An elementary proof that walk dimension is greater than two for Brownian motion on Sierpiński carpets

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Abstract

We give an elementary self-contained proof of the fact that the walk dimension of the Brownian motion on an *arbitrary* generalized Sierpiński carpet is greater than two, no proof of which in this generality had been available in the literature. Our proof is based solely on the self-similarity and hypercubic symmetry of the associated Dirichlet form and on several very basic pieces of functional analysis and the theory of regular symmetric Dirichlet forms. We also present an application of this fact to the singularity of the energy measures with respect to the canonical self-similar measure (uniform distribution) in this case, proved first by M. Hino in [*Probab. Theory Related Fields* **132** (2005), no. 2, 265–290].

1. *Introduction*

It is an established result in the field of analysis on fractals that, on the *Sierpiński carpet* and certain generalizations of it called *generalized Sierpiński carpets* (see Figure 1 below), there exists a canonical diffusion process $\{X_t\}_{t\in[0,\infty)}$ which is symmetric with respect to the canonical self-similar measure (uniform distribution) μ and satisfies the following estimates for its transition density (heat kernel) $p_t(x,y)$:

$$\begin{aligned}\frac{c_1}{\mu(B(x,t^{1/d_\mathrm{w}}))}\exp\biggl(-\Bigl(\frac{\rho(x,y)^{d_\mathrm{w}}}{c_2 t}\Bigr)^{\frac{1}{d_\mathrm{w}-1}}\biggr) &\leq p_t(x,y)\\ &\leq \frac{c_3}{\mu(B(x,t^{1/d_\mathrm{w}}))}\exp\biggl(-\Bigl(\frac{\rho(x,y)^{d_\mathrm{w}}}{c_4 t}\Bigr)^{\frac{1}{d_\mathrm{w}-1}}\biggr)\end{aligned} \tag{1.1}$$

for any points x,y and any $t\in(0,\infty)$, where $c_1,c_2,c_3,c_4\in(0,\infty)$ are some constants, ρ is the Euclidean metric, $B(x,s)$ denotes the open ball of radius s centered at x, and $d_\mathrm{w}\in[2,\infty)$ is a characteristic of the diffusion called its *walk dimension*. This result was obtained by M. T. Barlow and R. F. Bass in their series of papers [**1**, **3**, **4**] (see also [**2**, **28**, **5**, **6**]), its direct analog was proved also for the Sierpiński gasket in [**8**], for nested fractals in [**25**] and for affine nested fractals in [**14**], and it is believed for essentially all the known examples, and has been verified for many of them, that the walk dimension d_w is *strictly greater than two*. Therefore (1.1) implies in particular that a typical distance the diffusion travels by time t is of order t^{1/d_w} and is much smaller than the order $t^{1/2}$ of such a distance for the Brownian motion on the Euclidean spaces. The estimates (1.1) with $d_\mathrm{w}>2$ are called *sub-Gaussian estimates* for this reason, and are also known to imply a number of other anomalous features of the diffusion, one of the most important among which is the *singularity* of the associated *energy measures* with respect to the reference measure μ, proved recently in [**23**, Theorem 2.13-(a)]; see also [**26**, **27**, **9**, **16**, **18**] for earlier results on singularity of energy measures for diffusions on fractals.

