \mu)$ of metrics $\theta \in J(X, d)$ and measures $\mu \in A(X, d, m, \mathcal{E}, \mathcal{F})$ such that the corresponding time-changed MMD space $(X, \theta, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})$ satisfies $\text{PHI}(2)$.

We describe two examples of self-similar fractals for which a positive answer to the attainment problem is known. Kigami has shown that Brownian motion on the two-dimensional Sierpiński gasket attains the infimum, where $\mu$ is the Kusuoka measure and $\theta$ is the corresponding intrinsic metric [Kig08]. Further examples of admissible measures that attain the infimum for the two-dimensional Sierpiński gasket is described in [Kaj12]; see Theorem 6.30 below and the references in its proof for details. In retrospect, Kigami’s measurable Riemannian structure on the Sierpiński gasket is the first evidence towards the implication (a) $\Longrightarrow$ (c) in Theorem 2.9. Another example of a fractal that attains the infimum in (1.4) is the two-dimensional snowball described in [Mur19]. The “snowball”
fractal can be viewed as a limit of Riemann surfaces and is a two-dimensional analog of the von Koch snowflake. In this example, the answer to the attainment problem is obtained by considering a limit of uniformizing maps to $S^2$ and using the conformal invariance of Brownian motion. Our terminology “Gaussian uniformization problem” is inspired by this example and the classical fact from [Stu96] (see also Proposition 2.10 and Theorem 4.5 below) that PHI(2) is equivalent to Gaussian heat kernel estimates.

Nevertheless, the infimum in (1.4) need not be attained in general. We show in Subsection 6.3 that the Vicsek set and the $N$-dimensional Sierpiński gasket with $N \geq 3$ fails to attain the infimum in (1.4). The examples with non-attainment of $d_{cw}$ rely on the following result (Theorems 6.16 and 6.49). For a “regular” self-similar fractal, if the infimum in (1.4) is attained by a quasisymmetric metric $\theta$ and an admissible measure $\mu$, then it is possible to choose $\mu$ as the energy measure of a function that is harmonic outside a canonical boundary. This result immediately implies the non-attainment of $d_{cw}$ for the Vicsek set, since the energy measure of any such harmonic function fails to have full support. The non-attainment of $d_{cw}$ for the $N$-dimensional Sierpiński gasket with $N \geq 3$ requires a more delicate analysis of the intrinsic metric (see Definition 2.3) associated to the energy measure.

Next, we mention some progress towards the Gaussian uniformization problem. If $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies PHI(2), then it easily follows by combining the results in [KM] and [Mur20] (see Proposition 2.10-(a)) that the metric $\theta$ is bi-Lipschitz equivalent to the intrinsic metric of $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ and in particular that the metric $\theta$ is determined by the measure $\mu$ up to a bi-Lipschitz change. Therefore, in order to find a metric $\theta \in \mathcal{J}(X,d)$ and a measure $\mu \in \mathcal{A}(X,d,m,\mathcal{E},\mathcal{F})$ in the Gaussian uniformization problem, it is enough to find an appropriate measure $\mu$. Furthermore, by [KM, Propositions 4.5 and 4.7], we know that any such $\mu$ is a minimal energy-dominant measure, i.e., mutually absolutely continuous with respect to the whole family of energy measures (see Definition 2.2), so that any two admissible measures that arise in the Gaussian uniformization problem are mutually absolutely continuous. We strengthen this result by showing in Section 5 that any two admissible measures that arise in the Gaussian uniformization problem are $A_{\infty}$-related in $(X,d)$ in the sense of Muckenhoupt (Theorem 2.11).

This last result on $A_{\infty}$-relation between admissible measures in the Gaussian uniformization problem and its proof are inspired by a similar result for Ahlfors regular conformal dimension on Loewner spaces [HK, Theorem 7.11]. The combinatorial description of conformal gauge used in the proof of Theorem 2.9 was developed in [Car13] for studying Ahlfors regular conformal dimension. Therefore, we find it appropriate to recall the definition of Ahlfors regular conformal dimension and discuss some related questions.

Given a metric space $(X,d)$ and a Borel measure $\mu$ on $X$, we say that $\mu$ is $p$-Ahlfors regular if there exists $C > 0$ such that

$$C^{-1}r^p \leq \mu(B(x,r)) \leq Cr^p \quad \text{for all } x \in X \text{ and } r > 0 \text{ such that } B(x,r) \neq X.$$ 

It is easy to verify that if a $p$-Ahlfors regular measure $\mu$ exists on $(X,d)$, then the $p$-dimensional Hausdorff measure $\mathcal{H}^p$ is also $p$-Ahlfors regular and the Hausdorff dimension of $(X,d)$ is $p$. Therefore, the existence of a $p$-Ahlfors regular measure is a property of the
metric $d$. The *Ahlfors regular conformal dimension* of a metric space $(X,d)$ is defined as

$$d_{ARC}(X,d) = \inf \left\{ p > 0 \mid \text{there exists } \theta \in \mathcal{J}(X,d) \text{ and a } p\text{-Ahlfors regular measure } \mu \text{ on } (X,\theta) \right\}. \quad (1.6)$$

The attainment problem for the Ahlfors regular conformal dimension of the standard (two-dimensional) Sierpiński carpet is a well-known open question [BK05, Problem 6.2]. An important motivation for studying this attainment problem is Cannon’s conjecture in geometric group theory. Cannon’s conjecture states that every finitely generated, Gromov-hyperbolic group $G$ whose boundary (in the sense of Gromov) is homeomorphic to the 2-sphere is isomorphic to a Kleinian group, i.e., a discrete group of Möbius transformations on the Riemann sphere. Bonk and Kleiner have shown that Cannon’s conjecture is equivalent to the attainment of Ahlfors regular conformal dimension of the boundary of such a group [BK05, Theorem 1.1]. Our results and proof techniques will make it clear that there are similarities between the attainment problems for Ahlfors regular conformal dimension and conformal walk dimension. We hope that some of the methods we develop towards the attainment problem for conformal walk dimension has applications to the analogous attainment problem for Ahlfors regular conformal dimension.

Our work suggests that it would be useful to develop a theory of non-linear Dirichlet forms to study Ahlfors regular conformal dimension of fractals. In particular, Theorems 6.16 and 6.49 show that if the infimum in (1.4) is attained on a self-similar fractal, then an optimal admissible measure can be chosen to be the energy measure of a harmonic function. This result and its proof suggest that one might be able to construct an optimal Ahlfors regular measure attaining the Ahlfors regular conformal dimension as the “energy measure” of a $p$-harmonic function. However, the notions of energy measure and $p$-harmonic functions for non-linear Dirichlet energy remain to be developed on fractals (non-linear Dirichlet energy can be formally viewed as $\int |\nabla f|^p$ with $p \neq 2$ and the corresponding $p$-harmonic functions can be viewed as minimizers of the non-linear Dirichlet energy). There is a well-developed non-linear potential theory in smooth settings (see [HKM] and references therein) but a similar theory is yet to be developed on fractals.

## 2 Framework and results

In this section, we recall the background definitions and state our main results.

### 2.1 Metric measure Dirichlet space and energy measure

Throughout this paper, we consider a metric space $(X,d)$ in which $B(x,r) := B_d(x,r) := \{ y \in X \mid d(x,y) < r \}$ is relatively compact (i.e., has compact closure) for any $(x,r) \in X \times (0,\infty)$, and a Radon measure $m$ on $X$ with full support, i.e., a Borel measure $m$ on $X$ which is finite on any compact subset of $X$ and strictly positive on any non-empty open subset of $X$. We always assume that $X$ contains at least two elements, and such a triple
Definition 2.1. Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space. The energy measure \(\Gamma(f, f)\) of \(f \in \mathcal{F}\) associated with \((X, d, m, \mathcal{E}, \mathcal{F})\) is defined, first for \(f \in \mathcal{F} \cap L^\infty(X, m)\) as the unique \([0, \infty]-valued\) Borel measure on \(X\) such that

\[
\int_X g \, d\Gamma(f, f) = \mathcal{E}(f, fg) - \frac{1}{2} \mathcal{E}(f^2, g)
\]

for all \(g \in \mathcal{F} \cap C_c(X)\), \hspace{1cm} (2.1)

and then by \(\Gamma(f, f)(A) := \lim_{n \to \infty} \Gamma((-n) \vee (f \wedge n), (-n) \vee (f \wedge n))(A)\) for each Borel subset \(A\) of \(X\) for general \(f \in \mathcal{F}\).

We recall the notion of minimal energy-dominant measure.

Definition 2.2 ([Hin10, Definition 2.1]). Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space. A \(\sigma\)-finite Borel measure \(\nu\) on \(X\) is called a minimal energy-dominant measure of \((\mathcal{E}, \mathcal{F})\) if the following two conditions are satisfied:

(i) (Domination) For every \(f \in \mathcal{F}\), \(\Gamma(f, f) \ll \nu\).

(ii) (Minimality) If another \(\sigma\)-finite Borel measure \(\nu'\) on \(X\) satisfies condition (i) with \(\nu\) replaced by \(\nu'\), then \(\nu \ll \nu'\).

Note that by [Hin10, Lemmas 2.2, 2.3 and 2.4], a minimal energy-dominant measure of \((\mathcal{E}, \mathcal{F})\) always exists. By the minimality property, we note that any two minimal energy dominant measures are mutually absolutely continuous.
We recall the definition of intrinsic metric.

**Definition 2.3.** Let \((X,d,m,\mathcal{E},\mathcal{F})\) be an MMD space. We define its *intrinsic metric* \(d_{\text{int}} : X \times X \to [0, \infty]\) by

\[
d_{\text{int}}(x,y) := \sup\{f(x) - f(y) \mid f \in \mathcal{F}_{\text{loc}} \cap C(X), \Gamma(f, f) \leq m\},
\]

where \(C(X) := \{f \mid f : X \to \mathbb{R}, \ f \text{ is continuous}\}\),

\[
\mathcal{F}_{\text{loc}} := \left\{ f \mid f \text{ is an } m\text{-equivalence class of } \mathbb{R}\text{-valued Borel measurable functions on } X \text{ such that } f1_V = f^#1_V \text{ m-a.e. for some } f^\# \in \mathcal{F} \right\}
\]

and the energy measure \(\Gamma(f, f)\) of \(f \in \mathcal{F}_{\text{loc}}\) associated with \((X,d,m,\mathcal{E},\mathcal{F})\) is defined as the unique Borel measure on \(X\) such that \(\Gamma(f, f)(A) = \Gamma(f^#, f^#)(A)\) for any relatively compact Borel subset \(A\) of \(X\) and any \(V, f^#\) as in (2.3) with \(A \subset V\); note that \(\Gamma(f^#, f^#)(A)\) is independent of a particular choice of such \(V, f^#\).

### 2.2 Harnack inequalities

We recall the definition of harmonic and caloric functions.

**Definition 2.4.** Let \((X,d,m,\mathcal{E},\mathcal{F})\) be an MMD space and let \(\mathcal{F}_e\) extended Dirichlet space corresponding to \((X,d,m,\mathcal{E},\mathcal{F})\). Recall that the extended Dirichlet space is the space of \(m\)-equivalence class of functions \(f\) such that there exists an \(E\)-Cauchy sequence \((f_n)_{n \in \mathbb{N}}\) in \(F\) such that \(\lim_{n \to \infty} f_n = f\) \(m\)-almost everywhere in \(X\) [FOT, Theorem 1.5.2].

A function \(h \in \mathcal{F}_e\) is said to be \(E\)-harmonic on an open subset \(U\) of \(X\), and

\[
\mathcal{E}(h, f) = 0
\]

for all \(f \in \mathcal{F} \cap C_c(X)\) with \(\text{supp}_{m}[f] \subset U\).

Let \(I\) be an open interval in \(\mathbb{R}\). We say that a function \(u : I \to L^2(X, m)\) is *weakly differentiable* at \(t_0 \in I\) if for any \(f \in L^2(X, m)\) the function \(t \mapsto \langle u(t), f \rangle\) is differentiable at \(t_0\), where \(\langle \cdot, \cdot \rangle\) denotes the inner product in \(L^2(X, m)\). By the uniform boundedness principle, there exists a (unique) function \(w \in L^2(X, m)\) such that

\[
\lim_{t \to t_0} \frac{u(t) - u(t_0)}{t - t_0}, f = \langle w, f \rangle, \quad \text{for all } f \in L^2(X, m).
\]

We say that the function \(w\) is the *weak derivative* of the function \(u\) at \(t_0\) and write \(w = u'(t_0)\).

Let \(I\) be an open interval in \(\mathbb{R}\) and let \(\Omega\) be an open subset of \(X\). A function \(u : I \to \mathcal{F}\) is said to be *caloric* in \(I \times \Omega\) if \(u\) is weakly differentiable in the space \(L^2(\Omega)\) at any \(t \in I\), and for any \(f \in \mathcal{F} \cap C_c(\Omega)\), and for any \(t \in I\),

\[
\langle u', f \rangle + \mathcal{E}(u, f) = 0.
\]

11
Definition 2.5 (Harnack inequalities). We say that an MMD space \((X, d, m, \mathcal{E}, \mathcal{F})\) satisfies the elliptic Harnack inequality (abbreviated as EHI), if there exist \(C > 1\) and \(\delta \in (0,1)\) such that for all \(x \in X, r > 0\) and for any non-negative harmonic function \(h \in \mathcal{F}_e\) on the ball \(B(x, r)\), we have

\[
\text{ess sup}_{B(x, \delta r)} h \leq C \text{ess inf}_{B(x, \delta r)} h. \tag{EHI}
\]

We say that an MMD space \((X, d, m, \mathcal{E}, \mathcal{F})\) satisfies the parabolic Harnack inequality with walk dimension \(\beta\) (abbreviated as PHI(\(\beta\))), if there exist \(0 < C_1 < C_2 < C_3 < C_4 < \infty, C_5 > 1\) and \(\delta \in (0,1)\) such that for all \(x \in X, r > 0\) and for any non-negative bounded caloric function \(u\) on the space-time cylinder \(Q = (a, a + C_4 r^\beta) \times B(x, r)\), we have

\[
\text{ess sup}_{Q_-} u \leq C_5 \text{ess inf}_{Q_+} u, \tag{PHI(\(\beta\))}
\]

where \(Q_- = (a + C_1 r^\beta, a + C_2 r^\beta) \times B(x, \delta r)\) and \(Q_+ = (a + C_3 r^\beta, a + C_4 r^\beta) \times B(x, \delta r)\).

2.3 Admissible measures and time-changed Dirichlet space

Given an MMD space \((X, d, m, \mathcal{E}, \mathcal{F})\) and a Borel set \(A\), we define its 1-capacity as

\[
\text{Cap}(A) = \inf \{\mathcal{E}(f, f) + \|f\|_2^2 : f \in \mathcal{F} \cap C(X), f \equiv 1 \text{ on a neighborhood of } A\},
\]

where \(\|f\|_2\) denotes the \(L^2(X, m)\) norm. For disjoint Borel sets \(A, B\) such that \(B\) is closed and \(\overline{A} \in \mathcal{B}^c\) (by \(A \in \mathcal{B}^c\), we mean that \(\overline{A}\) is compact and \(\overline{A} \subset \mathcal{B}^c\)), we define \(\mathcal{F}(A, B)\) as the set of function \(\phi \in \mathcal{F}\) such that \(\phi \equiv 1\) in an open neighborhood of \(A\), and \(\text{supp}\phi \subset \mathcal{B}^c\). For such sets \(A\) and \(B\), we define the capacity between them as

\[
\text{Cap}(A, B) = \inf \{\mathcal{E}(f, f) \mid f \in \mathcal{F}(A, B)\}.
\]

Definition 2.6 (Smooth measures). Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space. A Radon measure \(\mu\) on \(X\) is smooth if \(\mu\) charges no set of zero capacity (that is \(\text{Cap}_1(A) = 0\) implies \(\mu(A) = 0\) for any Borel set \(A\)).

An increasing sequence \(\{F_k : k \geq 1\}\) of closed subsets of an MMD space \((X, d, m, \mathcal{E}, \mathcal{F})\) is said to be a nest if \(\bigcup_{k \geq 1} F_k\) is \(\sqrt{\mathcal{E}}\)-dense in \(\mathcal{F}\), where \(\mathcal{F}_{F_k} := \{f \in \mathcal{F} : f = 0\text{ m-a.e. on } X \setminus F_k\}\) and \(\mathcal{E}_1(f, f) = \mathcal{E}(f, f) + \|f\|_2^2\). Recall that \(D \subset X\) is quasi-open if there exists a nest \(\{F_n\}\) such that \(D \cap F_n\) is an open subset of \(F_n\) in the relative topology for each \(n \in \mathbb{N}\). We are interested in the class of admissible measures, namely smooth Radon measures having full quasi-support in the sense defined as follows.

Definition 2.7 (Full quasi-support, admissible measures; [FOT, (4.6.3) and (4.6.4)]). Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space. A smooth Radon measure \(\mu\) on \(X\) is said to have full quasi-support if for any quasi-closed set \(F\) with \(\mu(X \setminus F) = 0\) we have \(\text{Cap}_1(X \setminus F) = 0\). A Borel measure \(\mu\) on \(X\) is said to be admissible if \(\mu\) is a smooth Radon measure on \(X\) with full quasi-support, and the set of admissible measures with respect to \((X, d, m, \mathcal{E}, \mathcal{F})\) is denoted by \(\mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})\).
Definition 2.8 (Time-changed Dirichlet form). Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space. If \(\mu\) is a smooth Radon measure, it defines a time change of the process whose associated Dirichlet form is called the trace Dirichlet form and denoted by \((\mathcal{E}^\mu, \mathcal{F}^\mu)\) (see [FOT, Section 6.2] and [CF, Section 5.2]). Assume \(\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})\). Then the trace Dirichlet form \((\mathcal{E}^\mu, \mathcal{F}^\mu)\) is given by

\[
\mathcal{F}^\mu = \mathcal{F}_e \cap L^2(X, \mu) \quad \text{and} \quad \mathcal{E}^\mu(u, v) = \mathcal{E}(u, v) \quad \text{for } u, v \in \mathcal{F}^\mu, \tag{2.6}
\]

and \((\mathcal{E}^\mu, \mathcal{F}^\mu)\) is a strongly local, regular symmetric Dirichlet form on \(L^2(X, \mu)\) by [FOT, Theorems 5.1.5, 6.2.1 and Exercise 3.1.1]. We also note that by [CF, Theorem 5.2.11],

\[
\mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F}) = \mathcal{A}(X, d, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu). \tag{2.7}
\]

In probabilistic terms, \((\mathcal{E}^\mu, \mathcal{F}^\mu)\) is the Dirichlet form of the time-changed process \((\omega, t) \mapsto Y_{\tau_{\omega}}(\omega)\), where \((Y_t)_{t \geq 0}\) is an \(m\)-symmetric diffusion on \(X\) whose Dirichlet form is \((\mathcal{E}, \mathcal{F})\) and \(\tau_{\omega}\) is the right-continuous inverse of the positive continuous additive functional \((A_t)_{t \geq 0}\) of \((Y_t)_{t \geq 0}\) with Revuz measure \(\mu\); see [FOT, Section 6.2] and [CF, Section 5.2] for details.

2.4 Main results

Our first main result is that the value of the conformal walk dimension is an invariant for spaces satisfying the EHI. We recall that the conformal walk dimension of an MMD space \((X, d, m, \mathcal{E}, \mathcal{F})\) is the infimum over all \(\beta > 0\) such that there exist an admissible measure \(\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})\) and a metric \(\theta \in \mathcal{J}(X, d)\) such that the time-changed MMD space \((X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)\) satisfies the parabolic Harnack inequality \(\text{PHI}(\beta)\); note here that \((X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)\) is indeed an MMD space since, for any \((x, r) \in X \times (0, \infty)\), \(\text{diam}(B_\theta(x, r), d) < \infty\) by \(\theta \in \mathcal{J}(X, d)\) and [Hei, Proposition 10.8] and hence \(B_\theta(x, r)\) is relatively compact in \(X\).

Theorem 2.9 (Universality of conformal walk dimension). Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space such that \((X, d)\) satisfies the metric doubling property. Let \(d_{cw}\) denote the conformal walk dimension of \((X, d, m, \mathcal{E}, \mathcal{F})\). Then the following are equivalent:

\[
(a) \ (X, d, m, \mathcal{E}, \mathcal{F}) \text{ satisfies the elliptic Harnack inequality.}
\]

\[
(b) \ d_{cw} < \infty.
\]

\[
(c) \ d_{cw} = 2.
\]

The next question is whether or not the infimum in the definition of \(d_{cw}\) in (1.4) is attained. To this end we first describe the metric and measure. The following result is essentially contained in [KM]; see the beginning of Subsection 5.1 for the proof.

Proposition 2.10. Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space that satisfies \(\text{PHI}(2)\).

\[
(a) \ Then the metric \(d\) is bi-Lipschitz equivalent to the intrinsic metric \(d_{int}\).
\]
(b) The symmetric measure \( m \) is a minimal energy-dominant measure.

By Proposition 2.10(a), in order to find a metric \( \theta \in \mathcal{J}(X,d) \) and a measure \( \mu \in \mathcal{A}(X,d,m,\mathcal{E},\mathcal{F}) \) in the Gaussian uniformization problem, it is enough to find an appropriate measure \( \mu \) as the metric \( \theta \) is essentially determined by the measure up to a bi-Lipschitz transform. Since constructing measures are typically easier than constructing metrics, it is useful to restrict our attention to finding suitable measures in the Gaussian uniformization problem. By Proposition 2.10(b), any such measure is determined uniquely up to a mutually absolutely continuous change of measure. In fact, we have the following improvement of Proposition 2.10. Any two such measures \( \mu_1, \mu_2 \) that attain the infimum in the definition of \( d_{cw} \) are \( A_\infty \)-related in the sense of Muckenhoupt (see Definition 5.4).

**Theorem 2.11.** Let \((X,d,m,\mathcal{E},\mathcal{F})\) be an MMD space that satisfies the elliptic Harnack inequality. Let \((X,\theta_i,\mu_i,\mathcal{E}^{\mu_i},\mathcal{F}^{\mu_i}), i = 1,2\) be two time-changed MMD spaces that satisfy PHI(2) such that \( \theta_i \in \mathcal{J}(X,d) \), \( \mu_i \in \mathcal{A}(X,d,m,\mathcal{E},\mathcal{F}) \) for \( i = 1,2 \). Then the measures \( \mu_1 \) and \( \mu_2 \) are \( A_\infty \)-related in \((X,d)\).

Using Proposition 2.10, Theorem 2.11 and sharp constants of Poincaré inequalities in [CW], we answer the Gaussian uniformization problem for Brownian motion on \( \mathbb{R} \). We do not know the answer to Gaussian uniformization problem for Brownian motion in higher dimensional Euclidean space \( \mathbb{R}^n, n \geq 2 \).

### 2.5 Outline of the proof of Theorem 2.9

By changing metric and measure, we may assume that \((X,d,m,\mathcal{E},\mathcal{F})\) satisfies PHI(\( \gamma \)) for some \( \gamma > 2 \). Let \( \beta > 2 \) be arbitrary. We wish to construct a metric \( \theta \in \mathcal{J}(X,d) \) and a measure \( \mu \in \mathcal{A}(X,d,m,\mathcal{E},\mathcal{F}) \) such that the trace Dirichlet form \((X,\theta,\mu,\mathcal{E}^{\mu},\mathcal{F}^{\mu})\) satisfies PHI(\( \beta \)).

To sketch the main ideas, we further assume that \((X,d)\) is compact and the diameter of \((X,d)\) is normalized to \( \frac{1}{3} \). The non-compact case follows by the same argument as the compact case, by considering \( X \) as a limit of compact sets. Thanks to the known characterizations of PHI(\( \beta \)) as stated in Theorem 4.5 and the preservation of EHI between \((X,d,m,\mathcal{E},\mathcal{F})\) and \((X,\theta,\mu,\mathcal{E}^{\mu},\mathcal{F}^{\mu})\) as stated in Lemma 4.8, it is enough to construct a metric \( \theta \in \mathcal{J}(X,d) \) and a measure \( \mu \in \mathcal{A}(X,d,m,\mathcal{E},\mathcal{F}) \) such that

\[
\mu(B_\theta(x,r)) \asymp r^\beta \text{Cap}(B_\theta(x,r),B_\theta(x,2r)^c), \quad \text{for all } x \in X, r \lesssim \text{diam}(X,\theta). \tag{2.8}
\]

The above estimate relating the measure \( \mu \) and capacity implies that \( \mu \) is a smooth measure with full quasi-support and satisfies the following volume doubling and reverse volume doubling properties (see Proposition 4.14): there exists \( C_D > 0 \) and \( c_D \in (0,1) \) such that

\[
\frac{\mu(B_d(x,2r))}{\mu(B_d(x,r))} \leq C_D, \quad \text{for all } x \in X, r > 0, \text{ and} \tag{2.9}
\]

\[
\frac{\mu(B_d(x,2r))}{\mu(B_d(x,2r))} \leq c_D, \quad \text{for all } x \in X, 0 < r \lesssim \text{diam}(X,d). \tag{2.10}
\]
The estimate (2.8) along with Theorem 4.5, implies \( \text{PHI}(\beta) \) because volume doubling, reverse volume doubling, and EHI are preserved by the quasisymmetric change of the metric from \( d \) to \( \theta \).

The construction of metric \( \theta \) and measure \( \mu \) is a modification of [Car13], but instead of the Ahlfors regularity required in [Car13] we need to establish (2.8). Following [Car13], we construct the metric \( \theta \) and measure \( \mu \) that satisfy (2.8) using a multi scale argument. This part of the argument relies on theory of Gromov hyperbolic spaces. The basic idea behind the approach is to construct a graph (called the hyperbolic filling) that is Gromov hyperbolic, whose boundary (in the sense of Gromov) corresponds to the given metric space \((X, d)\). A well-known result in Gromov hyperbolic spaces asserts that any metric in the conformal gauge \( \mathcal{J}(X, d) \) up to bi-Lipschitz equivalence can be obtained by a bounded perturbation of edge weights on the hyperbolic filling. We recall the basic results about Gromov hyperbolic spaces and their boundaries in Subsection 3.1.

We first sketch the construction of this hyperbolic space postponing a more precise definition to Subsection 3.2. We choose a parameter \( a > 10^2 \) and cover the space \( X \) using a covering \( \mathcal{S}_n \) with balls of radii \( 2a^{-n} \) such that for any two distinct balls \( B(x_1, 2a^{-n}), B(x_2, 2a^{-n}) \in \mathcal{S}_n \), we have \( B(x_1, a^{-n}/2) \cap B(x_1, a^{-n}/2) = \emptyset \) (we think of these balls as ‘approximately pairwise disjoint covering’ at scale \( a^{-n} \)). Therefore the covering \( \mathcal{S}_n \) corresponds to scale \( a^{-n} \) for all \( n \in \mathbb{N}_0 \). In what follows, for a ball \( B \), we denote by \( x_B \) and \( r_B \) the center and radius of \( B \). For a ball \( B \) and \( \lambda > 0 \), we denote by \( \lambda \cdot B \), the ball \( B(x_B, \lambda r_B) \).

We define a tree of vertical edges with vertex set \( \coprod_{n \geq 0} \mathcal{S}_n \) by choosing for each ball \( B \in \mathcal{S}_n, n \geq 1 \) a ‘parent ball’ \( B' \in \mathcal{S}_{n-1} \) such that \( x_B \) is a closest point to \( x_B \) in the set \( \{ x_C : C \in \mathcal{S}_{n-1} \} \). By the assumption on the diameter, we may assume that \( \mathcal{S}_0 \) is a singleton. The edges in this tree are called vertical edges. We choose another parameter \( \lambda \geq 10 \) such to define another set of edges on \( \coprod_{n \geq 0} \mathcal{S}_n \) called the horizontal edge. Two distinct balls \( B, \tilde{B} \in \mathcal{S}_n, n \geq 0 \) share a horizontal edge if and only if \( \lambda \cdot B \cap \lambda \cdot \tilde{B} = \emptyset \). The edges of the hyperbolic filling is the union of horizontal and vertical edges.

In our construction, the vertical edge weights play a more central role and the value of horizontal edge weights are less important. The weight of the vertical edge between \( B \in \mathcal{S}_n, B' \in \mathcal{S}_{n-1} \) can be interpreted as the relative diameter under the \( \theta \)-metric. More precisely, let us define the relative diameter of \( B \in \mathcal{S}_n, n \geq 1 \) as

\[
\rho(B) := \frac{\text{diam}(B, \theta)}{\text{diam}(B', \theta)},
\]

where \( B' \in \mathcal{S}_{n-1} \) is such that there is a vertical edge between \( B' \) and \( B \) (\( B' \) is the parent of \( B \) in the tree of vertical edges). It turns out that the ‘relative diameter’ in (2.11) contains enough information about \( \theta \) to reconstruct the metric \( \theta \) (up to bi-Lipschitz equivalence). Hence we could reduce the problem of construction of \( \theta \in \mathcal{J}(X, d) \) to constructing the function \( \rho(\cdot) \) on \( \coprod_{n \geq 1} \mathcal{S}_n \); see Theorem 3.14.

It is therefore enough to construct \( \rho(\cdot) \) in (2.11). Next, we describe two key conditions that the relative diameter \( \rho(\cdot) \) defined in (2.11) must satisfy. For a ball \( B \in \mathcal{S}_{n-1}, n \geq 1 \),
let us denote by $\Gamma_n(B)$ the set of horizontal paths in $S_n$ defined by

$$\Gamma_n(B) = \left\{ (B_i)_{i=0}^N \mid N \in \mathbb{N}, B_i \in S_n \text{ for all } i = 0, \ldots, N; B_i \text{ and } B_{i+1} \text{ share a horizontal edge for all } i = 0, \ldots, N-1, x_{B_0} \in B \right\}.$$ 

The first condition on $\rho(\cdot)$ is

$$\sum_{i=1}^N \rho(B_i) \gtrsim 1, \quad \text{for all } (B_i)_{i=0}^N \in \Gamma_n(B) \text{ and } B \in S_{n-1}, n \geq 1. \quad (2.12)$$

The condition (2.12) is a consequence of the fact that $\theta \in J(X,d)$ and that $(X,d)$ is a uniformly perfect metric space. We like to think of (2.12) as a 'no shortcuts condition' as it disallows the possibility of short cuts in the $\theta$-metric from $B_d(x,r)$ to $B_d(x,2r)^c$.

The second condition arises from the estimate (2.8). For any ball $B \in S_k$, $k \geq 0$, by $D_{k+1}(B)$, we denote its descendants in $S_{k+1}$; that is $D_{k+1}(B)$ is the set of elements $B' \in S_{k+1}$ such that $B'$ and $B$ share a vertical edge. The second condition is that $\rho$ must satisfy the following estimate:

$$\sum_{B' \in D_{k+1}(B)} \rho(B')^\beta \text{Cap}(B', (2 \cdot B')^c) \lesssim \text{Cap}(B, (2 \cdot B)^c), \quad \text{for all } B \in S_k \text{ and } k \text{ large enough}. \quad (2.13)$$

To explain (2.13), we first observe that the volume doubling property (2.9) implies that

$$\sum_{B' \in D_{k+1}(B)} \mu(B') \lesssim \mu(B), \quad \text{for all } B \in S_k.$$ 

By (2.8), $\theta \in J(X,d)$ and comparing estimates using EHI, one obtains $\mu(B) \asymp \text{diam}(B, \theta)^\beta \text{Cap}(B, (2 \cdot B)^c)$ for all $B \in S_k$ and for all large enough $k$. Combining these estimates and with (2.9), we obtain (2.13). To summarize, the conditions (2.12) and (2.13) arise from the metric and measure respectively. It turns out that the necessary conditions (2.12) and (2.13) on $\rho(\cdot)$ are 'almost sufficient' to construct $\rho$; see Theorem 3.24.

We note that there is a tension between the estimates (2.12) and (2.13). On the one hand, in order to satisfy (2.12), the function $\rho(\cdot)$ must be large enough, whereas (2.13) imposes that $\rho(\cdot)$ can not be too large. Next, we sketch how to construct $\rho$ that satisfies these seemingly conflicting requirements in (2.12) and (2.13). Let $B \in S_{k+1}$, $u \in C_c(X)$ be such that

$$u \equiv 1 \text{ on } B(x_B, 1.1r_B), \quad u \equiv 0 \text{ on } B(x_B, 1.9r_B)^c. \quad \mathcal{E}(u,u) \asymp \frac{m(B)}{r_B^\gamma}. \quad (2.14)$$

Such a function exists because of (2.8) for $(X,d,m,\mathcal{E},\mathcal{F})$ with $\beta$ replaced by $\gamma$ and a covering argument. It is helpful to think of $u$ as the equilibrium potential corresponding
to $\text{Cap}(B(x_B, 1.1 r_B), B(x_B, 1.9 r_B)^c)$. Let us define the functions $u_B, \rho_B : S_{k+1} \to [0, \infty)$ as

$$u_B(B') := \int_{B'} u \, dm = \frac{1}{m(B')} \int_{B'} u \, dm,$$

$$\rho_B(B'') = \sum_{B'' \in S_{k+1}, B'' \sim B'} |u_B(B'') - u_B(B')|,$$

where $B'' \sim B'$ means that $B''$ and $B'$ share a horizontal edge (or equivalently, $\lambda \cdot B'' \cap \lambda \cdot B' \neq \emptyset$). From the definitions it is clear that $u_B$ is a discrete version of $u$ and $\rho_B$ is a discrete version of the gradient of $u$. Using a Poincaré inequality on $(X, d, m, \mathcal{E}, \mathcal{F})$ and the bound $\text{Cap}(B', (2 \cdot B')^c) \asymp \frac{m(B')}{\rho(B')}$, we obtain the following estimate (see (4.22)):

$$\sum_{B' \in D_{k+1}(B)} \rho_B(B')^2 \text{Cap}(B', (2 \cdot B')^c) \lesssim \mathcal{E}(u, u) \lesssim \text{Cap}(B, (2 \cdot B)^c). \quad (2.15)$$

Since $u_B(B_0) = 1$ for any $B_0 \in S_{k+1}$ with $x_{B_0} \in B$ and $u_B(B_N) = 0$ for any $B_N \in S_{k+1}$ with $x_{B_N} \notin (2 \cdot B)^c$, by the triangle inequality

$$\sum_{i=1}^N \rho_B(B_i) \geq 1, \quad \text{for all } (B_i)_{i=0}^N \in \Gamma_{k+1}(B). \quad (2.16)$$

Clearly, (2.16) and (2.15) are local versions of (2.12) and (2.13) respectively with $\beta = 2$. Here, by “local version” we mean that the estimates (2.12) and (2.13) are satisfied for a fixed value of $B$. To ensure (2.12) and (2.13) for all scales and locations, we define

$$\rho(B') = \sup_{B \in S_k} \rho_B(B'), \quad \text{for all } B' \in S_{k+1}, \quad (2.17)$$

where $\rho_B$ is defined as above at all locations and scales. This ensures (2.12) and (2.13) at all scales with $\beta = 2$.

However, $\rho$ should satisfy further additional conditions that the above construction need not obey. Since $\theta \in \mathcal{J}(X, d)$, one obtains that

$$\rho(B) \gtrsim 1 \quad \text{for all } B \in S_k, \ k \geq 1. \quad (2.18)$$

However, the function $\rho(\cdot)$ defined in (2.17) need not satisfy the estimate (2.18). This requires us to increase $\rho$ further if necessary. We define the ‘diameter function’

$$\pi(B) = \prod_{k=0}^n \pi(B_i), \quad \text{for all } B \in S_n, \quad (2.19)$$

where $B_i \in S_i$ for all $i = 0, \ldots, n$ and $B_n = B$ and there is a vertical edge between $B_i$ and $B_{i+1}$ for all $i = 0, \ldots, n - 1$. If $\rho$ is given by (2.11), then clearly $\pi(B) = \text{diam}(B, \theta)/\text{diam}(X, \theta)$. By quasisymmetry, $\text{diam}(B, \theta) \asymp \text{diam}(B', \theta)$ whenever $B$ and $B'$ share a horizontal edge. This suggests the following condition on $\rho$:

$$\pi(B) \asymp \pi(B'), \quad \text{whenever } B \text{ and } B' \text{ share a horizontal edge}, \quad (2.20)$$

...
where \( \pi \) is defined as given in (2.19). Similarly, for constructing measure, we need to ensure that \( \rho \) satisfies
\[
\pi(B)^\beta \text{Cap}(B,(2\cdot B)^c) \simeq \sum_{B' \in \mathcal{D}_n(B)} \pi(B')^\beta \text{Cap}(B',(2\cdot B')^c), \quad \text{for all } B \in S_k \text{ and } n > k \geq 0,
\]
where \( \mathcal{D}_n(B) \) denotes the descendants of \( B \) in \( S_n \). The conditions (2.20) and (2.21) are rather delicate because the value of \( \pi \) can change drastically if we change \( \rho \) by a bounded multiplicative factor. This is due to the fact that the multiplicative ‘errors’ in \( \rho \) accumulate as we move to finer and finer scales. This suggests that we need to control the constants very carefully.

To achieve this we need to consider \( \beta > 2 \) (instead of \( \beta = 2 \) considered above). We choose \( \rho \) defined in (2.17) uniformly small by picking a function \( u \) that satisfies (2.14) along with an additional scale invariant Hölder continuity estimate (see (4.27), (4.28) and (4.29)). Then using
\[
\sum_{B' \in \mathcal{D}_{k+1}(B)} \rho_B(B')^\beta \text{Cap}(B',(2\cdot B')^c) \leq \|\rho\|^{\beta-2}_\infty \sum_{B' \in \mathcal{D}_{k+1}(B)} \rho_B(B')^2 \text{Cap}(B',(2\cdot B')^c)
\]

we obtain enough control on the constants in (2.13) to ensure (2.18) and (2.20) after further modification of \( \rho \). By the Hölder continuity estimate on \( u \), \( \|\rho\|_\infty \) can be made arbitrarily small by increasing the vertical parameter \( a \).

### 3 Metric and measure via hyperbolic filling

Given a metric space, it is often useful to view the space as the boundary of a Gromov hyperbolic space. Such a viewpoint is prevalent but often implicit in various multi-scale arguments in analysis and probability. Roughly speaking, a metric space viewed simultaneously at different locations and scales has a natural hyperbolic structure. A nice introduction to this viewpoint can be found in [Sem01]. This will be made precise by the notion of hyperbolic filling in Subsection 3.2. The main tool for the construction of metric is Theorem 3.14(a), and the construction of measure uses Lemma 3.20. To describe the construction, we recall the definition of hyperbolic space in the sense of Gromov.

#### 3.1 Gromov hyperbolic spaces and their boundary

We briefly recall the basics of Gromov hyperbolic spaces and refer the reader to [CDP, GH90, Gro, Vä05] for a detailed exposition.

Let \((Z, d)\) be a metric space. Given three points \(x, y, p \in Z\), we define the Gromov product of \(x\) and \(y\) with respect to the base point \(w\) as
\[
(x|y)_w = \frac{1}{2}(d(x,w) + d(y,w) - d(x,y)).
\]
By the triangle inequality, Gromov product is always non-negative. We say that a metric space \((Z, d)\) is \(\delta\)-hyperbolic, if for any four points \(x, y, z, w \in Z\), we have
\[
(x|z)_w \geq (x|y)_w \wedge (y|z)_w - \delta.
\]
We say that \((Z, d)\) is hyperbolic (or \(d\) is a hyperbolic metric), if \((Z, d)\) is hyperbolic for some \(\delta \in [0, \infty)\). If the above condition is satisfied for a fixed base point \(w \in Z\), and arbitrary \(x, y, z \in Z\), then \((Z, d)\) is \(2\delta\)-hyperbolic [CDP, Proposition 1.2].

Next, we recall the notion of the boundary of a hyperbolic space. Let \((Z, d)\) be a hyperbolic space and let \(w \in Z\). A sequence of points \(\{x_i\} \subset Z\) is said to converge at infinity, if
\[
\lim_{i,j \to \infty} (x_i|x_j)_w = \infty.
\]
The above notion of convergence at infinity does not depend on the choice of the base point \(w \in Z\), because by the triangle inequality \(|(x|y)_w - (x|y)_w'| \leq d(w, w')\).

Two sequences \(\{x_i\}, \{y_i\}\) that converge at infinity are said to be equivalent, if
\[
\lim_{i \to \infty} (x_i|y_i)_w = \infty.
\]
This defines an equivalence relation among all sequences that converge at infinity. As before, is easy to check that the notion of equivalent sequences does not depend on the choice of the base point \(w\). The boundary \(\partial Z\) of \((Z, d)\) is defined as the set of equivalence classes of sequences converging at infinity under the above equivalence relation. If there are multiple hyperbolic metrics on the same set \(Z\), to avoid confusion, we denote the boundary of \((Z, d)\) by \(\partial(Z, d)\) (see Lemma 3.13(d)). The notion of Gromov product can be defined on the boundary as follows: for all \(a, b \in \partial Z\)
\[
(a|b)_w = \sup \left\{ \liminf_{i \to \infty} (x_i|y_i)_w : \{x_i\} \in a, \{y_i\} \in b \right\},
\]
and similarly, for \(a \in \partial Z, y \in Z\), we define
\[
(a|y)_w = \sup \left\{ \liminf_{i \to \infty} (x_i|y)_w : \{x_i\} \in a \right\}.
\]
The boundary \(\partial Z\) of the hyperbolic space \((Z, d)\) carries a family of metrics. Let \(w \in Z\) be a base point. A metric \(\rho : \partial Z \times \partial Z \to [0, \infty)\) on \(\partial Z\) is said to be a visual metric with visual parameter \(\alpha > 1\) if there exists \(k_1, k_2 > 0\) such that
\[
k_1 \alpha^{-\rho(a, b)} \leq \rho(a, b) \leq k_2 \alpha^{-\rho(a, b)}.
\]
Note that visual metrics depend on the choice of the base point, and on the visual parameter \(\alpha\). If a visual metric with base point \(w\) and visual parameter \(\alpha\) exists, then it can be chosen to be
\[
\rho_{\alpha, w}(a, b) := \inf \sum_{i=1}^{n-1} \alpha^{-\rho(a_i, a_{i+1})}_w,
\]
where the infimum is over all finite sequences \(\{a_i\}_{i=1}^{n} \subset \partial Z, n \geq 2\) such that \(a_1 = a, a_n = b\).
Visual metrics exist as we recall now. A metric space \((Z, d)\) is said to be \textit{proper} if all closed balls are compact. For any \(\delta\)-hyperbolic space \((Z, d)\), there exists \(\alpha_0 > 1\) (\(\alpha\) depends only on \(\delta\)) such that if \(\alpha \in (1, \alpha_0)\), then there exists a visual metric with parameter \(\alpha\) [GH90, Chapitre 7], [BS, Lemma 6.1]. It is well-known that quasi-isometry between almost geodesic hyperbolic spaces induces a quasisymmetry on their boundaries. Since this plays a central role in our construction of metric, we recall the relevant definitions and results below.

We say that a map (not necessarily continuous) \(f : (X_1, d_1) \to (X_2, d_2)\) between two metric spaces is a \textit{quasi-isometry} if there exists constants \(A, B > 0\) such that

\[
A^{-1}d_1(x, y) - A \leq d_2(f(x), f(y)) \leq Ad_1(x, y) + A,
\]

for all \(x, y \in X_1\), and

\[
\sup_{x_2 \in X_2} d(x_2, f(X_1)) = \sup_{x_2 \in X_2} \inf_{x_1 \in X_1} d(x_2, f(x_1)) \leq B.
\]

**Definition 3.1.** A \textit{distortion function} is a homeomorphism of \([0, \infty)\) onto itself. Let \(\eta\) be a distortion function. A map \(f : (X_1, d_1) \to (X_2, d_2)\) between metric spaces is said to be \(\eta\)-\textit{quasisymmetric}, if \(f\) is a homeomorphism and

\[
\frac{d_2(f(x), f(a))}{d_2(f(x), f(b))} \leq \eta \left( \frac{d_1(x, a)}{d_1(x, b)} \right)
\]

for all triples of points \(x, a, b \in X_1, x \neq b\). We say \(f\) is a \textit{quasisymmetry} if it is \(\eta\)-quasisymmetric for some distortion function \(\eta\). We say that metric spaces \((X_1, d_1)\) and \((X_2, d_2)\) are quasisymmetric, if there exists a quasisymmetry \(f : (X_1, d_1) \to (X_2, d_2)\). We say that metrics \(d_1\) and \(d_2\) on \(X\) are \textit{quasisymmetric} (or, \(d_1\) is quasisymmetric to \(d_2\)), if the identity map \(\text{Id} : (X, d_1) \to (X, d_2)\) is a quasisymmetry. A quasisymmetry \(f : (X_1, d_1) \to (X_2, d_2)\) is said to be a \textit{power quasisymmetry}, if there exists \(\alpha > 0, \lambda \geq 1\) such that \(f\) is \(\eta_{\alpha, \lambda}\)-quasisymmetric, where

\[
\eta_{\alpha, \lambda}(t) = \begin{cases} 
\lambda t^{1/\alpha}, & \text{if } 0 \leq t < 1, \\
\lambda t^{\alpha}, & \text{if } t \geq 1.
\end{cases}
\]

Recall from Definition 1.1 that the \textit{conformal gauge} of a metric space \((X, d)\) is defined as

\[
\mathcal{J}(X, d) := \{ \theta : X \times X \to [0, \infty) | \theta \text{ is a metric on } X, d \text{ is quasisymmetric to } \theta \}.
\]

Bi-Lipschitz maps are the simplest examples of quasi-symmetric maps. Recall that a map \(f : (X_1, d_1) \to (X_2, d_2)\) is said to be \textit{bi-Lipschitz}, if there exists \(C \geq 1\),

\[
C^{-1}d_1(x, y) \leq d_2(f(x), f(y)) \leq Cd_1(x, y), \quad \text{for all } x, y \in X_1.
\]

Two metrics \(d_1, d_2 : X \times X \to [0, \infty)\) on \(X\) are said to be \textit{bi-Lipschitz equivalent} if the identity map \(\text{Id} : (X, d_1) \to (X, d_2)\) is bi-Lipschitz.
We collect a few useful facts about quasisymmetric maps.

**Proposition 3.2** ([MT10, Lemma 1.2.18] and [Hei, Proposition 10.8]). Let the identity map \( \text{Id} : (X, d_1) \to (X, d_2) \) be an \( \eta \)-quasisymmetry for some distortion function \( \eta \). By \( B_i(x, r) \) we denote the open ball in \((X, d_i)\) with center \( x \) and radius \( r > 0 \), for \( i = 1, 2 \).

(a) For all \( A \geq 1, x \in X, r > 0 \), there exists \( s > 0 \) such that

\[
B_2(x, s) \subset B_1(x, r) \subset B_1(x, Ar) \subset B_2(x, \eta(A)s).
\]  

In (3.2), \( s \) can be defined as

\[
s = \sup \{ 0 \leq s_2 < 2\text{diam}(X, d_2) : B_2(x, s_2) \subset B_1(x, r) \} \tag{3.3}
\]

Conversely, for all \( A > 1, x \in X, r > 0 \), there exists \( t > 0 \) such that

\[
B_1(x, r) \subset B_2(x, t) \subset B_2(x, At) \subset B_1(x, A_1r), \tag{3.4}
\]

where \( A_1 = \tilde{\eta}(A) \) and \( \tilde{\eta}(t) \) is the distortion function given by \( \tilde{\eta}(t) = 1/\eta^{-1}(t^{-1}) \). In (3.4), \( t \) can be defined as

\[
At = \sup \{ 0 \leq t_2 < 2\text{diam}(X, d_2) : B_2(x, Ar_2) \subset B_1(x, A_1r) \}. \tag{3.5}
\]

(b) If \( A \subset B \subset X \) such that \( 0 < \text{diam}(A, d_1) \leq \text{diam}(B, d_1) < \infty \), then \( 0 < \text{diam}(A, d_2) \leq \text{diam}(B, d_2) < \infty \) and

\[
\frac{1}{2\eta \left( \frac{\text{diam}(B, d_1)}{\text{diam}(A, d_1)} \right)} \leq \frac{\text{diam}(A, d_2)}{\text{diam}(B, d_2)} \leq \eta \left( \frac{2\text{diam}(A, d_1)}{\text{diam}(B, d_1)} \right). \tag{3.6}
\]

**Definition 3.3.** A metric space \((X, d)\) is \( k \)-almost geodesic, if for every \( x, y \in X \) and every \( t \in [0, \text{d}(x, y)] \), there is some \( z \in X \) with \( \text{d}(x, z) - t \leq k \) and \( \text{d}(y, z) - (\text{d}(x, y) - t) \leq k \). We say that a metric space is almost geodesic if it is \( k \)-almost geodesic for some \( k \geq 0 \). We recall that quasi-isometry between almost geodesic hyperbolic spaces induces a quasisymmetry between their boundaries.

**Proposition 3.4** ([BS, Theorem 6.5 and Proposition 6.3]). Let \((Z_1, d_1)\) and \((Z_2, d_2)\) be two almost geodesic, \( \delta \)-hyperbolic metric spaces. Let \( f : (Z_1, d_1) \to (Z_2, d_2) \) denote quasi-isometry.