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The main concern of this paper is the proof of the strict inequality $d_{\mathrm{w}} > 2$ for an *arbitrary* generalized Sierpiński carpet (see Framework 2.1 and Definition 2.2 below for its definition). In fact, the existing proof of $d_{\mathrm{w}} > 2$ for this case due to Barlow and Bass in [4, Proof of Proposition 5.1-(a)] requires a certain extra geometric assumption on the generalized Sierpiński carpet (see Remark 2.8 below), and there is no proof of it in the literature that is applicable to *any* generalized Sierpiński carpet although they claimed to have one in [4, Remarks 5.4-1.]. The purpose of the present paper is to give such a proof as is also elementary, self-contained and based solely on the self-similarity and hypercubic symmetry of the associated Dirichlet form (see Theorem 2.4 below) and on several very basic pieces of functional analysis and the theory of regular symmetric Dirichlet forms in [15, Section 1.4]. This minimality of the requirements in our method is crucial for potential future applications; in fact, our proof has been adapted by R. Shimizu in his recent preprint [32] to show the counterpart of $d_{\mathrm{w}} > 2$ for a canonical self-similar p-energy form on the Sierpiński carpet, whose detailed properties are mostly unknown (except in the case of $p = 2$, where his energy form coincides with the Dirichlet form of the canonical diffusion). As an important consequence of $d_{\mathrm{w}} > 2$, we also see that [23, Theorem 2.13-(a)] applies and recovers M. Hino's result in [16, Subsection 5.2] that for any generalized Sierpiński carpet the energy measures are singular with respect to the reference measure μ.

This paper is organized as follows. In Section 2, we first introduce the framework of a generalized Sierpiński carpet and the canonical Dirichlet form on it, then give the precise statement of our main theorem on the strict inequality $d_{\mathrm{w}} > 2$ (Theorem 2.7) and deduce the singularity of the associated energy measures (Corollary 2.10). Finally, we give our elementary self-contained proof of Theorem 2.7 in Section 3.

Notation. Throughout this paper, we use the following notation and conventions.

(1) The symbols $\subset$ and $\supset$ for set inclusion *allow* the case of the equality.
(2) $\mathbb{N} := \{n \in \mathbb{Z} \mid n > 0\}$, i.e., $0 \notin \mathbb{N}$.
(3) The cardinality (the number of elements) of a set A is denoted by $\#A$.
(4) We set $a \vee b := \max\{a, b\}$, $a \wedge b := \min\{a, b\}$ and $a^+ := a \vee 0$ for $a, b \in [-\infty, \infty]$, and we use the same notation also for $[-\infty, \infty]$-valued functions and equivalence classes of them. All numerical functions in this paper are assumed to be $[-\infty, \infty]$-valued.
(5) Let K be a non-empty set. We define $\mathbf{1}_A = \mathbf{1}_A^K \in \mathbb{R}^K$ for $A \subset K$ by $\mathbf{1}_A(x) := \mathbf{1}_A^K(x) := \left\{\begin{smallmatrix} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A, \end{smallmatrix}\right.$ and set $\|u\|_{\sup} := \|u\|_{\sup,K} := \sup_{x \in K} |u(x)|$ for $u \colon K \to [-\infty, \infty]$.
(6) Let K be a topological space. The interior and closure of $A \subset K$ in K are denoted by $\operatorname{int}_K A$ and $\overline{A}^K$, respectively. We set $\mathcal{C}(K) := \{u \mid u \colon K \to \mathbb{R}, u \text{ is continuous}\}$ and $\operatorname{supp}_K[u] := \overline{K \setminus u^{-1}(0)}^K$ for $u \in \mathcal{C}(K)$.
(7) For $d \in \mathbb{N}$, we equip $\mathbb{R}^d$ with the Euclidean norm denoted by $|\cdot|$ and set $\mathbf{0}_d := (0)_{k=1}^d \in \mathbb{R}^d$.

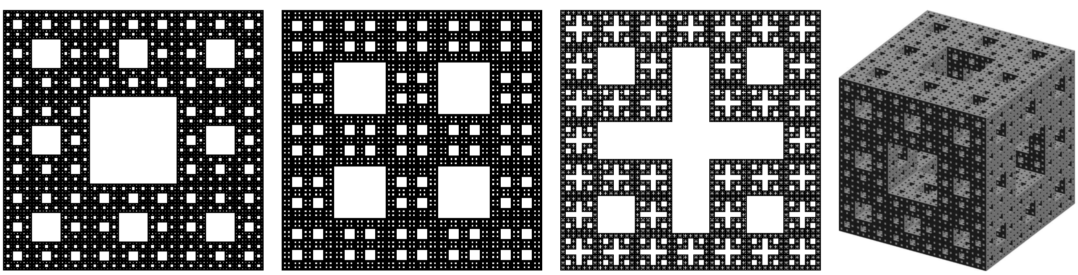

Figure 1. *Sierpiński carpet, two other generalized Sierpiński carpets with $d = 2$ and Menger sponge*