(a) If \( \{x_i\} \subset Z_1 \) converges at infinity, then \( \{f(x_i)\} \subset Y \) converges at infinity. If \( \{x_i\} \) and \( \{y_i\} \) are equivalent sequences in \( X \) converging at infinity, then \( \{f(x_i)\} \) and \( \{f(y_i)\} \) are also equivalent.

(b) If \( a \in \partial Z_1 \) and \( \{x_i\} \in a \), let \( b \in \partial Z_2 \) be the equivalence class of \( \{f(x_i)\} \). Then \( \partial f : \partial Z_1 \to \partial Z_2 \) is well-defined, and is a bijection.
(c) Let $p_1 \in Z_1$ be a base point in $Z_1$, and let $f(p_1)$ be a corresponding base point in $Z_2$. Let $\rho_1, \rho_2$ denote visual metrics (with not necessarily the same visual parameter) on $\partial Z_1, \partial Z_2$ with base points $p_1, f(p_1)$ respectively. Then the induced boundary map $\partial f : (\partial Z_1, \rho_1) \to (\partial Z_2, \rho_2)$ is a power quasisymmetry.

Remark 3.5. The distortion function $\eta$ for the quasisymmetry $\partial f$ in (c) above can be chosen to depend only on the constants associated with the quasi-isometry $f : Z_1 \to Z_2$ and the constants associated with the properties of being almost geodesic and Gromov hyperbolic for $Z_1, Z_2$.

3.2 Hyperbolic filling of a compact metric space

Given a compact metric space $(X, d)$, one can construct an almost geodesic, hyperbolic space whose boundary equipped with a visual metric can be identified with $(X, d)$. We assume further that $(X, d)$ is doubling and uniformly perfect. Recall that a metric space $(X, d)$ is $K_D$-doubling, if any ball $B(x, r)$ can be covered by $K_D$ balls of radius $r/2$. A metric space is $K_P$-uniformly perfect, if for any ball $B(x, r)$ such that $X \setminus B(x, r) \neq \emptyset$, then $B(x, r) \setminus B(x, r/K_P) \neq \emptyset$.

We recall the notion of hyperbolic filling due to Bourdon and Pajot [BP], based on a similar construction due to Élek [Ele]. We recall the definition in [Car13]. Let $(X, d)$ be a compact, doubling, uniformly perfect, metric space. For a ball $B$ similar construction due to Elek [Ele]. We recall the definition in [Car13]. Let $(X, d)$ be a compact, doubling, uniformly perfect, metric space. For a ball $B$ similar construction due to Elek [Ele]. We recall the definition in [Car13]. Let $(X, d)$ be a compact, doubling, uniformly perfect, metric space.

We construct a graph whose vertex set is $\mathcal{S} = \coprod_{n=0}^{\infty} \mathcal{S}_n$. Next, we construct a tree structure of vertical edges on $\mathcal{S}$. For each $n \geq 0$, we partition $\mathcal{S}_{n+1}$ into pairwise disjoint sets $\{ T_n(B) : B \in \mathcal{S}_n \}$ indexed by $\mathcal{S}_n$, with $\mathcal{S}_{n+1} = \coprod_{B \in \mathcal{S}_n} T_n(B)$ satisfying the following property:

\[
\text{if } B' \in T_n(B) \text{ for some } B \in \mathcal{S}_n, B' \in \mathcal{S}_{n+1}, \text{ then } d(x_{B'}, A_n) = d(x_{B'}, x_B). \tag{3.10}
\]
In other words, if \( B' \in T_n(B) \), then \( x_B \in A_n \) is a minimizer to the distance between \( x_{B'} \) and \( A_n \). Since such a minimizer always exists, there exists a (not necessarily unique) partition \( \{T_n(B) : B \in S_n\} \) of \( S_{n+1} \) for all \( n \geq 0 \). We call the elements of \( T_n(B) \) as the children of \( B \). From now on, let us fix one such partition \( \{T_n(B) : B \in S_n\} \) for each \( n \geq 0 \). We say that there exist a vertical edge between two sets \( B, B' \in S \), if there exists \( n \geq 0 \) such that either \( B \in S_n, B' \in S_{n+1} \cap T_n(B) \) or \( B' \in S_{n+1}, B \in S_{n+1} \cap T_n(B') \); in other words, one of them is a child of the other. Note that the vertical edges form a tree on the vertex set \( S \), with base point (or root) \( w \), where \( S_0 = \{w\} \). The unique path from the base point to a vertex \( B \in S \) denotes the genealogy \( g(B) \). More precisely, we define the genealogy \( g(B) \) as \( (B) \) as

\[
g(B) = \begin{cases} (B), & \text{if } B \in S_0 \\ (B_0, B_1, \ldots, B_n), & \text{if } B = B_n \in S_n, n \geq 1, \text{ and } B_{i+1} \in T(B_i), \text{ for } i = 0, \ldots, n - 1. \end{cases}
\]

In the above definition, if \( 0 \leq i < n \), we denote the vertex \( B_i \in S_i \) by \( g(B)_i \). If \( B \in S_n \), and \( l > n \), we define \( D_l(B) \) as the descendants of \( B \) in the generation \( l \)

\[
D_l(B) := \{B' \in S_l : g(B')_n = B\}. \tag{3.11}
\]

For \( B \in S_n \), we denote \( \cup_{l \geq n+1} D_l(B) \) by \( D(B) \) which are the descendants of \( B \).

Using the horizontal parameter \( \lambda \geq 3 \), we define another family of edges on the vertex set \( S \) call the horizontal edges. We say \( B \sim B' \) if there exists \( n \geq 0 \) such that \( B, B' \in S_n \) and \( \lambda \cdot B \cap \lambda \cdot B' \neq \emptyset \). We say that there is a horizontal edge between \( B, B' \in S \), if \( B \sim B' \) and they are distinct (so as to avoid self-loops).

**Definition 3.6** (Hyperbolic filling). Let \( S_d = (S, E) \) denote the graph with vertices in \( S \) and whose edges \( E \) are obtained by the taking the union of horizontal and vertical edges. With a slight abuse of notation, we often view \( S_d \) as a metric space equipped with the (combinatorial) graph distance, which we denote by \( D_S : S \times S \to \mathbb{Z}_{\geq 0} \). The metric space \( S_d = (S, D_S) \) is almost geodesic and hyperbolic [BP, Proposition 2.1]. The metric space \( S_d \) is said to be a hyperbolic filling of \( (X, d) \).

We refer to Subsection 3.3 for a construction of hyperbolic filling. Note that the hyperbolic filling is not unique as we make an arbitrary choice of covering. Even if the covering is fixed, the choice of children \( T_n(B) \) is not necessarily unique. Nevertheless, any two hyperbolic fillings (with possibly different parameters) of a metric space are quasi-isometric to each other [BP, Corollaire 2.4].

We fix the base point of \( S_d \) to be \( w \in S \), where \( \{w\} = S_0 \). We now define a map \( p : X \to \partial S_d \) that identifies \( X \) with the boundary of \( S_d \) as follows. For each \( x \in X \), choose a sequence \( \{B_i\} \) with \( x \in B_i \in S_i, i \in \mathbb{N} \). Then it is easy to see that the sequence \( \{B_i\} \) converges at infinity. Let \( p(x) \in \partial S_d \) denote the equivalence class containing \( \{B_i\} \).

The map \( p \) is a bijection and its inverse \( p^{-1} : \partial S_d \to X \) can be described as follows. For any \( a \in \partial S_d \), and for any \( \{B_i\} \in a \), the corresponding sequence of centers \( \{x_{B_i}\} \) is a convergent sequence in \( X \), and the limit is \( p^{-1}(a) = \lim_{i \to \infty} x_{B_i} \). The map \( p^{-1} \) is
well-defined; that is, if \( \{B_i\} \) and \( \{B'_i\} \) are equivalent sequences that converge at infinity, then \( \lim_{i \to \infty} x_{B_i} = \lim_{i \to \infty} x_{B'_i} \).

We summarize the properties of the hyperbolic filling \( S_d \) and its boundary \( \partial S_d \) as follows:

**Proposition 3.7** ([BP, Proposition 2.1]). Let \((X,d)\) denote a compact, doubling, uniformly perfect metric space. Let \( S_d \) denote a hyperbolic filling with vertical parameter \( a > 1 \), and horizontal parameter \( \lambda \geq 3 \). Then \( S_d \) is almost geodesic, Gromov hyperbolic space. The map \( p : X \to \partial S_d \) is a homeomorphism between \( X \) and \( \partial S_d \). If we choose the base point \( w \in S_d \) as the unique vertex in \( S_0 \), then there exists \( K > 1 \) such that

\[
K^{-1}a^{-1}d(p(x),p(y)w) \leq d(x,y) \leq K a^{-1}d(p(x),p(y)w)
\]

for all \( x, y \in X \).

By the above proposition we can recover the metric space \((X,d)\) from its hyperbolic filling \( S_d \) with horizontal parameter \( \lambda \) and vertical parameter \( a \) (up to bi-Lipschitz equivalence) as the boundary \( \partial S_d \) equipped with a visual metric with base point \( w \) and visual parameter \( a \).

For technical reasons following [Car13, (2.8)], we will often assume that

\[
\lambda \geq 32, \quad a \geq 24(\lambda \vee K_P),
\]

where \( K_P \) is such that \((X,d)\) is \( K_P \)-uniformly perfect.

### 3.3 Construction of hyperbolic fillings

Since the metric spaces we deal with need not be compact, we need a suitable substitute for hyperbolic fillings. To circumvent this difficulty, we view the metric space as an increasing union of compact spaces and construct a sequence of hyperbolic fillings. Quasisymmetric maps and doubling measures have nice compactness properties that persist under such limits.

We recall the notion of net in a metric space.

**Definition 3.8.** Let \((X,d)\) be a metric space and let \( \varepsilon > 0 \). A subset \( N \) of \( X \) is called an \( \varepsilon \)-net in \((X,d)\) if the following two conditions are satisfied:

1. (Separation) \( N \) is \( \varepsilon \)-separated in \((X,d)\), i.e., \( d(x,y) \geq \varepsilon \) for any \( x, y \in N \) with \( x \neq y \).
2. (Maximality) If \( N \subset M \subset X \) and \( M \) is \( \varepsilon \)-separated in \((X,d)\), then \( M = N \).

In the lemma below, we recall a standard construction of hyperbolic filling and some of its properties.
Lemma 3.9 (Cf. [Car13, Lemma 2.2] and [KRS, Theorem 2.1]). Let \((X, d)\) be a complete, \(K_P\)-uniformly perfect, \(K_D\)-doubling metric space such that either \(\text{diam}(X, d) = \frac{1}{2}\) or \(\infty\). Let \(a > 8\) and let \(x_0 \in X\). Let \(N_0\) be a 1-net in \((X, d)\) such that \(x_0 \in N_0\). Define inductively the sets \(N_k\) for \(k \in \mathbb{N}\) such that

\[N_{k-1} \subset N_k, \text{ and } N_k \text{ is a}^{-k}\text{-net in } (X, d), \text{ for all } k \in \mathbb{N},.\]

For \(k < 0\) and \(k \in \mathbb{Z}\), we define \(N_k\) to be a \(a^{-k}\)-net in \((N_{k+1}, d)\) such that \(x_0 \in N_k\) for all \(k \in \mathbb{Z}\) (Note that \(N_k\) need not be \(a^{-k}\)-net in \((X, d)\) for \(k < 0\)). For each \(x \in N_k\) and \(k \in \mathbb{Z}\), we pick a predecessor \(y \in N_{k-1}\) such that \(y\) is a closest point to \(x\) in \(N_{k-1}\) (by making a choice if there is more than one closest point); that is \(y \in N_{k-1}\) satisfies

\[d(x, y) = \min_{z \in N_{k-1}} d(x, z).\]

For any \(x \in N_k, k \in \mathbb{Z}\), we denote its predecessor as defined above by \(P(x) \in N_{k-1}\).

(a) For all \(k \in \mathbb{Z}\), and for any two distinct points \(x, y \in N_k\), we have

\[B(x, a^{-k}/2) \cap B(y, a^{-k}/2) = \emptyset.\]  \hspace{1cm} (3.13)

We have the following covering property:

\[\bigcup_{x \in N_k} B(x, a^{-k}) = X, \text{ for all } k \geq 0,\]  \hspace{1cm} (3.14)
\[\bigcup_{x \in N_k} B(x, (1 - a^{-1})^{-1}a^{-k}) = X, \text{ for all } k \in \mathbb{Z}.\]  \hspace{1cm} (3.15)

In particular, if \(\text{diam}(X, d) = \frac{1}{2}\), the coverings \(S_n = \{B(y, (1 - a^{-1})^{-1}a^{-n}) \mid y \in N_n\}\) for all \(n \geq 0\) is a covering that satisfies (3.7) and (3.8). For any \(n \geq 0\) and for any \(B = B(x_B, a^{-n}) \in S_n\), the sets

\[T_n(B) = \{B(y, a^{-n-1}) \mid y \in N_{k+1}\text{such that } x_B = P(y)\}\]

forms a partition of \(S_{n+1}\) as required by (3.10).

(b) Let \(a, \lambda\) satisfy (3.12). Let \(y \leq N_{k+1}\) be such that

\[B(y, (1 - a^{-1})^{-1}a^{-k-1}) \cap B(P(y), a^{-k}/3) \neq \emptyset.\]

Then for any \(z \in N_{k+1}\) such that \(d(y, z) < 2(1 - a^{-1})^{-1}a^{-k-1}\), we have \(P(y) = P(z)\). (In other words, \(y\) corresponds to the center of a non-peripheral ball in \(S_{k+1}\) as given in Definition 3.23).

(c) Let \(k \in \mathbb{Z}\) and \(y \in N_k\). Let \(D_k(y)\) denote the set of descendants of \(y\) defined by

\[D_k(y) = \{y\} \cup \{z \in N_l \mid \text{such that } l > k \text{ and } P^{l-k}(z) = y\}.\]  \hspace{1cm} (3.16)

Then

\[\overline{B}(y, (1 - a^{-1})^{-1}a^{-k}) \supset \overline{D_k(y)} \supset B(y, 2^{-1} - (a - 1)^{-1}a^{-k}).\]  \hspace{1cm} (3.17)

The space \(\overline{D_k(y)}\) with the restricted metric \(d\) is \(K_D^2\)-doubling and \(K'_P\)-uniformly perfect, where \(K'_P = 2aK_P(1 - a^{-1})^{-1}(2^{-1} - (a - 1)^{-1})^{-1}\).
Proof. (a) The properties (3.13) and (3.14) follow from the separation and maximality properties of the \( a^{-k} \)-net \( N_k \) in \( X \) respectively. We use the notation \( P^k(y) \) denote the \( k \)-predecessor of \( y \) (for example, \( P^2(y) = P(P(y)) \)). To show (3.15), by (3.14) it suffices to consider the case \( k < 0 \). By (3.14), for any \( y \in X \) there exists \( y_0 \in N_0 \) such that \( d(y_0, y) < 1 \). Define \( y_l = P^{-l}(y) \) for all \( l < 0 \). Since \( d(y_l, y_{l+1}) < a^{-l} \) for all \( l < 0 \), we have

\[
d(y_k, y) \leq d(y_0, y) + \sum_{l=0}^{k} d(y_l, y_{l+1}) < \sum_{l=0}^{k} a^{-l} = (1-a^{-1})^{-1}(a^{-k}a^{-1}) < (1-a^{-1})^{-1}a^{-k}.
\]

Since \( y \in X \) is arbitrary and \( y_k \in N_k \), we have (3.15).

(b) By the triangle inequality, we have

\[
d(z, P(y)) \leq d(z, y) + d(y, P(y)) \leq (2\lambda + 1)(1-a^{-1})^{-1}a^{-k-1} + \frac{1}{3} a^{-k}
\]

< \( a^{-k}/2 \) (by (3.12))

By (3.14) and \( d(z, P(y)) < a^{-k}/2 \), we conclude that \( P(z) = P(y) \).

(c) By (3.14) and triangle inequality, we have

\[
d(y, z) \geq a^{-k}/2, \quad \text{for all } z \in N_{k+1} \setminus D(y) \text{ and for all } y \in N_k.
\]

By (3.18), we have

\[
\overline{D(z)} \subset \overline{B(z, (1-a^{-1})^{-1}a^{-k-1})}, \quad \text{for all } z \in N_{k+1}.
\]

Since \( \bigcup_{w \in N_l} \overline{D(w)} \) is dense and closed (by the doubling property), we have

\[
\bigcup_{w \in N_l} \overline{D(w)} = X, \quad \text{for all } l \in \mathbb{Z}.
\]

Combining (3.19), (3.20), (3.21) and using triangle inequality, we obtain (3.17).

Next, we show that \( \overline{D(y)} \) is \( K_r \)-doubling. More generally, we show that any subset \( Y \subset X \) is \( K_r \)-doubling. Let \( B(x, r) \cap Y, x \in Y \) be an arbitrary ball in \( Y \). Since \( (X, d) \) is \( K_r \)-doubling, the ball \( B(x, r) \) can be covered by \( N \) balls \( B(x_i, r/4), i = 1, \ldots, N \), where \( N \geq K_r^2 \). If \( B(x_i, r/4) \cap Y \neq \emptyset \), we choose \( y_i \in B(x_i, r/4) \cap Y \), so that \( B(x_i, r/4) \subset B(y_i, r/2) \). Hence all such balls \( B(y_i, r/2) \cap Y \) cover \( B(x, r) \cap Y \).

Let \( B(x, r) \cap \overline{D(y)} \) be an arbitrary ball in \( \overline{D(y)} \) such that \( x \in \overline{D(y)} \) and \( B(x, r) \cap \overline{D(y)} \neq \overline{D(y)} \). Let \( n \in \mathbb{Z} \) be the unique integer such that

\[
(1-a^{-1})^{-1}a^{-n} < r \leq (1-a^{-1})^{-1}a^{-n+1}.
\]

Since \( \overline{D(y)} = \bigcup_{z \in N_n \cap D(y)} \overline{D(z)} \) for all \( n \geq k \), by (3.17), there exists \( z \in D(y) \cap N_n \) such that

\[
d(z, x) \leq (1-a^{-1})^{-1}a^{-n} < r.
\]
Since \((X, d)\) is \(K_P\)-uniformly perfect, and using (3.17) and \(B(z, (2^{-1} - (a - 1)^{-1})a^{-n}) \neq X\), there exists \(w \in \overline{D(y)}\) such that

\[
(2^{-1} - (a - 1)^{-1})^{-1}a^{-n} > d(w, z) \geq \frac{1}{K_P}(2^{-1} - (a - 1)^{-1})a^{-n}.
\]

(3.23)

We consider two cases, depending on whether or not \(d(z, x) < \frac{1}{2}a^{-n}\). If \(d(z, y) \geq \frac{1}{2}a^{-n}\), then

\[
r > d(z, x) \geq \frac{1}{2}a^{-n} \geq \frac{r}{2(a((1 - a^{-1})^{-1}).}
\]

(3.24)

On the other hand, if \(d(z, x) < \frac{1}{2}a^{-n}\), then

\[
d(w, x) \vee d(z, x) \leq d(z, w) + d(z, x) < (2^{-1} - (a - 1)^{-1})^{-1}a^{-n} + \frac{1}{2}a^{-n} < a^{-n} < r.
\]

Hence by (3.23), if \(d(z, x) < \frac{1}{2}a^{-n}\), we have

\[
d(w, x) \vee d(z, x) \geq \frac{1}{2}d(w, z) \geq \frac{1}{2K_P}(2^{-1} - (a - 1)^{-1})a^{-n}
\]

\[
> \frac{r}{2aK_P(1 - a^{-1})^{-1}(2^{-1} - (a - 1)^{-1})^{-1}}.
\]

(3.25)

By (3.24) and (3.25), \(\overline{D(y)}\) is \(2aK_P(1 - a^{-1})^{-1}(2^{-1} - (a - 1)^{-1})^{-1}\)-uniformly perfect. \(\square\)

**Definition 3.10 (Extended hyperbolic filling).** Let \((X, d)\) be a complete, \(K_P\)-uniformly perfect doubling metric space such that either \(\text{diam}(X, d) = \frac{1}{2}\) or \(\infty\). Let \(a > 8\), \(\lambda \geq 32\) be constants that satisfy (3.12). Let \(x_0 \in X\) and consider the sets \(N_k, k \in \mathbb{Z}\) as defined in Lemma 3.9. Define

\[
S_k = \left\{ B(x, 2a^{-k}) : x \in N_k \right\}, k \in \mathbb{Z}.
\]

For any \(k \in \mathbb{Z}\) and for any pair of distinct balls \(B, B' \in S_k\), we say that there is a horizontal edge between \(B\) and \(B'\) (denoted as \(B \sim B'\)) if and only if \(\lambda \cdot B \cap \lambda \cdot B' \neq \emptyset\). For any \(k \in \mathbb{Z}\) and for any \(B(x, 2a^{-k}) \in S_k, B(y, 2a^{-k-1}) \in S_{k+1}\), we say that there is a vertical edge between \(B(x, 2a^{-k})\) and \(B(y, 2a^{-k-1})\), if \(x\) is the predecessor of \(y\) (as defined in Lemma 3.9). We define a graph \((V, E)\) with vertex set \(V = \bigcup_{k \in \mathbb{Z}} S_k\) and the edge set \(E\) defined by the union of horizontal and vertical edges. This graph is called the extended hyperbolic filling of \((X, d)\) with horizontal parameter \(\lambda\) and vertical parameter \(a\).

If \((X, d)\) is compact, the subgraph of the extended hyperbolic filling induced by \(S = \bigcup_{k \in \mathbb{Z} \geq 0} S_k\) forms a hyperbolic filling as given in Definition 3.6.

On the other hand, if \((X, d)\) is non-compact, we view \(X\) as an increasing limit of compact spaces \(\overline{D_l(x_0)}\) as \(l \to -\infty\), where \(D_k(x_0)\) is as defined in (3.16). For any \(k, l \in \mathbb{Z}, l \leq 0, k \geq l\), we define

\[
S_k^l = \left\{ B(x, 2a^{-k}) \cap \overline{D_l(x_0)} : x \in N_k \cap D_l(x_0) \right\}.
\]
We define a graph with vertex set \( S' = \bigcup_{k \geq l, k \in \mathbb{Z}} S'_k \), whose edges are the union of horizontal and vertical edges. In this case, the vertical edges are defined using predecessor relation as above and the horizontal edges are defined with respect to the space \( D_l(x_0) \). That is \( B \cap D_l(x_0), B' \cap D_l(x_0) \in S'_k \) share a horizontal edge if and only if \( \lambda \cdot B \cap \lambda \cdot B' \cap D_l(x_0) \neq \emptyset \).

This graph with vertex set \( S' \) can be viewed as a hyperbolic filling of the compact space \( D_l(x_0) \).

### 3.4 Combinatorial description of the conformal gauge

The purpose of this section is to recall a combinatorial description of the conformal gauge essentially due to M. Carrasco Piaggo [Car13]. In this section, we fix a compact, doubling, uniformly perfect metric space \((X, d)\) and a hyperbolic filling \( S_d = (S, D_S) \) with horizontal parameter \( \lambda \geq 8 \) and vertical parameter \( a > 1 \) that satisfies (3.9).

Propositions 3.4 and 3.7 suggest the following strategy construct metrics that are in the conformal gauge of \((X, d)\). By changing the metric of the hyperbolic filling \( S_d \) to another metric that is almost geodesic and bi-Lipschitz (in particular, quasi-isometric), every visual metric of its boundary is changed to a metric in the conformal gauge of \((X, d)\). Perhaps surprisingly, all metrics in the conformal gauge can be obtained in this manner (up to a bi-Lipschitz map) as explained in Theorem 3.14.

The change of metric in a hyperbolic filling is done using a weight function \( \rho : S \rightarrow (0, 1) \) on its vertex set. We define

\[
\pi(B) = \prod_{B' \in g(B)} \rho(B').
\]  

(3.26)

A path \( \gamma = \{B_i\}_{i=1}^N \) in \( S_d \) is a sequence of vertices such that there is an edge between \( B_i \) and \( B_{i+1} \) for all \( i = 1, \ldots, N - 1 \). In this case, we say that \( \gamma \) is a path from \( B_1 \) to \( B_N \). A path is said to be simple, if no two vertices in the path are the same. A path is said to be horizontal (resp. vertical), if all the edges in the path are horizontal (resp. vertical).

We define the \( \rho \)-length of a path \( \gamma = \{B_i\}_{i=1}^N \) by

\[
L_\rho(\gamma) = \sum_{i=1}^N \pi(B_i),
\]  

(3.27)

where \( \pi \) is as defined in (3.26). For points \( x, y \in X \) and \( n \in \mathbb{N} \), then the set of paths \( \Gamma_n(x, y) \) is defined as

\[
\Gamma_n(x, y) = \left\{ \gamma = \{B_i\}_{i=1}^k : \gamma \text{ is a path from } B_1 \text{ to } B_k, x \in B_1, y \in B_k, B_1 \in S_n, B_k \in S_n \right\}.
\]  

(3.28)

We remark that a path \( \gamma \in \Gamma_n(x, y) \) need not be a horizontal path.

For two distinct points \( x, y \in X \) and \( \alpha \geq 2 \), we define

\[
m_\alpha(x, y) = \max \{k : B \in S_k, x \in \alpha \cdot B, y \in \alpha \cdot B\},
\]

\[
c_\alpha(x, y) = \{B \in S_k : k = m_\alpha(x, y), x \in \alpha \cdot B, y \in \alpha \cdot B\},
\]

\[
\pi(c_\alpha(x, y)) = \max_{B \in c_\alpha(x, y)} \pi(B).
\]  

(3.29)
Assumption 3.11. A weight function $\rho : \mathcal{S} \to (0, 1)$ may satisfy some of the following hypotheses:

(H1) (Quasi-isometry) There exist $0 < \eta_- \leq \eta_+ < 1$ so that $\eta_- \leq \rho(B) \leq \eta_+$ for all $B \in \mathcal{S}$.

(H2) (Gromov product) There exists a constant $K_0 \geq 1$ such that for all $B, B' \in \mathcal{S}$ with $B \sim B' \in \mathcal{S}$, we have
\[
\pi(B) \leq K_0 \pi(B'),
\]
where $\pi$ is as defined in (3.26).

(H3) (Visual parameter) There exists $\alpha \in [2, \lambda/4]$ and a constant $K_1 \geq 1$ such that for any pair of points $x, y \in \mathcal{X}$, there exists $n_0 \geq 1$ such that if $n \geq n_0$ and $\gamma$ is a path in $\Gamma_n(x, y)$, then
\[
L(\rho(\gamma)) \geq K_1^{-1} \pi(c_\alpha(x, y)),
\]
where $\Gamma_n(x, y), L(\rho), \pi(c_\alpha(x, y))$ are as defined in (3.28), (3.27), and (3.29) respectively.

The following observation concerns the stability of the above assumption under ‘finite perturbations’.

Remark 3.12. Let $\rho, \rho' : \mathcal{S} \to (0, 1)$ are two different weight functions such that the set \{ $B \in \mathcal{S} : \rho(B) \neq \rho'(B)$ \} is finite. Then if $\rho$ satisfies the hypotheses (H1), (H2), and (H3), then so does $\rho'$ (with possibly different constants).

The weight function $\rho$ can be used to define a metric on $\mathcal{S}$ that is bi-Lipschitz equivalent to $D_S$ as we recall below. We summarize the properties of the metric below.

Lemma 3.13 ([Car13, Lemma 2.3 and Proposition 2.4]). Let $(X, d)$ be a compact, doubling, uniformly perfect metric space with $\text{diam}(X, d) = \frac{1}{2}$, and let $\mathcal{S}_d = (\mathcal{S}, D_S)$ denote a hyperbolic filling with parameters $\lambda, a$ satisfying (3.12) and (3.9). Let $\rho : \mathcal{S} \to (0, 1)$ be a weight function that satisfies (H1) and (H2). Then there exists a metric $D_\rho$ on $\mathcal{S}$ such that:

(a) $D_\rho$ is bi-Lipschitz equivalent to $D_S$; that is there exists $\Lambda \geq 1$ such that
\[
\Lambda^{-1}D_S(B, B') \leq D_\rho(B, B') \leq \Lambda D_S(B, B'), \quad \text{for all } B, B' \in \mathcal{S};
\]

(b) any simple vertical path $\gamma = \{B_i\}_{i=1}^n$ joining $B \in \mathcal{S}_m$ and $B' \in \mathcal{S}_{m'}$ satisfies
\[
D_\rho(B, B') = \sum_{i=1}^{n-1} D_\rho(B_i, B_{i+1}) = \left| \log \frac{1}{\pi(B)} - \log \frac{1}{\pi(B')} \right|;
\]

(c) $(\mathcal{S}, D_\rho)$ is almost geodesic and Gromov hyperbolic.
(d) The identity map \( \text{Id} : (S, D_S) \to (S, D_\rho) \) induces the identity map on their boundaries as described in Proposition 3.4. That is, a sequence \( \{B_i\} \) converges at infinity in \((S, D_S)\) if and only if it converges at infinity in \((S, D_\rho)\), and any two sequences that converge at infinity in \((S, D_S)\) are equivalent if and only if they are equivalent in \((S, D_\rho)\). In particular, the bijection \( p : X \to \partial(S, D_S) \) described before Proposition 3.4 can be viewed as a bijection \( \tilde{p} : X \to \partial(S, D_\rho) \) by composing with the induced identity map above.

(e) Assume in addition that \((H3)\) is also satisfied. Let \((\cdot|\cdot)_\rho\) denote the Gromov product on \((S, D_\rho)\) with base point \(w \in S_0\) extended to its boundary. Define \( \tilde{\theta}_\rho : \partial(S, D_\rho) \times \partial(S, D_\rho) \to [0, \infty) \) as

\[
\tilde{\theta}_\rho(\tilde{p}(x), \tilde{p}(y)) = \inf_{n=1}^{n-1} e^{-(\tilde{p}(x_i)\tilde{p}(x_{i+1}))_\rho},
\]

where the infimum is over all finite sequence of points \( \{x_i\}_{i=1}^n \) in \(X\) such that \( n \in \mathbb{N}, x_1 = x, \text{ and } x_n = y \). Then \( \tilde{\theta}_\rho \) is a visual metric on \( \partial(S, D_\rho) \) with visual parameter \( e \). Moreover, there exists \( K > 1 \) such that

\[
K^{-1} e^{-((\tilde{p}(x)|\tilde{p}(y))_\rho)} \leq \tilde{\theta}_\rho(\tilde{p}(x), \tilde{p}(y)) \leq K e^{-(\tilde{p}(x)|\tilde{p}(y))_\rho},
\]

\[
K^{-1} \pi(c_\alpha(x, y)) \leq \tilde{\theta}_\rho(\tilde{p}(x), \tilde{p}(y)) \leq K \pi(c_\alpha(x, y)).
\]

Sketch of the proof. We briefly recall the construction of the metric \( D_\rho \). Let \( E \) denote the edge set of the hyperbolic filling and let \( \eta_-, \eta_+ , K_0 \) denote the constants in hypotheses \((H1)\), and \((H2)\). Define a function \( \ell_\rho : E \to (0, \infty) \) as

\[
\ell_\rho(e) = \begin{cases} 
2 \max \{- \log(\eta_+), - \log(\eta_-), \log(K_0)\}, & \text{if } e \text{ is a horizontal edge}, \\
\frac{\log(\pi(B'))}{\pi(B)}, & \text{if } e = (B', B) \text{ is a vertical edge}.
\end{cases}
\]

Then the distance \( D_\rho : S \times S \to [0, \infty) \) is defined as

\[
D_\rho(B, B') = \inf_{\gamma} \sum_{i=1}^{N-1} \ell_\rho(e_i),
\]

where the infimum is taken over all paths \( \gamma = \{B_i\}_{i=1}^N \) where \( N \) varies over \( \mathbb{N}, B_1 = B, B_N = B', \) and \( e_i = (B_i, B_{i+1}) \) is an edge for each \( i = 1, \ldots, N-1 \).

Part (a) is immediate from the definition of \( D_\rho \). Part (b) and (c) are proved in [Car13, Lemma 2.3]. Part (d) follows from (a),(c) and Proposition 3.4. Part (e) follows from [Car13, Proposition 2.4].

The following theorem provides a combinatorial description of the conformal gauge \( \mathcal{J}(X, d) \). In [Car13, Theorem 1.1] provides a combinatorial description of the Ahlfors regular conformal gauge

\[
\mathcal{J}_{AR}(X, d) = \{ \theta \in \mathcal{J}(X, d) : \text{there exists an Ahlfors regular measure } \mu \text{ on } (X, \theta) \}.
\]
In [Car13, Theorem 1.1] the hypotheses (H1), (H2), (H3) corresponds to a combinatorial description of $\mathcal{J}(X,d)$, whereas the hypothesis (H4) corresponds to the existence of an Ahlfors regular measure. This theorem is essentially contained in [Car13].

**Theorem 3.14** (Cf. [Car13, Theorem 1.1]). Let $(X,d)$ be a compact, doubling, uniformly perfect metric space.

(a) Let $S_d = (\mathcal{S}, D_S)$ denote a hyperbolic filling with parameters $\lambda, a$ satisfying (3.12) and (3.9). Let $\rho : S \to (0,1)$ be a weight function that satisfies the conditions (H1), (H2), and (H3). Define the metric $\theta_\rho : X \times X \to [0, \infty)$ as

$$\theta_\rho(x,y) = \hat{\theta}_\rho(p(x), p(y)) \quad \text{for } x,y \in X,$$

where $\hat{\theta}_\rho$ is as defined in (3.30). Then $\theta_\rho$ satisfies the following properties:

(i) $\theta_\rho \in \mathcal{J}(X,d)$; that is $\theta_\rho$ is quasisymmetric to $d$.

(ii) there exists $C > 0$ such that

$$C^{-1}\pi(c_\alpha(x,y)) \leq \theta_\rho(x,y) \leq C\pi(c_\alpha(x,y)),$$

where $\alpha$ is the constant in (H3). Furthermore, there exists $K > 1$ such that

$$K^{-1}\pi(B) \leq \text{diam}(B, \theta_\rho) \leq K\pi(B) \quad \text{for all } B \in S.$$

(iii) $\theta_\rho$ is a visual metric of the hyperbolic space $(\mathcal{S}, D_\rho)$ constructed in Lemma 3.13 in the following sense: there exists $C > 0$ such that

$$C^{-1}\theta_\rho(x,y) \leq e^{-(\hat{p}(x)|\hat{p}(y)\rho)} \leq C\theta_\rho(x,y),$$

where $\hat{p} : X \to \partial(S, D_\rho)$ is the bijection described in Lemma 3.13(d), and $(\cdot|\cdot)_\rho$ denotes the Gromov product (extended to the boundary) on the hyperbolic space $(\mathcal{S}, D_\rho)$ with base point $w \in S_0$.

(iv) The distortion function $\eta$ of the power quasisymmetry $\text{Id} : (X,d) \to (X,\theta)$ can be chosen to depend only on the constants in (H1), (H2), and (H3).

(b) Conversely, let $\theta \in \mathcal{J}(X,d)$ be any metric in the conformal gauge. Then there exists a hyperbolic filling $S_d = (\mathcal{S}, D_S)$ of $(X,d)$ with horizontal parameter $\lambda$, vertical parameter $a$, and a weight function $\rho : S \to (0,1)$ that satisfies the hypotheses (H1), (H2), (H3), and such that the metric $\theta_\rho$ defined in (3.31) is bi-Lipschitz equivalent to $\theta$.

**Proof.** We begin with the proof of (b).

(b) Let $\text{Id} : (X,d) \to (X,\theta)$ be an $\eta$-quasisymmetry for some distortion function $\eta$.

The definition of the weight function $\rho$ in [Car13] uses an Ahlfors regular measure. Since there is no such measure available in our setting, the following definition is more
suit for our purposes. We normalize the metric $\theta$, so that $\text{diam}(X, \theta) = \frac{1}{2}$. We will define the weight function $\rho : S \to (0, 1)$ so that

$$\pi(B) = \text{diam}_\theta(B),$$

for all $B \in S$, where $S$ is a hyperbolic filling of $(X, d)$ with parameters $\lambda, a$. Fix any $\lambda \geq 32$. The vertical parameter $a > 1$ will be determined later in the proof. Hence we define $\rho : S \to (0, 1)$ as

$$\rho(B) = \begin{cases} \frac{1}{2} \frac{\text{diam}_\theta(B)}{\text{diam}_\theta(g(B))_{n-1}} & \text{if } B \in S_0, \\ \text{diam}_\theta(B) & \text{if } B \in S_n, n \geq 1. \end{cases}$$

First, we show (H2). Let $B \sim B'$ with $B, B' \in S_n$. Then choose $y \in \lambda \cdot B \cap \lambda \cdot B'$. By triangle inequality,

$$B \subset B_d(x_B, 2a^{-n}) \subset B_d(y, (\lambda + 2)a^{-n}), \quad B' \subset B_d(y, (\lambda + 2)a^{-n}).$$

By uniform perfectness, and triangle inequality, for any $r < \frac{1}{2}, r/k_p \leq \text{diam}_d(B_d(x, r)) \leq 2r$. Therefore by (3.6), we obtain

$$\frac{1}{2\eta(4(\lambda + 2)k_p)} \leq \frac{\text{diam}_\theta(B)}{\text{diam}_\theta(B_d(y, (\lambda + 2)a^{-n}))} \leq \eta(8k_p/\lambda + 2)).$$

Since the same inequality holds with $B$ replaced with $B'$, we have (H2) with constant

$$K_0 = 2\eta(4(\lambda + 2)k_p)\eta(8k_p/\lambda + 2),$$

that depends only on the distortion function $\eta$, the constant $k_p$ of uniform perfectness, and the horizontal parameter $\lambda$ (in particular, does not depend on the vertical parameter $a$).

Next, we show (H1), which again relies on (3.6). We will choose $a > 2(\lambda + 1)$ large enough so that $\eta_+ = \frac{1}{2}$ in (H1). Clearly this choice works when $B \in S_0$. If $B = S_n, n \geq 1$, and by denoting $B' = g(B)_{n-1}$, we have $x_B \in B'$. For $n \geq 2$, we write (the case $n = 1$ is easier and left to the reader)

$$\rho(B) = \frac{\text{diam}_\theta(B)}{\text{diam}_\theta(B')} = \frac{\text{diam}_\theta(B)}{\text{diam}_\theta((4a) \cdot B)} \frac{\text{diam}_\theta((4a) \cdot B)}{\text{diam}_\theta(B')}.$$  

Each of the terms can be estimated (from above and below) using (3.6), since by the triangle inequality and $d(x_B, x_{B'}) < 2a^{-n+1}$ we have $B \subset (4a) \cdot B$, and $(4a) \cdot B \supset B(x_B, 2a^{-n-1}) \supset B'$. Hence, we obtain

$$\rho(B) \leq 2\eta \left( \frac{2\text{diam}_d(B)}{\text{diam}_d((4a) \cdot B)} \right) \eta \left( \frac{\text{diam}_d((4a) \cdot B)}{\text{diam}_d(B')} \right) \leq 2\eta(K_p/a) \eta(16K_p)$$

$$\rho(B) \geq \left[ 2\eta \left( \frac{\text{diam}_d((4a) \cdot B)}{\text{diam}_d(B)} \right) \eta \left( \frac{\text{diam}_d(B)}{\text{diam}_d(B')} \right) \right]^{-1} \geq \left[ 2\eta(8K_p) \eta(2K_p/a) \right]^{-1}.$$
First we choose \( a \) large enough so that \( 2\eta(K_P/a)\eta(K_P) \leq \frac{1}{2} \) and (3.12) are satisfied. We set \( \eta = [2\eta(K_P)\eta(K_P/a)]^{-1} \). Hence we obtain (H1).

For (H3), we once again use (3.6), to see that \( \pi(B) = \text{diam}_\theta(B) \) is comparable to \( \text{diam}_\theta(\lambda \cdot B) \) for all \( B \in \mathcal{S} \). More precisely, we have

\[
\text{diam}_\theta(B) \leq \text{diam}_\theta(\lambda \cdot B) \leq 2\eta(2\lambda K_P)\text{diam}_\theta(B)
\]

for all \( B \in \mathcal{S} \). For any path \( \gamma = \{B_i\}_{i=1}^m \in \Gamma_n(x, y) \), we choose points \( x_i \in \lambda \cdot B_i \cap \lambda \cdot B_{i+1}, i = 1, \ldots, m-1 \), \( x_0 = x, x_m = y \) so that

\[
\theta(x, y) \leq \sum_{i=0}^{m-1} \theta(x_i, x_{i+1}) \leq \sum_{i=1}^m \text{diam}_\theta(\lambda \cdot B_i) \leq \sum_{i=1}^m \text{diam}_\theta(B_i) = 2\eta(2\lambda K_P)\text{dist}_\rho(\gamma). \tag{3.34}
\]

Fixing \( \alpha = 2 \), and let \( C \in c_2(x, y) \) such that \( \pi(c_2(x, y)) = \pi(C) \). Let \( m = m_2(x, y) \). Let \( B \in \mathcal{S}_{m+1} \) be such that \( x \in B \). By definition of \( m_2(x, y) \), \( y \notin 2 \cdot B \). Therefore \( d(x, y) \geq d(x_B, y) - d(x_B, x) \geq a^{-m-1} \). By (3.6), and \( \pi(c_2(x, y)) \leq \text{diam}_\theta(2 \cdot C) \) we have

\[
\pi(c_2(x, y)) \leq 2\eta \left( \frac{\text{diam}_e(2 \cdot C)}{d(x, y)} \right) \theta(x, y) \leq 2\eta \left( \frac{8a^{-m}}{a^{-m-1}} \right) \theta(x, y) = 2\eta(8a)\theta(x, y). \tag{3.35}
\]

Combining (3.34) and (3.35) yields (H3) with \( \alpha = 2 \).

(a) This part is essentially contained in [Car13]. The hypotheses (H1) and (H2) are used to construct a metric \( D_\rho \) on \( \mathcal{S} \) as given in Lemma 3.13. If \( \theta_\rho \) were defined using (3.31), it clearly satisfies the symmetry \( \theta_\rho(x, y) = \theta_\rho(y, x) \), and triangle inequality. The role of (H3) is to show that \( \theta_\rho(x, y) \) is at least \( e^{-\langle \check{\rho}(x)\rangle} \langle \check{\rho}(y) \rangle \) (up to a constant factor) as explained in Lemma 3.13(e). The fact that \( \theta_\rho \) is quasisymmetric to \( d \) follows from Lemma 3.13, Propositions 3.7 and 3.4(c). The statement about the dependence of distortion function \( \eta \) on the constants follow from Remark 3.5.

The estimate (3.33) is also implicitly contained in [Car13] and is a consequence of (3.32). Choose \( x, y \in B \) such that \( d(x, y) \geq \text{diam}(B, d)/2 \). Since \( \text{Id} : (X, d) \to (X, \theta_\rho) \) is an \( \eta \)-quasisymmetry, by (3.6), there exists \( C_1 > 0 \) such that

\[
\theta_\rho(x, y) \leq \text{diam}(B, \theta_\rho) \leq C_1 \theta_\rho(x, y).
\]

Since \( d(x, y) \geq \text{diam}(B)/2 \), \( B \) is at a bounded distance in \( (\mathcal{S}, D_\mathcal{S}) \) from any set \( C \in c_\alpha(x, y) \). Combining these estimates along with (3.32) and (3.6), we obtain

\[
\text{diam}(B, \theta_\rho) \approx \theta_\rho(x, y) \approx \pi(c_\alpha(x, y)).
\]

\[\square\]

### 3.5 Construction of measure using the hyperbolic filling

As in Subsection 3.2, we fix a compact, doubling, uniformly perfect metric space \( (X, d) \) with \( \text{diam}(X, d) = \frac{1}{2} \), and a hyperbolic filling \( \mathcal{S}_d = (\mathcal{S}, D_\mathcal{S}) \) with horizontal parameter \( \lambda \) and vertical parameter \( a \) that satisfy (3.12).
Definition 3.15 (gentle function). Let \( C : S \to (0, \infty) \) and \( K \geq 1 \). We say that \( C \) is \( K \)-gentle if
\[
K^{-1}C(B') \leq C(B) \leq KC(B),
\]
whenever there is an edge between \( B \) and \( B' \). We say that \( C : S \to (0, \infty) \) is gentle if it is \( K \)-gentle for some \( K \geq 1 \). The notion of \( K \)-gentle function extends to any function \( f : V \to (0, \infty) \) on a graph \( G = (V, E) \). In other words, a function \( f : V \to (0, \infty) \) on is \( K \)-gentle if \( \log f \) is \( (\log K) \)-Lipschitz with respect to the graph distance metric.

We sometimes need to distinguish between the horizontal and vertical edges (see Theorem 3.24). We say \( C : S \to (0, \infty) \) is \( (K_h, K_v) \)-gentle if
\[
K_h^{-1}C(B') \leq C(B) \leq K_hC(B'),
\]
whenever \( B \) and \( B' \) share a horizontal edge, and
\[
K_v^{-1}C(B') \leq C(B) \leq K_vC(B'),
\]
whenever \( B \) and \( B' \) share a vertical edge. Therefore every \( (K_h, K_v) \)-gentle function is \( (K_h \lor K_v) \)-gentle.

Given a hyperbolic filling \( S \), we need to approximate a ball \( B(x, r) \) by a ball in the filling \( S \). We introduce this notion in the following definition.

Definition 3.16. Let \((X, d)\) be a doubling metric space. Let \((S, D_S)\) be a hyperbolic filling of \((X, d)\) with parameters \(a, \lambda\) that satisfy (3.12) as constructed in Lemma 3.9(a). By Lemma 3.9(a), given a ball \( B(x, r) \neq X \), there exists \( n \in \mathbb{Z} \) and \( B \in S_n \) such that
\[
2a^{-n-1} \leq r < 2a^{-n}, \quad \text{and} \quad d(x_B, x) < 2a^{-n}. \tag{3.36}
\]
We define
\[
A_S(B(x, r)) = \{ B \in S : n \in \mathbb{Z}_{\geq 0} \text{ and } B \in S_n \text{ satisfy (3.36)} \}.
\tag{3.37}
\]
We remark that \( B, B' \in A_S(B(x, r)) \), then \( x \in B \cap B' \neq \emptyset \) and hence \( B \) and \( B' \) share a horizontal edge.

Often, the measures in this work will satisfy the following volume doubling and reverse volume doubling properties.

Definition 3.17 (Volume doubling and Reverse volume doubling properties). We say that a non-zero Radon measure \( \mu \) on a metric space \((X, d)\) satisfies the volume doubling property \( \text{VD} \), if there exists \( C_D \in (1, \infty) \) such that
\[
\mu(B(x, 2r)) \leq C_D \mu(B(x, r)), \quad \text{for all } x \in X, r \in (0, \infty). \tag{VD}
\]
A non-zero Radon measure \( \mu \) on a metric space \((X, d)\) satisfies the reverse volume doubling property \( \text{RVD} \), if there exist \( C_1, C_2 \in (1, \infty) \), \( \alpha \in (0, \infty) \) such that
\[
\mu(B(x, R)) \geq C_1^{-1} \left( \frac{R}{r} \right)^{\alpha} \mu(B(x, r)), \quad \text{for all } 0 < r \leq R < \text{diam}(X, d)/C_2. \tag{RVD}
\]
Remark 3.18. We recall the following connection between the doubling and uniform perfectness properties on a metric space \((X,d)\) to the volume doubling and reverse volume doubling properties.

(a) If \(\mu\) satisfies VD on \((X,d)\), then \((X,d)\) is a doubling metric space. Conversely, every complete doubling metric space admits a measure that satisfies VD [Hei, Theorem 13.3]. The constant 2 in the definition of VD is essentially arbitrary, as VD implies

\[
\frac{\mu(B(x,R))}{\mu(B(x,r))} \leq C_D \left( \frac{R}{r} \right)^\alpha, \quad \text{for all} \ x \in X, \ 0 < r \leq R, \ \text{where} \ \alpha = \log_2 C_D. \tag{3.38}
\]

(b) Let \(\mu\) be a measure that satisfies VD on \((X,d)\). Then \(\mu\) satisfies RVD if and only if \((X,d)\) is uniformly perfect [Hei, Exercise 13.1].

We introduce a hypothesis on a weight function \(\rho : S \to (0, \infty)\) that plays an important role in the construction of a measure.

Assumption 3.19. Let \(C : S \to (0, \infty)\) be a gentle function, and let \(\beta > 0\). A weight function \(\rho : S \to (0,1)\) is said to be \((\beta,C)\)-compatible if there exists \(K_2 \geq 1\) such that for all \(B \in S_m\), and \(n > m\),

\[
K_2^{-1} \pi(B)^\beta C(B) \leq \sum_{B' \in \mathcal{D}_n(B)} \pi(B')^\beta C(B') \leq K_2 \pi(B)^\beta C(B),
\]

where \(\mathcal{D}_n(B)\) denotes the descendants of \(B\) of generation \(n\) as defined in (3.11).

The above assumption is similar to (H4) in [Car13]. The following lemma is an analogue of [Car13, Lemma 2.7].

Lemma 3.20. Let \((S, D_S)\) be a hyperbolic filling of a doubling, \(K_P\)-uniformly perfect, compact metric space as given in Lemma 3.9(a). Let \(\rho : S \to (0,1)\) be a weight function that satisfies (H1) and (H2). Let \(C : S \to (0, \infty)\) be a gentle function, and let \(\beta > 0\), such that \(\rho\) is \((\beta,S)\)-compatible. For \(n \geq 0\), denote

\[
\mu_n := \sum_{B \in S_n} \pi(B)^\beta C(B) \delta_{xB},
\]

where \(\delta_{xB}\) denotes the Dirac measure at \(xB\). Let \(\mu\) be any weak* subsequential limit of \(\mu_n\). Then there exists \(C_1 > 1\) such that, for all \(x \in X, r \leq \text{diam}(X)/2\), and for all \(B \in \mathcal{A}_S(B(x,r))\), we have

\[
C_1^{-1} \pi(B)^\beta C(B) \leq \mu((B(x,r))) \leq C_1 \pi(B)^\beta C(B), \tag{3.39}
\]

where \(\mathcal{A}_S\) is as given in Definition 3.16. Furthermore, \(\mu\) is satisfies VD on \((X,d)\).
Combining these estimates, we obtain the desired upper bound.

Sketch of the proof. We only sketch the proof and skip the details as it follows from almost the same argument as [Car13, Lemma 2.7].

Let \( x \in X, r \leq \text{diam}(X)/2, B \in \mathcal{A}_S(B(x, r)) \) and \( B = B(x_B, 2a^{-m}) \). Choose \( B_1 \in S_{m+2} \) such that \( x \in B_1 = B(x_{B_1}, 2a^{-m-2}) \). By [Car13, (2.10)], the centers of all the descendants of \( B_1 \) belong to \( \overline{B}(x, r/2) \). This along with \((\beta, C)\)-compatibility implies that

\[
\mu(B(x, r)) \geq \mu(\overline{B}(x, r/2)) \geq \liminf_{n \to \infty} \mu_n \left( \{ x_C : C \in \mathcal{D}_n(B_1) \} \right) \quad \text{(since } \overline{B}(x, r/2) \supseteq \{ x_C : C \in \mathcal{D}_n(B_1) \}) \]
\[
\geq \liminf_{n \to \infty} \sum_{B' \in \mathcal{D}_n(B_1)} \pi(B')^\beta \mathcal{C}(B') \geq \pi(B_1)^\beta \mathcal{C}(B_1) \geq \pi(B)^\beta \mathcal{C}(B) \quad \text{(by } (\beta, C)\)-compatibility and gentleness of } \mathcal{C}.
\]

By the argument in [Car13, proof of Lemma 2.7], for any \( B' \in \mathcal{S}_n, n \geq m \) satisfying \( x_{B'} \in B(x, r + a^{-m}) \), we have \( g(B')_n \sim g(B_1)_m \), where \( B_1 \) is as defined above. For the upper bound, for any \( B' \in \mathcal{S}_n, n \geq m \) such that \( x_{B'} \in B(x, r + a^{-m}) \), we have that \( g(B')_m \sim B \). Therefore, we estimate

\[
\mu(B(x, r)) \leq \mu(B(x, r + a^{-m})) \leq \limsup_{n \to \infty} \mu_n(B(x, r + a^{-m})) \leq \sum_{C \sim g(B_1)_m} \sum_{B' \in \mathcal{D}_n(C)} \pi(B')^\beta \mathcal{C}(B') \leq \sum_{C \sim g(B_1)_m} \pi(C)^\beta \mathcal{C}(C).
\]

Since \( C \sim g(B_1)_m \) and \( B \sim g(B_1)_m \), by gentleness of \( \mathcal{C} \), we have \( \mathcal{C}(C) \preceq \mathcal{C}(B) \). By (H2), we have \( \pi(C) \preceq \pi(B) \). Furthermore, by doubling the number of such \( C \in \mathcal{S}_m \) such that \( C \sim g(B_1)_m \) is bounded by a constant that depends only on the parameters of the filling. Combining these estimates, we obtain the desired upper bound \( \mu(B(x, r)) \lesssim \pi(B)^\beta \mathcal{C}(B) \). This completes the proof of (3.39).

The conclusion that \( \mu \) satisfies VD follows from (3.39) and the gentleness of \( \mathcal{C} \).

In the following proposition, we express the measure in Lemma 3.20 using the metric in Theorem 3.14(a).

**Proposition 3.21.** Let \((X, d)\) be a compact, doubling, uniformly perfect metric space. Let \((\mathcal{S}, D_S)\) be a hyperbolic filling with parameters \(\lambda, a\) satisfying (3.12), (3.9) as given in Lemma 3.9(a). Let \( \mathcal{C} : \mathcal{S} \to (0, \infty) \) be a gentle function and let \( \beta > 0 \). Let \( \rho : \mathcal{C} \to (0, 1) \) be a weight function that satisfies (H1), (H2), (H3), and \((\beta, \mathcal{C})\)-compatibility. Let \( \theta = \theta_\rho \in \mathcal{F}(X, d) \) denote the metric in Theorem 3.14(a) and \( \mu \) denote the measure on \( X \) constructed in Lemma 3.20. Then, there exist \( C_1 > 1 \) such that

\[
C_1^{-1} r^\beta \mathcal{C}(B) \leq \mu(B_\theta(x, r)) \leq C_1 r^\beta \mathcal{C}(B), \quad \text{for all } x \in X, r < \text{diam}(X, \theta), B \in \mathcal{A}_S(B_\theta(x, s)),
\]

where \( s \) is the largest number in \([0, 2\text{diam}(X, d)]\) such that \( B_\theta(x, s) \subseteq B_\theta(x, r) \) (as defined in (3.3)) and \( \mathcal{A}_S(B_\theta(x, s)) \) is as given in Definition 3.16.
Proof. By an easy covering argument using the metric doubling property, it suffices to consider the case \( r < \text{diam}(X, \theta)/2 \), so that \( B_\theta(x, r) \neq X \).

Let \( x \in X, 0 < r < \text{diam}(X, \theta)/2 \) and let \( s = \sup s_1 > 0 : B_d(x, s_1) < B_\theta(x, r) \). By Lemma 3.20, \( \mu \) satisfies VD in \((X, d)\). By (3.2) and in (3.38), \( \mu \) satisfies VD in \((X, \theta)\) and there exists \( C_2 > 1 \) such that

\[
C_2^{-1} \mu(B_d(x, s)) \leq \mu(B_\theta(x, r)) \leq C_2 \mu(B_d(x, s)).
\] (3.41)

By (3.2), there exists \( A_1 > 1 \) such that \( B_\theta(x, r) \subset A_1 \cdot B \) and \( B \subset B_\theta(x, A_1 r) \) for all \( B \in \mathcal{A}_S(B_d(x, s)) \). Hence by (3.6) and uniform perfectness, there exists \( C_3 > 1 \) such that

\[
C_3^{-1} r \leq \text{diam}(B, \theta) \leq C_3 r, \quad \text{for all } B \in \mathcal{A}_S(B_d(x, s)).
\] (3.42)

By (3.33), (3.41), and (3.42), we obtain (3.40).

\[\square\]

3.6 Simplified hypotheses for construction of metric and measure

The goal of this section is to present an analogue of [Car13, Theorem 1.2] that will be used in the construction of metric measure space. Some of the main ideas in the proof of [Car13, Theorem 1.2] are inspired by the ‘weight-loss program’ of Keith and Laakso [KL, §5.2].

We continue to consider a compact, doubling, uniformly perfect metric space \((X, d)\), and a hyperbolic filling \( S_d = (\mathcal{S}, D_S) \) with horizontal parameter \( \lambda \geq 8 \) and vertical parameter \( a > 1 \) that satisfy (3.9). We consider \( \beta > 0, \ C : \mathcal{S} \to (0, \infty) \) such that \( C \) is gentle. Theorem 3.24 provides simpler sufficient conditions \((S1), (S2)\) that allows us to construct a weight function that satisfies \((H1), (H2), (H3)\), and is \((\beta, C)\)-compatible. To state the sufficient conditions, we recall the following definition.

Definition 3.22. For \( B \in \mathcal{S}_k, k \geq 0 \), we define \( \Gamma_{k+1}(B) \) as the set of horizontal paths \( \gamma = \{B_i\}_{i=1}^N, N \geq 2 \) such that \( B_i \in \mathcal{S}_{k+1} \) for all \( i = 1, \ldots, N \), \( B_i \sim B_{i+1} \) for all \( i = 1, \ldots, N - 1 \), \( x_{B_1} \in B, x_{B_N} \notin 2 \cdot B \), and \( x_{B_i} \in 2 \cdot B \) for all \( i = 1, \ldots, N - 1 \).

We introduce a subadditive estimate based on [BM, Proposition 3.15].

Definition 3.23. We say that \( B \in \mathcal{S}_k, k \geq 1 \) is non-peripheral if every horizontal neighbour of \( B \) descends from the same parent. More precisely, \( B \in \mathcal{S}_k, k \geq 1 \) is non-peripheral if

\[
B \sim B' \implies g(B)_{k-1} = g(B')_{k-1}.
\]

By \( \mathcal{N} \) we denote the set of all non-peripheral vertices in \( \mathcal{S} \). We say that a function \( C : \mathcal{S} \to (0, \infty) \) satisfies \((E)\) if it obeys the following estimate:

\[\text{(E)} \text{ there exists } \delta \in (0, 1) \text{ such that } \]

\[
C(B) \leq (1 - \delta) \sum_{B' \in \mathcal{N} \cap \mathcal{D}_{k+1}(B)} C(B')
\]

for all \( B \in \mathcal{S}_k, k \geq 1 \).
In particular, the condition (E) implies \( \mathcal{N} \cap D_{k+1}(B) \neq \emptyset \) for all \( k \geq 1, B \in S_k \).

The following result is an analogue of [Car13, Theorem 1.2].

**Theorem 3.24.** Let \((X, d)\) be a compact, \(K_D\)-doubling, \(K_P\)-uniformly perfect metric space and let \( \beta > 0 \). Consider a hyperbolic filling \( S_d = (S, D_S) \) with horizontal parameter \( \lambda \geq 8 \) and vertical parameter \( a > 1 \) that satisfies (3.9). Let \( C : S \to (0, \infty) \) be a \((K_h, K_v)\)-gentle function that satisfies (E). Then, there exists \( \eta_0 \in (0, 1/4) \) that depends only on \( \beta, K_D, K_h, \lambda \) (but not on the vertical constants \( a, K_v \) or uniform perfectness constant \( K_P \)) such that the following is true. If there exists a function \( \sigma : S \to [0, 1/4) \) that satisfies:

(S1) for all \( B \in S_k, k \geq 0 \), if \( \gamma = \{B_i : 1 \leq i \leq N\} \) is a path in \( \Gamma_{k+1}(B) \) (as given in Definition 3.22), then

\[
\sum_{i=1}^{N} \sigma(B_i) \geq 1,
\]

(S2) and for all \( k \geq 0 \), and all \( B \in S_k \), we have

\[
\sum_{B' \in D_{k+1}(B)} \sigma(B') C(B') \leq \eta_0 C(B),
\]

then there exists a weight function \( \rho : S \to (0, 1) \) that satisfies (H1), (H2), (H3), and is \((\beta, C)\)-compatible.

We recall some results from [Car13] that goes into the proof of Theorem 3.24. Let \( \rho : S \to (0, \infty) \) be a function, we define \( \rho^* : S \to (0, \infty) \) as

\[
\rho^*(B) = \min_{B' \sim B} \rho(B'), \quad \text{for } B \in S.
\]

We recall that \( B \sim B' \) if there exists \( k \geq 0 \) such that \( B, B' \in S_k \) and \( (\lambda \cdot B) \cap (\lambda \cdot B') \neq \emptyset \).

If \( \gamma = \{B_i\}_{i=1}^{N} \) is a horizontal path, we define

\[
L_h(\gamma, \rho) = \sum_{j=1}^{N-1} \rho^*(B_j) \wedge \rho^*(B_{j+1}).
\]

**Proposition 3.25 ([Car13, Proposition 2.9]).** Let \((X, d)\) be a compact, doubling, and uniformly perfect metric space. Let \( S \) be a hyperbolic filling with parameters \( a \) and \( \lambda \) satisfying (3.12). Assume that \( \rho : S \to (0, 1) \) satisfies (H1), (H2), and also the condition (H3') for all \( k \geq 1 \), for all \( B \in S_k \) and for all \( \gamma \in \Gamma_{k+1}(B) \), it holds \( L_h(\gamma, \rho) \geq 1 \).

Then the function \( \rho \) also satisfies (H3).

[Car13, Propostion 2.9] also assumes an additional assumption (H4) which was not used in the proof. In [Car13], the condition (H3') was stated for \( k \geq 0 \) but it is equivalent to the above condition because \( \Gamma_1(B) = \emptyset \) for \( B \in S_0 \).
Lemma 3.26 ([Car13, Lemma 2.13]). Suppose we have a function \( \pi_0 : S_k \to (0, \infty) \) such that
\[ \forall B \sim B' \in S_k, \quad \frac{1}{K} \leq \frac{\pi_0(B)}{\pi_0(B')} \leq K, \]
where \( K \geq 1 \) is a constant. Suppose that we have a function \( \pi_1 : S_{k+1} \to (0, \infty) \) which satisfies the following property:
\[ \forall B \in S_{k+1}, \exists A \in S_k \text{ with } d(x_B, x_A) \leq 4a^k \text{ and } 1 \leq \frac{\pi_0(A)}{\pi_1(B)} \leq K. \]
Define \( \hat{\pi}_1 : S_{k+1} \to (0, \infty) \) as
\[ \hat{\pi}_1(B') = \pi_1(B') \vee \left( \frac{1}{K} \max \{ \pi_1(B) : B \sim B' \} \right). \]
Then, for all \( B \sim B' \in S_{k+1} \), we have
\[ \frac{1}{K} \leq \frac{\hat{\pi}_1(B)}{\hat{\pi}_1(B')} \leq K. \]

The following is a slight modification of [Car13, Lemma 2.14].

Lemma 3.27 (Cf. [Car13, Lemma 2.14]). Let \( G = (V, E) \) be a graph whose vertices have degree bounded by \( D \). Let \( C : V \to (0, \infty) \) be \( K \)-gentle; that is,
\[ K^{-1} \leq \frac{C(z)}{C(z')} \leq K, \text{ whenever there is an edge between } z \text{ and } z'. \]
Let \( \Gamma \) be a family of paths in \( G \) and let \( \beta > 0 \). Suppose that \( \tau : V \to (0, \infty) \) is a function satisfying
\[ \sum_{i=1}^{N-1} \tau(z_i) \geq 1, \text{ for all paths } \gamma = \{z_i\}_{i=1}^{N} \in \Gamma. \quad (3.43) \]
Define \( \tilde{\tau} : V \to (0, \infty) \) as
\[ \tilde{\tau}(x) = 2 \max \{ \tau(y) : y \in V_2(x) \}, \]
where \( V_2(x) \) denotes the set of all vertices whose graph distance from \( x \) is less than or equal to 2. Then \( \tilde{\tau} \) satisfies
\[ \sum_{i=1}^{N-1} \tilde{\tau}^*(z_i) \wedge \tilde{\tau}^*(z_{i+1}) \geq 1 \text{ for all paths } \gamma = \{z_i\}_{i=1}^{N} \in \Gamma, \]
where \( \tilde{\tau}^*(x) = \min \{ \tilde{\tau}(y) : y \sim x \} \), and such that
\[ \sum_{z \in V} \tilde{\tau}(z)^{\beta}C(z) \leq 2^\beta D^2 K^2 \sum_{z \in V} \tau(z)^{\beta}C(z). \quad (3.44) \]
Proof. As shown in [Car13, Lemma 2.14] the function $\tilde{\tau}$ satisfies (3.43).

Since $C$ is $K$-gentle and $\sup_{x \in V} |V_2(x)| \leq D^2$, we obtain

$$\sum_{x \in V} \tilde{\tau}(x)^\beta C(x) \leq 2^\beta \sum_{x \in V} \sum_{z \in V_2(x)} \tau(z)^\beta C(x)$$

\begin{align*}
&\leq 2^\beta K^2 \sum_{x \in V} \sum_{z \in V_2(x)} \tau(z)^\beta C(z) = 2^\beta \sum_{z \in X} \sum_{x \in V_2(z)} \tau(z)^\beta C(z) \\
&\leq 2^\beta K^2 D^2 \sum_{z \in X} \tau(z)^\beta C(z). \quad \Box
\end{align*}

Proof of Theorem 3.24. Let $D_h$ be such that

$$D_h \geq \sup_{k \geq 0} \max_{B \in S_k} \{|B' \in S_k : B' \sim B|$$

By $K_D$-doubling, $D_h$ and can be chosen to depend only on $\lambda$ and $K_D$ [Hei, Exercise 10.17]. Similarly, the number of children can be bounded by a constant $D_v$ that depends only on $a$ and $K_D$ with

$$D_v \geq \sup_{k \geq 0} \max_{B \in S_k} |D_{k+1}(B)|. \quad (3.46)$$

Take $\eta_0 \in (0, 1)$ be a constant which will be fixed later, and set

$$\eta_- := \left(\eta_0 K_v^{-1} D_v^{-1}\right)^{1/\beta} \wedge \frac{1}{4}.$$

Let $\sigma : S \to [0, \frac{1}{4})$ satisfy (S1) and (S2). Define $\tau = \sigma \lor \eta_-$. Then

$$\sum_{B' \in D_{k+1}(B)} \tau(B')^\beta C(B') \leq \sum_{B' \in D_{k+1}(B)} \sigma(B')^\beta C(B') + \eta_- D_v K_v C(B) \leq 2\eta_0 C(B).$$

For $B \in S_k$ define $V_{2,k}(B) = \{B' \in S_k : \exists B'' \in S_k \text{ such that } B \sim B'' \sim B\}$. The by Lemma 3.27, the function

$$\tilde{\tau}(B) = 2 \max \{\tau(B') : B' \in V_{2,k}(B)\}, \quad \text{for all } B \in S_k$$

satisfies (H3’) and

$$\sum_{B' \in D_{k+1}(B)} \tilde{\tau}(B')^\beta C(B') \leq 2^\beta K_h^2 D_h^2 \sum_{B' \in D_{k+1}(B)} \tau(B')^\beta C(B') \leq 2^\beta K_h^2 D_h^2 \eta_0 C(B), \quad (3.47)$$

for all $B \in S_k$.

We construct $\hat{\rho} : S \to (0, 1)$ satisfying

(1) $\hat{\rho} \geq \tilde{\tau}$. In particular $\hat{\rho}$ satisfies (H3’) and $\hat{\rho}(B) \geq \eta_-$ for all $B \in S$.

(2) (H2) with constant $K$, where $K = \eta_-^{-1}$. 40
(3) \( \hat{\rho}(B) \leq \max \{ \bar{\tau}(B') : B' \sim B \} \). In particular, \( \hat{\rho}(B) \leq \frac{1}{2} \) for all \( B \in S \).

We briefly recall the construction in [Car13]. Set \( \hat{\rho}(w) = \frac{1}{2} \), where \( w \in S_0 \). Note that \( \bar{\tau} \leq \frac{1}{2} \leq 1 \) (since \( \eta_- \leq \frac{1}{2} \) and \( \sigma \leq \frac{1}{2} \)). We construct \( \hat{\rho} \) inductively on \( S_k \). Suppose we have constructed \( \hat{\rho}_i \) for \( i = 1, \ldots, k \). We construct \( \hat{\rho}_{k+1} \) using Lemma 3.26. We denote

\[
\pi_0(A) = \prod_{i=0}^{k} \hat{\rho}_i(g(A)_i) \quad \text{for} \ A \in S_k \quad \text{and} \quad \pi_1(B) = \bar{\tau}(B)\pi_0(g(B)_j) \quad \text{for} \ B \in S_{k+1}.
\]

By the induction hypothesis along with Lemma 3.26, we obtain a function \( \hat{\tau}_1 : S_{j+1} \to (0, \infty) \) that satisfies \( K^{-1} \hat{\tau}_1(B') \leq \hat{\tau}_1(B) \leq K \hat{\tau}_1(B') \) for all \( B \sim B' \in S_{j+1} \). We define \( \hat{\rho} : S_{j+1} \to (0, \infty) \) as

\[
\hat{\rho}_{k+1}(B) = \frac{\hat{\tau}_1(B)}{\pi_0(g(B)_j)}.
\]

Carrasco Piaggio’s proof of [Car13, Theorem 1.2] shows that \( \hat{\rho} \) satisfies properties (1), (2), and (3) above. For any \( B \in S_k, k \geq 0 \), using (3.47) we estimate

\[
\sum_{B' \in D_{k+1}} \hat{\rho}(B')^\beta C(B') \\
\leq \sum_{B' \in D_{k+1}} \sum_{B'' \sim B'} \bar{\tau}(B'')^\beta C(B') \quad \text{(by property (3) above)} \\
\leq K_h \sum_{B' \in D_{k+1}} \sum_{B'' \sim B'} \bar{\tau}(B'')^\beta C(B'') \\
\leq K_h D_h \sum_{C \sim B} \sum_{B'' \in D_{k+1}(C)} \bar{\tau}(B'')^\beta C(B'') \quad (\because B'' \sim B' \implies g(B'') \sim B) \\
\leq 2^{\beta+1} K_h^3 D_h^3 \eta_0 \sum_{C \sim B} C(C) \leq 2^{\beta+1} K_h^4 D_h^4 \eta_0 C(B). \tag{3.48}
\]

Now choose \( \eta_0 \)

\[
2^{\beta+1} K_h^3 D_h^3 \eta_0 = \frac{1}{2}, \tag{3.49}
\]

so that (3.48) yields

\[
\sum_{B' \in D_{k+1}} \hat{\rho}(B')^\beta C(B') \leq \frac{1}{2} C(B) \quad \text{for all} \ B \in S_k, k \geq 0. \tag{3.50}
\]

Note from (3.49) that \( \eta_0 \) depends only on \( \beta, K_h, K_D, \lambda \) but not on constants \( K_v, a, K_p \).

Next, we modify \( \hat{\rho} \) so that it becomes \((\beta, C)\)-compatible. For each \( B \in S_k, k \geq 0 \), we choose \( \omega_B \geq 0 \) such that

\[
\rho(B') = \begin{cases} 
\omega_B \lor \hat{\rho}(B') & \text{if} \ B' \in D_{k+1}(B) \cap N, \\
\hat{\rho}(B') & \text{if} \ B' \in D_{k+1}(B) \setminus N.
\end{cases} \tag{3.51}
\]

41
satisfies
\[ \sum_{B' \in D_{k+1}(B)} \rho(B')\beta \mathcal{C}(B') = \mathcal{C}(B). \] (3.52)

The existence of an \( \omega_B \in (0, \infty) \) that satisfies (3.52) follows from the intermediate value theorem. In particular, we use (3.50), the continuity of the map

\[ \omega_B \mapsto \sum_{B' \in D_{k+1} \cap \mathcal{N}} (\omega_B \vee \rho(B'))^\beta \mathcal{C}(B') + \sum_{B' \in D_{k+1} \setminus \mathcal{N}} \hat{\rho}(B')^\beta \mathcal{C}(B'), \]

along with the fact that \( D_{k+1} \cap \mathcal{N} \) is non-empty. The equality (3.52) implies that \( \rho \) is \((\beta, \mathcal{C})\)-compatible since

\[ \sum_{B' \in D_n(B)} \pi(B')^\beta \mathcal{C}(B') = \pi(B)^\beta \mathcal{C}(B) \]

for all \( B \in S_k \) and for all \( n \geq k \).

It remains to show that \( \rho \) satisfies (H1), (H2) and (H3). We start by verifying (H1). Clearly \( \rho(B) \geq \hat{\rho}(B) \geq \eta_\ast \) for all \( B \in S_k \). On the other hand, (E) implies that \( \omega_B \leq (1 - \delta)^{1/\beta} \), since

\[ \sum_{B' \in D_{k+1}(B) \cap \mathcal{N}} \omega_B^\beta \mathcal{C}(B') \leq \sum_{B' \in D_{k+1}(B)} \rho(B')^\beta \mathcal{C}(B') = \mathcal{C}(B) \leq (1 - \delta) \sum_{B' \in D_{k+1}(B) \cap \mathcal{N}} \mathcal{C}(B'). \]

This combined with \( \sigma \leq \frac{1}{4} \) and property (3) of \( \hat{\rho} \) implies that

\[ \eta_\ast \leq \rho(B) \leq (1/2) \vee (1 - \delta)^{1/\beta}. \]

By setting \( \eta_\ast = (1/2) \vee (1 - \delta)^{1/\beta} \in (0, 1) \), we obtain (H1).

Since \( \rho \geq \hat{\rho} \), \( \rho \) satisfies (H3'). Therefore by Proposition 3.25 it suffices to show (H2).

Let \( B \sim B' \in S_k, k \geq 1 \). We consider two cases.

**Case 1:** \( g(B)_{k-1} = g(B')_{k-1} \). Then \( \frac{\pi(B)}{\pi(B')} = \frac{\rho(B)}{\rho(B')} \) which thanks to (H1) satisfies

\[ \eta_\ast \leq \frac{\pi(B)}{\pi(B')} \leq \eta_\ast^{-1}. \]

**Case 2:** \( g(B)_{k-1} \neq g(B')_{k-1} \). Let \( n \geq 0 \) be the maximal integer such that \( g(B)_n = g(B')_n \). In this case for \( i = n + 1, \ldots, k \), we have \( g(B)_i \sim g(B')_i \). Hence for \( i = n + 2, \ldots, k \), \( g(B)_i \) and \( g(B')_i \) must both be peripheral (belong to \( \mathcal{N}^c \)). Therefore

\[ \frac{\pi(B)}{\pi(B')} = \frac{\rho(g(B)_{n+1})}{\rho(g(B')_{n+1})} \prod_{i=n+2}^{k} \frac{\rho(g(B)_i)}{\rho(g(B')_i)} = \frac{\rho(g(B)_{n+1})}{\rho(g(B')_{n+1})} \prod_{i=n+2}^{k} \frac{\hat{\rho}(g(B)_i)}{\hat{\rho}(g(B')_i)} \]

\[ = \frac{\rho(g(B)_{n+1})}{\rho(g(B')_{n+1})} \frac{\hat{\rho}(B)}{\hat{\rho}(B')} \frac{\hat{\rho}(g(B')_{n+1})}{\hat{\rho}(g(B)_{n+1})} \]

42
By combining property (2) of $\hat{\rho}$ to estimate $\frac{\hat{\pi}(B)}{\pi(B)}$ and $\eta_3 \leq \hat{\rho} \leq \rho \leq 1$ for the remaining terms, we obtain

$$\eta^3 \leq \frac{\pi(B)}{\pi(B')} \leq \eta^{-3}.$$ 

Combining the two cases, we obtain (H2) with constant $K_0 = \eta^{-3}$.  

\vspace{0.2cm}

Remark 3.28. One of the key differences between the construction in [Car13] and our work is the proof of Theorem 3.24. In the construction in [Car13], a similar modification as defined in (3.51) was done but $\mathcal{N}$ was chosen to be a singleton set. However, that choice does not work in our context because we need to ensure that $\omega_B \leq \eta +$, where $\eta_+ \in (0, 1)$. This is because $C(B')$ can be strictly smaller than $C(B)$. The construction in [Car13] can be interpreted as the particular case $C(B) = 1$ for all $B \in \mathcal{S}$. The requirement $\eta_+ \in (0, 1)$ is the motivation behind the notion of non-peripheral vertices and the enhanced subadditive estimate (E).

The following ‘patching lemma’ allows us to combine functions that satisfy local versions of (S1) and (S2) into a global one. This is an adaptation of the construction in [Car13, pp. 533–534].

Lemma 3.29 (Patching lemma). Let $\mathcal{S}$ denote a hyperbolic filling of a $K_D$-doubling, uniformly perfect, compact metric space, and let $\beta, \eta_1 > 0$. Let $\mathcal{S}_d = (\mathcal{S}, D_{\mathcal{S}})$ be a hyperbolic filling with horizontal parameter $\lambda \geq 8$ and vertical parameter $a > 1$ that satisfies (3.9). Let $C : \mathcal{S} \to (0, \infty)$ be a $(K_h, K_v)$-gentle function. Assume that for all $B \in \mathcal{S}_k, k \geq 1$, there exists $\sigma_B : \mathcal{S}_{k+1} \to [0, \frac{1}{4})$ such that

(a) if we set $V_B = \{B' \in \mathcal{S}_{k+1} : B' \cap 3 \cdot B \neq \emptyset\}$, then $\sigma_B(B') = 0$ for all $B' \in \mathcal{S}_{k+1} \setminus V_B$.

(b) for any path $\gamma = \{B_i\}_{i=1}^N \in \Gamma_{k+1}(B)$, we have

$$\sum_{i=1}^N \sigma_B(B_i) \geq 1,$$

(c) and $\sum_{B' \in \mathcal{S}_{k+1}} \sigma_B(B')\beta C(B') \leq \eta_1 C(B)$.

Let $\sigma : \mathcal{S} \to [0, \frac{1}{4})$ be defined as

$$\sigma(B') = \max \{\sigma_A(B') : A \in \mathcal{S}_k\}$$

for all $B' \in \mathcal{S}_{k+1}$ and for all $k \geq 1$, and $\sigma(B') = 0$ for all $B' \in \mathcal{S}_0 \cup \mathcal{S}_1$. Then there exists $C_{3.29} \geq 1$ that depends only on $K_D, K_h$ such that $\sigma$ satisfies (S1) and the estimate

$$\sum_{B' \in \mathcal{D}_{k+1}(B)} \sigma(B')\beta C(B') \leq C_{3.29} \eta_1 C(B).$$
Proof. For any path $\gamma = \{B_i\}_{i=1}^N \in \Gamma_{k+1}(B), B \in \mathcal{S}_k$, we have $\sum_{i=1}^N \sigma(B_i) \geq \sum_{i=1}^N \sigma_B(B_i) \geq 1$. Therefore $\sigma$ satisfies (S1).

For any $B \in \mathcal{S}_{k+1}$, we have

$$\sum_{B' \in \mathcal{D}_{k+1}(B)} \sigma(B')^\beta \mathcal{C}(B') = \sum_{B' \in \mathcal{D}_{k+1}(B)} \max \{ \sigma_A(B')^\beta : A \in \mathcal{S}_k \} \mathcal{C}(B') \leq \sum_{B' \in \mathcal{D}_{k+1}(B), A : B' \in \mathcal{V}_A} \sigma_A(B')^\beta \mathcal{C}(B') \leq \sum_{A : \mathcal{V}_B \cap \mathcal{V}_A \neq \emptyset} \eta_A \mathcal{C}(B) \leq \sum_{A : A \sim B} \eta_A \mathcal{C}(B) \leq D_h K_h \eta_1 \mathcal{C}(B),$$

where $D_h$ is chosen as (3.45). \qed

4 Universality of the conformal walk dimension

4.1 Consequences of Harnack inequalities

In this subsection, we recall some previous results concerning the elliptic and parabolic Harnack inequalities. We start with recalling the definition of the heat kernel and its sub-Gaussian estimates.

Definition 4.1 (HKE($\beta$)). Let $(X,d,m,\mathcal{E},\mathcal{F})$ be an MMD space, and let $\{P_t\}_{t \geq 0}$ denote its associated Markov semigroup. A family $\{p_t\}_{t \geq 0}$ of non-negative Borel measurable functions on $X \times X$ is called the heat kernel of $(X,d,m,\mathcal{E},\mathcal{F})$, if $p_t$ is the integral kernel of the operator $P_t$ for any $t > 0$, that is, for any $t > 0$ and for any $f \in L^2(X,m)$,

$$P_t f(x) = \int_X p_t(x,y) f(y) \, dm(y) \quad \text{for m-a.e. } x \in X.$$ 

Let $\beta \in (1, \infty)$. We say that $(X,d,m,\mathcal{E},\mathcal{F})$ satisfies the heat kernel estimates HKE($\beta$) with walk dimension $\beta$, if there exist $C_1, c_1, c_2, \delta \in (0, \infty)$ and a heat kernel $\{p_t\}_{t \geq 0}$ such that for any $t > 0$,

$$p_t(x,y) \leq \frac{C_1}{m(B(x,t^{1/\beta}))} \exp\left(-c_1(d(x,y)^{\beta}/t)^{1/(\beta-1)}\right) \quad \text{for m-a.e. } x,y \in X, \quad (4.1)$$

$$p_t(x,y) \geq \frac{c_2}{m(B(x,t^{1/\beta}))} \quad \text{for m-a.e. } x,y \in X \text{ with } d(x,y) \leq \delta t^{1/\beta}. \quad (4.2)$$

44
The following condition is the key to establishing the parabolic Harnack inequality in the presence of the elliptic Harnack inequality.

**Definition 4.2 (Capacity estimate).** Let $\beta \in (1, \infty)$. We say that an MMD space $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the *capacity estimate* $\cap(\beta)$ if there exist $C_1, A_1, A_2 > 1$ such that for all $R \in (0, \text{diam}(X, d)/A_2)$, $x \in X$

$$C_1^{-1} \frac{m(B(x, R))}{R^\beta} \leq \text{Cap}(B(x, R), B(x, A_1 R)^c) \leq C_1 \frac{m(B(x, R))}{R^\beta}. \quad \text{cap(\beta)}$$

Poincaré and cutoff Sobolev inequalities are important functional inequalities for obtaining the stability of Harnack inequalities, which we recall below.

**Definition 4.3.** Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space and let $\beta \in (1, \infty)$.

(i) We say that $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the *Poincaré inequality* $\text{PI(\beta)}$, if there exist constants $C > 0$ and $A \geq 1$ such that for all $x \in X$, $R \in (0, \infty)$ and $f \in \mathcal{F}$,

$$\int_{B(x, R)} (f - \bar{f})^2 \, dm \leq CR^\beta \int_{B(x, AR)} d\Gamma(f, f), \quad \text{PI(\beta)}$$

where $\bar{f} = \frac{1}{\mu(B(x, R))} \int_{B(x, R)} f \, dm$.

(ii) Let $B_1 \subset B_2$ be open subsets of $X$. We say that $\varphi \in \mathcal{F}$ is a *cutoff function* for $B_1 \subset B_2$ if $0 \leq \varphi \leq 1$ m-a.e., $\varphi \equiv 1$ m-a.e. on $B_1$ and $\text{supp}_m[\varphi] \subset B_2$.

(iii) We say that $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies the *cutoff Sobolev inequality* $\text{CS(\beta)}$, if there exist $C_1, C_2, C_3, \eta > 0$ and $A > 1$ such that the following holds. For all $x \in X$, $R > 0$ with $B_1 = B(x, R)$, $B_2 = B(x, AR)$, there exists a cutoff function $\varphi \in \mathcal{F} \cap C(X)$ for $B_1 \subset B_2$ such that for any $u \in \mathcal{F}$,

$$\int_X u^2 \, d\Gamma(\varphi, \varphi) \leq C_1 \int_{B_2 \setminus B_1} \Gamma(u, u) + \frac{C_2}{R^\beta} \int_{B_2 \setminus B_1} u^2 \, dm, \quad \text{CS(\beta)}$$

and such that the following scale invariant Hölder continuity estimate holds:

$$|\varphi(x_1) - \varphi(x_2)| \leq C_3 \left( \frac{d(x_1, x_2)}{R} \right)^\eta \quad (4.3)$$

for all $x_1, x_2 \in X$.

The following lemma shows that, under the above Poincaré and cutoff Sobolev inequalities, the extended Dirichlet space $\mathcal{F}_e$ is contained in the space $\mathcal{F}_{\text{loc}}$ as defined in (2.3) of functions locally in the domain $\mathcal{F}$ of the Dirichlet form $(\mathcal{E}, \mathcal{F})$.

**Lemma 4.4.** Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies $\text{VD}$, $\text{RVD}$, $\text{PI(\beta)}$, $\text{CS(\beta)}$ for some $\beta \in (1, \infty)$. Then $\mathcal{F}_e \subset \mathcal{F}_{\text{loc}}$. 

45
Proof. Let $g \in \mathcal{F}_e$. Then there exists an $\mathcal{E}$-Cauchy sequence $\{g_n\}_n \subset \mathcal{F}$ such that $g_n$ converges to $g$ $m$-almost everywhere. Let $B = B(x,R)$ be any ball. By the Poincaré inequality $\text{PI}(\beta)$ the sequence $g_n - (g_n)_B$ is $L^2(B,m)$-Cauchy. Since $g_n$ converges to $g$ $m$-almost everywhere and $g_n - (g_n)_B$ is $L^2(B,m)$-Cauchy, we have that $\lim_{n \to \infty} (g_n)_B = g_B$ and that $g_n$ converges to $g$ in $L^2(B,m)$.

Let $A > 1$ be as in $\text{CS}(\beta)$ and let $\varphi$ be a cutoff function for $B = B(x,R) \subset B(x,AR)$ as in $\text{CS}(\beta)$. By [FOT, Theorem 2.1.7] we may assume that $g_n$ is bounded, so that $g_n \varphi \in \mathcal{F}$ by [FOT, Theorem 1.4.2(ii)]. By the Leibniz rule [FOT, Lemma 3.2.5] and the Cauchy Schwarz inequality [FOT, Lemma 5.6.1] for $\Gamma(\cdot,\cdot)$, we obtain

$$\mathcal{E}(\varphi(g_n - g_m), \varphi(g_n - g_m)) \leq 2 \left( \mathcal{E}(g_n - g_m, g_n - g_m) + \int_X (g_n - g_m)^2 \, d\Gamma(\varphi, \varphi) \right),$$

which together with $\text{CS}(\beta)$ for $u = g_n - g_m$ shows that $g_n \varphi$ is a Cauchy sequence in $(\mathcal{F}, \mathcal{E}_1)$. Its limit has to be $g \varphi$ by the previous paragraph with $B(x,AR)$ in place of $B = B(x,R)$, and thus $g \varphi \in \mathcal{F}$. Since $g \varphi = g$ in $B$ and $B = B(x,R)$ is an arbitrary ball in $(X,d)$, we conclude that $g \in \mathcal{F}_{\text{loc}}$. \qed

We record the following theorem which relates the elliptic and parabolic Harnack inequalities. The equivalence of (a), (b) and (c) is due to Grigor’yan and Telcs [GT02, Theorem 3.1] in the context of random walks on graphs. This was later extended to the MMD space setting by several authors. The equivalence between (a) and (d) is due to Barlow and Bass [BB04] for random walks on graphs and was extended to the current setting in [BBK].

**Theorem 4.5.** Let $(X,d,m,\mathcal{E},\mathcal{F})$ be an MMD space and let $\beta \in (1,\infty)$. Then the following are equivalent:

(a) $(X,d,m,\mathcal{E},\mathcal{F})$ satisfies $\text{PHI}(\beta)$.

(b) $(X,d,m,\mathcal{E},\mathcal{F})$ satisfies $\text{VD}$, $\text{RVD}$, $\text{EHI}$ and $\text{cap}(\beta)$.

(c) $(X,d,m,\mathcal{E},\mathcal{F})$ satisfies $\text{VD}$ and $\text{HKE}(\beta)$.

(d) $(X,d,m,\mathcal{E},\mathcal{F})$ satisfies $\text{VD}$, $\text{RVD}$, $\text{PI}(\beta)$ and $\text{CS}(\beta)$.

**Proof.** The equivalence between (b) and (c) is contained in [GHL15, Theorem 1.2]. That (d) implies (c) follows from [GHL15, Theorem 1.2] along with [BM, Proposition 5.11 and Remark 5.12]. Conversely, that (c) implies (d) also follows from [GHL15, Theorem 1.2] except for the Hölder continuity estimate (4.3) in cutoff Sobolev inequality. The construction of cutoff functions satisfying (4.3) has been achieved in [BBK, Section 3].

In [GHL15, Theorem 1.2] the condition $\text{EHI}$ is stated for $f \in \mathcal{F}$, but by using Lemma 4.4 we obtain the version in our definition, where $f \in \mathcal{F}_e$.

---

3We note that the proof of the Hölder continuity estimate of the resolvent kernel stated in [BBK, Lemma 3.3], from which (4.3) follows, has a gap which has been resolved in the arXiv version. The proof of cutoff Sobolev inequality works also in the compact setting with minor modifications and it does not use the assumption that the metric $d$ is geodesic.
The implication from (b) to \( \text{HKE}(\beta) \) follows from [GHL15, Theorem 1.2]. Under the assumption of VD, the equivalence between \( \text{PHI}(\beta) \) and \( \text{HKE}(\beta) \) is established in [BGK, Theorem 3.1]. This completes the proof that (b) implies (a).

For the implication from (a) to (b), we first show this under the additional assumption of VD. By [GH14, Proposition 5.6], VD and EHI imply that the space \((X, d)\) is uniformly perfect and hence \(m\) satisfies RVD by [Hei, Exercise 13.1]. The combination of \( \text{PHI}(\beta) \) and VD implies \( \text{HKE}(\beta) \) by [BGK, Theorem 3.1], and the combination of VD, RVD, \( \text{HKE}(\beta) \) and Lemma 4.4 implies EHI by [GT12, Theorem 7.4]. By [GHL15, Theorem 1.2], VD, RVD and \( \text{HKE}(\beta) \) implies \( \text{cap}(\beta) \). Therefore, it suffices to show that \( \text{PHI}(\beta) \) implies VD.

The implication from \( \text{PHI}(\beta) \) to VD follows from [BGK, Theorem 3.2] under the additional assumption that the metric is geodesic. However, this additional assumption is not necessary and we modify the proof in [BGK] as follows. By [BGK, Lemma 4.6], there exists a heat kernel \( p_t(x, y) \) such that \((t, x, y) \mapsto p_t(x, y)\) is continuous on \((0, \infty) \times X \times X\).

For \( \text{PHI}(\beta) \), we obtain

\[
\sup_{x, y \in B(x_0, r)} p_t(x, y) \geq \frac{c_1}{m(B(x_0, r))} \exp\left(-\frac{c_2 t}{r^\beta}\right) \quad \text{for all } x_0 \in X, r > 0, t > 0. \tag{4.4}
\]

Let \(0 < C_1 < C_2 < C_3 < C_4, \delta \in (0, 1)\) and \(C_5 > 1\) denote the constants in \( \text{PHI}(\beta) \). Define \( K = \frac{C_4 + C_2}{C_1 + C_2} \in (1, \infty) \).

Let \(x_0 \in X, r > 0\) be arbitrary. Fix \(t > 0\) such that \(t = (C_1 + C_2)\delta^{-\beta}r^{\beta}/2\). Using (4.4), we choose \(y \in B(x_0, r)\) such that \(\sup_{x \in B(x_0, r)} p_t(x, y) \geq \frac{1}{2m(B(x_0, r))} \exp\left(-\frac{c_2 t}{r^\beta}\right)\). By \( \text{PHI}(\beta) \), we obtain

\[
p_{Kt}(x_0, y) \geq \frac{C_5^{-1} c_1}{2m(B(x_0, r))} \exp\left(-\frac{c_2 t}{r^\beta}\right) \quad \text{for some } y \in B(x_0, r). \tag{4.5}
\]

By \( \text{PHI}(\beta) \) for the caloric function \((t, z) \mapsto p_t(x_0, z)\) on the cylinder \((0, C_5\delta^{-\beta}r_1^{\beta}) \times B(x, \delta^{-1}r_1)\), where \(r_1 > 0\) satisfies \((C_1 + C_2)\delta^{-\beta}r_1^{\beta}/2 = Kt\) (or equivalently, \(r_1 = K^{1/\beta}r\)) and (4.5), we obtain

\[
p_{K^{2t}}(x_0, z) \geq \frac{C_5^{-2} c_1}{2m(B(x_0, r))} \exp\left(-\frac{c_2 t}{r^\beta}\right) \quad \text{for all } z \in B(x_0, K^{1/\beta}r). \tag{4.6}
\]

Using \(\int_X p_{K^{2t}}(x_0, z) m(dz) \leq 1\) and \(t = (C_1 + C_2)\delta^{-\beta}r^{\beta}/2\) and (4.6), there exists \(C_6 > 1\) such that

\[
\frac{m(B(x_0, K^{1/\beta}r))}{m(B(x_0, r))} \leq C_6, \quad \text{for all } x_0 \in X, r > 0.
\]

By iterating the above estimate \(\lceil \beta \log 2/\log K \rceil\) times, we obtain the volume doubling property VD.

\[\square\]

**Remark 4.6.** Theorem 4.5 can be generalized to the case where the space-time scaling function \(\Psi(r) = r^\beta\) is replaced with a homeomorphism \(\Psi : [0, \infty) \to [0, \infty)\) satisfying the
following estimates: there exist \( C_1, \beta_1, \beta_2 \in (0, \infty) \) with \( 1 < \beta_1 \leq \beta_2 \) such that
\[
C_1^{-1} \left( \frac{R}{r} \right)^{\beta_1} \leq \frac{\Psi(R)}{\Psi(r)} \leq C_1 \left( \frac{R}{r} \right)^{\beta_2}
\]
for all \( r, R \in (0, \infty) \) with \( r \leq R \).

The generalized version of the relevant properties like \( \text{PHI}(\beta) \) and \( \text{cap}(\beta) \) for such space-time scale functions can be found in [BGK, GHL15].

Combining Theorem 4.5 with the main result of [Mur20], we have the following alternative proof that \( \text{PHI}(\beta) \) cannot hold for \( \beta \in (0, 2) \) and thereby that (1.5) holds.

Lemma 4.7. If \( \beta > 0 \) and an MMD space \((X, d, m, \mathcal{E}, \mathcal{F})\) satisfies \( \text{PHI}(\beta) \), then \( \beta \geq 2 \).

Proof. Assume to the contrary that \( \beta < 2 \), so that \( \rho := d^{\beta/2} \) would be a metric on \( X \) and \((X, \rho, m, \mathcal{E}, \mathcal{F})\) would satisfy \( \text{PHI}(2) \). For \( \varepsilon > 0 \), we define the \( \varepsilon \)-chain metric as
\[
d_{\varepsilon}(x, y) = \inf \sum_{i=0}^{N-1} d(x_i, x_{i+1}),
\]
where the infimum is taken over all finite collection of points \( \{x_i\}_{i=0}^N \subset X \), \( N \in \mathbb{N} \), such that \( x_0 = x, x_N = y \) and \( d(x_i, x_{i+1}) < \varepsilon \) for all \( i = 0, 1, \ldots, N - 1 \). We define \( \rho_{\varepsilon}(x, y) \) analogously for any \( x, y \in X \). Then since \((X, \rho, m, \mathcal{E}, \mathcal{F})\) would satisfy \( \text{cap}(2) \) and \( \text{PI}(2) \) by Theorem 4.5, it follows from [Mur20, Theorem 1.6 and Remark 1.7(a)] that there would exist \( C > 0 \) such that
\[
\rho_{\varepsilon}(x, y) \leq C \rho(x, y) \quad \text{for all } x, y \in X \text{ and } \varepsilon > 0. \tag{4.7}
\]
On the other hand, since \( d(x_i, x_{i+1}) < \varepsilon \) is equivalent to \( \rho(x_i, x_{i+1}) \leq \varepsilon^{\beta/2} \), we would have
\[
d(x, y) \leq d_{\varepsilon}(x, y) \leq \varepsilon^{1-\beta/2} \rho_{2^{\beta/2}}(x, y) \quad \text{for all } x, y \in X \text{ and } \varepsilon > 0. \tag{4.8}
\]
Combining (4.7) and (4.8), we would obtain
\[
d(x, y) \leq C \varepsilon^{1-\beta/2} \rho(x, y) = C \varepsilon^{1-\beta/2} d(x, y)^{\beta/2}
\]
for all \( x, y \in X \) and \( \varepsilon > 0 \), and letting \( \varepsilon \downarrow 0 \) would yield \( \text{diam}(X, d) = 0 \), which would contradict our standing assumption that \( X \) contains at least two elements.

We next collect some properties of \( \text{EHI} \) in relation to time changes and quasisymmetric changes of the metric. The following lemma was observed first by Jun Kigami and taught through [Kig16] to the authors of [BM] and of the present paper.

Lemma 4.8 ([Kig16], cf. [BM, Lemma 5.3]). Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space, let \( \mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F}) \) and let \( \theta \in \mathcal{F}(X, d) \). Then \((X, d, m, \mathcal{E}, \mathcal{F})\) satisfies \( \text{EHI} \) if and only if \((X, \theta, \mu, \mathcal{E}^{\mu}, \mathcal{F}^{\mu})\) satisfies \( \text{EHI} \).
Proof. \(((\mathcal{F}_e^\mu), \mathcal{E}_\mu) = (\mathcal{F}_e, \mathcal{E})\) by [CF, Corollary 5.2.12], \(\mathcal{F}_e^\mu \cap C_c(X) = \mathcal{F}_e \cap C_c(X) = \mathcal{F} \cap C_c(X)\) by (2.6) and the equality \(\mathcal{F}_e \cap L^2(X, m) = \mathcal{F}\) from [CF, Theorem 1.1.5(iii)], and therefore for each open subset \(U\) of \(X\) we have

\[
\{h \in (\mathcal{F}_e^\mu)_e \mid h \text{ is } \mathcal{E}_\mu\text{-harmonic on } U\} = \{h \in \mathcal{F}_e \mid h \text{ is } \mathcal{E}\text{-harmonic on } U\}. \tag{4.9}
\]

Note also that by [CF, Theorem 5.2.11] we have the following (see [FOT, Section 2.1] and [CF, Sections 1.2, 1.3 and 2.3] for the definition and basic properties of quasi-continuous functions):

(i) A subset \(N\) of \(X\) has 1-capacity zero with respect to \((X, d, m, \mathcal{E}, \mathcal{F})\), i.e., satisfies \(\text{Cap}_1(N) = 0\), if and only if \(N\) has 1-capacity zero with respect to \((X, \theta, \mu, \mathcal{E}_\mu, \mathcal{F}_\mu)\).