2. *Framework, the main theorem and an application*

The following presentation, up to Theorem 2.4 below, is a brief summary of the corresponding part in [22, Section 4]; see [21, Section 5] and the references therein for further details.

We fix the following setting throughout this and the next sections.

Framework 2.1. Let $d, l \in \mathbb{N}$, $d \geq 2$, $l \geq 3$ and set $Q_0 := [0,1]^d$. Let $S \subsetneq \{0,1,\dots,l-1\}^d$ be non-empty, define $f_i \colon \mathbb{R}^d \to \mathbb{R}^d$ by $f_i(x) := l^{-1}i + l^{-1}x$ for each $i \in S$ and set $Q_1 := \bigcup_{i\in S} f_i(Q_0)$, so that $Q_1 \subsetneq Q_0$. Let K be the *self-similar set* associated with $\{f_i\}_{i\in S}$, i.e., the unique non-empty compact subset of $\mathbb{R}^d$ such that $K = \bigcup_{i\in S} f_i(K)$, which exists and satisfies $K \subsetneq Q_0$ thanks to $Q_1 \subsetneq Q_0$ by [24, Theorem 1.1.4], and set $F_i := f_i|_K$ for each $i \in S$ and $\mathrm{GSC}(d,l,S) := (K, S, \{F_i\}_{i\in S})$. Let $\rho \colon K \times K \to [0,\infty)$ be the Euclidean metric on K given by $\rho(x,y) := |x-y|$, set $d_{\mathrm{f}} := \log_l \#S$, and let μ be the *self-similar measure* on $\mathrm{GSC}(d,l,S)$ with weight $(1/\#S)_{i\in S}$, i.e., the unique Borel probability measure on K such that $\mu = (\#S)\mu \circ F_i$ (as Borel measures on K) for any $i \in S$, which exists by [24, Propositions 1.5.8, 1.4.3, 1.4.4 and Corollary 1.4.8]. Let $\langle \cdot,\cdot\rangle$ and $\|\cdot\|_2$ denote the inner product and norm on $L^2(K,\mu)$, respectively.

Recall that d_{f} is the Hausdorff dimension of (K,ρ) and that μ is a constant multiple of the d_{f}-dimensional Hausdorff measure on (K,ρ); see, e.g., [24, Proposition 1.5.8 and Theorem 1.5.7]. Note that $d_{\mathrm{f}} < d$ by $S \subsetneq \{0,1,\dots,l-1\}^d$.

The following definition is due to Barlow and Bass [4, Section 2], except that the non-diagonality condition in [4, Hypotheses 2.1] has been strengthened later in [6] to fill a gap in [4, Proof of Theorem 3.19]; see [6, Remark 2.10-1.] for some more details of this correction.

Definition 2.2 (Generalized Sierpiński carpet, [6, Subsection 2.2]). $\mathrm{GSC}(d,l,S)$ is called a *generalized Sierpiński carpet* if and only if the following four conditions are satisfied:

(GSC1) (Symmetry) $f(Q_1) = Q_1$ for any isometry f of $\mathbb{R}^d$ with $f(Q_0) = Q_0$.

(GSC2) (Connectedness) Q_1 is connected.

(GSC3) (Non-diagonality) $\mathrm{int}_{\mathbb{R}^d}\bigl(Q_1 \cap \prod_{k=1}^d [(i_k - \varepsilon_k)l^{-1}, (i_k+1)l^{-1}]\bigr)$ is either empty or connected for any $(i_k)_{k=1}^d \in \mathbb{Z}^d$ and any $(\varepsilon_k)_{k=1}^d \in \{0,1\}^d$.