In other words, the notion of holding q.e., i.e., holding outside a set of 1-capacity zero, with respect to \((X, d, m, \mathcal{E}, \mathcal{F})\) is equivalent to that with respect to \((X, \theta, \mu, \mathcal{E}_\mu, \mathcal{F}_\mu)\).

(ii) A function \(u : X \setminus N \to [-\infty, \infty]\) defined q.e. on \(X\), where \(N\) is a subset of \(X\) with \(\text{Cap}_1(N) = 0\), is quasi-continuous with respect to \((X, d, m, \mathcal{E}, \mathcal{F})\) if and only if \(u\) is quasi-continuous with respect to \((X, \theta, \mu, \mathcal{E}_\mu, \mathcal{F}_\mu)\).

Now choose a distortion function \(\eta\) so that \(\text{Id} : (X, \theta) \to (X, d)\) is an \(\eta\)-quasisymmetry, and suppose that \((X, d, m, \mathcal{E}, \mathcal{F})\) satisfies EHI with the constants \(C > 1\) and \(\delta \in (0, 1)\). Let \(B_d(x, r)\) and \(B_\theta(x, r)\) denote open balls of radius \(r\) centered at \(x\) in \((X, d)\) and \((X, \theta)\), respectively. Let \(x \in X\), \(r > 0\), and let \(h \in (\mathcal{F}_\mu)_e = \mathcal{F}_e\) be \(\mathcal{E}_\mu\)-harmonic on \(B_\theta(x, r)\) and satisfy \(h \geq 0\) \(\mu\)-a.e. on \(B_\theta(x, r)\), where we consider only quasi-continuous \(m\)-versions of \(h \in \mathcal{F}_e\) with respect to \((X, d, m, \mathcal{E}, \mathcal{F})\). Then \(h\) is \(\mathcal{E}\)-harmonic on \(B_\theta(x, r)\) by (4.9), \(h \geq 0\) q.e. on \(B_\theta(x, r)\) by [FOT, Lemma 2.1.4] applied to \((X, \theta, \mu, \mathcal{E}_\mu, \mathcal{F}_\mu)\), hence \(h \geq 0\) \(m\)-a.e. on \(B_\theta(x, r)\), and \(B_\theta(x, \delta' r) \subset B_d(x, \delta t) \subset B_d(x, t) \subset B_\theta(x, r)\) for some \(t > 0\) by (3.4) with \(\delta' r\) in place of \(r\) and \(A = \delta^{-1}\), where \(\delta' := \eta^{-1}(\delta)\). Thus we obtain

\[
m\text{-ess sup } h \leq C \cdot \text{m-ess inf } h \quad \text{by EHI for } (X, d, m, \mathcal{E}, \mathcal{F}) \text{ applied to } B_d(x, t) \text{ and } h.
\]

On the other hand, we also have

\[
\text{m-ess inf } h \leq h(y) \leq \text{m-ess sup } h,
\]

first for \(m\)-a.e. \(y \in B_\theta(x, \delta' r)\), then for q.e. \(y \in B_\theta(x, \delta' r)\) by [FOT, Lemma 2.1.4] and hence for \(\mu\)-a.e. \(y \in B_\theta(x, \delta' r)\), and therefore from (4.10) we conclude that

\[
h(y) \leq C \cdot \text{m-ess inf } h \leq C h(z) \quad \text{for } \mu\text{-a.e. } y, z \in B_\theta(x, \delta' r),
\]

proving EHI for \((X, \theta, \mu, \mathcal{E}_\mu, \mathcal{F}_\mu)\). The converse implication from EHI for \((X, \theta, \mu, \mathcal{E}_\mu, \mathcal{F}_\mu)\) to EHI for \((X, d, m, \mathcal{E}, \mathcal{F})\) is proved in exactly the same way by noting that \(\text{Id} : (X, d) \to (X, \theta)\) is a \(\tilde{\eta}\)-quasisymmetry with the distortion function \(\tilde{\eta}\) given by \(\tilde{\eta}(t) := 1/\eta^{-1}(t^{-1})\).

As mentioned in the introduction, Delmotte has constructed a space that satisfies EHI but fails to satisfy VD and hence fails to satisfy PHI(\(\beta\)) for any \(\beta > 0\) [Del]; see also [BCM, Example 8.3] for a similar construction. Nevertheless, it is possible to obtain PHI(\(\beta\)) after a time change and a change of the metric. We recall the characterization of EHI in [BM, BCM].
Theorem 4.9 ([BM, BCM]). Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space such that \((X, d)\) satisfies the metric doubling property. Then the following are equivalent:

1. \((X, d, m, \mathcal{E}, \mathcal{F})\) satisfies EHI.

2. There exist \(\gamma > 2, \mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})\) and \(\theta \in \mathcal{J}(X, d)\) such that the time-changed MMD space \((X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)\) satisfies \(\text{PHI}(\gamma)\). In other words, \(d_{cw} < \infty\).

Either of the two equivalent conditions imply that \((X, d)\) is uniformly perfect.

Proof. This follows from [BCM, Theorem 7.9], Theorem 4.5, and [GHL15, Theorem 1.2]. □

The following elementary lemma is used to verify that the function defined in (4.11) on a hyperbolic filling is gentle and satisfies the enhanced subadditive estimate (E).

Lemma 4.10. Let \((X, d, m)\) be a metric measure space that satisfies VD and let \(\gamma > 0\). For any ball \(B(x, r)\), we define

\[
\mathcal{C}(B(x, r)) = \frac{m(B(x, r))}{r^\gamma}.
\]

(a) Let \(\lambda \geq 1\). There exists \(C_1 > 0\) (that depends on the only the constant of VD and \(\lambda\)) such that for any \(x, y \in X\) satisfying \(B(x, \lambda r) \cap B(y, \lambda r) \neq \emptyset\), we have

\[
\mathcal{C}(B(x, r)) \leq C_1 \mathcal{C}(B(y, r)).
\]

(b) Let \(a > 1\). There exists \(C_2 \geq 1\) (that depends on the only the constant of VD, \(\gamma\) and \(\lambda\)) such that for any \(x, y \in X\) satisfying \(y \in B(x, r)\), we have

\[
C_2^{-1} \mathcal{C}(B(y, r/a)) \leq \mathcal{C}(B(x, r)) \leq C_2 \mathcal{C}(B(y, r/a)).
\]

(c) There exists \(C_3 > 1\) such that the following estimate holds: for all \(a > 1, x \in X, r > 0\) and \(z_1, \ldots, z_k, k \in \mathbb{N}\) such that \(d(z_i, z_j) \geq r/(2a)\) for all \(1 \leq i < j \leq k\) and satisfying \(\cup_{i=1}^k B(z_i, r/a) \supseteq B(x, r/6)\), we have that

\[
\mathcal{C}(B(x, r)) \leq C_3 a^{-\gamma} \sum_{i=1}^k \mathcal{C}(B(z_i, r/a)).
\]

Proof. We denote \(m(B(x, r))\) by \(V(x, r)\) in this proof.

(a) Let \(C_D \in (1, \infty)\) denote the constant associated with VD and let \(\alpha = \log_2 C_D\). By using the volume doubling finitely for balls of the form \(B(x, 2^i r), i \in \mathbb{Z}_{\geq 0}\), we obtain

\[
\frac{V(x, R)}{V(x, r)} \leq C_D \left(\frac{R}{r}\right)^\alpha, \quad \text{for all } 0 < r \leq R \text{ and } x \in X.
\]

Let \(z \in B(x, \lambda r) \cap B(y, \lambda r)\). By using \(B(x, r) \subset B(z, (\lambda + 1)r)\), \(B(z, r) \subset B(y, (\lambda + 1)r)\) and (4.13), we obtain

\[
V(x, r) \leq V(z, (\lambda + 1)r) \leq C_D(\lambda + 1)^\alpha V(z, r) \leq C_D^2(\lambda + 1)^{2\alpha} V(y, r).
\]
(b) Since $B(x, r) \subset B(y, 2r)$ and $B(y, r/a) \subset B(x, 2r)$, by (4.13) we have

$$V(x, r) \leq V(y, 2r) \leq C_D(2a)\alpha V(y, r/a), \quad V(y, r/a) \leq V(x, 2r) \leq C_D V(x, r).$$

Therefore

$$C(B(x)) \leq C_D 2^\alpha a^\alpha C(B(y, r/a)), \quad C(B(y, r/a)) \leq C_D a^\alpha C(B(x, r)).$$

(c) By VD and $\cup_{i=1}^k B(z_i, r/a) \supset B(x, r/3)$, we have

$$V(x, r) \leq C_D^3 V(x, r/8) \leq C_D^2 \sum_{i=1}^k V(x, r/a).$$

Dividing both sides by $r^\gamma$, we obtain (4.12) with $C_3 = C_D^3$.

The elliptic Harnack inequality implies that the capacities across annuli with similar locations and scales are comparable as we recall below.

**Lemma 4.11** ([BCM, Lemmas 5.21 and 5.22]). Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies EHI, where $(X, d)$ is a doubling metric space. For any $A_1, A_2 > 1$, there exists $C_1, C_2 > 1$ and $\gamma > 0$ such that for all $x, \in X$, and for any $0 < s \leq r < \text{diam}(X, d)/C_1$, we have

$$C_2^{-1} \left( \frac{r}{s} \right)^{-\gamma} \leq \frac{\text{Cap}(B(x, r), B(x, A_2 r)^c)}{\text{Cap}(B(x, s), B(x, A_1 s)^c)} \leq C_2 \left( \frac{r}{s} \right)^{\gamma}.$$

**Proof.** This follows immediately from [BCM, Theorem 5.4, Lemmas 5.21 and 5.22].

Using this lemma, we obtain the following comparison of capacity across annuli under a quasisymmetric change of metric.

**Proposition 4.12.** Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies PHI$(\gamma)$, where $\gamma \geq 2$. Let $\theta \in \mathcal{F}(X, d)$ and $a > 1$. Then there exists $C, A > 0$ such that the following property holds. For any $x, \tilde{x} \in X, 0 < r < \text{diam}(X, \theta)/A, s > 0, n \in \mathbb{Z}$ such that

$$s = \sup \{0 < t < 2\text{diam}(X, d) : B_d(x, t) \subset B_\theta(x, r)\},$$

and

$$2a^{-n-1} \leq s < 2a^{-n}, d(\tilde{x}, x) < 2a^{-n},$$

we have

$$C^{-1} \frac{m(B_d(\tilde{x}, 2a^{-n}))}{[2a^{-n}]^\gamma} \leq \text{Cap}(B_\theta(x, r), B_\theta(x, 2r)^c) \leq C \frac{m(B_d(\tilde{x}, 2a^{-n}))}{[2a^{-n}]^\gamma}.$$ (4.15)
Proof. By Theorem 4.5, \((X,d)\) satisfies doubling, uniformly perfect metric space. By Proposition 3.2, there exists \(A_1, A_2, A_3 > 1\) such that for all \(x \in X, 0 < r < \text{diam}(X, \theta),\)

\[
B_d(x,s) \subset B_\theta(x,r) \subset B_d(x, A_1 s) \subset B_d(s, 2A_1 s) \subset B_\theta(x, A_2 r) \subset B_d(x, A_3 s),
\]

(4.16) where \(s > 0\) is as defined in (4.14). If \(B_d(x, A_3 s) \neq X\) in (4.16), we have

\[
\text{Cap}(B_d(x,s), B(x,A_3s)\cap) \leq \text{Cap}(B_\theta(x,r), B_\theta(x,2r)\cap) \leq \text{Cap}(B_d(x, A_1 s), B(x,2A_1 s)\cap).
\]

(4.17) By Lemma 4.11, Proposition 3.2(b), and Theorem 4.5, there exist \(C, A > 1\) such that for all \(x \in X, 0 < r < \text{diam}(X, \theta)/A,\) we have

\[
C^{-1} \text{Cap}(B_d(x,s), B(x, A_3 s)\cap) \leq \text{Cap}(B_\theta(x,r), B_\theta(x,2r)\cap) \leq C_1 \text{Cap}(B_d(x, s), B(x, A_3 s)\cap),
\]

and

\[
C^{-1} \frac{m(B_d(x,s))}{s^\gamma} \leq \text{Cap}(B_d(x,s), B(x,A_3 s)\cap) \leq C \frac{m(B_d(x,s))}{s^\gamma},
\]

(4.18) where \(s > 0\) is as given in (4.14). By (4.18), (4.19) and VD, we obtain (4.15). \(\square\)

We will use Theorem 3.24 to construct metrics. The following proposition plays a central role in constructing a function on the hyperbolic filling that satisfies the hypotheses (S1) and (S2) in Theorem 3.24.

Proposition 4.13. Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space that satisfies \(\text{PHI}(\gamma)\) for some \(\gamma > 2\) and let \(\lambda > 1\). There exists constants \(A, C_1, C_2 > 1, \eta > 0\) (that depend only on \(\lambda\) and the constants associated with \(\text{PHI}(\gamma)\)) such that for any \(\alpha > 1, x \in X, 0 < r < \text{diam}(X, d)/A,\) and for any collection of balls \(\mathcal{B} = \{B(y_i, r/a) : i \in I\}\) such that \(\bigcup_{i \in I} B(y_i, r/a) = X\) and \(\{B(y_i, r/(4a))\}\) is pairwise disjoint, there exists a function \(\sigma : \mathcal{B} \to [0, \infty)\) that obeys the following properties (note that \(\sigma\) depends on \(x \in X, r > 0\)):

(S1’) for any sequence of balls \(\gamma = \{B_i : 1 \leq i \leq N\}\) in \(\mathcal{B}\) such that \(x_{B_i} \in B(x, r), x_{B_N} \notin B(x,2r)\) and \(\lambda \cdot B_i \cap \lambda \cdot B_{i+1} \neq \emptyset\) for all \(i = 1, \ldots, N - 1\), we have

\[
\sum_{i=1}^{N} \sigma(B_i) \geq 1,
\]

(4.20) and

\[
\sigma(B) = 0, \quad \text{for any ball } B \in \mathcal{B} \text{ such that } x_B \notin B(x,2r).
\]

(4.21)

(S2’) \(\sigma : \mathcal{B} \to (0, \infty)\) satisfies the following estimates

\[
\sum_{B \in \mathcal{B}} \sigma(B)^2 \frac{m(B)}{(r/a)^\gamma} \leq C_1 \frac{m(B(x,r))}{r^\gamma}
\]

(4.22) and

\[
\sup_{B \in \mathcal{B}} \sigma(B) \leq C_2 \alpha^{-\eta}.
\]

(4.23)
In particular, for any $\beta > 2$, we have

$$
\sum_{B \in \mathcal{B}} \sigma(B)^{\beta} \frac{m(B)}{(r/a)^{\gamma}} \leq C_1 C_2^{\beta-2} a^{-(\beta-2)\eta} \frac{m(B(x, r))}{r^{\gamma}}. \tag{4.24}
$$

**Proof.** For a function $u \in \mathcal{F} \cap C(X)$ and $B \in \mathcal{B}$, we define its ‘discretization’ $u_d : \mathcal{B} \to \mathbb{R}$ as

$$
u_d(B) := \int_B u \, dm = \frac{1}{m(B)} \int_B u \, dm, \tag{4.25}
$$

and its ‘discrete gradient’ $\sigma_u : \mathcal{B} \to [0, \infty)$

$$
\sigma_u(B) := \sum_{B' \in \mathcal{B} : \lambda(B') \cap \lambda(B) \neq \emptyset} |u(B') - u(B)|. \tag{4.26}
$$

Our construction of $\sigma$ is the discrete gradient $\sigma_u$ of a well chosen function $u$. In particular, we choose a function $u \in \mathcal{F} \cap C_c(X)$ that satisfies the following properties: there exists $C_3 > 1, \eta > 0$ (that depends only on the constant associated with $\text{PHI}(\gamma)$) such that for all $x \in X, r < \text{diam}(X, d)/A$, we have

$$
u \equiv 1 \text{ on } B(x, 1.1r) \text{ and } \nu \equiv 0 \text{ on } B(x, 1.9r) \tag{4.27}
$$

$$
\mathcal{E}(\nu, \nu) \leq C_3 \frac{m(B(x, r))}{r^{\gamma}}, \tag{4.28}
$$

$$
|\nu(y) - \nu(z)| \leq C_3 \left( \frac{d(y, z)}{r} \right)^{\eta} \text{ for all } y, z \in X. \tag{4.29}
$$

The existence of a function $u \in \mathcal{F} \cap C_c(X)$ satisfying the above properties follows from the cutoff Sobolev inequality $\text{CS}(\beta)$, Theorem 4.5 and a standard covering argument as we recall below. By Theorem 4.5 we have that $m$ is a doubling measure in $(X, d)$ and hence $(X, d)$ is $K_D$-doubling metric space for some $K_D > 1$. Therefore there exists $N_D \in \mathbb{N}$ that depends only on $K_D$ and $y_1, \ldots, y_{N_D} \in B(x, 1.1r)$ such that $\bigcup_{i=1}^{N_D} B(y_i, r/10) \supset B(x, 1.1r)$. By the construction cutoff functions in [BBK, Section 3], there exists $C_4 > 0, \eta > 0$ such that for each $i = 1, \ldots, N_D$ satisfies

$$
\phi_i \equiv 1 \text{ on } B(y_i, r/10), \quad \phi_i \equiv 0 \text{ on } B(y_i, r/5)^c,
$$

$$
\mathcal{E}(\phi_i, \phi_i) \leq C_4 \frac{m(B(y_i, r/10))}{r^{\gamma}}, \quad |\phi_i(y) - \phi_i(z)| \leq C_4 \left( \frac{d(y, z)}{r} \right)^{\eta} \text{ for all } y, z \in X.
$$

By choosing $u = \max_{1 \leq i \leq N_D} \phi_i$ and using the above estimates along with triangle inequality, $\mathcal{E}(\nu, \nu) \leq \sum_{i=1}^{N_D} \mathcal{E}(\phi_i, \phi_i), |\nu(y) - \nu(z)| \leq \max_{1 \leq i \leq N_D} |\phi_i(y) - \phi_i(z)|$, we obtain the desired properties (4.27), (4.28) and (4.29).

Let us show that the function $\sigma = \sigma_u$ as defined by (4.25) and (4.26) satisfies the desired conditions (S1') and (S2'). To this end, we note the following properties of $u_d$:
\[ \mathcal{B} \to \mathbb{R}: \]
\[ u_d(B(x_B, r/a)) = 1 \quad \text{for any } x_B \in B(x, r/10) \quad \text{(since } B(x_B, r/a) \subset B(x, 1.1r)) \]
\[ u_d(B(y, r/a)) = 0 \quad \text{for any } y \in X \text{ such that } d(y, y') \leq 2\lambda r/a \text{ where } y' \in B(x, 2r)^c \]
\[ (B(y, r/a) \subset B(x, 1.9r)^c \text{ because } (2\lambda + 1)r/a < 0.1r \text{ by (3.12))} \]
\[ \sum_{i=1}^{N} \sigma_{u}(B_i) \geq \sum_{i=1}^{N-1} |u_d(B_i) - u_d(B_{i+1})| \geq |u_d(B_1) - u_d(B_N)| \]

for any sequence of balls \( B_1, \ldots, B_N \in \mathcal{B} \) such that \( \lambda \cdot B_i \cap \lambda \cdot B_{i+1} \neq \emptyset \) for all \( i = 1, \ldots, N - 1 \). The above equations immediately imply (4.20) and (4.21).

Since the balls \( B(y_i, r/(4a)), i \in I \) are disjoint, by doubling property of \((X, d)\), there exists \( C_5 > 1 \) that depends only on \( \lambda \) and the doubling constant (but not on \( a \)) such that
\[ \# \{ B' \in \mathcal{B} : \lambda \cdot B \cap \lambda \cdot B' \neq \emptyset \} \leq C_5, \quad \text{for all } B \in \mathcal{B}. \tag{4.30} \]

For any two balls \( B, B' \in \mathcal{B} \) such that \( \lambda \cdot B \cap \lambda \cdot B' \neq \emptyset \), by (4.29) we have
\[ |u_d(B) - u_d(B')| \leq \sup_{y, z \leq 2(\lambda + 1)r/a} |u(y) - u(z)| \leq C_3(2(\lambda + 1)^\gamma a^{-\gamma}). \tag{4.31} \]

Combining (4.26), (4.30) and (4.31), we obtain (4.23) for \( \sigma = \sigma_u \).

It remains to show that \( \sigma = \sigma_u \) satisfies (4.23). To this end, we recall the following Poincaré inequality (follows from [GHL15, Theorem 1.2] and Theorem 4.5): there exist \( C_P, A > 1 \) such that
\[ \frac{1}{2m(B(y, s))} \int_{B(y, s)} \int_{B(y, s)} (f(z) - f(w))^2 \, dm(z) \, dm(w) \leq C_P s^\gamma \int_{B(y, As)} d\Gamma(f, f), \tag{4.32} \]

for any \( f \in \mathcal{F}, y \in X, s > 0 \). Using this Poincaré inequality, the following comparison estimate between discrete and continuous energies is standard [CS, BB04]. Similar to \( \S 3.2 \), for any two balls \( B, B' \in \mathcal{B} \) by \( B' \sim B \) we mean that \( \lambda \cdot B \cap \lambda \cdot B' \neq \emptyset \). We obtain (4.22) by the following estimates:

\[ \sum_{B \in \mathcal{B}} \sigma_u^2(B) \frac{m(B)}{(r/a)^\gamma} \]
\[ \lesssim \sum_{B, B' \in \mathcal{B}, B' \sim B} |u_d(B') - u_d(B)|^2 \frac{m(B)}{(r/a)^\gamma} \quad \text{(by (4.30) and the Cauchy-Schwarz inequality)} \]
\[ \lesssim \sum_{B, B' \in \mathcal{B}, B' \sim B} \frac{1}{(r/a)^\gamma m(B')} \int_B \int_B (u(y) - u(z))^2 \, dm(y) \, dm(z) \quad \text{(by Jensen’s inequality)} \]
\[ \lesssim \sum_{B \in \mathcal{B}} \frac{1}{m((2\lambda + 1) \cdot B)(r/a)^\gamma} \int_B \int_{(2\lambda + 1) \cdot B} (u(y) - u(z))^2 \, dm(y) \, dm(z) \quad \text{(by VD)} \]
\[ \lesssim \sum_{B \in \mathcal{B}} (2\lambda + 1)^\gamma \int_{A(2\lambda + 1) \cdot B} d\Gamma(u, u) \quad \text{(by (4.32))} \]
\[ \lesssim \mathcal{E}(u, u) \quad \text{(since } (X, d) \text{ is } K_D\text{-doubling, we have } \sum_{B \in \mathcal{S}_n} 1_{(2\lambda A + A) \cdot B} \lesssim 1). \]
\[ \lesssim \frac{m(B(x, r))}{r^\gamma} \quad \text{(by (4.28)).} \]
Finally, (4.24) follows from (4.22),(4.23), and

$$\sum_{B \in \mathcal{B}} \sigma(B)^{\beta} \frac{m(B)}{(r/a)^{\gamma}} \leq \left( \sup_{B \in \mathcal{B}} \sigma(B) \right)^{\beta - 2} \sum_{B \in \mathcal{B}} \sigma(B)^{2} \frac{m(B)}{(r/a)^{\gamma}}.$$

The following proposition provides a convenient sufficient conditions for a measure $\mu$ to be smooth and have full quasi-support.

**Proposition 4.14.** Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies PHI($\gamma$) for some $\gamma \geq 2$ and let $\theta \in \mathcal{J}(X, d)$. Let $\beta > 2$ and $\mu$ be a measure that satisfies the following estimate: there exists $C_1, A > 0$ such that for any $x \in X$, $0 < r < \text{diam}(X, \theta)/A$, we have

$$C_1^{-1} \frac{\mu(B_\theta(x, r))}{r^\beta} \leq \text{Cap}(B_\theta(x, r), B_\theta(x, 2r)^c) \leq C_1 \frac{\mu(B_\theta(x, r))}{r^\beta}.$$

Then $\mu$ is a smooth measure with full quasi-support; or equivalently $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$. Furthermore $\mu$ satisfies VD and RVD on $(X, \theta)$.

**Proof.** By [BM, Lemma 6.3], the MMD space $(X, \theta, m, \mathcal{E}, \mathcal{F})$ satisfies EHI. Since $(X, d)$ is a doubling, uniformly perfect metric space, so is $(X, \theta)$ [Hei, Theorem 10.18 and Exercise 11.2]. The volume doubling property VD of $\mu$ in $(X, \theta)$ follows from [BCM, Lemma 6.3]. The RVD property for $\mu$ follows from the fact that $(X, \theta)$ is uniformly perfect [Hei, Exercise 13.1]. That $\mu$ is a smooth measure follows from [BCM, Proposition 6.17 and Theorem 4.6]. By [BCM, Proposition 5.17 and Theorem 5.4], $\mu$ has full quasi-support.

### 4.2 Completion of the proof of Theorem 2.9

Now we are in the stage of completing the proof of our first main theorem (Theorem 2.9).

**Proof of Theorem 2.9.** By Theorem 4.9, it suffices to show that (a) implies (c). By Lemma 4.7, it suffices to show that $d_{cw} \leq 2$. To this end, we fix an arbitrary $\beta > 2$. We shall construct a metric $\theta \in \mathcal{J}(X, d)$ and a measure $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$ such that the time-
changed MMD space $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies PHI($\beta$).

By Theorem 4.9, (1.3), (2.7), and by changing the metric and measure if necessary, we may assume that $(X, d, m, \mathcal{E}, \mathcal{F})$ satisfies $\text{VD}$, $\text{RVD}$, $\text{EHI}$ and $\text{cap}(\gamma)$.

If $(X, d)$ is bounded, we scale the metric so that $\text{diam}(X, d) = \frac{1}{2}$.

Fix $\lambda \geq 32$ and let $a$ be an arbitrary constant that satisfies (3.12). The choice of $a$ will be made later in the proof. Let $x_0 \in X$. Let $\mathcal{S} = \prod_{k \in \mathbb{Z}_{\geq 0}} \mathcal{S}_k$ denote the vertex set of the hyperbolic filling as defined in Definition 3.10, where $\mathcal{S}_k = \{ B(x, 2a^{-k}) : x \in N_k \}$, where $N_k, k \in \mathbb{Z}$ is a sequence of $a^{-k}$-separated sets such that $N_k \subset N_{k+1}$ and $x_0 \in N_k$ for all $k \in \mathbb{Z}$.
We define a function $C : \prod_{k \in \mathbb{Z}} S_k \to (0, \infty)$ on the extended hyperbolic filling by

$$C(B(x, 2a^{-k})) = \frac{m(B(x, 2a^{-k}))}{(2a^{-k})^\gamma}, \text{for any } k \in \mathbb{Z} \text{ and for any } B(x, 2a^{-k}) \in S_k.$$  \hspace{1cm} (4.33)

Let us verify that $C$ is gentle and satisfies the enhanced subadditivity property. By Lemma 4.10(a,b), there exist $K_h, K_v$ such that $K_v$ depends only on $a$ and the constant associated with VD, $K_h$ depends only on $\lambda$ and the constant associated with VD such that

$$C(B_1) \leq K_v C(B_2), \quad \text{whenever } B_1 \text{ and } B_2 \text{ share a horizontal edge,}$$

$$C(B_1) \leq K_h C(B_2), \quad \text{whenever } B_1 \text{ and } B_2 \text{ share a vertical edge.} \hspace{1cm} (4.34)$$

Recall that for every ball $B \in S_k, k \in \mathbb{Z}$, there exists an unique ball $g(B)_{k-1} \in S_{k-1}$ such that there is a vertical edge between $g(B)_{k-1}$. We say that a ball $B \in S_k, k \in \mathbb{Z}$ is non-peripheral if

$$B \sim B' \text{ implies that } g(B)_{k-1} = g(B')_{k-1}. \hspace{1cm} (4.37)$$

Note that is $B \sim B'$ and $B, B' \in S_{k+1}$, then by (3.12), we have $d(x_B, x_{B'}) \leq 2\gamma^{-1} \leq \frac{1}{12}a^{-k}$. We denote the set of all non-peripheral elements of $\prod_{k \in \mathbb{Z}} S_k$ by $\mathcal{N}$. Hence if $C = g(B)_k$ and $d(x_C, x_B) \leq \frac{1}{6}(2a^{-k}+2a^{-k-1})$, then $d(x_C, x_{B'}) < (3^{-1}+2a^{-1}+12^{-1})a^{-k} < \frac{1}{2}a^{-k}$, and hence $B \in \mathcal{N}$. This along with Lemma 4.10(c) imply that, there exists $C_1 > 0$ such that

$$C(B) \leq C_1 a^{-\gamma} \sum_{B' \in \mathcal{N} \cap D_k(B)} C(B'), \quad \text{for all } B \in S_k, k \in \mathbb{Z}. \hspace{1cm} (4.35)$$

For $k \in \mathbb{Z}$ and $B \in S_k$, let $\Gamma_{k+1}(B)$ denote the set of horizontal paths $\gamma = \{B_i\}_{i=1}^{N}$, $N \geq 2$ such that $B_i \in S_{k+1}$ for all $i = 1, \ldots, N$, $B_i \sim B_{i+1}$ for all $i = 1, \ldots, N-1$, $x_{B_1} \in B$, $x_{B_N} \notin 2 \cdot B$, and $x_{B_i} \in 2 \cdot B$ for all $i = 1, \ldots, N-1$ (as given in Definition 3.22). If $\text{diam}(X, d) = \frac{1}{2}$, we note that $\Gamma_k(B) = \emptyset$ for all $k \leq 0$. If $\text{diam}(X, d) = \frac{1}{2}$, we assume that $a > 2A,$ \hspace{1cm} (4.36)

where $A$ is the constant in Proposition 4.13. If either $k \in \mathbb{Z}, \text{diam}(X, d) = \infty$ or if $k \in \mathbb{N}, \text{diam}(X, d) = \frac{1}{2}$, for any $B \in S_k$, we define $\sigma_B : S_{k+1} \to (0, \infty)$ as the function defined in Proposition 4.13, that satisfies

$$\sum_{i=1}^{N} \sigma_B(B_i) \geq 1, \quad \text{for any } \{B_i\}_{i=1}^{N} \in \Gamma_{k+1}(B).$$

Otherwise if $\text{diam}(X, d) = \frac{1}{2}$ and $k \geq 0$, we simply define $\sigma_B : S_{k+1} \to [0, \infty)$ as $\sigma_B \equiv 0$ for all $B \in S_k$. For any $k \in \mathbb{Z}$ and for any $B \in S_k$, we define

$$\sigma(B) = \max_{C \in S_{k+1}} \sigma_C(B), \quad \text{for any } k \in \mathbb{Z}, B \in S_k. \hspace{1cm} (4.37)$$

Evidently, by Proposition 4.13, we have

$$\sum_{i=1}^{N} \sigma(B_i) \geq 1, \quad \text{for any } \{B_i\}_{i=1}^{N} \in \Gamma_{k+1}(B) \text{ and for any } k \in \mathbb{Z}, B \in S_k. \hspace{1cm} (4.38)$$
In the compact case, the above statement is vacuously true for \( k \leq 0 \). By Proposition 4.13 and the argument in Lemma 3.29, there exist \( C_2, \eta > 0 \) such that
\[
\sum_{B' \in \mathcal{D}_{k+1}(B)} \sigma(B') \beta \mathcal{C}(B') \leq C_2 \alpha^{-(\beta-2)\eta} \mathcal{C}(B), \quad \text{for any } k \in \mathbb{Z}, B \in \mathcal{S}_k, \tag{4.39}
\]
where \( \mathcal{D}_{k+1}(B) \) denotes the set of descendants of \( B \) in \( \mathcal{S}_{k+1} \) (that is, \( \mathcal{D}_{k+1}(B) \) is the set of elements in \( \mathcal{S}_{k+1}(B) \) that share a vertical edge with \( B \)).

We consider two cases.

**Case 1:** \((X, d)\) is bounded. Let \( \mathcal{S} = \bigsqcup_{k \geq 0} \mathcal{S}_k \) denote the vertex set of the hyperbolic filling. In this case by (4.35), we can ensure the enhanced subadditivity estimate (E) by choosing \( a \) large enough. Similarly by (4.38) and (4.39), the function \( \sigma \) defined above satisfies the hypotheses (S1) and (S2) of Theorem 3.24 for all large enough \( a \). Therefore by Theorems 3.24 and 3.14, and Proposition 3.21, there exist a metric \( \theta \in \mathcal{F}(X, d) \) and a measure \( \mu \) on \( X \) that satisfies the following estimate: there exists \( C_3 > 0 \) such that
\[
C_3^{-1} r^\beta \mathcal{C}(B) \leq \mu(B_\theta(x,r)) \leq C_3 r^\beta \mathcal{C}(B) \quad \text{for all } x \in X, r < \text{diam}(X, \theta), B \in \mathcal{A}_\mathcal{S}(B_d(x,s)) \tag{4.40}
\]
where \( s \) is the largest number in \([0, 2\text{diam}(X, d)]\) such that \( B_d(x, s) \subset B_\theta(x, r) \) (as defined in (3.3)) and \( \mathcal{A}_\mathcal{S}(B_d(x, s)) \) is as given in Definition 3.16. Combining (4.40) and Proposition 4.12, we see that there exist \( A_1, C_4 > 0 \) such that
\[
C_4^{-1} \frac{\mu(B_\theta(x,r))}{r^\beta} \leq \text{Cap}(B_\theta(x,r), B_\theta(x,2r)\setminus), \tag{4.41}
\]
for any \( x \in X, 0 < r < \text{diam}(X, d)/A_1 \). By (4.41) and Proposition 4.14, \( \mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F}), (X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu) \) satisfies VD, RVD and \( \text{cap}(\beta) \), and by Lemma 4.8 it also satisfies EHI. Thus by Theorem 4.5 the MMD space \((X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)\) satisfies PHI(\(\beta\)). Since \( \beta > 2 \) is arbitrary, we conclude that the conformal walk dimension is two.

**Case 2:** \((X, d)\) is unbounded.

The approach in the unbounded case is to construct metrics and measures on an increasing sequence of compact sets that cover \( X \), and to take suitable sub-sequential limit. Let \( x_0 \in X \) be the point such that \( x_0 \in N_k \) for all \( k \in \mathbb{Z} \) as given in Definition 3.10. We consider the sequence of subsets
\[
X_n = \overline{D_{-n}(x_0)}, \quad \text{for any } n \in \mathbb{N}, \tag{4.42}
\]
where \( D_{-n}(x_0) \) is as defined in (3.16). By Lemma 3.9(c), \( X_n \) is compact and satisfies
\[
B(x_0, (2^{-1} - (a-1)^{-1})a^n) \subset X_n \subset \overline{B}(x_0, (1-a^{-1})^{-1}a^n). \tag{4.43}
\]
For any \( k \geq -n \), we define
\[
\mathcal{S}_k^{(n)} = \{ B(x, 2a^{-k}) \cap X_n : x \in N_k \cap D_{-n}(x_0) \}.
\]
By Lemma 3.9, \( \mathcal{S}^{(n)} := \bigsqcup_{k \in \mathbb{Z}, k \geq -n} \mathcal{S}_k^{(n)} \) is a hyperbolic filling of the compact space with the same vertical edges induced from the extended hyperbolic filling. Similarly, \( B \cap X_n, B' \cap \)
$X_n \in S^{(n)}_k$ share a horizontal edge if and only if $(\lambda \cdot B \cap X_n) \cap (\lambda \cdot B' \cap X_n) \neq \emptyset$. We define $C_n : S^{(n)} \to (0, \infty)$, $\sigma_n : S^{(n)} \to [0, \infty)$ as

$$C_n(B \cap X_n) = C(B), \quad \sigma_n(B \cap X_n) = \sigma(B),$$

for any $B = B \in \prod_{k \geq -n} S_k$, where $C$ is as given in (4.33) and $\sigma$ is as given in (4.37). Similar to the compact case, by choosing $a > 1$ large enough, by (4.35), we obtain the enhanced subadditivity estimate (E) for $C_n$ uniformly over $n$ (that is, the constant $\delta$ in associated with (E) does not depend on $n$). Similarly, by increasing $a$ is necessary, and by (4.38) and (4.39), the function $\sigma_n$ defined above satisfies the hypotheses (S1) and (S2) of Theorem 3.24 uniformly in $n$ (that is, the constant $\eta$ in associated with (S2) does not depend on $n$).

Similar to the compact case, by Theorems 3.24 and 3.14, there exist metrics $\theta_n \in \mathcal{J}(X_n, d)$ for each $n \in \mathbb{N}$, and a distortion function $\eta : [0, \infty) \to [0, \infty)$ such that the identity map

$$\text{Id} : (X_n, d) \to (X_n, \theta_n)$$

is an $\eta$-quasisymmetry for each $n \in \mathbb{N}$. (4.44)

By Proposition 3.21, there exist measures $\mu_n$ on $X_n$ for each $n \in \mathbb{N}$, constant $C_0 > 0$ such that

$$C^{-1}_0 \mu_n(B_\theta(x, r)) \leq \mu_n(B_\theta(x, r)) \leq C_0 \mu_n(B_\theta(x, r)),$$

for all $x \in X_n$, $r < \text{diam}(X_n, \theta), B \in A_\delta(B_d(x, r))$, where $s$ is the largest number in $[0, 2\text{diam}(X_n, d)]$ such that $B_d(x, s) \cap X_n \subset B_\theta(x, r)$ (as defined in (3.3)) and $A_\delta(B_d(x, s) \cap X_n)$ is as given in Definition 3.16 corresponding to the hyperbolic filling $S^{(n)}_k$ of $X_n$.

Next, normalize the metrics and measures by choosing a pair of sequences $\beta_n, \gamma_n > 0$ such that

$$\hat{\theta}_n = \beta_n \theta_n, \hat{\mu}_n = \gamma_n \mu_n$$

satisfy

$$\text{diam}(B_d(x_0, 1), \hat{\theta}_n) = 1, \quad \hat{\mu}_n(B_d(o, 1)) = 1.$$ (4.46)

By (4.43) and (3.12), we note that $B(x_0, 1) \subset X_n$ for all $n \in \mathbb{N}$.

We choose $p \in X$ such that $d(x_0, p) = \frac{1}{2}$. Since $\text{Id} : (X_n, d) \to (X_n, \hat{\theta}_n)$ is a $\eta$-quasisymmetry, by comparing the ratio of the diameter of the sets $\{x_0, p\} \subset X_1$ in the metrics $d$ and $\theta_n$ using (3.6), there exists $C_{x_0, p}^{-1} > 1$ such that

$$C_{x_0, p}^{-1} \leq \hat{\theta}_n(o, p) \leq C_{x_0, p}^{-1},$$

for all $n \in \mathbb{N}$.

We estimate $\hat{\theta}_n(x, y)/\hat{\theta}_n(x_0, p)$ by writing it as $\theta_n(x, y) / \theta_n(x_0, p)$ and using $\eta$-quasisymmetry to estimate each of the factors and their reciprocals. This yields the following estimate; for any $x, y \in X_n, n \in \mathbb{N}$, there exists $C_{x, y} > 1$ ($C_{x, y}$ depends only on $d(x_0, x), d(x, y)$ and $\eta$) such that

$$C_{x, y}^{-1} \leq \hat{\theta}_n(x, y) \leq C_{x, y}^{-1},$$

for all $n \in \mathbb{N}$ such that $\{x, y\} \subset X_n$. (4.47)

By a similar computation, for any $(x, y) \in X \times X$ and for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $(x', y') \in X \times X$ with $d(x, x') \vee d(y, y') < \delta$, we have

$$\left| \hat{\theta}_n(x, y) - \hat{\theta}_n(x', y') \right| \leq \hat{\theta}_n(x, x') + \hat{\theta}_n(y, y') < \varepsilon,$$

for all $n \in \mathbb{N}$ such that $x, x', y, y' \in X_n$. (4.48)
By (4.48), (4.47) and Arzela-Ascoli theorem on the product space $X_n \times X_n$ equipped with the product metric $d_{\infty}((x, y), (x', y')) = d(x, x') \vee d(y, y')$, the sequence of functions $\hat{\theta}_m, m \geq n$, has a subsequence that converges uniformly to a metric $\theta$ in $X_n$. By a diagonalization argument, we obtain a subsequence of $\{\hat{\theta}_n : n \in \mathbb{N}\}$, that converges uniformly in compact subsets of $X \times X$. The limit metric $\theta \in \mathcal{J}(X, d)$ and $\text{Id} : (X, d) \to (X, \theta)$ is an $\eta$-quasisymmetry.

The measures $\hat{\mu}_n$ constructed using Lemma 3.20 are uniformly doubling in the following sense: there exists $C_D \geq 1$ such that for all $n \in \mathbb{N}, x \in X_n, 0 < r < \text{diam}(X_n, \hat{\theta}_n)$, we have

$$\frac{\hat{\mu}_n(B_{\hat{\theta}_n}(x, 2r))}{\hat{\mu}_n(B_{\hat{\theta}_n}(x, r))} \leq C_D.$$ By the argument in [LuS, Theorem 1], by a further diagonalization argument using weak*-compactness of $\{\hat{\mu}_m : m \geq n\}$ on $X_n$ for all $n \in \mathbb{N}$, we obtain a measure $\mu$ on $X$. By (4.45), Proposition 4.12, there exist constants $C_6 > 1, A_2 > 1$ such that

$$C_6^{-1} \frac{\mu(B_{\theta}(x, r))}{r^\beta} \leq \text{Cap}(B_{\theta}(x, r), B_{\theta}(x, 2r)^c) \leq C_4 \frac{\mu(B_{\theta}(x, r))}{r^\beta},$$

(4.49)

for any $x \in X, r > 0$. The remainder of the proof is exactly same as the compact case. Hence, we conclude that the conformal walk dimension is two.

5 The attainment and Gaussian uniformization problems

In this section, we introduce the attainment problem for the conformal walk dimension and the Gaussian uniformization problem. Then we discuss partial progress towards them.

Let $(X, d, m, \mathcal{E}, \mathcal{F})$ be an MMD space that satisfies the metric doubling property and EHI. Recall that the Gaussian uniformization problem ask for a description of all metrics $\theta \in \mathcal{J}(X, d)$ and measures $\mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F})$ such that the corresponding time-changed MMD space $(X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)$ satisfies PHI(2). For any $\beta > 0$, we define

$$\mathcal{G}_\beta(X, d, m, \mathcal{E}, \mathcal{F}) = \left\{ \mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F}) \mid \text{there exists } \theta \in \mathcal{J}(X, d) \text{ such that } (X, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu) \text{ satisfies PHI}(\beta) \right\}. \quad (5.1)$$

We define the Gaussian admissible measures as

$$\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) := \mathcal{G}_2(X, d, m, \mathcal{E}, \mathcal{F}). \quad (5.2)$$

By Theorem 2.9, we have

$$\mathcal{G}_\beta(X, d, m, \mathcal{E}, \mathcal{F}) = \emptyset \text{ for any } \beta < 2, \text{ and } \mathcal{G}_\beta(X, d, m, \mathcal{E}, \mathcal{F}) \neq \emptyset \text{ for any } \beta > 2.$$ This raises the following questions:
1. **Attainment problem**: Is \( \mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) \neq \emptyset \)? Or equivalently, is the infimum in (1.4) attained?

2. **Gaussian uniformization problem**: Describe all measures in the set \( \mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) \).

By Proposition 2.10(a), the *Gaussian admissible measures* can be described as

\[
\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) = \left\{ \mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F}) \mid \text{\( d^\mu_{\text{int}} \in \mathcal{J}(X, d) \) and \text{\( (X, d^\mu_{\text{int}}, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu) \) satisfies PHI(2)} \right\},
\]

where \( d^\mu_{\text{int}} \) denotes the intrinsic metric of the MMD space \( (X, d, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu) \) (recall Definition 2.3).

In this section, we prove Theorem 2.11 and discuss its consequences for the Gaussian uniformization problem. In particular, Theorem 2.11 shows that any two measures \( \mu_1, \mu_2 \in \mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) \) must be \( A_\infty \)-related in \( (X, d) \) (provided such measures exist).

### 5.1 Consequences of PHI(2)

We begin with the proof of Proposition 2.10, which is essentially contained in [KM, Section 4]. This states that any measure on \( \mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) \) must be a minimal energy-dominant measure and that the metric must be bi-Lipschitz equivalent to the intrinsic metric of the time-changed MMD space.

**Proof of Proposition 2.10.** By Theorem 4.5, we have all of the equivalent conditions in [GHL15, Theorem 1.2].

(a) We use [Mur20, Theorem 1.6 and Remark 1.7(a)] and [KM, Proposition 4.8] to obtain (a).

(b) This follows from [KM, Propositions 4.5 and 4.7].

**Definition 5.1.** Let \( (X, d) \) be a metric space and let \( u \) be a Borel measurable function. We define the *pointwise Lipschitz constant* as

\[
\text{Lip } u(x) := \limsup_{y \to x} \frac{|u(x) - u(y)|}{d(x, y)},
\]

and \( \text{Lip}(X) \) denotes the collection of all measurable functions \( u \) with

\[
\|u\|_{\text{Lip}(X)} := \sup_{x, y \in X, x \neq y} \frac{|u(x) - u(y)|}{d(x, y)} < \infty.
\]

When it is necessary, we also write Lip as Lip\(_d\) to specify the metric \( d \).

We recall the notion of upper gradient and its variants. We refer the reader to [HKST, Hei] for a comprehensive account.
**Definition 5.2.** Let \((X,d,m)\) be a metric measure space and let \(u : X \to \mathbb{R}\) be a Borel measurable function. A non-negative Borel measurable function \(g\) is called a *upper gradient*\(^4\) if

\[
|u(x) - u(y)| \leq \int_{\gamma} g \, ds,
\]

for every rectifiable curve \(\gamma\). A non-negative Borel measurable function \(g\) is called a *\(p\)-weak upper gradient* of \(u\) with \(p \in [1, \infty)\) if

\[
|u(x) - u(y)| \leq \int_{\gamma} g \, ds,
\]

for all \(\gamma \in \Gamma_{\text{rect}} \setminus \Gamma_0\), where \(x\) and \(y\) are the endpoints of \(\gamma\), \(\Gamma_{\text{rect}}\) denotes the collection of non-constant compact rectifiable curves and \(\Gamma_0\) has \(p\)-modulus zero in the sense that

\[
\inf \left\{ \|\rho\|_{L^p(X)}^p : \rho \text{ is non-negative, Borel measurable, } \int_{\gamma} \rho \, ds \geq 1 \text{ for all } \gamma \in \Gamma_0 \right\} = 0.
\]

We denote \(N^{1,p}(X)\) the collection of functions \(u \in L^p(X)\) that have a \(p\)-weak upper gradient \(g \in L^p(X)\), and define \(\|u\|_{N^{1,p}(X)} = \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)}\), where \(g\) is taken over all \(p\)-weak upper gradients of \(u\). We denote by \(N^{1,p}_{\text{loc}}(X)\) the class of functions \(u \in L^p_{\text{loc}}(X)\) that have a \(p\)-weak upper gradient that belongs to \(L^p(B)\) for each ball \(B\). If necessary, we denote the spaces \(N^{1,p}(X)\) and \(N^{1,p}_{\text{loc}}(X)\) by \(N^{1,p}(X,d,m)\) and \(N^{1,p}_{\text{loc}}(X,d,m)\) respectively.

We say that \((X,d,m)\) is said to support a \((1,p)\)-Poincaré inequality with \(p \in [1, \infty)\) if there exists constants \(K \geq 1, C > 0\) such that for all \(u \in \text{Lip}(X), x \in X\) and \(r > 0\),

\[
\int_{B(x,r)} |u - u_{B(x,r)}| \, dm \leq Cr \left[ \int_{B(x,Kr)} (\text{Lip}(u))^p \, dm \right]^{1/p},
\]

where \(\int_A f \, dm\) denotes \(\frac{1}{m(A)} \int_A f \, dm\) and \(u_{B(x,r)} = \frac{1}{m(B(x,r))} \int_{B(x,r)} u \, dm\). It is known that \((X,d,m)\) supports a \((1,p)\)-Poincaré inequality if and only if there exists constants \(K \geq 1, C > 0\) such that for every function \(u\) that is integrable on balls and for any upper gradient \(g\) of \(u\) in \(X, x \in X\) and \(r > 0\),

\[
\int_{B(x,r)} |u - u_{B(x,r)}| \, dm \leq Cr \left[ \int_{B(x,Kr)} g^p \, dm \right]^{1/p},
\]

where \(u_{B(x,r)}\) is as above [HKST, Theorem 8.4.2].

We need the following self-improvement of Poincaré inequality.

**Proposition 5.3.** Let \((X,d,m,\mathcal{E},\mathcal{F})\) be an MMD space that satisfies PHI(2). Then we have

(a) (Cf. [KZ12, Theorem 2.2]) \(\mathcal{F} = N^{1,2}(X)\) with equivalent norms, \(\text{Lip}(X) \cap C_c(X)\) is dense in \(\mathcal{F}\) and \(\mathcal{F}_{\text{loc}} = N^{1,2}_{\text{loc}}(X)\).

\(^4\)this notion is called *very weak gradient* in [HK]. Our terminology is borrowed from [HKST].
(b) (Cf. [KZ08, Theorem 1.0.1]) \((X,d,m)\) satisfies \((1,p)\)-Poincaré inequality for some \(p \in [1,2)\).

**Proof.** (i) Let \(d_{\text{int}}\) denote the intrinsic metric corresponding to the MMD space \(\text{PHI}(2)\). Since \(\text{VD}\) and \(\Pi(2)\) are preserved under a bi-Lipschitz change of the metric (cf. [HKST, Lemma 8.3.18]), by Proposition 2.10(a), the MMD space \((X,d_{\text{int}},m,\mathcal{E},\mathcal{F})\) also satisfies \(\text{VD}\) and \(\Pi(2)\). Therefore by [KZ12, Theorem 2.2], \(\mathcal{F} = \mathcal{N}_{1,2}^{1,2}(X,d_{\text{int}},m)\) with equivalent norms and \(\text{Lip}_{d_{\text{int}}}(X) \cap C_c(X)\) is dense in \(\mathcal{F}\) and \(\mathcal{F}_{\text{loc}} = \mathcal{N}_{1,2}^{1,2}(X,d_{\text{int}},m)\). Since \(d\) and \(d^\prime\) are bi-Lipschitz equivalent, \(\text{Lip}_{d_{\text{int}}}(X) = \text{Lip}_{d}(X)\) and \(\text{Lip}_d(u)\) is comparable to \(\text{Lip}_{d_{\text{int}}}(u)\); that is there exists \(C > 0\) such that

\[
C^{-1} \text{Lip}_d(u)(x) \leq \text{Lip}_{d_{\text{int}}}(u)(x) \leq C \text{Lip}_d(u)(x) \quad \text{for all } x \in X, u \in \text{Lip}_d(X).
\]

(ii) By (i), [KZ08, Proposition 2.1] and [HKST, Lemma 8.3.18], \((X,d,m)\) satisfies the \((1,2)\)-Poincaré inequality. By the self-improving property of [KZ08, Theorem 1.0.1], \((X,d,m)\) satisfies \((1,p)\)-Poincaré inequality for some \(p \in [1,2)\). \(\square\)

### 5.2 \(A_\infty\)-weights and the Gaussian uniformization problem

**Definition 5.4** \((A_\infty\) relation). Let \((X,d,m)\) be a complete metric measure space such that \(m\) is a doubling measure. Let \(m'\) be another doubling Borel measure on \(X\). Then \(m'\) is said to be \(A_\infty\)-related to \(m\) if for each \(\varepsilon > 0\) there exists \(\delta > 0\) such that

\[
m(E) < \delta m(B) \quad \text{implies} \quad m'(E) < \varepsilon m'(B)
\]

whenever \(E\) is a measurable subset of a ball \(B\). Evidently, if \(m'\) is \(A_\infty\)-related to \(m\), then \(m'\) is absolutely continuous with respect to \(m\), so that \(dm' = w dm\) for some nonnegative locally integrable weight function \(w\). It turns out that being \(A_\infty\)-related is a symmetric relation among doubling measures; that is, if \(m'\) is \(A_\infty\)-related to \(m\), then \(m\) is \(A_\infty\)-related to \(m'\) [ST, Chapter I].

Consider the following reverse Hölder inequality: There is a locally \(m\)-integrable function \(w\) in \(X\) together with constants \(C \geq 1\) and \(p > 1\) such that \(dm' = w dm\) and

\[
\left( \int_B w^p dm \right)^{1/p} \leq C \int_B w dm \quad (5.5)
\]

whenever \(B\) is a ball in \((X,d)\). It is well known that a doubling measure \(m'\) is \(A_\infty\)-related to \(m\) if and only if the reverse Hölder inequality (5.5) is satisfied.

The \(A_\infty\) relation among doubling measures is preserved under quasisymmetric change of metric as we show below.

**Lemma 5.5.** Let \(d_1, d_2\) two quasisymmetric metrics on \(X\) such that the metrics \(d_1, d_2\) are uniformly perfect. Let \(m_1, m_2\) be doubling Borel measures with respect to \(d_1\) such that \(m_1\) and \(m_2\) are \(A_\infty\) with respect to the metric \(d_1\). Then \(m_1\) and \(m_2\) are \(A_\infty\) related with respect to \(d_2\).
Proof. Let $B_i(x, r)$ denote the open ball in metric $d_i$ for $i = 1, 2$. By [MT10, Lemma 1.2.18], there exists $C > 0$ such that the following holds: for each $x \in X, r > 0$ and $i \in \{1, 2\}$, there exists $s > 0$ such that

$$B_{3-i}(x, C^{-1}s) \subseteq B_i(x, r) \subseteq B_{3-i}(x, Cs), \text{ for all } x \in X, r > 0. \tag{5.6}$$

Note that since $m_1$ and $m_2$ are doubling with respect to $d_1$, they are also doubling on $(X, d_2)$. Therefore, there exists $C_1 > 0$ such that

$$\frac{m_i(B_j(x, Cr))}{m_i(B_j(x, r))} \leq C_1, \text{ for all } x \in X, r > 0, \text{ and } i, j \in \{1, 2\}. \tag{5.7}$$

Since $m_1$ and $m_2$ are $A_{\infty}$-related in $(X, d_1)$, we have $m_2 \ll m_1, dm_2 = w dm_1$, where $w \geq 0$ is a Borel measurable function that satisfies the following reverse Hölder inequality: there exists $C_R \geq 1, p > 1$ such that

$$\left( \int_{B_1(x, r)} w^p dm_1 \right)^{1/p} \leq C_R \int_{B_1(x, r)} w dm_1, \text{ for all } x \in X, r > 0. \tag{5.8}$$

For all $x \in X, r > 0$, we estimate

$$\left( \int_{B_2(x, r)} w^p dm_1 \right)^{1/p} \leq \left( \int_{B_1(x,Cs)} w^p dm_1 \right)^{1/p} \left( \frac{m_1(B_1(x, Cs))}{m_1(B_1(x, C^{-1}s))} \right)^{1/p} \leq C_R C_1^{2/p} \int_{B_1(x,Cs)} w dm_1 \tag{by (5.6)}$$

$$= C_R C_1^{2/p} \frac{m_2(B_1(x, Cs))}{m_1(B_1(x, Cs))} \tag{by dm_2 = w dm_1}$$

$$\leq C_R C_1^{2/p} \frac{m_2(B_1(x, C^{-1}s))}{m_1(B_1(x, C^{-1}s))} \tag{by (5.6)}$$

$$\leq C_R C_1^{2/p} \frac{m_2(B_2(x, r))}{m_1(B_2(x, r))} \tag{by (5.7)}$$

$$\leq C_R C_1^{2/p} \int_{B_2(x, r)} w dm_1, \tag{by (5.6)}$$

$$\leq C_R C_1^{2/p} \int_{B_2(x, r)} w dm_1, \tag{since dm_2 = w dm_1}$$

\[ \square \]

Let $f : (X_1, d_1) \to (X_2, d_2)$ be a homeomorphism between two metric spaces. For all $x \in X, r > 0$, we define

$$L_f(x, r) = \sup \{ d_2(f(x), f(y)) : d_1(x, y) \leq r \}, \quad L_f(x) = \lim_{r \to 0} \frac{L_f(x, r)}{r}. \tag{5.9}$$

For $\varepsilon > 0$, we define

$$L^*_f(x) = \sup_{0 < r \leq \varepsilon} \frac{L_f(x, r)}{r}. \tag{5.10}$$
Clearly, $L_f$ decreases as $\varepsilon$ decreases and
\[
\lim_{\varepsilon \downarrow 0} L_f^\varepsilon(x) = L_f(x), \quad \text{for all } x \in X.
\]

**Lemma 5.6** ([HK, Lemma 7.16]). Let $f : (X_1, d_1) \to (X_2, d_2)$ be a $\eta$-quasisymmetry. Let $x_0 \in X$ and $0 < R < \text{diam}(X_1,d_1)$. There is a constant $C$ (that depends only on $\eta$) such that for all $\varepsilon > 0$, the function $CL_f^\varepsilon$ is an upper gradient of the function $u(x) = d_2(f(x), f(x_0))$ in $B(x_0, R)$.

We introduce the notion of $C$-approximation to compare balls in different metrics.

**Definition 5.7.** Let $d_1$ and $d_2$ be two metrics on $X$ such that the identity map $\text{Id} : (X, d_1) \to (X, d_2)$ is a $\eta$-quasisymmetry. Let $C \geq 1$ be a constant. We say that a ball $B_{d_2}(x_2, r_2)$ is a $C$-approximation of $B_{d_1}(x_1, r_1)$ if
\[
d_1(x_1, x_2) \leq Cr_1, \quad d_2(x_1, x_2) \leq Cr_2
\]
\[
B_{d_1}(x_2, C^{-1}r_1) \subset B_{d_2}(x_2, r_2) \subset B_{d_1}(x_2, Cr_1),
\]
\[
B_{d_2}(x_1, C^{-1}r_2) \subset B_{d_1}(x_1, r_1) \subset B_{d_2}(x_1, Cr_2)
\]

By the same argument as Proposition 4.12, we obtain the following comparison of capacities.

**Lemma 5.8.** Let $(X, d_i, m_i, \mathcal{E}, \mathcal{F}_i), i = 1, 2$, be two geodesic, MMD spaces that satisfy EHI such that the identity map $\text{Id} : (X, d_1) \to (X, d_2)$ is a quasisymmetry. Let $B_i(x, r)$ denote a open ball of radius $r$ and center $x$, for $i = 1, 2$. Let $C_1 \geq 1$ and $A_1, A_2 > 1$. There exists $C_2, A_3 > 1$ such that
\[
\text{Cap}(B_1(x_1, r_1), B_1(x_1, A_1r_1)) \leq C_2 \text{Cap}(B_2(x_2, r_2), B_2(x_2, A_2r_2))
\]
for all balls $B_1(x_1, r_1)$ and $B_2(x_2, r_2)$ such that $r_1 < \text{diam}(X, d_1)/A_3, r_2 < \text{diam}(X, d_2)/A_3$ and that $B_1(x_1, r_1)$ is a $C_1$-approximation of $B_2(x_2, r_2)$.

The following is an analogue of [HK, Lemma 7.19]

**Lemma 5.9.** Let $(X, d_i, m_i, \mathcal{E}, \mathcal{F}_e \cap L^2(m_i)), i = 1, 2$ be two MMD spaces that satisfy PHI(2). Furthermore, these MMD spaces are time changes of each other with full quasi-support. Let the identity map $f : (X, d_1) \to (X, d_2)$ be a $\eta$-quasisymmetry. Then the function $L_f^\varepsilon$ defined in (5.10) is in weak $L^2$ for any $\varepsilon < R/10$ and for any ball $B_{d_1}(x_0, R), R < \text{diam}(X, d_1)$. Furthermore, there exists $C \geq 1$ such that $L_f^\varepsilon$ satisfies the estimate
\[
m_1 \left( \left\{ x \in B_{d_1}(x_0, R) : L_f^\varepsilon(x) > t \right\} \right) \leq C t^{-2} m_2 \left( B_{d_1}(x_0, R) \right), \quad (5.11)
\]
for all $t > 0, 0 < R < \text{diam}(X, d_1), x_0 \in X$. Here $C \geq 1$ depends only on $\eta$ and the constants associated with the MMD spaces $(X, d_i, m_i, \mathcal{E}, \mathcal{F}_e \cap L^2(m_i)), i = 1, 2$. 

64
Proof. Let \( E_t \) denote the set
\[
E_t := \{ x \in B_{d_1}(x_0, R) : L_f^t(x) > t \}.
\]
Then by the 5B-covering lemma [Hei, Theorem 1.2] there exists a countable collection of disjoint balls \( B_i = B_{d_1}(x_i, r_i), i \in I \) such that \( 0 < r_i \leq \varepsilon \),
\[
\frac{L_f(x_i, r_i)}{r_i} > t
\]
and
\[
E_t \subset \bigcup_i 5B_i \subset 2B.
\]
Note that the metrics \( d_1, d_2 \) are uniformly perfect by Proposition 2.10(a). Define
\[
B'_i := B_{d_2}(x_i, L_f(x_i, r_i)/\eta(1)).
\]
Roughly speaking, the balls \( B'_i \) in \( d_2 \)-metric approximate the balls \( B_i \) in the \( d_1 \)-metric for each \( i \in I \). More precisely, since \( f \) is a \( \eta \)-quasisymmetry and \( d_1, d_2 \) are uniformly perfect, there exists \( C \geq 1 \) such that \( B_i \) is a \( C \)-approximation of \( B'_i \) for all \( i \in I \). In particular,
\[
C^{-1}B'_i \subset B_i \subset CB'_i, \quad C^{-1}B_i \subset B'_i \subset CB_i, \quad \text{for all } i \in I. \tag{5.13}
\]
Since \( (X, d_i, m_2, \mathcal{E}, \mathcal{F}_e \cap L^2(m_i)), i = 1, 2 \) satisfies PHI(2), there exists \( A_1, A_2 > 1 \) such that
\[
\text{Cap}(B_i, (A_1B_i)^c) \asymp \frac{r_i^2}{m_1(B_i)}, \quad \text{Cap}(B'_i, (A_2B'_i)^c) \asymp \frac{L_f(x_i, r_i)^2}{m_2(B'_i)} \quad \text{for all } i \in I. \tag{5.14}
\]
Furthermore, since by (5.13) and Lemma 5.8, we have
\[
\text{Cap}(B_i, (A_1B_i)^c) \asymp \text{Cap}(B'_i, (A_2B'_i)^c) \quad \text{for all } i \in I. \tag{5.15}
\]
We combine the above estimates, to obtain (5.11) as follows:
\[
m_1(E_t) \leq \sum_i m_1(5B_i) \lesssim \sum_i m(B_i) \quad \text{(since } m_1 \text{ is a doubling )}
\]
\[
\lesssim \sum_i r_i^2 \text{Cap}(B_i, (A_1B_i)^c) \quad \text{(by (5.14))}
\]
\[
\lesssim t^{-2} \sum_i L_f(x_i, r_i)^2 \text{Cap}(B_i, (A_1B_i)^c) \quad \text{(by (5.12))}
\]
\[
\lesssim t^{-2} \sum_i L_f(x_i, r_i)^2 \text{Cap}(B'_i, (A_2B'_i)^c) \quad \text{(by (5.15))}
\]
\[
\lesssim t^{-2} \sum_i m_2(B'_i) \quad \text{(by (5.14))}
\]
\[
\lesssim t^{-2} \sum_i m_2(C^{-1}B'_i) \quad \text{(by (VD) for } (X, d_2, m_2))
\]
\[
\lesssim t^{-2} \sum_i m_2(B_i) \quad \text{(since } C^{-1}B'_i \subset B_i \text{)}
\]
\[
\lesssim t^{-2} m_2(B_{d_1}(x_0, 2R)) \quad \text{(since } B_i \text{'s are disjoint and } \bigcup_i B_i \subset B_{d_1}(x_0, 2R)\text{)}
\]
\[
\lesssim t^{-2} m_2(B_{d_1}(x_0, R)) \quad \text{(since } (X, d_1, m_2) \text{ is doubling).}
\]

65
Corollary 5.10 ([HK, Corollary 7.21]). Let \((X,d_i,m_i,E,\mathcal{F} \cap L^2(m_i)), i = 1,2\) be two MMD spaces that satisfy PHI(2). Furthermore, these MMD spaces are time changes of each other. Let the identity map \(f : (X,d_1) \rightarrow (X,d_2)\) be a \(\eta\)-quasisymmetry. Then the function \(L_f^s\) defined in (5.10). Let \(L_f^s\) denote the function defined in (5.10). For all \(s \in [1,2]\) and \(x_0 \in X, 0 < \varepsilon < R/10, R < \text{diam}(X,d_1)\), the function \(L_f^s\) is in \(L^s(B_{d_1}(x_0,R),m_1)\) with

\[
\left(\int_{B_{d_1}(x_0,R)} |L_f^s|^s \, dm_1\right)^{1/s} \leq Cm_1(B(x_0,R))^{(2-s)/(2s)}m_2(B(x_0,R))^{1/s},
\]

where \(C\) only depends only on \(s,\eta\) and the constants associated with the two MMD spaces. By letting \(\varepsilon \downarrow 0\), a similar statement is true for \(L_f\).

Proof of Theorem 2.11. By Proposition 2.10(b), both \(m_1\) and \(m_2\) are minimal energy dominant measures. Therefore, \(m_1\) and \(m_2\) are mutually absolute continuous. By Proposition 2.10(a), both \(d_1\) and \(d_2\) are bi-Lipschitz equivalent to intrinsic metrics, and therefore by Lemma 5.5, we may assume that \(d_1\) and \(d_2\) are intrinsic metrics with respect to the symmetric measures \(m_1\) and \(m_2\) respectively.

Let \(f : (X,d_1) \rightarrow (X,d_2)\) denote the identity map that is a \(\eta\)-quasisymmetry. Then by the Lebesgue-Radon-Nikodym theorem, the volume derivative

\[
\mu_f(x) = \lim_{r \downarrow 0} \frac{m_2(B_{d_1}(x,r))}{m_1(B_{d_1}(x,r))} \tag{5.16}
\]

exists and is finite for \(m_1\)-almost every \(x \in X\). Since \(m_2 \ll m_1\), we have \(dm_2 = \mu_f dm_1\); that is \(m_2(E) = \int_E \mu_f \, dm_1\) for all measurable sets \(E\).

Since \((X,d_i,m_i,E,\mathcal{F} \cap L^2(m_i))\) satisfies PHI(2) for \(i = 1,2\), there exists constants \(A_1,A_2,C_1,C_2\) such that

\[
\text{Cap}(B_{d_i}(x,r),B_{d_i}(x,A_ir)^c) \asymp \frac{r^2}{m_i(B_{d_i}(x,r))}, \text{ for all } x \in X, r < \text{diam}(X,d_i)/C_i, i = 1,2. \tag{5.17}
\]

Similar to (5.13), there exists \(C \geq 1\) such that for all \(r < \text{diam}(X,d_1), x \in X, B_{d_1}(x,r)\) is a \(C\)-approximation of \(B_{d_2}(x,L_f(x,r))\). That is, for all \(r < \text{diam}(X,d_1), x \in X, B_{d_1}(x,r),\)

\[
B_{d_2}(x,C^{-1}L_f(x,r)) \subset B_{d_1}(x,r) \subset B_{d_2}(x,CL_f(x,r)),
B_{d_1}(x,C^{-1}r) \subset B_{d_2}(x,L_f(x,r)) \subset B_{d_1}(x,Cr). \tag{5.18}
\]

By (5.18), \(\eta\)-quasisymmetry of \(f\), and the same argument as Proposition 4.12, there exists \(C_3 > 1\) such that

\[
\text{Cap}(B_{d_1}(x,r),B_{d_1}(x,A_1r)^c) \asymp \text{Cap}(B_{d_2}(x,L_f(x,r)),B_{d_2}(x,A_2L_f(x,r))^c), \tag{5.19}
\]

where the dependence of the constant \(C\) in (5.11) follows from the above argument. \(\square\)
for all \( x \in X, r < \text{diam}(X, d_1)/C_3 \). Combining (5.17) and (5.19), 
\[
\frac{L_f(x,r)^2}{r^2} \leq \frac{m_2(B_{d_1}(x, L_f(x,r)))}{m_1(B_{d_1}(x,r))} \leq \frac{m_2(B_{d_1}(x,r))}{m_1(B_{d_1}(x,r))}, \text{ for all } x \in X, r < \text{diam}(X, d_1).
\]

Therefore
\[
\mu_f(x) \propto L_f(x)^2 \quad \text{for almost every } x \in X.
\]

Let \( p \in [1,2) \) be the constant in Proposition 5.3(b) so that \((X, d_1, m_1)\) satisfies \((1, p)\)-Poincaré inequality. We shall show that \( L_f \) satisfies the reverse Hölder inequality
\[
\left( \frac{1}{B_{d_1}(x_0,r)} \int_{B_{d_1}(x_0,r)} L_f^2 \, dm_1 \right)^{1/2} \leq C \left( \frac{1}{B_{d_1}(x_0,r)} \int_{B_{d_1}(x_0,r)} L_f^p \, dm_1 \right)^{1/p}, \text{ for all } x_0 \in X, r < \text{diam}(X, d_1).
\]

Then by Gehring’s lemma [HK, Lemma 7.3], Hölder inequality and (5.22), we obtain the following reverse Hölder inequality for the function \( \mu_f \): there exists \( \varepsilon > 0 \) such that
\[
\left( \frac{1}{B_{d_1}(x_0,r)} \int_{B_{d_1}(x_0,r)} \mu_f^{1+\varepsilon} \, dm_1 \right)^{1/(1+\varepsilon)} \leq C \int_{B_{d_1}(x_0,r)} \mu_f \, dm_1, \text{ for all } x_0 \in X, r < \text{diam}(X, d_1).
\]

By the equivalence between reverse Hölder inequality and \( A_\infty \) relation as explained in Definition 5.4, it suffices to show (5.22).

Since \((X, d_1, m_1)\) satisfies the \((1, p)\) Poincaré inequality, by Lemma 5.6 \( CL_f^{\varepsilon} \) is an upper gradient of \( u(x) = d_2(x, x_0) \) in \( B_{d_1}(x_0,r) \). Therefore by the Poincaré inequality we have
\[
\int_{B_{d_1}(x_0,K^{-1}r)} \left| u - u_{B_{d_1}(x_0,K^{-1}r)} \right| \, dm_1 \lesssim r \left( \frac{1}{B_{d_1}(x_0,r)} \int_{B_{d_1}(x_0,r)} (L_f^{\varepsilon})^p \, dm_1 \right)^{1/p}.
\]

We let \( \varepsilon \downarrow 0 \) and use Corollary 5.10, and dominated convergence theorem, to obtain
\[
\int_{B_{d_1}(x_0,K^{-1}r)} \left| u - u_{B_{d_1}(x_0,K^{-1}r)} \right| \, dm_1 \lesssim r \left( \frac{1}{B_{d_1}(x_0,r)} \int_{B_{d_1}(x_0,r)} |L_f|^p \, dm_1 \right)^{1/p}.
\]

By the uniform perfectness of \((X, d_1)\) and the volume doubling property, there exists \( K_1 \) such that \( m_1 \left( B(x_0, K^{-1}r) \setminus B(x_0, K_1^{-1}K^{-1}r) \right) \gtrsim m_1(B(x_0, r)) \). Using the quasisymmetry of \( f \), we obtain
\[
u_{B_{d_1}(x_0,K^{-1}r)} = \int_{B_{d_1}(x_0,K^{-1}r)} d_2(x, x_0) \, m_1(dx)
\geq \frac{1}{m_1(B_{d_1}(x_0,r))} \int_{B(x_0,K^{-1}r) \setminus B(x_0, K_1^{-1}K^{-1}r)} d_2(x, x_0) \, m_1(dx)
\gtrsim L_f(x_0, r) \frac{m_1 \left( B(x_0, K^{-1}r) \setminus B(x_0, K_1^{-1}K^{-1}r) \right)}{m_1(B(x_0, r))}
\gtrsim C_1^{-1} L_f(x_0, r),
\]
because
\[ L_f(x_0, r) \lesssim d_2(x, x_0) \]
by quasisymmetry of \( \eta \) and the uniform perfectness of \((X, d_1)\). For sufficiently small \( \delta > 0 \), we similarly have
\[ u(x) = d_2(x, x_0) \leq \eta(\delta K) L_f(x_0, r) \leq (2C_1)^{-1} L_f(x_0, r) \]
for all \( x \in B(x_0, \delta K^{-1} r) \). Consequently, using the above estimates and the volume doubling property we obtain
\[
\int_{B_{d_1}(x_0, K^{-1} r)} |u - u_{B_{d_1}(x_0, K^{-1} r)}| \, dm_1 \gtrsim \int_{B_{d_1}(x_0, \delta K^{-1} r)} |u - u_{B_{d_1}(x_0, K^{-1} r)}| \, dm_1 \gtrsim L_f(x_0, r).
\]
Combining the above estimate
\[
\left( \int_{B_{d_1}(x_0, r)} \frac{L_f^2}{m_2(B_{d_1}(x_0, r))} \, dm_1 \right)^{1/2} \lesssim \left( \int_{B_{d_1}(x_0, r)} \mu_f \, dm_1 \right)^{1/2} \quad \text{(by (5.21))}
\]
\[
\lesssim \left( \frac{m_2(B_{d_1}(x_0, r))}{m_2(B_{d_1}(x_0, r))} \right)^{1/2} \quad \text{(since } dm_2 = \mu_f \, dm_1)\]
\[
\lesssim \frac{L_f(x_0, r)}{r} \quad \text{(by (5.20))}
\]
\[
\lesssim \left( \int_{B_{d_1}(x_0, r)} |L_f|^p \, dm_1 \right)^{1/p} \quad \text{(by (5.24) and (5.23)).}
\]
This completes the proof of (5.22), and therefore Theorem 2.11.

Let \((X, d, m, \mathcal{E}, \mathcal{F})\) be an MMD space that satisfies EHI, where \((X, d)\) is a doubling metric space. If \( \mu \in \mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) \), then by Theorem 2.11
\[
\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) \subseteq \{ \tilde{\mu} : \tilde{\mu} \text{ is } A_\infty\text{-related to } \mu \}.
\]
One might ask if the inclusion in (5.25) is strict. For the Brownian motion on \( \mathbb{R}^n \), the above inclusion is strict if and only if \( n \geq 2 \) (see Theorem 5.17 and Example 5.13). We need the definition of a maximal semi-metric.

**Definition 5.11.** A function \( r : X \times X \to [0, \infty) \) is said to be a *semi-metric*, if it satisfies all the properties of a metric except possibly the property that \( r(x, y) = 0 \) implies \( x = y \).

Let \( h : X \times X \to [0, \infty) \) be an arbitrary function. Then there exists a unique maximal semi-metric \( d_h : X \times X \to [0, \infty) \) such that \( d_h(x, y) \leq h(x, y) \) for all \( x, y \in X \) [BBI, Lemma 3.1.23]. We say that \( d_h \) is the *maximal semi-metric induced by* \( h \). Equivalently, \( d_h \) can be defined as follows. Let \( \tilde{h}(x, y) = \min(h(x, y), h(y, x)) \). Then
\[
d_h(x, y) = \inf \left\{ \sum_{i=0}^{N-1} \tilde{h}(x_i, x_{i+1}) : N \in \mathbb{N}, x_0 = x, x_N = y \right\}.
\]
We provide a necessary condition for a measure to be in \( G(X, d, m, \mathcal{E}, \mathcal{F}) \). Using this necessary condition, below we obtain examples for which the inclusion (5.25) is strict.

**Lemma 5.12.** Let \((X, d, m, \mathcal{E}, \mathcal{F})\) satisfy \( \text{PHI}(\gamma) \) for some \( \gamma \geq 2 \). Let \( \mu \in G(X, d, m, \mathcal{E}, \mathcal{F}) \).

Define

\[
h(x, y) = \sqrt{\frac{\mu(B_d(x, d(x, y)))d(x, y)}{m(B_d(x, d(x, y)))}}, \quad \text{for any } x, y \in X \text{ with } x \neq y,
\]

and \( h(x, x) = 0 \) for any \( x \in X \). Let \( d_h \) denote the maximal semi-metric induced by \( h \). Then there exist \( C > 0 \) such that

\[
h(x, y) \leq C d_h(x, y) \quad \text{for all } x, y \in X.
\]

**Proof.** Let \( \theta \in \mathcal{J}(X, d) \) such that the MMD space \((X, \theta, \mathcal{E}^\mu, \mathcal{F}^\mu)\) satisfies \( \text{PHI}(2) \). It suffices to show the existence of \( C_1 > 0 \) such that

\[
C_1^{-1} \theta(x, y) \leq h(x, y) \leq C_1 \theta(x, y) \quad \text{for all } x, y \in X.
\]

(5.28)

In particular, (5.28) implies a similar inequality as eqref:ns1 with \( h(x, y) \) replaced with \( d_h(x, y) \), which immediately implies that \( h \) is comparable to \( d_h \).

By Theorem 4.5, \( m \) and \( \mu \) satisfy \( \text{VD} \) on \((X, d)\) and \((X, \theta)\). Using Proposition 3.2 and \( \text{VD} \), there exists \( C_2 > 0 \) such that

\[
C_2^{-1} \mu(B_\theta(x, \theta(x, y))) \leq \mu(B_d(x, d(x, y))) \leq C_2 \mu(B_\theta(x, \theta(x, y)))
\]

for all \( x, y \in X \), where \( x \neq y \). By an argument similar to the proof of Proposition 4.12, using Lemma 4.11 and Proposition 3.2, there exists \( C_3, A > 0 \) such that

\[
C_3^{-1} \leq \frac{\text{Cap}(B_\theta(x, \theta(x, y)), B_\theta(x, 2\theta(x, y)))}{\text{Cap}(B_d(x, d(x, y)), B_d(x, 2d(x, y)))} \leq C_3,
\]

(5.30)

for any pair \( x, y \in X \) such that \( 0 < \theta(x, y) < \text{diam}(X, \theta)/A \). By Theorem 4.5, Lemma 4.11 and Proposition 3.2 and increasing \( A \) if necessary, there exist \( C_4 > 0 \) such that

\[
C_4^{-1} \frac{\mu(B_\theta(x, \theta(x, y)))}{\theta(x, y)^2} \leq \text{Cap}(B_\theta(x, \theta(x, y)), B_\theta(x, 2\theta(x, y))) \leq C_4 \frac{\mu(B_\theta(x, \theta(x, y)))}{\theta(x, y)^2},
\]

(5.31)

and

\[
C_4^{-1} \frac{m(B_d(x, d(x, y)))}{d(x, y)^\gamma} \leq \text{Cap}(B_d(x, d(x, y)), B_d(x, 2d(x, y))) \leq C_4 \frac{m(B_d(x, d(x, y)))}{d(x, y)^\gamma},
\]

(5.32)

for any pair \( x, y \in X \) such that \( 0 < \theta(x, y) < \text{diam}(X, \theta)/A \). Combining (5.29), (5.30), (5.31), and (5.32), there exists \( C_5 > 0 \) such that

\[
C_5^{-1} \theta(x, y) \leq h(x, y) \leq C_5 \theta(x, y) \quad \text{for all } x, y \in X \text{ such that } 0 \leq \theta(x, y) < \text{diam}(X, \theta)/A.
\]

(5.33)

Since \((X, \theta)\) is uniformly perfect, by replacing \( y \) with a closer point \( \bar{y} \), and using (5.33), \( \text{VD} \), we obtain (5.28).
**Example 5.13.** Let \( n \geq 2 \) and let \((X, d, m, \mathcal{E}, \mathcal{F})\) denote the Dirichlet form corresponding to Brownian motion on \( \mathbb{R}^n \). If \( w(x) = |x_1|^t \), where \( t \in \mathbb{R}, x = (x_1, \ldots, x_n) \), then \( w \, dm \) is \( A_\infty \)-related to \( m \) if and only if \( t > -1 \) [Sem93, p. 222, Example (c)]. If \( t > 0, x = (0, \ldots, 0), y = (0, \ldots, 0, 1) \) and \( h \) is as given in (5.27) with \( \gamma = 2 \), then using (5.26) it is easy to check that \( d_h(x, y) = 0 \). This can be seen by choosing equally spaced points \( x_0, \ldots, x_n \) on the straight line joining \( x \) and \( y \) and letting \( n \to \infty \) in (5.26). Therefore, by Lemma 5.12 we obtain

\[
wdm \in \{ \tilde{\mu} : \tilde{\mu} \text{ is } A_\infty\text{-related to } \mu \} \setminus \mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}), \quad \text{for any } n \geq 2, t > 0.
\]

In other words, the inclusion in (5.25) is strict for the Brownian motion on \( \mathbb{R}^n, n \geq 2 \).

The above example and Lemma 5.12 illustrate that if a measure is too small in the neighborhood of a curve, then it will fail be in \( \mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) \). As we will see, a similar (but more subtle) phenomenon happens in higher dimensional Sierpiński gaskets.

We recall the definition of strong \( A_\infty \)-weights on \( \mathbb{R}^n \) introduced by David and Semmes in [DS] and show its relevance to the Gaussian uniformization problem for the Brownian motion on \( \mathbb{R}^2 \). The following definition is a slight reformulation of the one in [DS] and follows from [Sem93, Lemma 3.1].

**Definition 5.14.** Let \( d, m \) denote the Euclidean metric and Lebesgue measure on \( \mathbb{R}^n \) respectively. Let \( \mu = w \, dm \) be \( A_\infty \)-related to \( m \). Define

\[
h(x, y) = (\mu(B_{x,y}))^{1/n},
\]

where \( B_{x,y} \) is the Euclidean ball with center \( z = (x + y)/2 \) and radius \( d(x,y)/2 \). Let \( d_h \) denote the maximal metric. We say that \( \mu \) is **strong** \( A_\infty \)-related to \( m \), if there exists \( C > 0 \) such that

\[
d_h(x, y) \geq C^{-1} h(x, y), \quad \text{for all } x, y \in \mathbb{R}^n.
\]

The following relates the Gaussian uniformization problem in \( \mathbb{R}^2 \) in terms of strong \( A_\infty \)-weights.

**Proposition 5.15.** Let \((X, d, m, \mathcal{E}, \mathcal{F})\) denote the MMD space corresponding to the Brownian motion on \( \mathbb{R}^2 \). Then

\[
\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) = \{ \tilde{\mu} : \tilde{\mu} \text{ is strong } A_\infty\text{-related to } \mu \}.
\]  

(5.34)

**Proof.** By Lemma 5.12 and Theorem 2.11, we have the inclusion

\[
\mathcal{G}(X, d, m, \mathcal{E}, \mathcal{F}) \subseteq \{ \tilde{\mu} : \tilde{\mu} \text{ is strong } A_\infty\text{-related to } \mu \}.
\]

(5.35)

For the reverse inclusion, consider a measure \( \mu = w \, dm \) that is \( A_\infty \)-related to \( m \). Since \( \mu \) is mutually absolutely continuous with respect to \( m, \mu \in \mathcal{A}(X, d, m, \mathcal{E}, \mathcal{F}) \).

Let \( \theta \) denote the metric \( d_h \) in Definition 5.14. Since \( \mu \) is \( A_\infty \)-related to \( m \), \( \mu \) satisfies VD on \((X, d)\). Since \( \theta(x, y) \) is comparable to \( \sqrt{\mu(B_{x,y})} \), where \( B_{x,y} = B((x + y)/2, d(x,y)/2) \), by the VD and RVD for the measure \( \mu \), we obtain that

\[
\theta \in \mathcal{J}(X, d).
\]

(5.36)
By (5.36), Proposition 3.2 and VD, there exists $C_1 > 0$ such that
\[ C_1^{-1}r^2 \leq \mu(B_\theta(x,r)) \leq C_1r^2 \quad \text{for all } x \in X, r > 0. \]  
(5.37)

By Lemma 4.11, Proposition 3.2, and (5.36), there exist $C_2 > 0$ such that
\[ C_2^{-1} \leq \operatorname{Cap}(B_\theta(x,r), B_\theta(x,2r)) \leq C_2 \quad \text{for all } x \in X, r > 0. \]  
(5.38)

By [BM, Lemma 5.3], the time-changed MMD space $(X, \theta, \mu, \mathcal{E}, \mathcal{F})$ satisfies EHI. 

Combining (5.36), (5.37), (5.38), and using Theorem 4.5, we obtain that $\mu \in \mathcal{G}(X, d, m, \mathcal{E},\mathcal{F})$. 

Proposition 5.15 along with known results on strong $A_\infty$ related measures lead to many further examples of measures in $\mathcal{G}(X, d, m, \mathcal{E},\mathcal{F})$ for Brownian motion in $\mathbb{R}^2$. For instance, Bessel potentials can be used to construct strong $A_\infty$-measures [BHS, Theorem 3.1].

Unlike the case of Brownian motion on $\mathbb{R}^n$ with $n \geq 2$ treated so far, it turns out that the inclusion in (5.25) is an equality for Brownian motion on $\mathbb{R}$, which we prove in the rest of this subsection as a complete answer to the Gaussian uniformization problem for Brownian motion on $\mathbb{R}$. For this purpose, we need the following lemma characterizing $A_\infty$-related weights in terms of a reverse Hölder inequality.

Lemma 5.16. Let $(X, d)$ be a locally compact metric space and let $m$ be a non-zero Radon measure on $(X, d)$ satisfying VD and such that the function $(0, \infty) \ni r \mapsto m(B(x,r))$ is continuous for each $x \in X$. Then for each $[0, \infty)$-valued $w \in L^1_{\text{loc}}(X, m)$, the following are equivalent:

(a) $\mu = wd\, dm$ is $A_\infty$ related to $m$.

(b) $w \not\equiv 0$ and there exists $C > 1$ such that the following reverse Hölder inequality holds:
\[ \int_B w \, dm \leq C \left( \int_B \sqrt{w} \, dm \right)^2 \quad \text{whenever } B \text{ is a ball in } (X, d). \]

Proof. (b) $\implies$ (a): By Gehring’s lemma [HK, Lemma 7.3], there exist $\varepsilon > 0$ and $C_1 > 0$ such that
\[ \left( \int_B w^{1+\varepsilon} \, dm \right)^{1/(1+\varepsilon)} \leq C_1 \left( \int_B \sqrt{w} \, dm \right)^2 \leq \int_B w \, dm \]
for all balls $B$ in $(X, d)$. By [ST, Theorem 18 in Chapter I], $\mu$ is $A_\infty$-related to $m$.

(a) $\implies$ (b): By [ST, Theorem 18 in Chapter I], there exist $r > 1$ and $C_2 > 1$ such that
\[ \left( \int_B w^r \, dm \right)^{1/r} \leq C_2 \int_B w \, dm \quad \text{for all balls } B. \]  
(5.39)

Choose $\theta \in (0, 1)$ such that $\theta r^{-1} + 2(1 - \theta) = 1$. By Hölder’s inequality and (5.39),
\[ \int_B w \, dm \leq \left( \int_B w^r \, dm \right)^{\theta/r} \left( \int_B \sqrt{w} \, dm \right)^{2(1-\theta)} \leq C_2^\theta \left( \int_B w \, dm \right)^{\theta} \left( \int_B \sqrt{w} \, dm \right)^{2(1-\theta)} \]
for all balls $B$. This immediately implies (b) with $C = C_2^{\theta/(1-\theta)}$. 

\[ \Box \]
In the following result, we consider the case of Brownian motion in $\mathbb{R}$; that is, $$(X,d,m,E,F)$$ is given by $X = \mathbb{R}$, $d$ is the Euclidean distance, $m$ is the Lebesgue measure, $F = W^{1,2}$ and $\mathcal{E}(f,f) = \int |f|^2 \, dm$.

**Theorem 5.17.** Let $(X,d,m,E,F)$ denote the MMD space corresponding to the Brownian motion on $\mathbb{R}$. Then the family of Gaussian admissible measures is characterized by the reverse Hölder inequality as in Lemma 5.16-(b), i.e.,

$$
\mathcal{G}(X,d,m,E,F) = \{ \mu \mid \mu \text{ is } A_{\infty} \text{ related to } m \} = \{ w \, dm \mid w \in L^1_{\text{loc}}(X,m), w \text{ is } [0,\infty)-\text{valued and satisfies Lemma 5.16-(b)} \}.
$$

**Proof.** Set

$$
\tilde{\mathcal{G}} = \left\{ g \, dm \mid \text{there exists } C \geq 1 \text{ such that for all } a < b, \text{ we have} \right\}
$$

$$(b - a)^{1/2} \left( \int_a^b g \, dm \right)^{1/2} \leq C \int_a^b \sqrt{g} \, dm \text{ and } g \not\equiv 0. \right\}.
$$

By Theorem 2.11 and Lemma 5.16, it suffices to show that $\tilde{\mathcal{G}} \subset \mathcal{G}(X,d,m,E,F)$. Let $g \, dm \in \tilde{\mathcal{G}}$. Then consider the measures $\mu_1 = \sqrt{g} \, dm$ and $\mu = g \, dm$. For all $a < b$, an easy calculation shows that

$$
d_{\text{int}}^\mu(a,b) = \mu_1([a,b]) = \int_a^b \sqrt{g} \, dm. \quad (5.40)
$$

Since $\sqrt{g}$ satisfies a reverse Hölder inequality, by [ST, Lemma 12 and Theorem 17 in Chapter I], $\mu_1$ is a doubling measure on $(X,d)$. By the correspondence between doubling measures and quasisymmetric maps on $\mathbb{R}$ described in [Hei, Remark 13.20 (b)] and (5.40), we have

$$
d_{\text{int}}^\mu \in \mathcal{J}(X,d). \quad (5.41)
$$

By the reverse Hölder inequality assumption on $\sqrt{g}$, we have

$$
\mu_1([a,b]) \leq (b - a)^{1/2} (\mu([a,b]))^{1/2} \leq C \mu_1([a,b]).
$$

Since $\mu_1$ is a doubling measure on $(X,d)$, the above estimate shows that $\mu$ is also a doubling measure on $(X,d)$. Since $d_{\text{int}}^\mu \in \mathcal{J}(X,d)$, $\mu$ is a doubling measure in $(X,d_{\text{int}}^\mu)$.

By [ST, Theorem 18 of Chapter I] and Lemma 5.16, the measures $\mu$ and $m$ are mutually absolutely continuous. This implies that $\mu \in \mathcal{A}(X,d,m,E,F)$.

By [CW, Theorem 1.4], for any interval $I = [a,b]$ and for all $f \in W^{1,2}$, we have a Poincaré inequality

$$
\int_I \left( f(x) - \frac{1}{\mu(I)} \int_I f \, d\mu \right)^2 \mu(dx) \leq K_{\mu,I}^2 \int_I |f'(x)|^2 \, dm(x),
$$

where the optimal constant $K_{\mu,I}$ is satisfies the two-sided estimate

$$
K_{\mu,I} \asymp \frac{1}{\mu(I)} \left\{ \sup_{a < x < b} \left\{ \mu([x,b])^{1/2} \left( \int_a^x \mu([a,t])^2 \, dt \right)^{1/2} \right\} \right\}
$$

$$
+ \sup_{a < x < b} \left\{ \mu([a,x])^{1/2} \left( \int_x^b \mu([t,b])^2 \, dt \right)^{1/2} \right\} \quad (5.42)
$$

72
By using the bound $\mu(A) \leq \mu(I)$ for all $A \subset I$, (5.42) and reverse Hölder inequality, we have

$$K_{\mu,I}^2 \lesssim \mu(I)m(I) \lesssim \mu_1(I)^2.$$  \hfill (5.43)

By (5.40), $\mu_1(I)$ is the diameter of $I$ under the intrinsic metric $d_{\text{int}}^\mu$. Therefore by (5.42) and (5.43), we have the Poincaré inequality PI(2) for the MMD space $(X,d_{\text{int}}^\mu,\mu,\mathcal{E}^\mu,\mathcal{F}^\mu)$. Since $\mu$ is a doubling measure in $(X,d_{\text{int}}^\mu)$, we have that $\mu \in \mathcal{G}(X,d,m,\mathcal{E},\mathcal{F})$.

**Remark 5.18.** A major obstruction to determine the Gaussian admissible measures for multidimensional Brownian motion in $\mathbb{R}^n$, $n \geq 2$ is that the intrinsic metric with respect to $\mu = gdm$ does not admit a simple description unlike the one dimensional case where there is a simple formula (5.40). As noted in Example 5.13, the conclusion of Theorem 5.17 fails in higher dimensions.

## 6 The attainment problem for self-similar sets

In this section, we study the attainment problem, that of whether the infimum in (1.4) defining the conformal walk dimension $d_{cw} = 2$ is attained, in the case of a self-similar Dirichlet form $(\mathcal{E},\mathcal{F})$ on a post-critically finite self-similar set $K$. After introducing the framework of such a Dirichlet form in Subsection 6.1, we prove in Subsection 6.2 that $d_{cw} = 2$ is attained (if and) only if $(K,\theta_h,\Gamma(h,h),\mathcal{E},\mathcal{F})$ satisfies PHI(2) for some harmonic function $h \in \mathcal{F}$ and a metric $\theta_h$ on $K$ quasisymmetric to the resistance metric $R_\mathcal{E}$ of $(\mathcal{E},\mathcal{F})$, where $\Gamma(h,h)$ denotes the $\mathcal{E}$-energy measure of $h$. Then in Subsection 6.3 we present several examples, all of which are shown NOT to attain $d_{cw} = 2$ except for the two-dimensional standard Sierpiński gasket, which is known to attain $d_{cw} = 2$ by the results in [Kig08, Kaj12] as discussed in Theorem 6.30 and its proof below.

The restriction of the framework to post-critically finite self-similar sets is mainly for the sake of simplicity. In fact, all the results in Subsection 6.2 can be verified, with just slight modifications in the proofs, also for the canonical self-similar Dirichlet form on any generalized Sierpiński carpet, which forms essentially the only class of examples of infinitely ramified self-similar fractals where the theory of a canonical self-similar Dirichlet form has been established. We treat the case of generalized Sierpiński carpets in Subsection 6.4 and explain what changes are needed for the arguments in Subsections 6.1 and 6.2 to go through in this case.

### 6.1 Preliminaries

In this subsection, we first introduce our framework of a post-critically finite self-similar set and a self-similar Dirichlet form on it, for which we mainly follow the presentation of [Kaj14, Section 3], and then present preliminary facts.

Let us start with the standard notions concerning self-similar sets. We refer to [Kig01, Chapter 1] for details. Throughout this subsection, we fix a compact metrizable topological space $K$, a finite set $S$ with $\#S \geq 2$ and a continuous injective map $F_i : K \to K$ for each $i \in S$. We set $\mathcal{L} := (K,S,\{F_i\}_{i \in S})$. 

73
Definition 6.1. (1) Let \( W_0 := \{ \emptyset \} \), where \( \emptyset \) is an element called the empty word, let \( W_n := S^n = \{ w_1 \ldots w_n \mid w_i \in S \text{ for } i \in \{1, \ldots, n\} \} \) for \( n \in \mathbb{N} \) and let \( W_* := \bigcup_{n \in \mathbb{N} \cup \{0\}} W_n \). For \( w \in W_* \), the unique \( n \in \mathbb{N} \cup \{0\} \) with \( w \in W_n \) is denoted by \( |w| \) and called the length of \( w \). For \( i \in S \) and \( n \in \mathbb{N} \cup \{0\} \) we write \( i^n := i \ldots i \in W_n \).

(2) We set \( \Sigma := S^\mathbb{N} = \{ \omega_1 \omega_2 \omega_3 \ldots \mid \omega_i \in S \text{ for } i \in \mathbb{N} \} \), which is always equipped with the product topology of the discrete topology on \( S \), and define the shift map \( \sigma : \Sigma \to \Sigma \) by \( \sigma(\omega_1 \omega_2 \omega_3 \ldots) := \omega_2 \omega_3 \omega_4 \ldots \). For \( i \in S \) we define \( \sigma_i : \Sigma \to \Sigma \) by \( \sigma_i(\omega_1 \omega_2 \omega_3 \ldots) := i \omega_1 \omega_2 \omega_3 \ldots \). For \( \omega = \omega_1 \omega_2 \omega_3 \ldots \in \Sigma \) and \( n \in \mathbb{N} \cup \{0\} \), we write \( [\omega]_n := \omega_1 \ldots \omega_n \in W_n \).

(3) For \( w = w_1 \ldots w_n \in W_* \), we set \( F_w := F_{w_1} \circ \ldots \circ F_{w_n} \) \((F_\emptyset := \text{id}_K)\), \( K_w := F_w(K) \), \( \sigma_w := \sigma_{w_1} \circ \ldots \circ \sigma_{w_n} \) \((\sigma_\emptyset := \text{id}_\Sigma)\) and \( \Sigma_w := \sigma_w(\Sigma) \), and if \( w \neq \emptyset \) then \( w^\infty \in \Sigma \) is defined by \( w^\infty := w w w \ldots \) in the natural manner.

Definition 6.2. \( \mathcal{L} = (K,S,\{ F_i \}_{i \in S}) \) is called a self-similar structure if and only if there exists a continuous surjective map \( \pi : \Sigma \to K \) such that \( F_i \circ \pi = \pi \circ \sigma_i \) for any \( i \in S \).

Note that such \( \pi \), if it exists, is unique and satisfies \( \{ \pi(\omega) \} = \bigcap_{n \in \mathbb{N}} K_{[\omega]_n} \) for any \( \omega \in \Sigma \).

In the rest of this subsection we always assume that \( \mathcal{L} \) is a self-similar structure.