(GSC4) (Borders included) $[0,1] \times \{0\}^{d-1} \subset Q_1$.

As special cases of Definition 2.2, $\mathrm{GSC}(2,3,S_{\mathrm{SC}})$ and $\mathrm{GSC}(3,3,S_{\mathrm{MS}})$ are called the *Sierpiński carpet* and the *Menger sponge*, respectively, where $S_{\mathrm{SC}} := \{0,1,2\}^2 \setminus \{(1,1)\}$ and $S_{\mathrm{MS}} := \bigl\{(i_1,i_2,i_3) \in \{0,1,2\}^3 \bigm| \sum_{k=1}^3 \mathbf{1}_{\{1\}}(i_k) \leq 1\bigr\}$ (see Figure 1 above).

See [4, Remark 2.2] for a description of the meaning of each of the four conditions (GSC1), (GSC2), (GSC3) and (GSC4) in Definition 2.2. To be precise, (GSC3) is slightly different from the formulation of the non-diagonality condition in [6, Subsection 2.2], but they have been proved to be equivalent to each other in [20, Theorem 2.4]; see [20, §2] for some other equivalent formulations of the non-diagonality condition.

Throughout the rest of this paper, we assume that $\mathrm{GSC}(d,l,S) = (K,S,\{F_i\}_{i\in S})$ as introduced in Framework 2.1 is a generalized Sierpiński carpet as defined in Definition 2.2.

We next recall the result on the existence and uniqueness of a canonical diffusion (Brownian motion) on $\mathrm{GSC}(d,l,S)$, which can be presented most efficiently in the language of its associated (regular symmetric) Dirichlet form on $L^2(K,\mu)$ as follows; see [15, 11] for the basics of regular symmetric Dirichlet forms and associated symmetric Markov processes. Below we state only the final consequence of the unique characterizations of a canonical Dirichlet form on $\mathrm{GSC}(d,l,S)$ established in [6] (combined with some complementary discussions in [17, 21]), and we refer the reader to [6, Section 1] for a description of the earlier results on its existence in [1, 28, 4].

Definition 2.3. We define

$$\mathcal{G}_0 := \{f|_K \mid f \text{ is an isometry of } \mathbb{R}^d,\ f(Q_0) = Q_0\}, \tag{2.1}$$

which forms a finite subgroup of the group of homeomorphisms of K by virtue of (GSC1).

Theorem 2.4 ([6, Theorems 1.2 and 4.32], [17, Proposition 5.1], [21, Proposition 5.9]). *There exists a unique (up to constant multiples of $\mathcal{E}$) regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ satisfying $\mathcal{E}(u, u) > 0$ for some $u \in \mathcal{F}$, $\mathbf{1}_K \in \mathcal{F}$, $\mathcal{E}(\mathbf{1}_K, \mathbf{1}_K) = 0$, and the following:*

(GSCDF1) *If $u \in \mathcal{F} \cap \mathcal{C}(K)$ and $g \in \mathcal{G}_0$ then $u \circ g \in \mathcal{F}$ and $\mathcal{E}(u \circ g, u \circ g) = \mathcal{E}(u, u)$.*

(GSCDF2) *$\mathcal{F} \cap \mathcal{C}(K) = \{u \in \mathcal{C}(K) \mid u \circ F_i \in \mathcal{F}$ for any $i \in S\}$.*

(GSCDF3) *There exists $r \in (0, \infty)$ such that for any $u \in \mathcal{F} \cap \mathcal{C}(K)$,*

$$\mathcal{E}(u, u) = \sum_{i \in S} \frac{1}{r} \mathcal{E}(u \circ F_i, u \circ F_i). \tag{2.2}$$

Throughout the rest of this paper, we fix $(\mathcal{E}, \mathcal{F})$ and r as given in Theorem 2.4; note that r is uniquely determined by $(\mathcal{E}, \mathcal{F})$, since $\mathcal{E}(u, u) > 0$ for some $u \in \mathcal{F} \cap \mathcal{C}(K)$ by the existence of such $u \in \mathcal{F}$ and the denseness of $\mathcal{F} \cap \mathcal{C}(K)$ in the Hilbert space $(\mathcal{F}, \mathcal{E}_1 := \mathcal{E} + \langle \cdot, \cdot \rangle)$.

Definition 2.5. The regular symmetric Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(K, \mu)$ is called the *canonical Dirichlet form* on $\mathrm{GSC}(d, l, S)$, and the *walk dimension* d_{w} of $(\mathcal{E}, \mathcal{F})$ (or of $\mathrm{GSC}(d, l, S)$) is defined by $d_{\mathrm{w}} := \log_l(\#S/r)$.

Remark 2.6. The walk dimension d_{w} defined in Definition 2.5 coincides with the exponent d_{w} in (1.1) for the regular symmetric Dirichlet space $(K, \mu, \mathcal{E}, \mathcal{F})$ equipped with the Euclidean metric ρ; see the proof of Corollary 2.10 below and the references therein for details.

The main result of this paper is an elementary self-contained proof of the following theorem based solely on the setting and properties stated in Framework 2.1, Definitions 2.2, 2.3 and Theorem 2.4 (except the uniqueness of $(\mathcal{E}, \mathcal{F})$) and on several very basic pieces of functional analysis and the theory of regular symmetric Dirichlet forms in [15, Section 1.4]. To keep the whole treatment as elementary and self-contained as possible, in our proof of Theorem 2.7 we refrain from using any known properties of $(\mathcal{E}, \mathcal{F})$ other than those in Theorem 2.4.

Theorem 2.7 (Cf. [4, Remarks 5.4-1.]). $d_{\mathrm{w}} > 2$.

Remark 2.8. *No proof of Theorem 2.7 in the present generality had been available in the literature*, although Barlow and Bass claimed to have one in [4, Remarks 5.4-1.]. Its existing proof in [4, Proof of Proposition 5.1-(a)] requires the extra assumption on $\mathrm{GSC}(d, l, S)$ that

$$\#\{(i_k)_{k=1}^d \in S \mid i_1 = j\} \neq \#\{(i_k)_{k=1}^d \in S \mid i_1 = 0\} \quad \text{for some } j \in \{1, \ldots, l-1\}, \tag{2.3}$$

which holds for any generalized Sierpiński carpet with $d = 2$ but does fail for infinitely many examples of generalized Sierpiński carpets with fixed d for each $d \geq 3$; indeed, for each $d, l \in \mathbb{N}$ with $d \geq 3$ and $l \geq 2$, it is not difficult to see that $\mathrm{GSC}(d, 2ld, S_{d,l})$ with

$$S_{d,l} := \left\{ i \;\middle|\; \begin{array}{l} i = (i_k)_{k=1}^d \in \{0, 1, \ldots, 2ld-1\}^d, \text{and for any } j \in \{1, 3, \ldots, 2l-1\}, \\ \{|2i_k - 2ld + 1| \mid k \in \{1, 2, \ldots, d\}\} \neq \{j, j+2l, \ldots, j+2l(d-1)\} \end{array} \right\} \tag{2.4}$$

satisfies (GSC1), (GSC2), (GSC3) and (GSC4) in Definition 2.2 but not (2.3).

The proof of Theorem 2.7 is given in the next section. We conclude this section by presenting an application of Theorem 2.7 to the singularity with respect to μ of the energy measures associated with $(K, \mu, \mathcal{E}, \mathcal{F})$, which was proved first by Hino in [16, Subsection 5.2] via $d_{\mathrm{w}} > 2$ and is obtained here by combining [23, Theorem 2.13-(a)] with $d_{\mathrm{w}} > 2$.