Definition 6.3. (1) We define the critical set \( \mathcal{C}_\mathcal{L} \) and the post-critical set \( \mathcal{P}_\mathcal{L} \) of \( \mathcal{L} \) by
\[
\mathcal{C}_\mathcal{L} := \pi^{-1}(\bigcup_{i,j \in S, i \neq j} K_i \cap K_j) \quad \text{and} \quad \mathcal{P}_\mathcal{L} := \bigcup_{n \in \mathbb{N}} \sigma^n(\mathcal{C}_\mathcal{L}). \tag{6.1}
\]

\( \mathcal{L} \) is called post-critically finite, or p.-c.f. for short, if and only if \( \mathcal{P}_\mathcal{L} \) is a finite set.

(2) We set \( V_0 := \pi(\mathcal{P}_\mathcal{L}), V_n := \bigcup_{w \in W_n} F_w(V_0) \) for \( n \in \mathbb{N} \) and \( V_* := \bigcup_{n \in \mathbb{N} \cup \{0\}} V_n \).

\( V_0 \) should be considered as the “boundary” of the self-similar set \( K \); indeed, \( K_w \cap K_v = F_w(V_0) \cap F_v(V_0) \) for any \( w, v \in W_* \) with \( \Sigma_w \cap \Sigma_v = \emptyset \) by [Kig01, Proposition 1.3.5-(2)].

According to [Kig01, Lemma 1.3.11], \( V_{n-1} \subset V_n \) for any \( n \in \mathbb{N} \), and if \( V_0 \neq \emptyset \) then \( V_* \) is dense in \( K \). Also note that by [Kig01, Theorem 1.6.2], \( K \) is connected if and only if for any \( i, j \in S \) there exist \( n \in \mathbb{N} \) and \( \{i_k\}_{k=0}^n \subset S \) with \( i_0 = i \) and \( i_n = j \) such that \( K_{i_{k-1}} \cap K_{i_k} \neq \emptyset \) for any \( k \in \{1, \ldots, n\} \), and if \( K \) is connected then it is arcwise connected.

In the remainder of this subsection our self-similar structure \( \mathcal{L} = (K,S,\{ F_i \}_{i \in S}) \) is always assumed to be post-critically finite with \( K \) connected, so that \( 2 \leq \#V_0 < \infty \), \( K \neq \overline{V_0} = V_0 \) and \( V_* \) is countably infinite and dense in \( K \).

Next we briefly recall the construction and basic properties of a self-similar Dirichlet form on such \( \mathcal{L} \); see [Kig01, Chapter 3] for details. Let \( D = (D_{pq})_{p,q \in V_0} \) be a real symmetric matrix of size \( \#V_0 \) (which we also regard as a linear operator on \( \mathbb{R}^{V_0} \)) such that
\begin{enumerate}[(D1)]
\item \( \{ u \in \mathbb{R}^{V_0} \mid Du = 0 \} = \mathbb{R}1_{V_0} \),
\item \( D_{pq} \geq 0 \) for any \( p,q \in V_0 \) with \( p \neq q \).
\end{enumerate}

We define
\[
\mathcal{E}^{(0)}(u,v) := -\sum_{p,q \in V_0} D_{pq}u(q)v(p) = \frac{1}{2} \sum_{p,q \in V_0} D_{pq}(u(p) - u(q))(v(p) - v(q)), \quad u,v \in \mathbb{R}^{V_0}, \tag{6.2}
\]
so that \((\mathcal{E}^{(0)}, \mathbb{R}^{V_0})\) is a Dirichlet form on \(L^2(V_0, \#)\). Furthermore let \(r = (r_i)_{i \in S} \in (0, \infty)^S\) and define
\[
\mathcal{E}^{(n)}(u, v) := \sum_{w \in W_n} \frac{1}{r_w} \mathcal{E}^{(0)}(u \circ F_w|_{V_0}, v \circ F_w|_{V_0}), \quad u, v \in \mathbb{R}^{V_n}
\] (6.3)
for each \(n \in \mathbb{N}\), where \(r_w := r_{w_1}r_{w_2} \ldots r_{w_n}\) for \(w = w_1w_2 \ldots w_n \in W_n\) \((r_0 := 1)\).

**Definition 6.4.** The pair \((D, r)\) of a real symmetric matrix \(D = (D_{pq})_{p,q \in V_0}\) of size \(\#V_0\) with the properties (D1) and (D2) and \(r = (r_i)_{i \in S} \in (0, \infty)^S\) is called a harmonic structure on \(L\) if and only if \(\mathcal{E}^{(0)}(u, u) = \inf_{v \in \mathbb{R}^{V_1}, v|_{V_0} = u} \mathcal{E}^{(1)}(v, v)\) for any \(u \in \mathbb{R}^{V_0}\); note that then
\[
\mathcal{E}^{(n_1)}(u, u) = \min_{v \in \mathbb{R}^{V_{n_2}}, v|_{V_{n_1}} = u} \mathcal{E}^{(n_2)}(v, v) \quad (6.4)
\]
for any \(n_1, n_2 \in \mathbb{N} \cup \{0\}\) with \(n_1 \leq n_2\) and any \(u \in \mathbb{R}^{V_{n_1}}\) by [Kig01, Proposition 3.1.3]. If \(r \in (0, 1)^S\) in addition, then \((D, r)\) is called regular.

In the rest of this subsection, we assume that \((D, r)\) is a regular harmonic structure on \(L\). In this case, \(\{\mathcal{E}^{(n)}(u|_{V_n}, u|_{V_n})\}_{n \in \mathbb{N} \cup \{0\}}\) is non-decreasing and hence has the limit in \([0, \infty]\) for any \(u \in C(K)\). Then we define a linear subspace \(\mathcal{F}\) of \(C(K)\) and a non-negative definite symmetric bilinear form \(\mathcal{E} : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}\) by
\[
\mathcal{F} := \{u \in C(K) \mid \lim_{n \rightarrow \infty} \mathcal{E}^{(n)}(u|_{V_n}, u|_{V_n}) < \infty\}, \quad (6.5)
\]
\[
\mathcal{E}(u, v) := \lim_{n \rightarrow \infty} \mathcal{E}^{(n)}(u|_{V_n}, v|_{V_n}) \in \mathbb{R}, \quad u, v \in \mathcal{F}, \quad (6.6)
\]
so that \((\mathcal{E}, \mathcal{F})\) is easily seen to possess the following self-similarity properties (note that \(\mathcal{F} \cap C(K) = \mathcal{F}\) in the present setting):
\[
\mathcal{F} \cap C(K) = \{u \in C(K) \mid u \circ F_i \in \mathcal{F} \text{ for any } i \in S\}, \quad (6.7)
\]
\[
\mathcal{E}(u, v) = \sum_{i \in S} \frac{1}{r_i} \mathcal{E}(u \circ F_i, v \circ F_i), \quad u, v \in \mathcal{F} \cap C(K). \quad (6.8)
\]

By [Kig01, Proposition 2.2.4, Lemma 2.2.5, Theorem 2.2.6, Lemma 2.3.9, Theorems 2.3.10 and 3.3.4], \((\mathcal{E}, \mathcal{F})\) is a resistance form on \(K\) and its resistance metric \(R_\mathcal{E} : K \times K \rightarrow [0, \infty)\) is a metric on \(K\) compatible with the original topology of \(K\); here the former means that \((\mathcal{E}, \mathcal{F})\) has the following properties (see [Kig01, Definition 2.3.1] or [Kig12, Definition 3.1]):

- **(RF1)** \(\{u \in \mathcal{F} \mid \mathcal{E}(u, u) = 0\} = \mathbb{R}1_K\).
- **(RF2)** \((\mathcal{F}/\mathbb{R}1_K, \mathcal{E})\) is a Hilbert space.
- **(RF3)** \(\{u|_V \mid u \in \mathcal{F}\} = \mathbb{R}^V\) for any non-empty finite subset \(V\) of \(K\).
- **(RF4)** \(R_\mathcal{E}(x, y) := \sup_{u \in \mathcal{F}\setminus\mathbb{R}1_K} |u(x) - u(y)|^2/\mathcal{E}(u, u) < \infty\) for any \(x, y \in K\).
- **(RF5)** \(u^+ \wedge 1 \in \mathcal{F}\) and \(\mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) \leq \mathcal{E}(u, u)\) for any \(u \in \mathcal{F}\).

See [Kig01, Chapter 2] and [Kig12, Part 1] for further details of resistance forms.

In the present framework, the notion of harmonic functions is defined as follows.
Definition 6.5. Let \( n \in \mathbb{N} \cup \{0\} \). A continuous function \( h \in C(K) \) is called \( \mathcal{E} \)-harmonic on \( K \setminus V_n \), or \( n \)-harmonic for short, if and only if \( h \in \mathcal{F} \) and

\[
\mathcal{E}(h, h) = \inf_{v \in \mathcal{F}, v|_{V_n} = h|_{V_n}} \mathcal{E}(v, v), \quad \text{or equivalently,} \quad \mathcal{E}(h, v) = 0 \text{ for any } v \in \mathcal{F}^{K \setminus V_n}, \quad (6.9)
\]

where \( \mathcal{F}^{K \setminus V_n} := \{ u \in \mathcal{F} \mid u|_{V_n} = 0 \} \). We set \( \mathcal{H}_n := \{ h \in C(K) \mid h \text{ is } n\text{-harmonic} \} \).

It is obvious that \( \mathcal{H}_n \) is a linear subspace of \( \mathcal{F} \) and \( \mathbb{R}1_K \subset \mathcal{H}_n \subset \mathcal{H}_{n+1} \) for any \( n \in \mathbb{N} \cup \{0\} \). Moreover, we easily have the following proposition by [Kig01, Lemma 2.2.2 and Theorem 3.2.4], (6.3), (6.4), (6.7) and (6.8).

Proposition 6.6. Let \( n \in \mathbb{N} \cup \{0\} \).

1. For each \( u \in \mathbb{R}^{V_n} \) there exists a unique \( H_n(u) \in \mathcal{H}_n \) such that \( H_n(u)|_{V_n} = u \). Moreover, \( H_n : \mathbb{R}^{V_n} \to \mathcal{H}_n \) is linear (and hence it is a linear isomorphism).

2. It holds that

\[
\mathcal{H}_n = \{ h \in \mathcal{F} \mid \mathcal{E}(h, h) = \mathcal{E}^{(n)}(h|_{V_n}, h|_{V_n}) \} = \{ h \in C(K) \mid h \circ F_w \in \mathcal{H}_0 \text{ for any } w \in W_n \}. \quad (6.10)
\]

In particular, for each \( w \in W_n \), a linear map \( F_w^* : \mathcal{H}_0 \to \mathcal{H}_0 \) is defined by \( F_w^* h := h \circ F_w \).

Now we equip \( K \) with a measure to turn \( (\mathcal{E}, \mathcal{F}) \) into a Dirichlet form. Indeed, we have the following proposition.

Proposition 6.7. Let \( \mu \) be a Radon measure on \( K \) with full support. Then \( (\mathcal{E}, \mathcal{F}) \) is an irreducible, strongly local, regular symmetric Dirichlet form on \( L^2(K, \mu) \), and its extended Dirichlet space \( \mathcal{F}_e \) coincides with \( \mathcal{F} \). Moreover, the capacity \( \text{Cap}_1^\mu \) associated with \( (K, R_\mathcal{E}, \mu, \mathcal{E}, \mathcal{F}) \) satisfies \( \inf_{x \in K} \text{Cap}_1^\mu(\{x\}) > 0 \), and in particular (recall Definition 2.7)

\[
\mathcal{A}(K, R_\mathcal{E}, \mu, \mathcal{E}, \mathcal{F}) = \{ \nu \mid \nu \text{ is a Radon measure on } K \text{ with full support} \}. \quad (6.12)
\]

Proof. This proposition is well known to experts on Dirichlet forms on fractals, but we include a complete proof of it for the reader’s convenience. \( (\mathcal{E}, \mathcal{F}) \) is a regular symmetric Dirichlet form on \( L^2(K, \mu) \) by [Kig12, Corollary 6.4 and Theorem 9.4], strongly local by the same argument as [Hin05, Proof of Lemma 3.12] on the basis of (6.7), (6.8) and \( \mathcal{E}(1_K, 1_K) = 0 \), and irreducible by (RF1) above and [CF, Theorem 2.1.11]. The equality \( \mathcal{F}_e = \mathcal{F} \) is immediate from (RF1), (RF2) and (RF4). We also easily see from (RF4), \( \text{diam}(K, R_\mathcal{E}) < \infty \) and \( \mu(K) < \infty \) that \( \inf_{x \in K} \text{Cap}_1^\mu(\{x\}) > 0 \), so that a subset of \( K \) is quasi-closed with respect to \( (K, R_\mathcal{E}, \mu, \mathcal{E}, \mathcal{F}) \) if and only if it is closed in \( K \). In particular, any Radon measure \( \nu \) on \( K \) is smooth with respect to \( (K, R_\mathcal{E}, \mu, \mathcal{E}, \mathcal{F}) \) and the only quasi-support of \( \nu \) with respect to \( (K, R_\mathcal{E}, \mu, \mathcal{E}, \mathcal{F}) \) is the support of \( \nu \) in \( K \), which together imply (6.12). \( \square \)
Let $d_H \in (0, \infty)$ be such that $\sum_{i \in S} r_i^{d_H} = 1$, so that $d_H \geq 1$ since

$$\max_{x, y \in V_0} R_{\xi}(x, y) \leq \sum_{i \in S} \max_{x, y \in V_0} R_{\xi}(F_i(x), F_i(y)) \leq \left( \sum_{i \in S} r_i \right) \max_{x, y \in V_0} R_{\xi}(x, y)$$

(6.13)

by the connectedness of $K$, [Kig01, Theorem 1.6.2 and Lemma 3.3.5] and hence $\sum_{i \in S} r_i \geq 1 = \sum_{i \in S} r_i^{d_H}$. Let $m$ be the self-similar measure on $L$ with weight $(r_i^{d_H})_{i \in S}$, i.e., the unique Borel measure on $K$ such that $m(K_w) = r_w^{d_H}$ for any $w \in W_*$. The measure $m$ could be considered as the “uniform distribution” on $K$, and it is the most typical choice of the reference measure $\mu$ for $(E, F)$. It is well known that $(K, R_{\xi}, m, E, F)$ satisfies PHI($d_H+1$); more precisely, the following lemma and proposition hold.

**Lemma 6.8.** There exist $c_1, c_2 \in (0, \infty)$ such that for any $(x, s) \in K \times (0, \text{diam}(K, R_{\xi})]$,

$$c_1 s^{d_H} \leq m(B_{R_{\xi}}(x, s)) \leq c_2 s^{d_H}.$$  

(6.14)

**Proof.** This is immediate from Lemma 6.14 below and [Kig01, Lemma 4.2.3].

**Proposition 6.9.** $(K, R_{\xi}, m, E, F)$ satisfies PHI($d_H+1$).

**Proof.** Lemma 6.8 and [Kig12, Theorem 15.10] together imply that $(K, R_{\xi}, m, E, F)$ satisfies HKE($d_H+1$) as well as VD and RVD, and therefore it also satisfies PHI($d_H+1$) by Theorem 4.5.

We conclude this subsection with the following Proposition, which is essentially due to Kigami [Kig09, Kig12] and gives a simple equivalent condition for the validity of PHI($\beta$) after quasisymmetric change of the metric and time change.

**Proposition 6.10.** Let $\theta \in J(K, R_{\xi})$ (recall (1.2)), let $\mu$ be a Radon measure on $K$ with full support and let $\beta \in (1, \infty)$. Then the following conditions are equivalent:

(a) $(K, \theta, \mu, E, F)$ satisfies PHI($\beta$).

(b) There exists $C \in (1, \infty)$ such that

$$C^{-1}(\text{diam}(K_w, \theta))^\beta \leq r_w \mu(K_w) \leq C(\text{diam}(K_w, \theta))^\beta \quad \text{for any } w \in W_*.$$ 

(6.15)

Moreover, if either of these conditions holds, then $\mu(F_w(V_0)) = 0$ for any $w \in W_*$ and $\mu(\{x\}) = 0$ for any $x \in K$.

The rest of this subsection is devoted to the proof of Proposition 6.10, which requires the following lemmas and definitions.

**Lemma 6.11** ([Kig01, Lemma 3.3.5], [Kig04, Theorem A.1]). There exists $c_{R_{\xi}} \in (0, 1]$ such that for any $w \in W_*$ and any $x, y \in K$,

$$c_{R_{\xi}} r_w R_{\xi}(x, y) \leq R_{\xi}(F_w(x), F_w(y)) \leq r_w R_{\xi}(x, y).$$

(6.16)
Lemma 6.14. There exist \( \{ x \in K \} \) for the reader’s convenience. By the upper inequality in (6.16) we have \( \text{diam}(x, y) \leq r_{w_1} \text{diam}(K, R_\varepsilon) \) for any \( w \in W \), which implies the latter inclusion in (6.18) with \( \alpha_2 \in (2 \text{diam}(K, R_\varepsilon), \infty) \) arbitrary. On the other hand, by [Kig01, Proof of Lemma 4.2.4] there exists \( \alpha_1 \in (0, \infty) \) such that \( R_\varepsilon(x, y) \geq \alpha_1 s \) for any \( s \in (0, 1] \), any \( w, v \in \Lambda \) with \( K_w \cap K_v = \emptyset \) and any \( (x, y) \in K_w \times K_v \), which yields the former inclusion in (6.18). □

Proof of Theorem 6.10. This equivalence can be easily concluded by combining Theorem 4.5 and results in [Kig12, Kig09], as follows. First, by Theorem 4.5 and [Kig12, Theorem 15.10], under \( \theta \in \mathcal{J}(K, R_\varepsilon) \), (a) is equivalent to the following condition (c):

(c) \((K, \theta, \mu)\) is VD and there exists \( C \in (1, \infty) \) such that for any \( x, y \in K \) with \( x \neq y \),

\[
C^{-1} \theta(x, y)^\beta \leq R_\varepsilon(x, y) \mu(B_\theta(x, \theta(x, y))) \leq C \theta(x, y)^\beta. \tag{6.19}
\]

Next, by \( \theta \in \mathcal{J}(K, R_\varepsilon) \), (1.3) and (3.2) with \((d_1, d_2) \in \{(R_\varepsilon, \theta), (\theta, R_\varepsilon)\} \), \((K, \theta, \mu)\) is VD if and only if \((K, R_\varepsilon, \mu)\) is VD, which in turn is, by Lemma 6.14 and the compactness of \( K \), equivalent to the existence of \( C \in (1, \infty) \) such that

\[
\mu(U_s(x)) \leq C \mu(U_{s/2}(x)) \quad \text{for any } (s, x) \in (0, 1] \times K. \tag{6.20}
\]
Then by [Kig09, Theorem 1.3.5] and the fact that $S$ is \textit{locally finite} with respect to $\mathcal{L}$, i.e.,
\[
\sup_{(s,x) \in [0,1] \times K} \# \Lambda_{s,x}^{1} < \infty
\]
by [Kig01, Lemma 4.2.3] and [Kig09, Lemma 1.3.6], we have (6.20) if and only if there exists $C \in (1, \infty)$ such that the following hold:
\[
\mu(K_{wj}) \geq C^{-1} \mu(K_{w}) \quad \text{for any } (w, j) \in W_{s} \times S,
\]
(6.22)
\[
\mu(K_{w}) \leq C \mu(K_{v}) \quad \text{for any } s \in (0,1) \text{ and any } w, v \in \Lambda_{s} \text{ with } K_{w} \cap K_{v} \neq \emptyset.
\]
(6.23)
Moreover, by [Kig09, Theorem 1.2.4], (6.22) implies that
\[
\mu(F_{w}(V_{0})) = 0 = \mu(\{x\}) \quad \text{for any } w \in W_{s} \text{ and any } x \in K.
\]
(6.24)
(We remark that (6.24) is part of the assumptions of [Kig09, Theorem 1.3.5] but can be dropped; indeed, even without assuming (6.24), [Kig09, Proofs of Theorems 1.3.10 and 1.3.11] show that any one of the three conditions [Kig09, Theorem 1.3.5-(1),(2),(3)] implies (6.22), from which (6.24) also follows by [Kig09, Theorem 1.2.4].)

On the other hand, since the quasisymmetry of $R_{\varepsilon}$ to $\theta$ yields $\delta_{1}, \delta_{2} \in (0, \infty)$ such that $B_{0}(x, \delta_{1}\theta(x,y)) \subset B_{R_{\varepsilon}}(x, R_{\varepsilon}(x,y)) \subset B_{0}(x, \delta_{2}\theta(x,y))$ for any $x, y \in K$ with $x \neq y$ by (3.2), under VD of $(K, \theta, \mu)$ and $(K, R_{\varepsilon}, \mu)$ we have (6.19) if and only if there exists $C \in (0, \infty)$ such that for any $x, y \in K$ with $x \neq y$,
\[
C^{-1} \theta(x,y) \beta \leq R_{\varepsilon}(x,y) \mu(B_{R_{\varepsilon}}(x,R_{\varepsilon}(x,y))) \leq C \theta(x,y) \beta.
\]
(6.25)
Therefore (c) is equivalent to the following condition (d):
\[
\text{(d) There exists } C \in (1, \infty) \text{ such that (6.22), (6.23) and (6.25) hold.}
\]

Thus it remains to show that (d) is equivalent to (b). Indeed, let $w \in W_{s}$ and take $x, y \in K_{w}$ with the property $\text{diam}(K_{wj}, R_{\varepsilon}) = R_{\varepsilon}(x,y)$, so that $w \in \Lambda_{r_{w}}$, $R_{\varepsilon}(x,y)/\text{diam}(K, R_{\varepsilon}) \in [c_{R_{\varepsilon}} r_{w}, r_{w}]$ by Lemma 6.11, and $\theta(x,y)^{\beta} = (\text{diam}(\{x,y\}, \theta))^{\beta} \asymp (\text{diam}(K_{w}, \theta))^{\beta}$ by $\theta \in \mathscr{J}(K, R_{\varepsilon})$ and (3.6). If (d) holds, then since $(K, R_{\varepsilon}, \mu)$ is VD we easily see from Lemma 6.14, (6.21) and (6.23) that
\[
\mu(B_{R_{\varepsilon}}(x, \text{diam}(K_{w}, R_{\varepsilon}))) = \mu(B_{R_{\varepsilon}}(x, R_{\varepsilon}(x,y))) \asymp \mu(K_{w})
\]
(6.26)
and hence (6.25) implies (b). Conversely suppose that (b) holds. Then for any $j \in S$, Lemma 6.11 yields $\text{diam}(K_{wj}, R_{\varepsilon})/\text{diam}(K_{w}, R_{\varepsilon}) \in [c_{R_{\varepsilon}} \min_{k \in S} r_{k}, c_{R_{\varepsilon}}^{-1} \max_{k \in S} r_{k}]$, hence
\[
(\text{diam}(K_{wj}, \theta))^{\beta} \asymp (\text{diam}(K_{w}, \theta))^{\beta}
\]
(6.27)
by $\theta \in \mathscr{J}(K, R_{\varepsilon})$ and (3.6), and therefore (6.15) implies $\mu(K_{wj}) \asymp \mu(K_{w})$, i.e., (6.22) holds. Also for any $s \in (0,1)$ and any $v, \tau \in \Lambda_{s}$ with $K_{v} \cap K_{\tau} \neq \emptyset$, $\text{diam}(K_{v}, R_{\varepsilon}) \asymp \text{diam}(K_{v} \cup K_{\tau}, R_{\varepsilon}) \asymp \text{diam}(K_{v} \cup K_{\tau}, \theta)$ by Lemma 6.11 and hence
\[
(\text{diam}(K_{v}, \theta))^{\beta} \asymp (\text{diam}(K_{v} \cup K_{\tau}, \theta))^{\beta} \asymp (\text{diam}(K_{\tau}, \theta))^{\beta}
\]
(6.28)
by $\theta \in \mathscr{J}(K, R_{\varepsilon})$ and (3.6), which together with (6.15) implies $\mu(K_{v}) \asymp \mu(K_{\tau})$, proving (6.23). In particular, $(K, R_{\varepsilon}, \mu)$ is VD, and now it follows from Lemma 6.14, (6.21) and (6.23) that (6.26) holds, which together with (6.15) yields (6.25), proving (d). \hfill \Box
6.2 A necessary condition: attainment by the energy measure of some harmonic function

Throughout this subsection, we assume that \( \mathcal{L} = (K, S, \{ F_i \}_{i \in S}) \) is a post-critically finite self-similar structure with \( \# S \geq 2 \) and \( K \) connected and that \( (D, r) \) is a regular harmonic structure on \( \mathcal{L} \), and we follow the notation introduced in Subsection 6.1.

Proposition 6.10 with \( \beta = 2 \) justifies introducing the following set of pairs of metrics and Borel probability measures on \( K \).

**Definition 6.15.** We set \( \text{Homeo}^+ := \{ \eta \mid \eta : [0, \infty) \to [0, \infty), \eta \text{ is a homeomorphism} \} \), set \( \mathcal{P}(K) := \{ \mu \mid \mu \text{ is a Borel probability measure on } K \} \), which is equipped with the topology of weak convergence, and for each \( \eta \in \text{Homeo}^+ \) and \( C \in (1, \infty) \) we define

\[
\mathcal{G}(\eta, C) := \mathcal{G}_{\mathcal{L},(D,r)}(\eta, C) := \left\{ (\theta, \mu) \mid \theta \text{ is a metric on } K \eta\text{-quasisymmetric to } R_{\mathcal{E}}, \mu \in \mathcal{P}(K), C^{-1} \leq w_{\mu}(K_{w})/(\text{diam}(K_{w}, \theta))^2 \leq C \text{ for any } w \in W_{s} \right\},
\]

which is considered as a subset of \( C(K \times K) \times \mathcal{P}(K) \). We also set

\[
\mathcal{G} := \bigcup_{\eta \in \text{Homeo}^+, C \in (1, \infty)} \mathcal{G}(\eta, C).
\]

Since \( \mu \in \mathcal{A}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F}) \) for any \( (\theta, \mu) \in \mathcal{G} \) by (6.12) with \( m \) in place of \( \mu \) and (6.29), it follows from Proposition 6.10 with \( \beta = 2 \) and (1.3) that

\[
\mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F}) = \{ a\mu \mid (\theta, \mu) \in \mathcal{G}, a \in (0, \infty) \}.
\]

In particular,

\[
\mathcal{G}(K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F}) \neq \emptyset, \text{ i.e., the infimum in (1.4) is attained for } (K, R_{\mathcal{E}}, m, \mathcal{E}, \mathcal{F}), \text{ if and only if } \mathcal{G} \neq \emptyset, \text{ namely } \mathcal{G}(\eta, C) \neq \emptyset \text{ for some } (\eta, C) \in \text{Homeo}^+ \times (1, \infty).
\]

In fact, it turns out that in this case \( (\theta_{h}, \Gamma(h, h)) \in \mathcal{G} \) for some \( h \in \mathcal{H}_0 \setminus \mathbb{R}1_{K} \) and some \( \theta_{h} \in \mathcal{J}(K, R_{\mathcal{E}}) \), which is the main result of this subsection and stated as follows. We take arbitrary \( (\eta, C) \in \text{Homeo}^+ \times (1, \infty) \), define \( \tilde{\eta} \in \text{Homeo}^+ \) by \( \tilde{\eta}(t) := 1/\eta^{-1}(t^{-1}) \) for \( \tilde{\eta}(0) := 0 \) and fix them throughout the rest of this subsection.

**Theorem 6.16.** If \( \mathcal{G}(\eta, C) \neq \emptyset \), then there exist \( h \in \mathcal{H}_0 \setminus \mathbb{R}1_{K} \) and a metric \( \theta_{h} \) on \( K \) such that \( (\theta_{h}, \Gamma(h, h)) \in \mathcal{G}(c_{R_{\mathcal{E}}}^{-1}\eta, C) \), where \( c_{R_{\mathcal{E}}} \) is the constant in Lemma 6.11.

The rest of this subsection is devoted to the proof of Theorem 6.16, which is reduced to proving a series of propositions and lemmas concerning the set \( \mathcal{G}(\eta, C) \). We start with establishing its compactness. Note that \( \mathcal{P}(K) \) is a compact metrizable topological space by [Str, Theorems 9.1.5 and 9.1.9] and hence that \( C(K \times K) \times \mathcal{P}(K) \) is also metrizable.

**Proposition 6.17.** \( \mathcal{G}(\eta, C) \) is a compact subset of \( C(K \times K) \times \mathcal{P}(K) \).
Proof. Let \( \{ (\theta_n, \mu_n) \}_{n \in \mathbb{N}} \subset \mathcal{G}(\eta, C) \). By the metrizability of \( C(K \times K) \times \mathcal{P}(K) \) noted above, it suffices to show that there exists a subsequence of \( \{ (\theta_n, \mu_n) \}_{n \in \mathbb{N}} \) converging to some \( (\theta, \mu) \in \mathcal{G}(\eta, C) \) in \( C(K \times K) \times \mathcal{P}(K) \).

First, recalling that the compactness of \( K \) implies that of \( \mathcal{P}(K) \) by [Str, Theorem 9.1.9], we can choose \( \mu \in \mathcal{P}(K) \) and a subsequence \( \{ \mu_{n_k} \}_{k \in \mathbb{N}} \) of \( \{ \mu_n \}_{n \in \mathbb{N}} \) converging to \( \mu \) in \( \mathcal{P}(K) \), and therefore by considering \( \{ (\theta_{n_k}, \mu_{n_k}) \}_{k \in \mathbb{N}} \) instead of \( \{ (\theta_n, \mu_n) \}_{n \in \mathbb{N}} \) we may assume that \( \{ \mu_n \}_{n \in \mathbb{N}} \) itself converges to \( \mu \) in \( \mathcal{P}(K) \).

Next, \( \text{diam}(K, \theta_n) \in [C^{-1/2}, C^{1/2}] \) for any \( n \in \mathbb{N} \) by the inequalities in (6.29) and hence \( \{ \theta_n \}_{n \in \mathbb{N}} \) is uniformly bounded. Moreover, for each \( n \in \mathbb{N} \), since the \( \eta \)-quasisymmetry of \( \theta_n \) to \( R_\varepsilon \) yields the \( \tilde{\eta} \)-quasisymmetry of \( R_\varepsilon \) to \( \theta_n \) by [Hei, Proposition 10.6], it follows from (3.6) that for any \( x, y \in K \),

\[
\frac{R_\varepsilon(x, y)}{\text{diam}(K, R_\varepsilon)} = \frac{\text{diam}(\{x, y\}, R_\varepsilon)}{\text{diam}(K, R_\varepsilon)} \leq \eta \left( \frac{2\text{diam}(\{x, y\}, \theta_n)}{\text{diam}(K, \theta_n)} \right) \leq \eta \left( \frac{2C^{1/2}\theta_n(x, y)}{\text{diam}(K, \theta_n)} \right),
\]

(6.33)

\[
C^{-1/2}\theta_n(x, y) \leq \frac{\text{diam}(\{x, y\}, \theta_n)}{\text{diam}(K, \theta_n)} \leq \tilde{\eta} \left( \frac{2\text{diam}(\{x, y\}, R_\varepsilon)}{\text{diam}(K, R_\varepsilon)} \right) = \tilde{\eta} \left( \frac{2R_\varepsilon(x, y)}{\text{diam}(K, R_\varepsilon)} \right),
\]

(6.34)

which in turn implies that for any \( x_1, y_1, x_2, y_2 \in K \),

\[
|\theta_n(x_1, y_1) - \theta_n(x_2, y_2)| \leq \theta_n(x_1, x_2) + \theta_n(y_1, y_2) \leq C^{1/2} \left( \tilde{\eta} \left( \frac{2R_\varepsilon(x_1, x_2)}{\text{diam}(K, R_\varepsilon)} \right) \right) + \tilde{\eta} \left( \frac{2R_\varepsilon(y_1, y_2)}{\text{diam}(K, R_\varepsilon)} \right),
\]

(6.35)

so that \( \{ \theta_n \}_{n \in \mathbb{N}} \subset C(K \times K) \) is equicontinuous. Thus by the Arzelà-Ascoli theorem (see, e.g., [Rud, Theorem 11.28]) there exist \( \theta \in C(K \times K) \) and a subsequence \( \{ \theta_{n_k} \}_{k \in \mathbb{N}} \) of \( \{ \theta_n \}_{n \in \mathbb{N}} \) converging to \( \theta \) in \( C(K \times K) \), so that \( \{ (\theta_{n_k}, \mu_{n_k}) \}_{k \in \mathbb{N}} \) converges to \( (\theta, \mu) \) in \( C(K \times K) \times \mathcal{P}(K) \). Then \( \text{diam}(K, \theta) \in [C^{-1/2}, C^{1/2}] \) and for any \( x, y, z \in K \) we have (6.33) with \( \theta \) in place of \( \theta_n \), \( \theta(x, x) = 0 \), \( \theta(x, y) = \theta(y, x) \geq 0 \) and \( \theta(x, y) \leq \theta(x, z) + \theta(z, y) \) by the same properties of \( \theta_{n_k} \) for \( k \in \mathbb{N} \), whence \( \theta \) is a metric on \( K \). Furthermore letting \( k \to \infty \) in the \( \eta \)-quasisymmetry of \( \theta_{n_k} \) to \( R_\varepsilon \) as defined in (3.1) yields that of \( \theta \) to \( R_\varepsilon \).

To show the inequalities in (6.29) for \( (\theta, \mu) \), let \( w \in W_s \), choose \( x = x_w \in K_w \setminus F_w(V_0) \) and set \( s := r_w \), so that \( w \in \Lambda_s \) and \( \Lambda_{s,x} = \{ w \} \) by [Kig01, Proposition 1.3.5-(2)] (recall Definition 6.13). Note that \( K_w = F_w(K) \) is compact and hence closed in \( K \), that \( K_w \setminus F_w(V_0) = K \setminus (F_w(V_0) \cup \bigcup_{v \in W_s \setminus \{ w \}} K_v) \) by [Kig01, Proposition 1.3.5-(2)] and is thus open in \( K \), and that \( K_w \subset U^s_w(x) \) with \( U^s_w(x) \) the interior of \( U_s(x) \) in \( K \) by [Kig01, Proposition 1.3.6]. By using these facts and the convergence of \( \{ (\theta_{n_k}, \mu_{n_k}) \}_{k \in \mathbb{N}} \) to \( (\theta, \mu) \) in \( C(K \times K) \times \mathcal{P}(K) \) to let \( k \to \infty \) in the inequalities in (6.29) for \( (\theta_{n_k}, \mu_{n_k}) \), we obtain

\[
r_w \mu(K_w) \geq \limsup_{k \to \infty} r_w \mu_{n_k}(K_w) \geq C^{-1}(\text{diam}(K_w, \theta))^2,
\]

(6.36)

\[
r_w \mu(K_w \setminus F_w(V_0)) \leq \liminf_{k \to \infty} r_w \mu_{n_k}(K_w \setminus F_w(V_0)) \leq C(\text{diam}(K_w, \theta))^2,
\]

(6.37)

\[
r_w \mu(K_w) \leq r_w \mu(U^s_w(x)) \leq \liminf_{k \to \infty} r_w \mu_{n_k}(U^s_w(x)) \leq \sum_{v \in \Lambda^s_{s,x}} C r_v r_v^{-1}(\text{diam}(K_v, \theta))^2
\]

\[
\leq C \left( \min_{k \in \mathbb{N}} r_k \right)^{-1} \sum_{v \in \Lambda^s_{s,x}} (\text{diam}(K_v, \theta))^2 \asymp (\text{diam}(K_w, \theta))^2,
\]

(6.38)
where the last step in (6.38) follows from (6.28) and (6.21). We now conclude from (6.36), (6.38) and (6.27) that \( \mu(K_{w_j}) \asymp r_{w_j}^{-1}(\text{diam}(K_{w_j}, \theta))^2 \asymp r_{w}^{-1}(\text{diam}(K_{w}, \theta))^2 \asymp \mu(K_w) \) for any \( (w, j) \in W_s \times S \), which together with [Kig09, Theorem 1.2.4] implies that \( \mu(F_w(V_0)) = 0 \) and hence \( \mu(K_w \setminus F_w(V_0)) = \mu(K_w) \) for any \( w \in W_s \), so that (6.36) and (6.37) yield the inequalities in (6.29) for \( (\theta, \mu) \) and thus \( (\theta, \mu) \in \mathcal{G}(\eta, C) \). \( \square \)

**Corollary 6.18.** Let \( \{\theta_n, \mu_n\}_{n \in \mathbb{N}} \subset \mathcal{G}(\eta, C) \), \( \mu \in \mathcal{P}(K) \) and suppose that \( \{\mu_n\}_{n \in \mathbb{N}} \) converges to \( \mu \) in \( \mathcal{P}(K) \). Then there exists a metric \( \theta \) on \( K \) such that \( (\theta, \mu) \in \mathcal{G}(\eta, C) \).

**Proof.** Since \( \mathcal{G}(\eta, C) \) is a compact subset of \( C(K \times K) \times \mathcal{P}(K) \) by Proposition 6.17, there exist \( (\theta, \nu) \in \mathcal{G}(\eta, C) \) and a subsequence \( \{(\theta_{n_k}, \mu_{n_k})\}_{k \in \mathbb{N}} \) of \( \{(\theta_n, \mu_n)\}_{n \in \mathbb{N}} \) converging to \( (\theta, \nu) \) in \( C(K \times K) \times \mathcal{P}(K) \), but then \( \{\mu_{n_k}\}_{k \in \mathbb{N}} \) converges in \( \mathcal{P}(K) \) to both \( \mu \) and \( \nu \), hence \( \mu = \nu \) and thus \( (\theta, \mu) = (\theta, \nu) \in \mathcal{G}(\eta, C) \). \( \square \)

We now observe that the set \( \mathcal{G}(\eta, C) \) is almost invariant under the operation of pulling back by \( F_w \) followed by a suitable normalization, as stated in the following lemma.

**Lemma 6.19.** Let \( (\theta, \mu) \in \mathcal{G}(\eta, C) \), \( w \in W_s \) and define \( (\theta_w, \mu_w) \in C(K \times K) \times \mathcal{P}(K) \) by

\[
\theta_w(x, y) := \frac{\theta(F_w(x), F_w(y))}{\sqrt{r_w \mu(K_w)}} \quad \text{and} \quad \mu_w(A) := \frac{\mu(F_w(A))}{\mu(K_w)}, \tag{6.39}
\]

Then \( (\theta_w, \mu_w) \in \mathcal{G}(c_{R^\varepsilon}^{-1}\eta, C) \), where \( c_{R^\varepsilon} \) is the constant in Lemma 6.11.

**Proof.** It is immediate from \( (\theta, \mu) \in \mathcal{G}(\eta, C) \) that \( \theta_w \) and \( \mu_w \) can be defined by (6.39) and are a metric and a Borel probability measure on \( K \), respectively, and that \( (\theta_w, \mu_w) \) satisfies the inequalities in (6.29). Moreover, for any \( x, y, z \in K \) and \( t \in (0, \infty) \) with \( \theta_w(x, y) \leq t \theta_w(x, z) \), we have \( \theta(F_w(x), F_w(y)) \leq t \theta(F_w(x), F_w(z)) \) and hence it follows from the \( \eta \)-quasisymmetry of \( \theta \) to \( R^\varepsilon \) and Lemma 6.11 that

\[
c_{R^\varepsilon} r_w R^\varepsilon(x, y) \leq R^\varepsilon(F_w(x), F_w(y)) \leq \eta(t) R^\varepsilon(F_w(x), F_w(z)) \leq \eta(t) r_w R^\varepsilon(x, z),
\]

so that \( R^\varepsilon(x, y) \leq c_{R^\varepsilon}^{-1}\eta(t) R^\varepsilon(x, z) \), proving the \( c_{R^\varepsilon}^{-1}\eta \)-quasisymmetry of \( \theta_w \) to \( R^\varepsilon \). \( \square \)

The operation as in (6.39) of pulling back Borel measures on \( K \) by \( F_w \) is compatible with the analogous operation on \( \mathcal{F} \) (recall (6.7) and (6.8)) in the following sense.

**Lemma 6.20.** Let \( u \in \mathcal{F} \) and \( w \in W_s \). Then \( \Gamma(u, u)(F_w(A)) = r_w^{-1} \Gamma(u \circ F_w, u \circ F_w)(A) \) for any Borel subset \( A \) of \( K \), and in particular \( \Gamma(u, u)(K_w) = r_w^{-1} \mathcal{E}(u \circ F_w, u \circ F_w) \). Moreover, if \( \Gamma(u, u)(K_w) > 0 \), then for any Borel subset \( A \) of \( K \),

\[
\frac{\Gamma(u, u)(F_w(A))}{\Gamma(u, u)(K_w)} = \Gamma(u_w, u_w)(A), \quad \text{where} \quad u_w := \mathcal{E}(u \circ F_w, u \circ F_w)^{-1/2} u \circ F_w. \tag{6.40}
\]

**Proof.** Since \( F_w : K \to K_w \) is a homeomorphism, the first assertion is easily seen to be equivalent to [HN, Lemma 4-(i)], and the second follows by choosing \( A = K \) in the first.
Furthermore if \( \Gamma(u, u)(K_w) > 0 \), then we see from the first and second assertions and the bilinearity of \( \Gamma(f, g) \) in \( f, g \in \mathcal{F} \) that for any Borel subset \( A \) of \( K \),

\[
\frac{\Gamma(u, u)(F_w(A))}{\Gamma(u, u)(K_w)} = \frac{r_w^{-1}\Gamma(u \circ F_w, u \circ F_w)(A)}{r_w^{-1}\mathcal{E}(u \circ F_w, u \circ F_w)} = \Gamma(u_w, u_w)(A),
\]

completing the proof.

Recall that \( \mu \) is a minimal energy-dominant measure of \( (\mathcal{E}, \mathcal{F}) \) for any \( (\theta, \mu) \in \mathcal{G} \) by Proposition 6.10 and Proposition 2.10-(b), and hence in particular that \( \Gamma(u, u) \) is absolutely continuous with respect to \( \mu \) for any \( (\theta, \mu) \in \mathcal{G} \) and any \( u \in \mathcal{F} \). The following lemma is a special case of the well-known Lebesgue differentiation theorem.

**Lemma 6.21.** Let \( (\theta, \mu) \in \mathcal{G} \), \( u \in \mathcal{F} \) and set \( f := d\Gamma(u, u)/d\mu \). Then \( \mu \)-a.e. \( x \in K \) is an \((\mathcal{R}_\mathcal{E}, \mu)\)-Lebesgue point for \( f \), i.e., satisfies

\[
\lim_{s \downarrow 0} \frac{1}{\mu(B_{\mathcal{R}_\mathcal{E}}(x, s))} \int_{B_{\mathcal{R}_\mathcal{E}}(x, s)} |f(y) - f(x)| \, d\mu(y) = 0. \tag{6.41}
\]

**Proof.** We have \( \mathcal{R}_\mathcal{E} \in \mathcal{J}(K, \theta) \) by \( (\theta, \mu) \in \mathcal{G} \) and hence \( \theta \in \mathcal{J}(K, \mathcal{R}_\mathcal{E}) \) by (1.3). Thus (3.2) holds with \((d_1, d_2) = (\mathcal{R}_\mathcal{E}, \theta)\), and \( (K, \theta, \mu) \) is VD by Proposition 6.10 and Theorem 4.5, which together imply that \((K, \mathcal{R}_\mathcal{E}, \mu) \) is VD. Now since \( C(K) \) is dense in \( L^1(K, \mu) \) (see, e.g., [Rud, Theorem 3.14]) and \( f \in L^1(K, \mu) \), the claim follows by Lebesgue’s differentiation theorem [Hei, (2.8)] for \((K, \mathcal{R}_\mathcal{E}, \mu)\), which requires \((K, \mathcal{R}_\mathcal{E}, \mu) \) to be VD.

**Lemma 6.22.** Let \( (\theta, \mu) \in \mathcal{G} \), \( u \in \mathcal{F} \), let \( f : K \rightarrow [0, \infty) \) be a Borel measurable \( \mu \)-version of \( d\Gamma(u, u)/d\mu \) and let \( x \in K \) satisfy (6.41). Then for any \( \omega \in \pi^{-1}(x) \) and any \( w \in W_* \),

\[
\lim_{n \rightarrow \infty} \frac{\Gamma(u, u)(K_{[w]n, w})}{\mu(K_{[w]n, w})} = f(x). \tag{6.42}
\]

**Proof.** Let \( \omega \in \pi^{-1}(x) \), \( w \in W_* \), \( n \in \mathbb{N} \cup \{0\} \) and set \( s_n := \text{diam}(K_{[w]n, R_{\mathcal{E}}}) \). Then by (6.22), (6.26), and VD of \((K, \mathcal{R}_\mathcal{E}, \mu)\) noted in the above proof of Lemma 6.21, we have

\[
\mu(K_{[w]n, w}) \geq c^{[w]} \mu(K_{[w]n}) \geq c^{[w]} c' \mu(B_{\mathcal{R}_\mathcal{E}}(x, 2s_n)) \tag{6.43}
\]

for some \( c, c' \in (0, \infty) \) determined solely by \( \mathcal{L}, (\mathcal{D}, \mathcal{r}), (\theta, \mu) \). Now since \( K_{[w]n, w} \subset K_{[w]n} \subset B_{\mathcal{R}_\mathcal{E}}(x, 2s_n) \) and \( \lim_{n \rightarrow \infty} s_n = 0 \) by Lemma 6.11, it follows from (6.43) and (6.41) that

\[
\left| \frac{\Gamma(u, u)(K_{[w]n, w})}{\mu(K_{[w]n, w})} - f(x) \right| = \left| \frac{1}{\mu(K_{[w]n, w})} \int_{K_{[w]n, w}} (f(y) - f(x)) \, d\mu(y) \right|
\leq \frac{1}{\mu(K_{[w]n, w})} \int_{K_{[w]n, w}} |f(y) - f(x)| \, d\mu(y)
\leq \frac{(c^{[w]} c')^{-1}}{\mu(B_{\mathcal{R}_\mathcal{E}}(x, 2s_n))} \int_{B_{\mathcal{R}_\mathcal{E}}(x, 2s_n)} |f(y) - f(x)| \, d\mu(y) \xrightarrow{m \rightarrow \infty} 0,
\]

proving (6.42).
Taking an \((R_\xi, \mu)\)-Lebesgue point \(x \in K\) for \(d\Gamma(u, u)/d\mu\) with \((d\Gamma(u, u)/d\mu)(x) > 0\) and considering the enlargements of infinitesimally small cells containing \(x\) to the original scale as in Lemmas 6.19 and 6.20, we arrive at the following proposition.

**Proposition 6.23.** Let \((\theta, \mu) \in \mathcal{G}, u \in \mathcal{F}\), let \(f : K \to [0, \infty)\) be a Borel measurable \(\mu\)-version of \(d\Gamma(u, u)/d\mu\), let \(x \in K\) satisfy (6.41) and \(f(x) > 0\), and let \(\omega \in \pi^{-1}(x)\). For each \(n \in \mathbb{N} \cup \{0\}\), define \(\mu_n := \mu_{[\omega]_n} \in \mathcal{P}(K)\) by (6.39) with \(w = [\omega]_n\) and, noting that \(\Gamma(u, u)(K_{[\omega]_n}) > 0\) by (6.42), define \(u_n := u_{[\omega]_n} \in \mathcal{F}\) by (6.40) with \(w = [\omega]_n\). If \(v \in \mathcal{F}\) and \(\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}\) is strictly increasing and satisfies \(\lim_{k \to \infty} \mathcal{E}(v - u_{n_k}, v - u_{n_k}) = 0\), then \(\Gamma(v, v) \in \mathcal{P}(K)\) and \(\{\mu_{n_k}\}_{k \in \mathbb{N}}\) converges to \(\Gamma(v, v)\) in \(\mathcal{P}(K)\).

**Proof.** Let \(w \in W_*\). Then we see from (6.39), (6.40) and (6.42) that
\[
\frac{\Gamma(u_n, u_n)(K_w)}{\mu_n(K_w)} = \left(\frac{\Gamma(u, u)(K_{[\omega]_n})}{\mu(K_{[\omega]_n})}\right)^{-1} \frac{\Gamma(u, u)(K_{[\omega]_n} w)}{\mu(K_{[\omega]_n} w)} m \to \infty f(x)^{-1} f(x) = 1. \tag{6.44}
\]
On the other hand, since \(\Gamma\) is bilinear, symmetric and non-negative definite,
\[
|\Gamma(v, v)(K_w)^{1/2} - \Gamma(u_n, u_n)(K_w)^{1/2}|^2 \leq \Gamma(v - u_{n_k}, v - u_{n_k})(K_w) \leq \Gamma(v - u_{n_k}, v - u_{n_k})(K) = \mathcal{E}(v - u_{n_k}, v - u_{n_k}) \xrightarrow{k \to \infty} 0. \tag{6.45}
\]
It follows from (6.44) and (6.45) that
\[
\lim_{k \to \infty} \mu_{n_k}(K_w) = \Gamma(v, v)(K_w) \tag{6.46}
\]
and in particular that \(\Gamma(v, v)(K) = \lim_{k \to \infty} \mu_{n_k}(K) = 1\), namely \(\Gamma(v, v) \in \mathcal{P}(K)\). Note that \(\mu_n(F_w(V_0)) = \mu(K_{[\omega]_n})^{-1} \mu(F_{[\omega]_n} w(V_0)) = 0\) for any \(n \in \mathbb{N} \cup \{0\}\) by Proposition 6.10 and that \(\Gamma(v, v)(F_w(V_0)) = 0\) by \(\#F_w(V_0) < \infty\) and [CF, Theorem 4.3.8], and recall that \(K_w = F_w(K)\) is closed in \(K\) and \(K_w \setminus F_w(V_0)\) is open in \(K\) as noted in the last paragraph of the proof of Proposition 6.17. By using these facts and the equality \(K \setminus V_n = \bigcup_{v \in W_n} (K_v \setminus F_v(V_0))\), with the union disjoint, implied by [Kig01, Proposition 1.3.5-(2)] for any \(n \in \mathbb{N}\), we easily see that the validity of (6.46) for any \(w \in W_*\) is equivalent to the desired convergence of \(\{\mu_{n_k}\}_{k \in \mathbb{N}}\) to \(\Gamma(v, v)\) in \(\mathcal{P}(K)\), which completes the proof. \(\square\)

Now we can conclude the proof of the main result of this subsection (Theorem 6.16).

**Proof of Theorem 6.16.** By the assumption \(\mathcal{G}(\eta, C) \neq \emptyset\) we can take \((\theta, \mu) \in \mathcal{G}(\eta, C)\). Let \(u \in \mathcal{H}_0 \setminus \mathbb{R} I_K\), which exists by Proposition 6.6-(1) and \(\#V_0 \geq 2\), and let \(f : K \to [0, \infty)\) be a Borel measurable \(\mu\)-version of \(d\Gamma(u, u)/d\mu\). Then since \(\mu\left(f^{-1}(0, \infty)\right) > 0\) by \(\int_K f d\mu = \Gamma(u, u)(K) = \mathcal{E}(u, u) > 0\), Lemma 6.21 implies that there exists \(x \in K\) with the properties (6.41) and \(f(x) > 0\). Let \(\omega \in \pi^{-1}(x)\), and for each \(n \in \mathbb{N} \cup \{0\}\), as in Proposition 6.23 define \((\theta_n, \mu_n) := (\theta_{[\omega]_n}, \mu_{[\omega]_n}) \in C(K \times K) \times \mathcal{P}(K)\) by (6.39) with \(w = [\omega]_n\) and \(u_n := u_{[\omega]_n} \in \mathcal{F}\) by (6.40) with \(w = [\omega]_n\), so that \(\{(\theta_n, \mu_n)\}_{n \in \mathbb{N} \cup \{0\}} \subset \mathcal{G}(\xi \wedge C)\) by Lemma 6.19 and \(\{u_n\}_{n \in \mathbb{N} \cup \{0\}} \subset \{h \in \mathcal{H}_0 \mid \mathcal{E}(h, h) = 1\}\) by Proposition
6.6-(2). Noting that \( \mathcal{H}_0/\mathbb{R}1_K \) is a finite-dimensional linear space with inner product \( \mathcal{E} \) by (RF1) and hence that \( \{ h \in \mathcal{H}_0/\mathbb{R}1_K \mid \mathcal{E}(h, h) = 1 \} \) is a compact subset of \( \mathcal{H}_0/\mathbb{R}1_K \), we get \( h \in \mathcal{H}_0 \setminus \mathbb{R}1_K \) and a strictly increasing sequence \( \{ n_k \}_{k \in \mathbb{N}} \subset \mathbb{N} \) satisfying \( \mathcal{E}(h, h) = 1 \) and \( \lim_{k \to \infty} \mathcal{E}(h - u_{n_k}, h - u_{n_k}) = 0 \). Then \( \Gamma(h, h) \in \mathcal{P}(K) \) and \( \{ \mu_{n_k} \}_{k \in \mathbb{N}} \) converges to \( \Gamma(h, h) \) in \( \mathcal{P}(K) \) by Proposition 6.23, and it follows from \( \{ (\theta_{n_k}, \mu_{n_k}) \}_{k \in \mathbb{N}} \subset \mathcal{G}(c_{R\xi}^{-1} \eta, C) \) and Corollary 6.18 that \( (\theta_h, \Gamma(h, h)) \in \mathcal{G}(c_{R\xi}^{-1} \eta, C) \) for some metric \( \theta_h \) on \( K \). \( \square \)

6.3 Examples

In this subsection, we shall show that the infimum in (1.4) defining the conformal walk dimension \( d_{cw} = 2 \) fails to be attained for some concrete examples of post-critically finite self-similar sets. In view of (6.32) and Theorem 6.16, for the proof of the non-attainment \( \mathcal{G}(K, R\xi, m, \mathcal{E}, \mathcal{F}) = \emptyset \) it suffices to verify that the conclusion of Theorem 6.16 cannot hold for any \( h \in \mathcal{H}_0 \setminus \mathbb{R}1_K \). We start with providing a further characterization of \( h \in \mathcal{H}_0 \setminus \mathbb{R}1_K \) as in the conclusion of Theorem 6.16. In the following definition and proposition, we continue to assume the setting specified in the first paragraph of Subsection 6.2.

Definition 6.24. Let \( \mu \) be a Borel measure on \( K \). Recalling Definition 2.3, we define the \( \mu \)-intrinsic metric \( d_{int}^\mu : K \times K \to [0, \infty] \) of \( (\mathcal{E}, \mathcal{F}) \) by

\[
d_{int}^\mu(x, y) := \sup \{ u(x) - u(y) \mid u \in \mathcal{F}, \Gamma(u, u) \leq \mu \}. \tag{6.47}
\]

Proposition 6.25. (1) There exists \( C \in (1, \infty) \) such that for any \( h \in \mathcal{H}_0 \) and any \( w \in W_\ast \),

\[
C^{-1}(\text{diam}(K_w, d_{int}^{\Gamma(h,h)}(h)))^2 \leq r_w \Gamma(h, h)(K_w) \leq C(\text{diam}(K_w, d_{int}^{\Gamma(h,h)}(h)))^2. \tag{6.48}
\]

(2) Let \( h \in \mathcal{H}_0 \). Then \( \Gamma(h, h) \in \mathcal{G}(K, R\xi, m, \mathcal{E}, \mathcal{F}) \) if and only if \( d_{int}^{\Gamma(h,h)} \in \mathcal{J}(K, R\xi) \).

(3) \( \mathcal{G}(K, R\xi, m, \mathcal{E}, \mathcal{F}) = \emptyset \) if and only if \( d_{int}^{\Gamma(h,h)} \notin \mathcal{J}(K, R\xi) \) for any \( h \in \mathcal{H}_0 \).

Proof. (1) Let \( h \in \mathcal{H}_0 \) and \( w \in W_\ast \). Since \( |h(x) - h(y)| \leq d_{int}^{\Gamma(h,h)}(x, y) \) for any \( x, y \in K \) by (6.47) with \( \mu = \Gamma(h, h) \), we see from Lemma 6.20, \( h \in \mathcal{H}_0 \), (6.11), (6.10) and (6.2) that

\[
r_w \Gamma(h, h)(K_w) = \mathcal{E}(h \circ F_w, h \circ F_w) = \mathcal{E}(0)(h \circ F_w|_V, h \circ F_w|_V) \leq C(\text{diam}(K_w, d_{int}^{\Gamma(h,h)}(h)))^2,
\]

where \( C := \frac{1}{2} \sum_{p \neq q \in V_0} D_{pq} \). On the other hand, setting \( C' := \text{diam}(K, R\xi) \), for any \( x, y \in K \) and any \( u \in \mathcal{F} \) with \( \Gamma(u, u) \leq \Gamma(h, h) \), by (RF4) and Lemma 6.20 we have

\[
|u(x) - u(y)|^2 = |(u \circ F_w^{-1}(x)) - (u \circ F_w^{-1}(y))|^2 \\
\leq C' \mathcal{E}(u \circ F_w, u \circ F_w) = C'r_w \Gamma(h, h)(K_w) \leq C'r_w \Gamma(h, h)(K_w),
\]

therefore taking the supremum over such \( u \) yields \( d_{int}^{\Gamma(h,h)}(x, y) \leq C'r_w \Gamma(h, h)(K_w) \) by (6.47) with \( \mu = \Gamma(h, h) \) and thus \( (\text{diam}(K_w, d_{int}^{\Gamma(h,h)}(h)))^2 \leq C'r_w \Gamma(h, h)(K_w) \), proving (6.48).

(2) If \( \Gamma(h, h) \in \mathcal{G}(K, R\xi, m, \mathcal{E}, \mathcal{F}) \) then \( d_{int}^{\Gamma(h,h)} \in \mathcal{J}(K, R\xi) \) by Proposition 2.10-(a) (recall (5.3)), and conversely if \( d_{int}^{\Gamma(h,h)} \in \mathcal{J}(K, R\xi) \) then \( \Gamma(h, h) \in \mathcal{G}(K, R\xi, m, \mathcal{E}, \mathcal{F}) \) by (6.48) and Proposition 6.10 with \( \beta = 2 \).

(3) This is immediate from (2) and the fact that, by (6.32), Theorem 6.16 and (6.31), \( \mathcal{G}(K, R\xi, m, \mathcal{E}, \mathcal{F}) = \emptyset \) if and only if \( \Gamma(h, h) \notin \mathcal{G}(K, R\xi, m, \mathcal{E}, \mathcal{F}) \) for any \( h \in \mathcal{H}_0 \). \( \square \)
6.3.1 The Vicsek set

Example 6.26 (Vicsek set). Set \( S := \{0, 1, 2, 3, 4\} \), define \( \{q_i\}_{i \in S} \subset \mathbb{R}^2 \) by \( q_0 := (0, 0) \), \( q_1 := (1, 1) \), \( q_2 := (-1, 1) \), \( q_3 := (-1, -1) \) and \( q_4 := (1, -1) \), and define \( f_i : \mathbb{R}^2 \to \mathbb{R}^2 \) for each \( i \in S \) by \( f_i(x) := q_i + \frac{1}{2}(x - q_i) \). Let \( K \) be the self-similar set associated with \( \{f_i\}_{i \in S} \), i.e., the unique non-empty compact subset of \( \mathbb{R}^2 \) such that \( K = \bigcup_{i \in S} f_i(K) \), which exists and satisfies \( K \subseteq [-1, 1]^2 \) thanks to \( \bigcup_{i \in S} f_i([-1, 1]^2) \subseteq [-1, 1]^2 \) by [Kig01, Theorem 1.1.4], and set \( F_i := f_i|_K \) for each \( i \in S \). Then \( \mathcal{L} := (K, S, \{F_i\}_{i \in S}) \) is a self-similar structure by [Kig01, Theorem 1.2.3] and called the Vicsek set (see Figure 1 below), and it easily follows from \( K \subset [-1, 1]^2 \) that \( \mathcal{P} = \{1^\infty, 2^\infty, 3^\infty, 4^\infty\} \) and \( V_0 = \{q_1, q_2, q_3, q_4\} \), so that \( \mathcal{L} \) is post-critically finite and \( K \) is connected.

Let \( r \in (0, \frac{1}{2}) \), set \( \mathbf{r} = (r_i)_{i \in S} := (1 - 2r, r, r, r) \), and define \( D = (D_{pq})_{p,q \in V_0} \) and \( D' = (D'_{pq})_{p,q \in V_0 \cup \{q_0\}} \) by

\[
D_{pq} := \begin{cases} 
1 & \text{if } p \neq q, \\
-3 & \text{if } p = q,
\end{cases} \quad D'_{pq} := \begin{cases} 
4 & \text{if } \{p, q\} = \{q_0, q_i\} \text{ for some } i \in S \setminus \{0\}, \\
0 & \text{if } \{p, q\} = \{q_i, q_j\} \text{ for some } i, j \in S \setminus \{0\}, \\
-16 & \text{if } p = q = q_0, \\
-4 & \text{if } p = q = q_i \text{ for some } i \in S \setminus \{0\}.
\end{cases}
\]

Then setting \( \mathcal{E}^{(0)}(u, v) := -\sum_{p,q \in V_0 \cup \{q_0\}} D'_{pq} u(q) v(p) \) for \( u, v \in \mathbb{R}^{V_0 \cup \{q_0\}} \), we immediately see that \( \mathcal{E}^{(0)}(u, u) = \inf_{i \in \mathbb{R}^{V_0 \cup \{q_0\}}} \mathcal{E}^{(0)}(v, v) \) for any \( u \in \mathbb{R}^{V_0 \cup \{q_0\}} \), which in turn easily implies that \( (D, \mathbf{r}) \) is a regular harmonic structure on \( \mathcal{L} \).

In the situation of Example 6.26, we have the following proposition.

Proposition 6.27. Let \( \mathcal{L} = (K, S, \{F_i\}_{i \in S}) \) and \( (D, \mathbf{r}) \) be as in Example 6.26 and consider the MMD space \( (K, R_\mathcal{L}, m, \mathcal{E}, \mathcal{F}) \) resulting from \( \mathcal{L}, (D, \mathbf{r}) \) as introduced in Subsection 6.1. Set \( I_{i,i+2} := \{(1-t)q_i + tq_{i+2} \mid t \in [0, 1]\} \) for \( i \in \{1, 2\} \). Then \( \Gamma(h, h)(K \setminus (I_{1,3} \cup I_{2,4})) = 0 \) for any \( h \in \mathcal{H}_0 \).

Proof. Let \( U \) be a connected component of \( K \setminus (I_{1,3} \cup I_{2,4}) \). Then it is immediate from \( K = \bigcup_{i \in S} F_i(K) \subseteq [-1, 1]^2 \) that \( \partial_K U \) consists of a unique element \( q_U \in I_{1,3} \cup I_{2,4} \), where \( \partial_K U \) denotes the boundary of \( U \) in \( K \). Note that \( u1_U, u1_{K \setminus U} \in \mathcal{F} \) and \( \mathcal{E}(u1_U, u1_{K \setminus U}) = 0 \) for any \( u \in \mathcal{F} \) with \( u(q_U) = 0 \); indeed, this is clear by [FOT, Exercise 1.4.1 and Theorem 1.4.2-(ii)] and the strong locality of \( \mathcal{E}, \mathcal{F} \) if \( q_U \notin \text{supp}_m[u] \), and the general case follows by approximating \( u \) by \( \left(u - \left(-\frac{1}{2}\right) \vee \left(u \wedge \frac{1}{2}\right)\right)_{n \in \mathbb{N}} \) on the basis of [FOT, Theorem 1.4.2-(iv)]. Therefore for any \( u \in \mathcal{F} \), the function \( u_U \in C(K) \) defined by \( u_U|_U := u(q_U)1_U \) and \( u_U|_{K \setminus U} := u|_{K \setminus U} \) satisfies \( u_U = (u - u(q_U)1_{K \setminus U} + u(q_U)1_K) \in \mathcal{F} \) and \( \mathcal{E}(u, u) \geq \mathcal{E}(u_U, u_U) \), where the equality holds if and only if \( u = u_U \) by (RF1). In particular, any \( u \in \mathcal{F} \) with \( u \neq u_U \) fails to be 0-harmonic by \( U \cap V_0 = \emptyset \), so that any \( h \in \mathcal{H}_0 \) satisfies \( h|_U = h(q_U)1_U \), hence \( \Gamma(h, h)(U) = 0 \) by [FOT, Corollary 3.2.1] and thus \( \Gamma(h, h)(K \setminus (I_{1,3} \cup I_{2,4})) = 0 \) since \( U \) is any one of the countably many connected components of \( K \setminus (I_{1,3} \cup I_{2,4}) \).

Corollary 6.28. Let \( \mathcal{L} = (K, S, \{F_i\}_{i \in S}) \) and \( (D, \mathbf{r}) \) be as in Example 6.26 and consider the MMD space \( (K, R_\mathcal{L}, m, \mathcal{E}, \mathcal{F}) \) resulting from \( \mathcal{L}, (D, \mathbf{r}) \) as introduced in Subsection 6.1. Then \( \mathcal{G}(K, R_\mathcal{L}, m, \mathcal{E}, \mathcal{F}) = \emptyset \), i.e., the infimum in (1.4) is not attained for \( (K, R_\mathcal{L}, m, \mathcal{E}, \mathcal{F}) \).
Figure 1: The Vicsek set and the $N$-dimensional Sierpiński gaskets ($N = 2, 3$)

**Proof.** For any $h \in \mathcal{H}_0$ we have $\Gamma(h, h) \notin \mathcal{A}(K, R_\mathcal{E}, m, \mathcal{E}, \mathcal{F})$ since $\Gamma(h, h)$ does not have full support by Proposition 6.27, and therefore $\Gamma(h, h) \notin \mathcal{G}(K, R_\mathcal{E}, m, \mathcal{E}, \mathcal{F})$ for any $h \in \mathcal{H}_0$, which is equivalent to $\mathcal{G}(K, R_\mathcal{E}, m, \mathcal{E}, \mathcal{F}) = \emptyset$ by (6.32), Theorem 6.16 and (6.31) as already noted in the proof of Proposition 6.25-(3). 

### 6.3.2 Higher-dimensional Sierpiński gaskets

**Example 6.29** ($N$-dimensional Sierpiński gasket). Let $N \in \mathbb{N}$, $N \geq 2$ and let $\{q_k\}_{k=0}^N \subset \mathbb{R}^N$ be the set of the vertices of a regular $N$-dimensional simplex $\triangle_N$, so that $\triangle_N$ is the convex hull of $\{q_k \mid k \in \{0, 1, \ldots, N\}\}$ in $\mathbb{R}^N$. We further set $S := \{0, 1, \ldots, N\}$ and for each $i \in S$ define $f_i : \mathbb{R}^N \to \mathbb{R}^N$ by $f_i(x) := q_i + \frac{1}{2}(x - q_i)$. Let $K$ be the self-similar set associated with $\{f_i\}_{i \in S}$, which exists and satisfies $K \subseteq \triangle_N$ thanks to $\bigcup_{i \in S} f_i(\triangle_N) \subseteq \triangle_N$ by [Kig01, Theorem 1.1.4], and set $F_i := f_i|_K$ for each $i \in S$. Then $\mathcal{L} := (K, S, \{F_i\}_{i \in S})$ is a self-similar structure by [Kig01, Theorem 1.2.3] and called the $N$-dimensional (standard) Sierpiński gasket (see Figure 1 above), and it easily follows from $K \subset \triangle_N$ that $\mathcal{P} = \{K^x \mid k \in S\}$ and $V_0 = \{q_k \mid k \in S\}$, so that $\mathcal{L}$ is post-critically finite and $K$ is connected.

Define $D = (D_{pq})_{p,q \in V_0}$ by $D_{pp} := -N$ and $D_{pq} := 1$ for $p, q \in V_0$ with $p \neq q$. By the symmetry of $\mathcal{L}$ and $D$, there exists a unique $r \in (0, \infty)$ such that $(D, r = (r_i)_{i \in S})$ with $r_i := r$ is a harmonic structure on $\mathcal{L}$, and an elementary calculation shows that $r = \frac{N+1}{N+3} < 1$, so that $(D, r)$ is a regular harmonic structure on $\mathcal{L}$.

In the rest of this subsection, we fix the setting of Example 6.29 and consider the MMD space $(K, R_\mathcal{E}, m, \mathcal{E}, \mathcal{F})$ resulting from $\mathcal{L}, (D, r)$ as introduced in Subsection 6.1. The following result, which is essentially due to Kigami [Kig08], was the starting point of the whole study of the present paper.

**Theorem 6.30** ([Kig08, Kaj12]). Assume that $N = 2$, and let $h_1, h_2 \in \mathcal{H}_0$ satisfy $\mathcal{E}(h_1, h_1) = \mathcal{E}(h_2, h_2) = 1$ and $\mathcal{E}(h_1, h_2) = 0$. Then $d_{\text{int}}^{\Gamma(h_1,h_1)+\Gamma(h_2,h_2)}, d_{\text{int}}^{(h_1,h_1)} \in \mathcal{J}(K, R_\mathcal{E})$ and $\Gamma(h_1, h_1) + \Gamma(h_2, h_2), \Gamma(h_1, h_1) \in \mathcal{G}(K, R_\mathcal{E}, m, \mathcal{E}, \mathcal{F})$.

**Proof.** Set $\mu_{h_1,h_2} := \Gamma(h_1, h_1) + \Gamma(h_2, h_2)$ and let $\mu \in \{\mu_{h_1,h_2}, \Gamma(h_1, h_1)\}$. Note that the results in [Kig08, Sections 5 and 6] and [Kaj12, Sections 3 and 4] are applicable to $d_{\text{int}}^\mu$ by
virtue of the identification of $d^\mu_{\text{int}}$ given in [Kaj12, Theorem 4.2]. By [Kig08, Theorem 5.11] for \( \mu = \mu_{h_1,h_2} \) and [Kaj12, Proposition 3.16-(1)] for \( \mu = \Gamma(h_1,h_1) \), $d^\mu_{\text{int}}$ is a metric on $K$ compatible with the original topology of $K$ and adapted in the sense of [Kig20, Definition 2.4.7 and Proposition 2.4.8]. Also by [Kig08, Theorem 5.11 and Proof of Theorem 3.2] for \( \mu = \mu_{h_1,h_2} \) and [Kaj12, Proposition 3.16-(1) and Lemma 3.9] for \( \mu = \Gamma(h_1,h_1) \), $d^\mu_{\text{int}}$ is exponential in the sense of [Kig20, Definitions 3.1.15 and 3.6.2] and gentle with respect to the Euclidean metric $d$ on $K$ in the sense of [Kig20, Definition 3.3.1]. Since Lemma 6.14 is easily seen to hold also with $d_{\log_2}^{(5/3)}$ in place of $R_\varepsilon$, it follows that $R_\varepsilon$ is bi-Lipschitz equivalent to $d_{\log_2}^{(5/3)}$ and hence belongs to $\mathcal{J}(K,d)$ and that $d$ is adapted as well as exponential in the sense of [Kig20], so that [Kig20, Theorem 3.6.6] is applicable to $d_{\text{int}}$ and $d$, and shows, in combination with (1.3), that $d^\mu_{\text{int}} \in \mathcal{J}(K,d) = \mathcal{J}(K,R_\varepsilon)$. Finally, \((K,d^\mu_{\text{int}},\mu,\mathcal{E},\mathcal{F})\) satisfies $\text{VD}$ and $\text{HKE}(2)$ by [Kig08, Theorems 6.2 and 6.3] for $\mu = \mu_{h_1,h_2}$ and by [Kaj12, Theorem 3.19 and Corollary 4.3] for $\mu = \Gamma(h_1,h_1)$, and it thus satisfies $\text{PHI}(2)$ by Theorem 4.5, proving $\mu \in \mathcal{G}(K,R_\varepsilon,m,\mathcal{E},\mathcal{F})$. \(\square\)

One of the key observations for the validity of Theorem 6.30 is that the energy measures of harmonic functions are volume doubling with respect to the resistance metric $R_\varepsilon$ (or equivalently, with respect to the Euclidean metric $d$ on $K$), which in fact extends to the $N$-dimensional Sierpiński gasket with $N \geq 3$ as follows.

**Proposition 3.31.** \((K,R_\varepsilon,\Gamma(h,h))\) is $\text{VD}$ for any $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathcal{I}_K$. More generally, if \(\{h_n\}_{n \in \mathbb{N}} \subset \mathcal{H}_0\) satisfies $\sum_{n \in \mathbb{N}} \mathcal{E}(h_n,h_n) \in (0,\infty)$, then \((K,R_\varepsilon,\sum_{n \in \mathbb{N}} \Gamma(h_n,h_n))\) is $\text{VD}$.

**Proof.** We follow [Kig08, Proof of Theorem 3.2]. Let $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathcal{I}_K$. As noted in the second paragraph of the proof of Proposition 6.10, \((K,R_\varepsilon,\Gamma(h,h))\) is $\text{VD}$ if and only if (6.22) and (6.23) with $\mu = \Gamma(h,h)$ hold. To verify (6.22) and (6.23), recalling Proposition 6.6-(1) and (RF1), for each $j \in S$ we choose a basis \(\{h^j_k\}_{k=0}^N\) of the linear space $\mathcal{H}_0$ such that $h^j_0 = 1_K$, $h^j_1|_{\mathcal{V}_0} = N^{-1/2} 1_{\mathcal{V}_0 \setminus \{q_j\}}$, $h^j_k(q_j) = 0$ for any $k \in \{2,\ldots,N\}$ and \(\{h^j_k\}_{k=2}^N\) is orthonormal in \((\mathcal{H}_0/\mathbb{R}\mathcal{I}_K,\mathcal{E})\). Then \(\{h^j_k\}_{k=0}^2\) is also orthonormal in \((\mathcal{H}_0/\mathbb{R}\mathcal{I}_K,\mathcal{E})\) by (6.10), and it easily follows by $\mathcal{H}_0 \subset \mathcal{H}_1$, (6.11) and solving [Kig01, (3.2.1)] that $h^j_k \circ F_j = \frac{N+1}{N+3} h^j_k$ and that $F_j \circ F_i = \frac{1}{N+3} h^j_k$ for any $k \in \{2,\ldots,N\}$. In particular, for each $w \in W_\ast$, a linear map $F^w_w : \mathcal{H}_0/\mathbb{R}\mathcal{I}_K \to \mathcal{H}_0/\mathbb{R}\mathcal{I}_K$ is defined by $F^w_w(h + \mathbb{R}\mathcal{I}_K) := h \circ F_w + \mathbb{R}\mathcal{I}_K$ and is invertible, and we set $h_w := \mathcal{E}(h \circ F_w, h \circ F_w)^{-1/2} h \circ F_w \in \mathcal{H}_0 \setminus \mathbb{R}\mathcal{I}_K$.

Let \((w,j) \in W_\ast \times S\). Then $h_w = \sum_{k=0}^N a_k h^j_k$ for some \((a_k)_{k=0}^N \in \mathbb{R}^{N+1}\). Since $h_w \circ F_j = a_0 h^j_0 + \frac{N+1}{N+3} a_1 h^j_1 + \frac{N+1}{N+3} \sum_{k=2}^N a_k h^j_k$ and $\sum_{k=1}^N a_k^2 = \mathcal{E}(h_w,h_w) = 1$, by Lemma 6.20 we have

$$\frac{\frac{N+1}{N+3} \Gamma(h,h)(K_{wj})}{\Gamma(h,h)(K_w)} = \mathcal{E}(h_w \circ F_j, h_w \circ F_j) = \frac{(N+1)^2 a_1^2 + \sum_{k=2}^N a_k^2}{(N+3)^2} \in \left[ \frac{1}{(N+3)^2}, \frac{(N+1)^2}{(N+3)^2} \right],$$

proving (6.22) with $\mu = \Gamma(h,h)$.

Next, to show (6.23), let $w,v \in W_\ast$ satisfy $|w| = |v|$ and $K_w \cap K_v \neq \emptyset$. We may assume $w \neq v$, so that there exist $\tau \in W_\ast$, $n \in \mathbb{N} \cup \{0\}$ and $i,j \in S$ with $i \neq j$ such that $w = \tau i^n$ and $v = \tau j^n$. Take \((b_k)_{k=0}^N, (c_k)_{k=0}^N \in \mathbb{R}^{N+1}\) such that $h_\tau \circ F_i = \sum_{k=0}^N b_k h^i_k$ and $h_\tau \circ F_j = \sum_{k=0}^N c_k h^j_k$. Then $b_0 = h_\tau(F_i(q_j)) = h_\tau(F_j(q_i)) = c_0$ by $F_i(q_j) = F_j(q_i)$,
\(Nb_0 + N^{1/2}b_1 + NC_0 + N^{1/2}c_1 = 2Nb_0\) by the harmonicity of \(h_r\) at \(F_i(q_i)\) (see [Kig01, (3.2.1)]) and thus \(b_1 = -c_1\). Moreover, by (6.49),

\[
\sum_{k=1}^{N} b_k^2 = \sum_{k=1}^{N} c_k^2 = \frac{\mathcal{E}(h_r \circ F_i, h_r \circ F_i)}{\mathcal{E}(h_r \circ F_j, h_r \circ F_j)} \leq \frac{(N+1)^2/(N+3)^2}{1/(N+3)^2} = (N+1)^2. \tag{6.50}
\]

Since \(h_r \circ F_{ij} = b_0h_0^0 + (\frac{N+1}{N+3})^n b_1h_1^0 + (\frac{1}{N+3})^n \sum_{k=2}^{N} b_kh_k^0\) and \(h_r \circ F_{jn} = c_0h_0^0 + (\frac{N+1}{N+3})^n c_1h_1^0 + (\frac{1}{N+3})^n \sum_{k=2}^{N} c_kh_k^0\), we see from Lemma 6.20, \(b_1 = -c_1\) and (6.50) that

\[
\frac{\Gamma(h, h)(K_w)}{\Gamma(h, h)(K_v)} = \frac{\mathcal{E}(h_r \circ F_{ij}, h_r \circ F_{ij})}{\mathcal{E}(h_r \circ F_{jn}, h_r \circ F_{jn})} = \frac{(N+1)^2b_1^2 + (\frac{1}{N+3})^2 \sum_{k=2}^{N} b_k^2}{(N+1)^2c_1^2 + (\frac{1}{N+3})^2 \sum_{k=2}^{N} c_k^2} \leq (N+1)^2, \tag{6.51}
\]

which proves (6.23) with \(\mu = \Gamma(h, h)\) and thereby that \((K, R_\xi, \Gamma(h, h))\) is VD.

Finally, if \(\{h_n\}_{n \in \mathbb{N}} \subset \mathcal{H}_0\) satisfies \(\sum_{n \in \mathbb{N}} \mathcal{E}(h_n, h_n) \in (0, \infty)\), then \(\sum_{n \in \mathbb{N}} \Gamma(h_n, h_n)\) is a Radon measure on \(K\), (6.22) and (6.23) with \(\mu = \sum_{n \in \mathbb{N}} \Gamma(h_n, h_n)\) hold by (6.49) and (6.51), and hence \((K, R_\xi, \sum_{n \in \mathbb{N}} \Gamma(h_n, h_n))\) is VD.

Despite Proposition 6.31, Theorem 6.30 does NOT extend to the case of \(N \geq 3\), which is the main result of this subsection and stated as follows; recall Proposition 6.25-(3).

**Theorem 6.32.** Assume that \(N \geq 3\). Then \(d_{\text{int}}^{\Gamma(h, h)} \notin \mathcal{J}(K, R_\xi)\) for any \(h \in \mathcal{H}_0\). Equivalently, \(\mathcal{G}(K, R_\xi, m, \mathcal{E}, \mathcal{F}) = \emptyset\), i.e., the infimum in (1.4) is not attained for \((K, R_\xi, m, \mathcal{E}, \mathcal{F})\).

Theorem 6.32 is proved by deducing a contradiction to the conclusion of the following proposition through a limiting argument based on Lemmas 6.20, 6.19 and Corollary 6.18.

**Proposition 6.33.** Let \(i, j \in S, i \neq j\) and let \(h^{i,j} \in \mathcal{H}_0\) be such that \(h^{i,j}|_{V_0} = 1_{\{q_i\}} - 1_{\{q_j\}}\). Then \(d_{\text{int}}^{\Gamma(h^{i,j}, h^{i,j})}(x, y) = 0\) for any \(k, l \in S \setminus \{i, j\}\) and any \(x, y \in \{(1-t)q_k + tq_l \mid t \in [0, 1]\}\). Thus if \(N \geq 3\), then \(d_{\text{int}}^{\Gamma(h^{i,j}, h^{i,j})}\) is not a metric on \(K\) and \(\Gamma(h^{i,j}, h^{i,j}) \notin \mathcal{G}(K, R_\xi, m, \mathcal{E}, \mathcal{F})\).

**Proof.** In the same way as in the proof of Proposition 6.31, it easily follows by \(\mathcal{H}_0 \subset \mathcal{H}_1, (6.11)\) and solving [Kig01, (3.2.1)] that \(h^{i,j} \circ F_k = \frac{1}{N+3} h^{i,j}\) for any \(k \in S \setminus \{i, j\}\). Let \(u \in \mathcal{F}\) satisfy \(\Gamma(u, u) \leq \Gamma(h^{i,j}, h^{i,j})\). Then setting \(C := \text{diam}(K, R_\xi)\), we see from Lemma 6.20 that for any \(w \in \bigcup_{n=1}^{\infty} (S \setminus \{i, j\})^n\) and any \(x, y \in K_w\),

\[
|u(x) - u(y)|^2 = |(u \circ F_w)(F_w^{-1}(x)) - (u \circ F_w)(F_w^{-1}(y))|^2 \leq C\mathcal{E}(u \circ F_w, u \circ F_w) = Cr_w \Gamma(u, u)(K_w) \leq Cr_w \Gamma(h^{i,j}, h^{i,j})(K_w) = C\mathcal{E}(h^{i,j} \circ F_w, h^{i,j} \circ F_w) = (N+3)^{-2|\omega|} C\mathcal{E}(h^{i,j}, h^{i,j}). \tag{6.52}
\]
Now let \( k, l \in S \setminus \{ i, j \} \) and \( x, y \in \{(1-t)q_k + tq_l \mid t \in [0,1]\} \). Then for any \( n \in \mathbb{N} \), taking \( v^{(n)}, r^{(n)} \in \{k,l\}^n \) such that \( x \in K_{v^{(n)}} \) and \( y \in K_{r^{(n)}} \), from (6.52) we obtain

\[
|u(x) - u(y)| \leq |u(x) - u(F_{v^{(n)}}(q_k))| + |u(F_{r^{(n)}}(q_l)) - u(F_{v^{(n)}}(q_k))| + |u(F_{v^{(n)}}(q_k)) - u(y)| \\
\leq 2(N + 3)^{-n} C^{1/2} \varepsilon (h^{i,j}, h^{i,j})^{1/2} + \sum_{w \in \{k,l\}^n} |u(F_w(q_k)) - u(F_w(q_l))| \\
\leq (2 + 2^n)(N + 3)^{-n} C^{1/2} \varepsilon (h^{i,j}, h^{i,j})^{1/2} \xrightarrow{n \to \infty} 0
\]

and thus \( u(x) - u(y) = 0 \). Since \( u \in \mathcal{F} \) satisfying \( \Gamma(u, u) \leq \Gamma(h^{i,j}, h^{i,j}) \) is arbitrary, it follows from (6.47) with \( \mu = \Gamma(h^{i,j}, h^{i,j}) \) that

\[d_{\text{int}}^{(h^{i,j}, h^{i,j})}(x, y) = 0.\]

Finally, if \( N \geq 3 \), then we can choose \( k, l \) as above so that \( k \neq l \), hence \( x, y \) as above so that \( x \neq y \), thus \( d_{\text{int}}^{(h^{i,j}, h^{i,j})} \) is not a metric on \( K \) by \( d_{\text{int}}^{(h^{i,j}, h^{i,j})}(x, y) = 0 \), and therefore \( \Gamma(h^{i,j}, h^{i,j}) \not\in \mathcal{G}(K, R_{\varepsilon}, m, \mathcal{E}, \mathcal{F}) \) by Proposition 6.25-(2).

**Proof of Theorem 6.32.** Let \( h \in \mathcal{H}_0 \). By Proposition 6.25-(2),(3) it suffices to show that \( \Gamma(h, h) \not\in \mathcal{G}(K, R_{\varepsilon}, m, \mathcal{E}, \mathcal{F}) \). Since this is clear for \( h \in \mathbb{R} I_K \) by \( \Gamma(h, h)(K) = \mathcal{E}(h, h) = 0 \), in the rest of this proof we assume that \( h \in \mathcal{H}_0 \setminus \mathbb{R} I_K \). Take \( j \in S \) such that \( h(q_j) = \min_{q \in \mathcal{V}_0} h(q) \), let \( \{h_k\}_{k=1}^\infty \) be the basis of \( \mathcal{H}_0 \) as introduced in the first paragraph of the proof of Proposition 6.31, and set \( \psi := h_1 \), so that \( \psi \circ F_j = \frac{N+1}{N+3}(h - a\psi - h(q_j)I_K) \), for a unique \( a \in \mathbb{R} \) we have \( (h - a\psi - h(q_j)I_K) \circ F_j = \frac{1}{N+3}(h - a\psi - h(q_j)I_K) \), and \( a > 0 \) by \( h(q_j) = \min_{q \in \mathcal{V}_0} h(q) \).

Setting \( h_j := \mathcal{E}(h \circ F_{j^n}, h \circ F_{j^n}^{-1/2} h \circ F_{j^n} \) for \( n \in \mathbb{N} \), we easily see from these observations and \( \mathcal{E}(\psi, \psi) = 1 \) that \( \lim_{n \to \infty} \mathcal{E}(\psi - h_jn, \psi - h_jn) = 0 \), which together with the same argument as (6.45) shows that

\[
\lim_{n \to \infty} \Gamma(h_jn, h_jn)(A) = \Gamma(\psi, \psi)(A) \quad \text{for any Borel subset } A \text{ of } K. \tag{6.53}
\]

Now suppose that \( \Gamma(h, h) \in \mathcal{G}(K, R_{\varepsilon}, m, \mathcal{E}, \mathcal{F}) \), which means, by (6.30) and (6.31), that \( (\theta, \Gamma(h_jn, h_jn)) \in \mathcal{G}(\eta, C) \) for some metric \( \theta \) on \( K \) and \( (\eta, C) \in \text{Homeo}^+(\times(1, \infty)) \). For each \( n \in \mathbb{N} \) we would define \( \theta_jn \) by (6.39) with \( \mu = \Gamma(h_jn, h_jn) \) and \( w = j^n \), so that \( \{\theta_jn, \Gamma(h_jn, h_jn)\}_{n \in \mathbb{N}} \subset \mathcal{G}(c_{R\varepsilon}^{-1}_n, C) \) by Lemmas 6.20 and 6.19. Then since \( \{\Gamma(h_jn, h_jn)\}_{n \in \mathbb{N}} \) converges to \( \Gamma(\psi, \psi) \) in \( \mathcal{P}(K) \) by (6.53), Corollary 6.18 would imply that \( (d_\psi, \Gamma(\psi, \psi)) \in \mathcal{G}(c_{R\varepsilon}^{-1}_n, C) \) for some metric \( d_\psi \) on \( K \). Next, we would further let \( i \in S \setminus \{j\} \), set \( \varphi := \mathcal{E}(\psi \circ F_i, \psi \circ F_i)^{-1/2}(\psi \circ F_i - N^{-1/2} \frac{N+3}{N+1} I_K) \) and define \( d_\varphi : K \times K \to [0, \infty) \) by \( d_\varphi(x, y) := \mathcal{E}(\psi \circ F_i, \psi \circ F_i)^{-1/2} d_\psi(F_i(x), F_i(y)) \). Then we would have \( \varphi = \frac{2N+2}{N+2} h^{i,j} \) with \( h^{i,j} \) as in Proposition 6.33 by \( \mathcal{H}_0 \subset \mathcal{H}_1 \), (6.11) and solving \([\text{Kig01}, (3.2.1)], \varphi \in \mathcal{G}(c_{R\varepsilon}^{-2}_n, C) \) by Lemmas 6.20 and 6.19 and thus \( \Gamma(h^{i,j}, h^{i,j}) = (2N+2) \Gamma(\varphi, \varphi) \in \mathcal{G}(K, R_{\varepsilon}, m, \mathcal{E}, \mathcal{F}) \) by (6.30) and (6.31), which would be a contradiction to Proposition 6.33 and thereby proves that \( \Gamma(h, h) \not\in \mathcal{G}(K, R_{\varepsilon}, m, \mathcal{E}, \mathcal{F}) \).

We conclude this subsection with the following theorem, which is an easy consequence of the conjunction of Proposition 6.31, [Kig12, Corollary 15.12] and Theorem 6.32.

**Theorem 6.34.** Assume that \( N \geq 3 \), let \( \{h_n\}_{n \in \mathbb{N}} \subset \mathcal{H}_0 \) satisfy \( \sum_{n \in \mathbb{N}} \mathcal{E}(h_n, h_n) \in (0, \infty) \) and set \( \mu := \sum_{n \in \mathbb{N}} \Gamma(h_n, h_n) \). Then there does not exist a metric \( \theta \) on \( K \) compatible with the original topology of \( K \) such that the MMD space \( (K, \theta, \mu, \mathcal{E}, \mathcal{F}) \) satisfies HKE(2).
Proof. Since \((K, R_\varepsilon, \mu)\) is VD by Proposition 6.31, \(\mu\) is a Radon measure on \(K\) with full support, hence \(\mu \in A(K, R_\varepsilon, m, \varepsilon, F)\) by (6.12) with \(m\) in place of \(\mu\), and [Kig12, Corollary 15.12] is applicable to \((K, R_\varepsilon, \mu, \varepsilon, F)\) and yields \(\rho \in \mathcal{J}(K, R_\varepsilon)\) and \(\beta \in (1, \infty)\) such that \((K, \rho, \mu, \varepsilon, F)\) has a continuous heat kernel \(p^\mu = p^\mu(t, x, y) : (0, \infty) \times K \times K \to \mathbb{R}\) and satisfies VD and HKE(\(\beta\)) with “\(\mu\)-a.e. \(x, y\)” in (4.1) and (4.2) replaced by “any \(x, y\)”.

Now suppose that there existed a metric \(\theta\) on \(K\) compatible with the original topology of \(K\) such that \((K, \theta, \mu, \varepsilon, F)\) satisfied HKE(2). Then for each \(t \in (0, \infty)\), we easily see from the continuity of \(p^\mu(t, x, y)\), the lower semi-continuity of \(\mu(B_\theta(t, 1/2))\), the upper semi-continuity of \(\mu(B_\theta(t, 1/2))\) and VD of \((K, \rho, \mu)\) that HKE(2) for \((K, \theta, \mu, \varepsilon, F)\) would also hold with the same heat kernel \(p^\mu\) and with “\(\mu\)-a.e. \(x, y\)” in (4.1) and (4.2) replaced by “any \(x, y\)”

6.4 The case of generalized Sierpiński carpets

In this subsection, we treat the case of the canonical self-similar Dirichlet form on an arbitrary generalized Sierpiński carpet and see that the arguments in Subsection 6.2 go through also in this case with just slight modifications required in the proofs. We closely follow the presentation of [Kaj14, Section 4] for the introduction of the framework of this subsection up to Theorem 6.45 below, which we repeat here for the reader’s convenience.

We fix the following setting throughout this subsection.

**Framework 6.35.** Let \(N, l \in \mathbb{N}, N \geq 2, l \geq 3\) and set \(Q_0 := [0, 1]^N\). Let \(S \subset \{0, 1, \ldots, l - 1\}^N\) be non-empty, define \(f_i : \mathbb{R}^N \to \mathbb{R}^N\) by \(f_i(x) := l^{-1}i + l^{-1}x\) for each \(i \in S\) and set \(Q_1 := \bigcup_{i \in S} f_i(Q_0)\), so that \(Q_1 \subsetneq Q_0\). Let \(K\) be the self-similar set associated with \(\{f_i\}_{i \in S}\), i.e., the unique non-empty compact subset of \(\mathbb{R}^N\) such that \(K = \bigcup_{i \in S} f_i(K)\), which exists and satisfies \(K \subsetneq Q_0\) thanks to \(Q_1 \subsetneq Q_0\) by [Kig01, Theorem 1.1.4], and set \(F_i := f_i|_K\) for each \(i \in S\), so that GSC\((N, l, S) := (K, S, \{F_i\}_{i \in S})\) is a self-similar structure by [Kig01, Theorem 1.2.3]. Let \(d : K \times K \to [0, \infty)\) be the Euclidean metric on \(K\) given by \(d(x, y) := \|x - y\|\), set \(d_t := \log_t S\), and let \(m\) be the self-similar measure on GSC\((N, l, S)\) with weight \((1/\#S)_{i \in S}\), i.e., the unique Borel probability measure on \(K\) such that \(m(K_w) = (\#S)^{-|w|}\) for any \(w \in \mathcal{W}_s\), which exists by [Kig01, Propositions 1.5.8, 1.4.3, 1.4.4 and Corollary 1.4.8].

Recall that \(d_t\) is the Hausdorff dimension of \((K, d)\) and that \(m\) is a constant multiple of the \(d_t\)-dimensional Hausdorff measure on \((K, d)\); see, e.g., [Kig01, Proposition 1.5.8 and Theorem 1.5.7]. Note that \(d_t < N\) by \(S \subsetneq \{0, 1, \ldots, l - 1\}^N\).
The following definition is essentially due to Barlow and Bass [BB99, Section 2]. In what follows, the interior of $A \subset \mathbb{R}^N$ in $\mathbb{R}^N$ is denoted by $\text{int}_{\mathbb{R}^N}(A)$.

**Definition 6.36 (Generalized Sierpiński carpet, [BBKT, Subsection 2.2]).** GSC($N, l, S$) is called a *generalized Sierpiński carpet* if and only if the following four conditions are satisfied:

(GSC1) (Symmetry) $f(Q_1) = Q_1$ for any isometry $f$ of $\mathbb{R}^N$ with $f(Q_0) = Q_0$.

(GSC2) (Connectedness) $Q_1$ is connected.

(GSC3) (Non-diagonality) $\text{int}_{\mathbb{R}^N}(Q_1 \cap \prod_{k=1}^{N} [(i_k - \varepsilon_k)l^{-1}, (i_k + 1)l^{-1}])$ is either empty or connected for any $(i_k)_{k=1}^{N} \in \mathbb{Z}^N$ and any $(\varepsilon_k)_{k=1}^{N} \in \{0, 1\}^N$.

(GSC4) (Borders included) $[0, 1] \times \{0\}^{N-1} \subset Q_1$.

As special cases of Definition 6.36, GSC($2, 3, S_{SC}$) and GSC($3, 3, S_{MS}$) are called the *Sierpiński carpet* and the *Menger sponge*, respectively, where $S_{SC} := \{0, 1, 2\}^2 \setminus \{(1, 1)\}$ and $S_{MS} := \{(1, 2, 3) \in \mathbb{Z}^3 \mid \sum_{k=1}^{3} \mathbf{1}_{\{1\}}(i_k) \leq 1\}$ (see Figure 2 above).

See [BB99, Remark 2.2] for a description of the meaning of each of the four conditions (GSC1), (GSC2), (GSC3) and (GSC4) in Definition 6.36. We remark that there are several equivalent ways of stating the non-diagonality condition, as in the following proposition.

**Proposition 6.37 ([Kaj10, §2]).** Set $|x|_1 := \sum_{k=1}^{N} |x_k|$ for $x = (x_k)_{k=1}^{N} \in \mathbb{R}^N$. Then (GSC3) is equivalent to any one of the following three conditions:

(ND)$_{N}$ $\text{int}_{\mathbb{R}^N}(Q_1 \cap \prod_{k=1}^{N} [(i_k - 1)l^{-n}, (i_k + 1)l^{-n}])$ is either empty or connected for any $n \in \mathbb{N}$ and any $(i_k)_{k=1}^{N} \in \{1, 2, \ldots, l^n - 1\}^N$.

(ND)$_{2}$ $\text{int}_{\mathbb{R}^N}(Q_1 \cap \prod_{k=1}^{N} [(i_k - 1)l^{-2}, (i_k + 1)l^{-2}])$ is either empty or connected for any $(i_k)_{k=1}^{N} \in \{1, 2, \ldots, l^2 - 1\}^N$.

(NDF) For any $i, j \in S$ with $f_i(Q_0) \cap f_j(Q_0) \neq \emptyset$ there exists $\{(n(k))_{k=0}^{i-j+1} \subset S$ such that $n(0) = i$, $n(|i-j|) = j$ and $|n(k) - n(k+1)|_1 = 1$ for any $k \in \{0, \ldots, |i-j| - 1\}$.

**Remark 6.38.** (1) Only the case of $n = 1$ of (ND)$_{N}$ had been assumed in the original definition of generalized Sierpiński carpets in [BB99, Section 2], but Barlow, Bass, Kumagai and Teplyaev [BBKT] later realized that it had been too weak for [BB99, Proof of Theorem 3.19] and had to be replaced by (ND)$_{N}$ (or equivalently, by (GSC3)).
(2) In fact, [BBKT, Subsection 2.2] assumes instead of (GSC2) the seemingly stronger condition that \(\text{int}_{\mathbb{R}^N} Q_1\) is connected, but it is implied by (GSC2) and (GSC3) in view of (NDF) in Proposition 6.37 and is thus equivalent to (GSC2) under the assumption of (GSC3).

In the rest of this subsection, we assume that \(\mathcal{L} := \text{GSC}(N, l, S) = (K, S, \{F_i\}_{i \in S})\) is a generalized Sierpiński carpet. Then we easily see the following proposition and lemma.

**Proposition 6.39.** Set \(S_{k, \varepsilon} := \{(i_n)_{n=1}^N \in S \mid i_k = (l-1)\varepsilon\}\) for \(k \in \{1, 2, \ldots, N\}\) and \(\varepsilon \in \{0, 1\}\). Then \(\mathcal{P}_L = \bigcup_{k=1}^N (S_{k,0}^N \cup S_{k,1}^N)\) and \(V_0 = V_1 = K \setminus (0, 1)^N \neq K\).

**Lemma 6.40.** There exist \(c_1, c_2 \in (0, \infty)\) such that for any \((x, s) \in K \times (0, \text{diam}(K, d)]\),
\[
c_1 s^{d_1} \leq m(B_d(x, s)) \leq c_2 s^{d_1}. \tag{6.54}
\]

We next recall some basics of the canonical self-similar Dirichlet form on \(\text{GSC}(N, l, S)\).

There are two established ways of constructing a non-degenerate \(m\)-symmetric diffusion without killing inside on \(K\), or equivalently, a non-zero, strongly local, regular symmetric Dirichlet form on \(L^2(K, m)\), one by Barlow and Bass [BB89, BB99] using the reflecting Brownian motions on the domains approximating \(K\), and the other by Kusuoka and Zhou [KZ92] based on graph approximations. It had been a long-standing open problem to prove that the constructions in [BB89, BB99] and in [KZ92] give rise to the same diffusion on \(K\), which Barlow, Bass, Kumagai and Teplyaev [BBKT] have finally solved by proving the uniqueness of a non-zero conservative regular symmetric Dirichlet form on \(L^2(K, m)\) possessing certain local symmetry. As a consequence of the results in [BBKT], after some additional arguments in [Hin13, Kaj13, Kaj20] we have the unique existence of a canonical self-similar Dirichlet form \((\mathcal{E}, \mathcal{F})\) on \(L^2(K, m)\), the heat kernel estimates \(\text{HKE}(d_w)\) with \(d_w > 2\), and hence \(\text{PHI}(d_w)\) by Lemma 6.40 and Theorem 4.5, as follows.

**Definition 6.41.** We define \(\mathcal{I}_0 := \{f|_K \mid f\) is an isometry of \(\mathbb{R}^N, f(Q_0) = Q_0\}\), which forms a subgroup of the group of isometries of \((K, d)\) by virtue of (GSC1).

**Theorem 6.42** ([BBKT, Theorems 1.2 and 4.32], [Hin13, Proposition 5.1], [Kaj13, Proposition 5.9]). There exists a unique (up to constant multiples of \(\mathcal{E}\)) regular symmetric Dirichlet form \((\mathcal{E}, \mathcal{F})\) on \(L^2(K, m)\) satisfying \(\mathcal{E}(u, u) > 0\) for some \(u \in \mathcal{F}\), \(\mathcal{E}(1_K, 1_K) = 0\), (6.7), (6.8) for some \(r \in (0, \infty)\), and the following:

\((\text{GSCDF})\) If \(u \in \mathcal{F} \cap C(K)\) and \(g \in \mathcal{I}_0\) then \(u \circ g \in \mathcal{F}\) and \(\mathcal{E}(u \circ g, u \circ g) = \mathcal{E}(u, u)\).

**Definition 6.43.** The regular symmetric Dirichlet form \((\mathcal{E}, \mathcal{F})\) on \(L^2(K, m)\) as in Theorem 6.42 is called the **canonical Dirichlet form** on \(\text{GSC}(N, l, S)\), and we set \(d_w := \log_2(\#S/r)\).

Note that \((\mathcal{E}, \mathcal{F})\) is also strongly local by the same argument as [Hin05, Proof of Lemma 3.12] based on (6.7), (6.8) and \(\mathcal{E}(1_K, 1_K) = 0\).

**Theorem 6.44** ([BB99, Remarks 5.4-1.], [Kaj20, Theorem 2.9]). \(d_w > 2\).
Theorem 6.45 ([BB99, Theorem 1.3], [BBKT, Theorem 4.30 and Remark 4.33]). There exists a (unique) continuous version $p = p_t(x, y) : (0, \infty) \times K \times K \to [0, \infty)$ of the heat kernel of $(K, d, m, E, F)$, and there exist $c_1, c_2, c_3, c_4 \in (0, \infty)$ such that for any $(t, x, y) \in (0, 1] \times K \times K$,

$$\frac{c_1}{t^{d/d_w}} \exp\left(-\left(\frac{(d(x, y)^{d_w}}{c_2 t}\right)^{\frac{1}{p_w - 1}}\right) \leq p_t(x, y) \leq \frac{c_3}{t^{d/d_w}} \exp\left(-\left(\frac{(d(x, y)^{d_w}}{c_4 t}\right)^{\frac{1}{p_w - 1}}\right).$$

(6.55)

In particular, $(K, d, m, E, F)$ satisfies HKE($d_w$) and PHI($d_w$) by Lemma 6.40 and Theorem 4.5.

As the counterpart of Proposition 6.10 for generalized Sierpiński carpets, we have the following characterization of the pair $(\theta, \mu)$ of $\theta \in \mathcal{J}(K, d)$ (recall (1.2)) and $\mu \in \mathcal{A}(K, d, m, E, F)$ (recall Definition 2.7) such that $(K, \theta, \mu, E^\mu, F^\mu)$ satisfies PHI($\beta$); see also [Kig19] for related results.

**Theorem 6.46.** Let $\theta \in \mathcal{J}(K, d)$, let $\mu$ be a Radon measure on $K$ with full support and let $\beta \in (1, \infty)$. Then the following conditions are equivalent:

(a) $\mu \in \mathcal{A}(K, d, m, E, F)$ and $(K, \theta, \mu, E^\mu, F^\mu)$ satisfies PHI($\beta$).

(b) There exists $C \in (1, \infty)$ such that

$$C^{-1}(\text{diam}(K_w, \theta))^\beta \leq r_w \mu(K_w) \leq C(\text{diam}(K_w, \theta))^\beta$$

for any $w \in W_*$, (6.56)

where $r_w := r_{|w|}$. Moreover, if either of these conditions holds, then $\mu(F_w(V_0)) = 0$ for any $w \in W_*$ and $\mu(\{x\}) = 0$ for any $x \in K$.

Theorem 6.46 follows by repeating the same arguments as the proof of Proposition 6.10, on the basis of the following proposition concluded from Theorem 4.5 with the help of [BCM, Lemma 5.21 and Proposition 6.17].

**Proposition 6.47.** Let $\theta \in \mathcal{J}(K, d)$, let $\mu$ be a Radon measure on $K$ with full support and let $\beta \in (1, \infty)$. Then Theorem 6.46-(a) is equivalent to the following condition:

(c) $(K, \theta, \mu)$ is VD and there exists $C \in (0, \infty)$ such that for any $x, y \in K$ with $x \neq y$,

$$C^{-1}\theta(x, y)^\beta \leq d(x, y)^{d_w-d_\mu}(B_\theta(x, \theta(x, y))) \leq C\theta(x, y)^\beta.$$  

(6.57)

**Proof.** Note first that, if $\mu$ is admissible with respect to $(K, d, m, E, F)$, then for any Borel subsets $A, B$ of $K$ with $B$ closed in $K$ and $\overline{A} \cap B = \emptyset$, by [CF, Theorem 5.2.11] and [FOT, Theorem 4.6.2 and Lemma 2.1.4] we have

$$\text{Cap}^\mu(A, B) = \inf\{E(f^+ \land 1, f^+ \land 1) \mid f \in F^\mu, \overline{A} \cap \text{supp}_\mu[f - 1_K] = \emptyset = B \cap \text{supp}_\mu[f]\}$$

$$= \inf\{E(f, f) \mid f \in F^\mu, 0 \leq f \leq 1 \text{-a.e., } \overline{A} \cap \text{supp}_\mu[f - 1_K] = \emptyset = B \cap \text{supp}_\mu[f]\}$$

$$= \inf\{E(f, f) \mid f \in F(A, B), 0 \leq f \leq 1 \text{-a.e.}\}$$

$$= \inf\{E(f^+ \land 1, f^+ \land 1) \mid f \in F(A, B)\} = \text{Cap}(A, B),$$

(6.58)
where $\text{Cap}^\mu(A, B)$ and $\text{Cap}(A, B)$ denote the capacity between $A, B$ with respect to $(K, \theta, \mu, {\mathcal E}, {\mathcal F})$ and $(K, d, m, {\mathcal E}, {\mathcal F})$, respectively. Next, since $(K, d, m, {\mathcal E}, {\mathcal F})$ satisfies (6.54), $\text{cap}(d_w)$ and EHI by Lemma 6.40, Theorems 6.45 and 4.5, there exist $C_1, A_1, A_2 \in (1, \infty)$ with $A_2 \geq 2$ such that for any $(x, s) \in K \times (0, \text{diam}(K, d)/A_2)$,

$$C_1^{-1} s^{d_1-d_w} \leq \text{Cap}(B_d(x, s), K \setminus B_d(x, A_1 s)) \leq C_1 s^{d_1-d_w},$$

(6.59)

and by the quasisymmetry of $\theta$ to $d$ and Lemma 4.8 we have EHI for $(K, \theta, m, {\mathcal E}, {\mathcal F})$, as well as for $(K, \theta, \mu, {\mathcal E}, {\mathcal F})$ provided $\mu \in {\mathcal A}(K, d, m, {\mathcal E}, {\mathcal F})$.

To prove the desired equivalence, assume Theorem 6.46-(a), so that $(K, \theta, \mu, {\mathcal E}, {\mathcal F})$ satisfies VD and $\text{cap}(\beta)$ by Theorem 4.5 and therefore in view of (6.58) there exist $C_2, A_3, A_4 \in (1, \infty)$ such that

$$C_2^{-1} \frac{\mu(B_\theta(x, s))}{s^\beta} \leq \text{Cap}(B_\theta(x, s), K \setminus B_\theta(x, A_3 s)) \leq C_2 \frac{\mu(B_\theta(x, s))}{s^\beta}$$

(6.60)

for any $(x, s) \in K \times (0, \text{diam}(K, \theta)/A_4)$. To verify (6.57), let $x, y \in K$ satisfy $x \neq y$. By the quasisymmetry of $\theta$ to $d$, (3.2) and (3.4), there exist $A_5 \in (1, \infty)$ determined solely by $d, \theta, A_1$ and $A_6 \in (1, \infty)$ determined solely by $d, \theta, A_3$ such that

$$B_\theta(x, \theta(x, y)/A_5) \subset B_d(x, d(x, y)) \subset B_d(x, A_1 d(x, y)) \subset B_\theta(x, A_5 \theta(x, y)),$$

(6.61)

$$B_d(x, d(x, y)/A_6) \subset B_\theta(x, \theta(x, y)) \subset B_\theta(x, A_3 \theta(x, y)) \subset B_d(x, A_6 d(x, y)).$$

(6.62)

Then by EHI for $(K, d, m, {\mathcal E}, {\mathcal F})$ and $(K, \theta, m, {\mathcal E}, {\mathcal F})$ and [BCM, Lemma 5.21] (note also [BCM, Theorem 5.4 and Lemma 5.2-(e)]), there exist $A_7 \in (A_2, \infty)$ and $A_8 \in (A_4, \infty)$ such that for any $(z, s) \in K \times (0, \infty)$,

$$\text{Cap}(B_d(z, s), K \setminus B_d(z, A_1 s)) \asymp \text{Cap}(B_d(z, s), K \setminus B_d(z, A_5^2 s)) \quad \text{if } s < \text{diam}(K, d)/A_7,$$

(6.63)

$$\text{Cap}(B_\theta(z, s), K \setminus B_\theta(z, A_3 s)) \asymp \text{Cap}(B_\theta(z, s), K \setminus B_\theta(z, A_5^2 s)) \quad \text{if } s < \text{diam}(K, \theta)/A_8.$$  

(6.64)

(To be precise, the definition of capacity between sets in [BCM, Section 5] is slightly different from ours, but they are easily seen to be equivalent to each other by virtue of [FOT, Lemma 2.2.7-(ii)].) Moreover, the quasisymmetry of $\theta$ to $d$ again and (3.6) show that by taking $A_8$ large enough we may further assume that $\theta(x, y) < \text{diam}(K, \theta)/A_8$ implies $d(x, y) < \text{diam}(K, d)/A_7$. Now, if $\theta(x, y) \geq \text{diam}(K, \theta)/A_8$, then (6.57) clearly holds for some sufficiently large $C \in (0, \infty)$ independent of $x, y$ since $\mu(B_\theta(x, \theta(x, y))) \asymp \mu(K)$ by VD of $(K, \theta, \mu)$ and $d(x, y) \asymp \text{diam}(K, d)$ by the quasisymmetry of $\theta$ to $d$ and (3.6). Otherwise $\theta(x, y) < \text{diam}(K, \theta)/A_8$, which implies $d(x, y) < \text{diam}(K, d)/A_7$, hence

$$C_2 \frac{\mu(B_\theta(x, \theta(x, y)))}{\theta(x, y)^\beta} \geq \text{Cap}(B_\theta(x, \theta(x, y)), K \setminus B_\theta(x, A_3 \theta(x, y))) \geq \text{Cap}(B_d(x, d(x, y)/A_6), K \setminus B_d(x, A_6 d(x, y))) \asymp d(x, y)^{d_1-d_w}$$

95
by (6.60), (6.62), (6.63) and (6.59), and similarly

\[ C_1 d(x, y)^{d_1-d_w} \geq \text{Cap}(B_d(x, d(x, y)), K \setminus B_d(x, A_1 d(x, y))) \]

\[ \geq \text{Cap}(B_\theta(x, \theta(x, y)/A_5), K \setminus B_\theta(x, A_5 \theta(x, y))) \times \frac{\mu(B_\theta(x, \theta(x, y)))}{\theta(x, y)^\beta} \]

by (6.59), (6.61), (6.64), (6.60) and VD of \((K, \theta, \mu)\), proving (6.57) and thereby (c).

Conversely, assume (c). Since \(K\) is connected, VD of \((K, \theta, \mu)\) implies RVD of \((K, \theta, \mu)\) by Remark 3.18-(b). Note that (6.61) remains valid and that for each \(A_3, A_4 \in (1, \infty)\) we still have (6.62), (6.63) and (6.64). Note also that by the connectedness of \(K\) and [Hei, Theorem 11.3] there exist \(\lambda, \alpha \in [1, \infty)\) such that \(\theta\) is \(\eta_{\alpha, \lambda}\)-quasisymmetric to \(d\) with \(\eta_{\alpha, \lambda}\) as in Definition 3.1. Let \(x \in K\) and let \(s_1, s_2 \in (0, \text{diam}(K, d)/A_2)\) satisfy \(s_1 \leq s_2\). Then \(d(x, y) = s_1\) and \(d(x, z) = s_2\) for some \(y, z \in K\) by the connectedness of \(K\).

\[ \mu(B_d(x, s_1)) \text{Cap}(B_d(x, s_1), K \setminus B_d(x, A_1 s_1)) \leq \frac{d(x, s_1)^{d_1-d_w} \mu(B_d(x, \theta(x, z)))}{d(x, s_2)^{d_1-d_w} \mu(B_d(x, \theta(x, y)))} \]

\[ \leq \left( \frac{\theta(x, z)}{\theta(x, y)} \right)^\beta \left[ \frac{1}{2\lambda} \left( \frac{s_2}{s_1} \right)^{1/\alpha} \right] \]

(6.65)

It follows from (6.65) that [BCM, Proposition 6.17] is applicable to \(\mu\) and implies that \(\mu\) is admissible with respect to \((K, d, m, \mathcal{E}, \mathcal{F})\). In particular, \((K, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)\) satisfies EHI. Finally, to show \(\text{cap}(\beta)\) for \((K, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)\), let \(A_3, A_4 \in (1, \infty)\) satisfy \(A_4 \geq 2\), and choose \(A_6, A_7, A_8 \in (1, \infty)\) with \(A_7 > A_2\) and \(A_8 > A_4\) so that (6.62), (6.63) and (6.64) hold. Again thanks to the quasisymmetry of \(\theta\) to \(d\) and (3.6), by taking \(A_8\) large enough we may further assume that \(d(x, y) < \text{diam}(K, d)/A_7\) for any \(x, y \in K\) with \(\theta(x, y) < \text{diam}(K, \theta)/A_8\). Let \((x, s) \in K \times (0, \text{diam}(K, \theta)/(A_5 A_8))\), so that \(\theta(x, y) = s = \theta(x, z)/A_5\) for some \(y, z \in K\) by the connectedness of \(K\). Then by (6.62), (6.63), (6.59) and (6.57),

\[ \text{Cap}(B_\theta(x, s), K \setminus B_\theta(x, A_3 s)) \geq \text{Cap}(B_d(x, d(x, y)/A_6), K \setminus B_d(x, A_6 d(x, y))) \]

\[ \geq d(x, y)^{d_1-d_w} \times \frac{\mu(B_\theta(x, \theta(x, y)))}{\theta(x, y)^\beta} = \frac{\mu(B_\theta(x, s))}{s^\beta}, \]

and similarly by (6.64), (6.61), (6.59), (6.57) and VD of \((K, \theta, \mu)\),

\[ \text{Cap}(B_\theta(x, s), K \setminus B_\theta(x, A_3 s)) \leq \text{Cap}(B_d(x, d(x, z)), K \setminus B_d(x, A_1 d(x, z))) \]

\[ \leq d(x, z)^{d_1-d_w} \times \frac{\mu(B_\theta(x, \theta(x, z)))}{\theta(x, z)^\beta} \times \frac{\mu(B_\theta(x, s))}{s^\beta}, \]

proving (6.60) for \((x, s) \in K \times (0, \text{diam}(K, \theta)/(A_5 A_8))\), namely \(\text{cap}(\beta)\) for \((K, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)\). Now \((K, \theta, \mu, \mathcal{E}^\mu, \mathcal{F}^\mu)\) satisfies PHI(\(\beta\)) by Theorem 4.5, showing Theorem 6.46-(a). □
Proof of Theorem 6.46. By Proposition 6.47, it suffices to prove the equivalence of Proposition 6.47-(c) and (b). We can verify it in exactly the same way as the proof of Proposition 6.10, by considering the scale \( S = \{ \Lambda_s \}_{s \in [0,1]} \) on \( \Sigma \) defined by \( \Lambda_1 := \{ \emptyset \} \) and
\[
\Lambda_s := \{ w \mid w = w_1 \ldots w_n \in W_* \setminus \{ \emptyset \}, l^{1-|w|} \geq s \}
\]
for each \( s \in (0,1) \), which clearly satisfies (6.21), and by using instead of \( R_E \) the Euclidean metric \( d \) on \( K \), which is easily seen to satisfy (6.18) with \( d \) in place of \( R_E \); note that since (6.19) needs to be replaced by (6.57) we also need to replace \( R_E(x,y) \) in (6.25) by \( d(x,y)^{d_w-d_f} \mu(B_d(x,d(x,y))) \). \( \square \)

By virtue of Theorem 6.46, the whole of Subsection 6.2 can be easily adapted for the present case with just slight modifications in the proofs, and below we explicitly state the details of the adaptation for concreteness.

Definition 6.48. For each \((\eta,C) \in \text{Homeo}^+ \times (1,\infty)\) we define \( \mathcal{G}(\eta,C) = \mathcal{G}_{N,l,S}(\eta,C) \) by
\[
\mathcal{G}(\eta,C) := \left\{ (\theta,\mu) \mid \begin{array}{l}
\theta \text{ is a metric on } K \text{ quasisymmetric to } d, \mu \in \mathcal{P}(K),

C^{-1} \leq r_w \mu(K_w)/(\text{diam}(K_w,\theta))^2 \leq C \text{ for any } w \in W_*
\end{array} \right\},
\]
which is considered as a subset of \( C(K \times K) \times \mathcal{P}(K) \), and we also set
\[
\mathcal{G} := \mathcal{G}_{N,l,S} := \bigcup_{\eta \in \text{Homeo}^+,C \in (1,\infty)} \mathcal{G}(\eta,C).
\]

Since \( \mu \) is a Radon measure on \( K \) with full support for any \( (\theta,\mu) \in \mathcal{G} \), it follows from Theorem 6.46 with \( \beta = 2 \) and (1.3) that
\[
\mathcal{G}(K,d,m,E,F) = \{ a\mu \mid (\theta,\mu) \in \mathcal{G}, a \in (0,\infty) \}.
\]
In particular,
\[
\mathcal{G}(K,d,m,E,F) \neq \emptyset, \text{ i.e., the infimum in (1.4) is attained for } (K,d,m,E,F),
\]
if and only if \( \mathcal{G} \neq \emptyset \), namely \( \mathcal{G}(\eta,C) \neq \emptyset \) for some \((\eta,C) \in \text{Homeo}^+ \times (1,\infty)\).

The following is the main result of this subsection. We take arbitrary \((\eta,C) \in \text{Homeo}^+ \times (1,\infty)\), define \( \tilde{\eta} \in \text{Homeo}^+ \) by \( \tilde{\eta}(t) := 1/\eta^{-1}(t^{-1}) \) \((\tilde{\eta}(0) := 0)\) and fix them throughout the rest of this subsection.

Theorem 6.49. If \( \mathcal{G}(\eta,C) \neq \emptyset \), then there exist \( h \in \mathcal{F} \) which is \( E \)-harmonic on \( K \setminus V_0 \) and a metric \( \theta_h \) on \( K \) such that \( (\theta_h,\Gamma(h,h)) \in \mathcal{G}(\eta,C) \).

The rest of this subsection is devoted to a brief sketch of the proof of Theorem 6.49, which goes in exactly the same way as that of Theorem 6.16 except for slight modifications explained in some detail below.

Proposition 6.50. \( \mathcal{G}(\eta,C) \) is a compact subset of \( C(K \times K) \times \mathcal{P}(K) \).
The proof of Proposition 6.17 remains valid also in this case, except that $R_{\mathcal{E}}$ needs to be replaced by $\mathcal{S}$ and that $s$ in the last paragraph needs to be defined as $s := l^{-|w|}$. □

**Corollary 6.51.** Let $\{\{\theta_n, \mu_n\}\}_{n \in \mathbb{N}} \subseteq \mathcal{G}(\eta, \mathcal{C})$, $\mu \in \mathcal{P}(K)$ and suppose that $\{\mu_n\}_{n \in \mathbb{N}}$ converges to $\mu$ in $\mathcal{P}(K)$. Then there exists a metric $\theta$ on $K$ such that $(\theta, \mu) \in \mathcal{G}(\eta, \mathcal{C})$.

**Proof.** The proof of Corollary 6.18 remains valid with Proposition 6.50 applied in place of Proposition 6.17. □

**Lemma 6.52.** Let $(\theta, \mu) \in \mathcal{G}(\eta, \mathcal{C})$, $w \in W_*$ and define $(\theta_w, \mu_w) \in C(K \times K) \times \mathcal{P}(K)$ by (6.39). Then $(\theta_w, \mu_w) \in \mathcal{G}(\eta, \mathcal{C})$.

**Proof.** The proof of Lemma 6.19 remains valid also in this case. □

**Lemma 6.53.** Let $w \in W_*$. Then $\int_K |u \circ F_w| \ dm = (\#S)^{|w|} \int_{K_w} |u| \ dm$ for any Borel measurable function $u : K \to [-\infty, \infty]$, and hence a bounded linear operator from $L^2(K, m)$ to itself is defined by $u \mapsto u \circ F_w$. Moreover, $u \circ F_w \in \mathcal{F}$ and (6.8) holds for any $u, v \in \mathcal{F}$.

**Proof.** The former assertions are immediate from $m = (\#S)^{|w|} m \circ F_w$, and the latter follows from (6.7), (6.8) and the regularity of $(\mathcal{E}, \mathcal{F})$; see [Kaj20, Proof of Lemma 3.3]. □

**Lemma 6.54.** Suppose that $\mathcal{G} \neq \emptyset$. Let $u \in \mathcal{F}$ and $w \in W_*$. Then $\Gamma(u, u)(F_w(A)) = r_w^{-1} \Gamma(u \circ F_w, u \circ F_w)(A)$ for any Borel subset $A$ of $K$, and in particular $\Gamma(u, u)(K_w) = r_w^{-1} \mathcal{E}(u \circ F_w, u \circ F_w)$. Moreover, if $\Gamma(u, u)(K_w) > 0$, then (6.40) holds for any Borel subset $A$ of $K$.

**Proof.** By (6.7), (6.8) and [Hin05, Lemma 3.11-(ii)], for any $v \in \mathcal{F}$ and any $n \in \mathbb{N} \cup \{0\}$,

$$\Gamma(v, v)(A) = \sum_{\tau \in \mathcal{W}_w} \frac{1}{r_{\tau}} \Gamma(v \circ F_{\tau}, v \circ F_{\tau})(F_{\tau}^{-1}(A))$$

for any Borel subset $A$ of $K$. (6.71)

By $\mathcal{G} \neq \emptyset$ we can take $(\theta, \mu) \in \mathcal{G}$, then $\mu$ is a minimal energy-dominating measure of $(\mathcal{E}, \mathcal{F})$ by Theorem 6.46 and Proposition 2.10-(b), thus $\Gamma(v, v) \ll \mu$ for any $v \in \mathcal{F}$, and hence

$$\Gamma(v, v)(V_0) = 0$$

for any $v \in \mathcal{F}$ (6.72)

since $\mu(V_0) = 0$ by Theorem 6.46. Recalling that for any $\tau \in W_{|w|} \setminus \{w\}$ we have $K_{\tau} \cap K_w = F_{\tau}(V_0) \cap F_w(V_0)$ by Proposition 6.39 and [Kig01, Proposition 1.3.5-(2)] and hence $F_{\tau}^{-1}(K_w) = F^{-1}_{\tau}(K_{\tau} \cap K_w) \subset V_0$, we see from (6.71) and (6.72) that for any Borel subset $A$ of $K$,

$$\Gamma(u, u)(F_w(A)) = \sum_{\tau \in \mathcal{W}_{|w|}} \frac{1}{r_{\tau}} \Gamma(u \circ F_{\tau}, u \circ F_{\tau})(F_{\tau}^{-1}(F_w(A))) = \frac{1}{r_w} \Gamma(u \circ F_w, u \circ F_w)(A).$$

The rest of the proof goes in exactly the same way as that of Lemma 6.20. □
Remark 6.55. In fact, (6.72) holds without supposing $G \neq \emptyset$ by [Hin13, Proposition 4.15] and hence so does Lemma 6.54. The proof of (6.72) presented in [Hin13, Section 5], however, is long and difficult, and since we later use Lemma 6.54 only under the supposition that $G \neq \emptyset$, we have decided to suppose it explicitly to keep our present arguments independent of the demanding result [Hin13, Proposition 4.15].

Lemma 6.56. Suppose that $G \neq \emptyset$, let $(\theta, \mu) \in G$, $u \in \mathcal{F}$ and set $f := d\Gamma(u, u)/d\mu$. Then $\mu$-a.e. $x \in K$ is a $(d, \mu)$-Lebesgue point for $f$, i.e., satisfies

$$\lim_{s \downarrow 0} \frac{1}{\mu(B_d(x, s))} \int_{B_d(x, s)} |f(y) - f(x)| \, d\mu(y) = 0. \quad (6.73)$$

Proof. The proof of Lemma 6.21 remains valid also in this case, except that $R_\varepsilon$ needs to be replaced by $d$. \qed

Lemma 6.57. Suppose that $G \neq \emptyset$, let $(\theta, \mu) \in G$, $u \in \mathcal{F}$, let $f : K \to [0, \infty)$ be a Borel measurable $\mu$-version of $d\Gamma(u, u)/d\mu$ and let $x \in K$ satisfy (6.73). Then (6.42) holds for any $\omega \in \pi^{-1}(x)$ and any $w \in W_*$.

Proof. The proof of Lemma 6.22 remains valid also in this case, except that $R_\varepsilon$ needs to be replaced by $d$. \qed

Proposition 6.58. Suppose that $G \neq \emptyset$, let $(\theta, \mu) \in G$, $u \in \mathcal{F}$, let $f : K \to [0, \infty)$ be a Borel measurable $\mu$-version of $d\Gamma(u, u)/d\mu$, let $x \in K$ satisfy (6.73) and $f(x) > 0$, and let $\omega \in \pi^{-1}(x)$. For each $n \in \mathbb{N} \cup \{0\}$, define $\mu_n := \mu_\omega \mid_n \in \mathcal{P}(K)$ by (6.39) with $w = [\omega]_n$ and, noting that $\Gamma(u, u)(K_\omega) > 0$ by Lemma 6.57, define $u_n := u_{\omega} \mid_n \in \mathcal{F}$ by (6.40) with $w = [\omega]_n$. If $v \in \mathcal{F}$ and $\{n_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$ is strictly increasing and satisfies $\lim_{k \to \infty} \mathcal{E}(v - u_{n_k}, v - u_{n_k}) = 0$, then $\Gamma(v, v) \in \mathcal{P}(K)$ and $\{\mu_{n_k}\}_{k \in \mathbb{N}}$ converges to $\Gamma(v, v)$ in $\mathcal{P}(K)$.

Proof. The proof of Proposition 6.23 remains valid also in this case, except that it is because of $G \neq \emptyset$, (6.72) and Lemma 6.54 that $\Gamma(v, v)(F_\omega(V_0)) = 0$ for any $w \in W_*$. \qed

To conclude the proof of Theorem 6.49, we need the following two lemmas.

Lemma 6.59. The inclusion map $\mathcal{F} \hookrightarrow L^2(K, m)$ is a compact linear operator from $(\mathcal{F}, \mathcal{E}_1)$ to $L^2(K, m)$, and there exists $C_P \in (0, \infty)$ such that for any $u \in \mathcal{F}$,

$$\int_K |u - \int_K u \, dm|^2 \, dm \leq C_P \mathcal{E}(u, u). \quad (6.74)$$

Proof. The compactness of the inclusion map $\mathcal{F} \hookrightarrow L^2(K, m)$ follows from Theorem 6.45 and [Dav, Corollary 4.2.3 and Exercise 4.2]. The existence of $C_P \in (0, \infty)$ satisfying (6.74) for any $u \in \mathcal{F}$ is implied by Theorems 6.45 and 4.5 as a special case of PI($d_\mu$), or more elementarily by [Dav, Theorems 4.5.1 and 4.5.3] and the fact that $\{u \in \mathcal{F} \mid \mathcal{E}(u, u) = 0\} = \mathcal{R} \mathcal{I}_K$ by Theorem 6.45 and [CF, Theorem 2.1.11]. \qed

99
**Lemma 6.60** (Reverse Poincaré inequality). There exists $C_{\text{RP}} \in (0, \infty)$ such that for any $(x, s) \in K \times (0, \infty)$, any $a \in \mathbb{R}$ and any function $h \in \mathcal{F} \cap L^\infty(K, m)$ that is $\mathcal{E}$-harmonic on $B_d(x, 2s)$,

$$\int_{B_d(x, s)} d\Gamma(h, h) \leq \frac{C_{\text{RP}}}{s^{d_m}} \int_{B_d(x, 2s) \setminus B_d(x, s)} |h - a|^2 \, dm. \quad (6.75)$$

**Proof.** This is a special case of [KM, Lemma 3.3] with $\Psi(\varphi) = s^{d_m}$, whose assumption $CS(\Psi)$ formulated in [KM, Definition 2.6-(b)] is implied by Lemma 6.40, Theorem 6.45 and [AB, Theorem 5.5]. \qed

**Proof of Theorem 6.49.** By the assumption $\mathcal{G}(\eta, C) \neq \emptyset$ we can take $(\theta, \mu) \in \mathcal{G}(\eta, C)$. Choose $u \in \mathcal{F} \cap L^\infty(K, m)$ which is $\mathcal{E}$-harmonic on $K \setminus V_0$ and satisfies $\mathcal{E}(u, u) > 0$; such $u$ exists, e.g., by [Kaj20, Propositions 3.8 and 3.11]. Let $f : K \rightarrow [0, \infty)$ be a Borel measurable $\mu$-version of $d\Gamma(u, u)/d\mu$. Then $\mu(f^{-1}((0, \infty))) > 0$ by $\int_K f \, d\mu = \Gamma(u, u)(K) = \mathcal{E}(u, u) > 0$, and therefore by Lemma 6.56 there exists $x \in K$ with the properties (6.73) and $f(x) > 0$. Let $\omega \in \pi^{-1}(x)$, and for each $n \in \mathbb{N} \cup \{0\}$, as in Proposition 6.58 define $(\theta_n, \mu_n) := (\theta_{[\omega]|n}, \mu_{[\omega]|n}) \in C(K \times K) \times \mathcal{P}(K)$ by (6.39) and $u_n := u_{[\omega]|n} \in \mathcal{F} \cap L^\infty(K, m)$ by (6.40) with $\omega = [\omega]_n$, so that $\{(\theta_n, \mu_n)\}_{n \in \mathbb{N} \cup \{0\}} \subset \mathcal{G}(\eta, C)$ by Lemma 6.52 and, for any $n \in \mathbb{N} \cup \{0\}$, $u_n$ is $\mathcal{E}$-harmonic on $K \setminus V_0$ by (6.7) and (6.8) and satisfies $\mathcal{E}(u_n, u_n) = 1$. Then setting $v_n := u_n - \int_K u_n \, dm \in \mathcal{F}$ for each $n \in \mathbb{N} \cup \{0\}$, by $\mathcal{E}_1(v_n, v_n) \leq C + 1$ for any $n \in \mathbb{N} \cup \{0\}$. Hence by [Yos, Chapter V, Section 2, Theorem 1] there exist $h \in \mathcal{F}$ and a strictly increasing sequence $\{n'_j\}_{j \in \mathbb{N}} \subset \mathbb{N}$ such that $\{v_{n'_j}\}_{j \in \mathbb{N}}$ converges weakly to $h$ in the Hilbert space $(\mathcal{F}, \mathcal{E}_1)$, which in combination with the compactness of the inclusion map from $(\mathcal{F}, \mathcal{E}_1)$ to $L^2(K, m)$ stated in Lemma 6.59 implies further that

$$\lim_{j \to \infty} \int_K |h - v_{n'_j}|^2 \, dm = 0. \quad (6.76)$$

Moreover, by $\{(\theta_{n'_j}, \mu_{n'_j})\}_{j \in \mathbb{N}} \subset \mathcal{G}(\eta, C)$, Proposition 6.50 and the metrizability of $C(K \times K) \times \mathcal{P}(K)$ we can choose a strictly increasing sequence $\{k_j\}_{k \in \mathbb{N}} \subset \mathbb{N}$ such that $\{(\theta_{n_k}, \mu_{n_k})\}_{k \in \mathbb{N}}$ converges to some $(\theta, \nu) \in \mathcal{G}(\eta, C)$ in $C(K \times K) \times \mathcal{P}(K)$, where $n_k := n'_{k_j}$. Now we claim that

$$\lim_{k \to \infty} \mathcal{E}(h - u_{n_k}, h - u_{n_k}) = 0. \quad (6.77)$$

If (6.77) holds, then $h$ is $\mathcal{E}$-harmonic on $K \setminus V_0$ since $u_{n_k}$ is so for any $k \in \mathbb{N}$. Proposition 6.58 implies that $\mu_{n_k} \in \mathcal{G}(\eta, C)$, which proves the desired assertion.

Thus it remains to show (6.77). To this end, recalling that $K \setminus V_0$ is open in $K$ and non-empty, let $\tau \in W_*$, $z_\tau \in K_\tau$ and $s_\tau \in (0, 1)$ satisfy $K_\tau \subset B_d(z_\tau, s_\tau) \subset B_d(z_\tau, 2s_\tau) \subset K \setminus V_0$. Then for any $v \in \mathcal{F} \cap L^\infty(K, m)$ that is $\mathcal{E}$-harmonic on $K \setminus V_0$, by Lemmas 6.54 and 6.60 we have

$$r_\tau^{-1} \mathcal{E}(v \circ F_\tau, v \circ F_\tau) = \Gamma(v, v)(K_\tau) \leq \Gamma(v, v)(B_d(z_\tau, s_\tau)) \leq \frac{C_{\text{RP}}}{s^{d_m}_\tau} \int_0^1 \left| v - \int_0^1 v \, dm \right|^2 \, dm, \quad (6.78)$$

100
which with \( v = v_{n_j} - v_{n_k} \) for \( j, k \in \mathbb{N} \), together with (6.76) and Lemma 6.53, shows that \( \{v_{n_k} \circ F_\tau\}_{k \in \mathbb{N}} \) is a Cauchy sequence in \((\mathcal{F}, \mathcal{E}_1)\) converging in \(L^2(K, m)\) to \( h \circ F_\tau\), whence

\[
\Gamma(h - u_{n_k}, h - u_{n_k})(K_\tau) = r_\tau^{-1} \mathcal{E}(h \circ F_\tau - v_{n_k} \circ F_\tau, h \circ F_\tau - v_{n_k} \circ F_\tau) \xrightarrow{k \to \infty} 0 \tag{6.79}
\]

with the equality implied by Lemma 6.54 and \( \mathcal{E}(1_K, 1_{K'}) = 0 \). On the other hand, for any \( w \in W_* \), combining (6.40) from Lemma 6.54, (6.42) from Lemma 6.57, \( f(x) > 0 \) and the convergence of \( \{\mu_{n_k}\}_{k \in \mathbb{N}} \) to \( \nu \) in \( \mathcal{P}(K) \), we obtain

\[
\limsup_{k \to \infty} \Gamma(u_{n_k}, u_{n_k})(K_w) = \limsup_{k \to \infty} \frac{\Gamma(u, u)(K_{[w]n_k})}{\Gamma(u, u)(K_{[w]n_k})} = \limsup_{k \to \infty} \frac{\mu(K_{[w]n_k})}{\Gamma(u, u)(K_{[w]n_k})} \mu_{n_k}(K_w) \leq \nu(K_w). \tag{6.80}
\]

Finally, let \( n \in \mathbb{N} \), set \( V_{0,n} := \bigcup_{w \in W_n, K_w \cap V_0 \neq \emptyset} K_w \) and \( W^1_n := \{\tau \in W_{n+1} \mid K_\tau \not\subset V_0\} \), so that \( K = V_{0,n} \cup \bigcup_{\tau \in W^1_n} K_\tau \) and each \( \tau \in W^1_n \) satisfies (6.79) since \( K_\tau \subset B_d(z, l^{-n-1}) \subset B_d(z, 2l^{-n-1}) \subset K \setminus V_0 \) for any \( z \in K_{\tau} \). Then recalling that \( \nu(F_w(V_0)) = 0 \) for any \( w \in W_* \) by \((\theta, \nu) \in \mathcal{G}(\eta, C)\) and Theorem 6.46 and that \( \Gamma(v, v)(F_w(V_0)) = 0 \) for any \( v \in \mathcal{F} \) and any \( w \in W_* \) by \( \mathcal{G}(\eta, C) \neq \emptyset \), (6.72) and Lemma 6.54, we see from \( K = V_{0,n} \cup \bigcup_{\tau \in W^1_n} K_\tau \), \([\text{Kig01, Proposition 1.3.5-(2)}], \) (6.79) for \( \tau \in W^1_n \) and (6.80) that

\[
\limsup_{k \to \infty} \mathcal{E}(h - u_{n_k}, h - u_{n_k}) = \limsup_{k \to \infty} \Gamma(h - u_{n_k}, h - u_{n_k})(K)
= \limsup_{k \to \infty} \left( \Gamma(h - u_{n_k}, h - u_{n_k})(V_{0,n}) + \sum_{\tau \in W^1_n} \Gamma(h - u_{n_k}, h - u_{n_k})(K_\tau) \right)
= \limsup_{k \to \infty} \Gamma(h - u_{n_k}, h - u_{n_k})(V_{0,n})
\leq \limsup_{k \to \infty} \left( 2\Gamma(h, h)(V_{0,n}) + 2\Gamma(u_{n_k}, u_{n_k})(V_{0,n}) \right)
= 2\Gamma(h, h)(V_{0,n}) + 2 \limsup_{k \to \infty} \sum_{w \in W_n, K_w \cap V_0 \neq \emptyset} \Gamma(u_{n_k}, u_{n_k})(K_w)
\leq 2\Gamma(h, h)(V_{0,n}) + 2 \limsup_{k \to \infty} \sum_{w \in W_n, K_w \cap V_0 \neq \emptyset} \Gamma(u_{n_k}, u_{n_k})(K_w)
\leq 2\Gamma(h, h)(V_{0,n}) + 2 \nu(K_w)
= 2\Gamma(h, h)(V_{0,n}) + 2\nu(V_{0,n}) \xrightarrow{n \to \infty} 2\Gamma(h, h)(V_0) + 2\nu(V_0) = 0,
\]

which shows (6.77) and thereby completes the proof of Theorem 6.49. \( \square \)

**Remark 6.61.** Note that the relatively short proof of (6.77) above is enabled by the supposition of the existence of \((\theta, \mu) \in \mathcal{G}\), which implies (6.72), and the property (6.73), which is guaranteed by VD of \((K, d, \mu)\) and yields (6.42) of Lemma 6.57 and thus (6.80). In fact, Hino [Hin13, Proposition 4.18] has obtained a result similar to (6.77) without assuming \( \mathcal{G} \neq \emptyset \), at the price of its long difficult proof presented in [Hin13, Section 5].
7 Further remarks and open questions

We collect some open questions that are related to our work.

**Problem 7.1.** Does $d_{cw} < \infty$ characterize the elliptic Harnack inequality for symmetric jump process?

It is not clear if the equivalence between (a) and (b) in Theorem 2.9 extends to jump processes. Despite the progress made in the diffusion case, the characterization and stability of the elliptic Harnack inequality is still open for jump processes.

For Problems 7.2, 7.3 and 7.4 below, let $N, K, d, m, V_0, (\mathcal{E}, \mathcal{F})$ be as introduced in Subsection 6.4 as pieces of the framework of the canonical self-similar Dirichlet form on an arbitrary generalized Sierpiński carpet $K$.

**Problem 7.2.** Is the conformal walk dimension attained for the canonical MMD space $(K, d, m, \mathcal{E}, \mathcal{F})$ over the generalized Sierpiński carpet $K$?

By Theorem 6.49, if the conformal walk dimension is attained then there exists a non-constant function $h \in \mathcal{F}$ which is $\mathcal{E}$-harmonic on $K \setminus V_0 = K \cap (0, 1)^N$ such that its energy measure $\Gamma(h, h)$ satisfies the volume doubling property with respect to the Euclidean metric $d$. This motivates the following question as a first step towards Problem 7.2.

**Problem 7.3.** Is there a non-constant function $h \in \mathcal{F}$ which is $\mathcal{E}$-harmonic on $K \setminus V_0$ on the generalized Sierpiński carpet $K$ such that its energy measure $\Gamma(h, h)$ satisfies the volume doubling property with respect to the Euclidean metric $d$?

Problem 7.3 appears very challenging since we do not know the answer to the following much simpler question.

**Problem 7.4.** Is there a function $h \in \mathcal{F}$ which is $\mathcal{E}$-harmonic on $K \setminus V_0$ on the generalized Sierpiński carpet $K$ such that its energy measure $\Gamma(h, h)$ has full support?

It is tempting to conjecture that the energy measure of every non-constant function $h \in \mathcal{F}$ that is $\mathcal{E}$-harmonic on $K \setminus V_0 = K \cap (0, 1)^N$ on the generalized Sierpiński carpet $K$ has full support. This can be viewed as a unique continuation principle for harmonic functions on a given generalized Sierpiński carpet. We also mention this question in a classical setting.

**Problem 7.5.** Characterize all the Gaussian admissible measures for the standard Brownian motion on $\mathbb{R}^n, n \geq 2$ (see (5.3) for definition).

We expect that there is a version of Theorem 6.16 and 6.49 for Ahlfors regular conformal dimension on self-similar spaces. In particular, we expect that if the Ahlfors regular conformal dimension $p > 1$ is attained on a self-similar space then there exists a “$p$-harmonic function” such that its “energy measure” is a $p$-Ahlfors regular measure with respect to a metric in the conformal gauge. This motivates the following problem.

**Problem 7.6.** Define non-linear analogs of Dirichlet space, energy measure and harmonic functions on self-similar spaces (for example, the Sierpiński carpet).
Using the theory of Dirichlet forms and its relationship to diffusion process we have a notion of Sobolev space $W^{1,2}$ on the Sierpiński carpet whose seminorm can be formally thought of as the integral $(\int |\nabla f|^2)^{1/2}$ on a dense subspace of $L^2$. For example in [KZ92], the $W^{1,2}$-seminorm and the Dirichlet energy $\int |\nabla f|^2$ is constituted as suitably renormalized version of discrete Dirichlet energy $\sum_{x \sim y} (\hat{f}(x) - \hat{f}(y))^2$, where $\hat{f}$ is a discretization of the function $f$ on a sequence of graph approximations $V_n$. A key ingredient in the constructions in [BB89] and [KZ92] is the following submultiplicative and supermultiplicative inequalities for resistance between opposite faces. Let $R_n$ denote the resistance between the opposite faces for the $n$-th level approximation of the Sierpiński carpet. Then the submultiplicative and supermultiplicative inequalities are given by $R_n R_m \gtrsim R_m + n$ and $R_n R_m \lesssim R_{m+n}$. These inequalities have been generalized in the non-linear context for the Sierpiński carpet by Bourdon and Kleiner for any $p > 1$ [BK13, Lemma 4.4]. This suggests that one could construct a non-linear version of Dirichlet form with a $W^{1,p}$-seminorm (formally denoted by $\|\nabla f\|_p$). This seminorm is defined on a dense subspace $\mathcal{F}_p$ of $L^p$ and is conjectured to satisfy the following properties:

1. (Closability) If $f_n$ is a Cauchy sequence in the $\|\nabla f\|_p$-seminorm and if $f_n \to 0$ in $L^p$ then $f_n$ converges to 0 in the $\|\nabla f\|_p$-seminorm.

2. (Regularity) Let $K$ denote the Sierpiński carpet. Then $\mathcal{F}_p \cap C(K)$ is dense with respect to the uniform norm in $C(K)$ and is dense with respect to the $f \mapsto \|\nabla f\|_p + \|f\|_p$ norm.

This yields the notion of $p$-harmonic functions which are defined as minimizers of the $p$-Dirichlet energy $\|\nabla f\|_p^p$. On the basis of Theorem 6.49, we have the following conjecture: if the Sierpiński carpet attains the Ahlfors regular conformal dimension, then there exists a $p$-harmonic function whose energy measure (formally written as the measure $A \mapsto \int_A |\nabla f|^p$) is an optimal Ahlfors regular measure, where $p$ is the Ahlfors regular conformal dimension of the Sierpiński carpet. The previous discussion on unique continuation question in the linear case ($p = 2$) also applies to $p$-harmonic functions due to the above mentioned relationship to the attainment problem for the Ahlfors regular conformal dimension. These conjectures serve as a motivation to develop a theory of non-linear Dirichlet forms on fractals and develop methods to obtain the elliptic Harnack inequality and quantitative unique continuation principle for $p$-harmonic functions.

Acknowledgements. The authors would like to thank Ryosuke Shimizu for his valuable comments on an earlier version of this paper.

References


[GH14] A. Grigor’yan and J. Hu, Heat kernels and Green functions on metric measure

[GHL15] A. Grigor’yan, J. Hu, K.-S. Lau. Generalized capacity, Harnack inequality and
(2015). MR3417504

[GT02] A. Grigor’yan, A. Telcs. Harnack inequalities and sub-Gaussian estimates for

[GT12] A. Grigor’yan and A. Telcs, Two-sided estimates of heat kernels on metric measure

75–263.

[Hei] J. Heinonen. Lectures on Analysis on Metric Spaces, Universitext. Springer-Verlag,

[HK] J. Heinonen, P. Koskela. Quasiconformal maps in metric spaces with controlled


metric measure spaces. An approach based on upper gradients. New Mathematical

[Hin02] M. Hino, On short time asymptotic behavior of some symmetric diffusions on

[Hin05] M. Hino, On singularity of energy measures on self-similar sets, Probab. Theory

[Hin10] M. Hino, Energy measures and indices of Dirichlet forms, with applications to

Theory Related Fields 156 (2013), no. 3-4, 739–793. MR3078285

[HN] M. Hino and K. Nakahara, On singularity of energy measures on self-similar sets II,

[KRS] A. Käenmäki, T. Rajala, V. Suomala. Existence of doubling measures via gener-

[Kaj10] N. Kajino, Remarks on non-diagonality conditions for Sierpiński carpets, in: M.
[Kaj12] N. Kajino, Heat kernel asymptotics for the measurable Riemannian structure on

[Kaj13] N. Kajino, Non-regularly varying and non-periodic oscillation of the on-diagonal
heat kernels on self-similar fractals, in: Fractal Geometry and Dynamical Systems in
Pure and Applied Mathematics II: Fractals in Applied Mathematics, Contemp. Math.,

[Kaj14] N. Kajino, Log-periodic asymptotic expansion of the spectral partition function

[Kaj20] N. Kajino, An elementary proof of walk dimension being greater than two for

[KM] N. Kajino and M. Murugan, On singularity of energy measures for symmetric dif-
fusions with full off-diagonal heat kernel estimates, Ann. Probab. 48 (2020), no. 6,
2920–2951. MR4164457

Art. ID 81415, 21 pp.


[KZ08] S. Keith, X. Zhong The Poincaré inequality is an open ended condition. Ann. of
Math. (2) 167 (2008), no. 2, 575–599. MR2415381

[KZ92] S. Kusuoka and X. Y. Zhou, Dirichlet forms on fractals: Poincaré constant and


no. 2, 399–444. MR2017320

[Kig08] J. Kigami, Measurable Riemannian geometry on the Sierpiński gasket: the
4, 781–804. MR2372738

[Kig09] J. Kigami, Volume doubling measures and heat kernel estimates on self-similar

[Kig12] J. Kigami, Resistance forms, quasisymmetric maps and heat kernel estimates,


Research Institute for Mathematical Sciences, Kyoto University, Kitashirakawa-Oiwake-cho, Sakyo-ku, Kyoto 606-8502, Japan.  
nkajino@kurims.kyoto-u.ac.jp

Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada.  
mathav@math.ubc.ca