Definition 2.9 (Cf. [15, (3.2.13), (3.2.14) and (3.2.15)]). The $\mathcal{E}$*-energy measure* $\mu_{\langle u \rangle}$ of $u \in \mathcal{F}$ is defined, first for $u \in \mathcal{F} \cap L^\infty(K, \mu)$ as the unique ($[0, \infty]$-valued) Borel measure on K such that

$$\int_K v \, d\mu_{\langle u \rangle} = \mathcal{E}(uv, u) - \frac{1}{2}\mathcal{E}(v, u^2) \qquad \text{for any } v \in \mathcal{F} \cap \mathcal{C}(K), \tag{2.5}$$

and then by $\mu_{\langle u \rangle}(A) := \lim_{n\to\infty} \mu_{\langle (-n)\vee(u\wedge n) \rangle}(A)$ for each Borel subset A of K for general $u \in \mathcal{F}$; note that $uv \in \mathcal{F}$ for any $u, v \in \mathcal{F} \cap L^\infty(K, \mu)$ by [15, Theorem 1.4.2-(ii)] and that $\{(-n) \vee (u \wedge n)\}_{n=1}^\infty \subset \mathcal{F}$ and $\lim_{n\to\infty} \mathcal{E}\bigl(u - (-n) \vee (u \wedge n), u - (-n) \vee (u \wedge n)\bigr) = 0$ by [15, Theorem 1.4.2-(iii)].

Corollary 2.10 ([16, Subsection 5.2]). *$\mu_{\langle u \rangle}$ is singular with respect to μ for any $u \in \mathcal{F}$.*

Proof. (Unlike the proof of Theorem 2.7, this proof is *not* meant to be self-contained.) $(\mathcal{E}, \mathcal{F})$ is local by [19, Lemma 3.4], whose proof is based only on (GSCDF2), (GSCDF3) and [15, Exercise 1.4.1 and Theorem 3.1.2], and is therefore strongly local since $\mathcal{E}(\mathbf{1}_K, v) = 0$ for any $v \in \mathcal{F}$ by $\mathcal{E}(\mathbf{1}_K, \mathbf{1}_K) = 0$ (see Lemma 3.4 below). We easily see that $c_5 s^{d_{\mathrm{f}}} \le \mu(B(x, s)) \le c_6 s^{d_{\mathrm{f}}}$ for any $(x, s) \in K \times (0, d]$ for some $c_5, c_6 \in (0, \infty)$, where $B(x, s) := \{y \in K \mid \rho(x, y) < s\}$. It is also immediate that (K, ρ) satisfies the chain condition as defined in [23, Definition 2.10-(a)], in view of the fact that by (GSC4), (GSC1) and (GSC2) there exists $c_7 \in (0, \infty)$ such that for any $x, y \in K$ there exists a continuous map $\gamma\colon [0, 1] \to K$ with $\gamma(0) = x$ and $\gamma(1) = y$ whose Euclidean length is at most $c_7 \rho(x, y)$. Finally, by [6, Theorem 4.30 and Remark 4.33] (see also [4, Theorem 1.3]) the heat kernel $p_t(x, y)$ of $(K, \mu, \mathcal{E}, \mathcal{F})$ exists and there exist $\beta_0 \in (1, \infty)$ and $c_1, c_2, c_3, c_4 \in (0, \infty)$ such that (1.1) with β_0 in place of d_{w} holds for μ-a.e. $x, y \in K$ for each $t \in (0, \infty)$, but then necessarily $\beta_0 = \log_l(\#S/r) = d_{\mathrm{w}}$ by (GSCDF2), (GSCDF3) and [6, Theorem 4.31] as shown in [21, Proof of Proposition 5.9, Second paragraph], whence $\beta_0 = d_{\mathrm{w}} > 2$ by Theorem 2.7. Thus $(K, \rho, \mu, \mathcal{E}, \mathcal{F})$ satisfies all the assumptions of [23, Theorem 2.13-(a)], which implies the desired claim. □

3. *The elementary proof of the main theorem*

This section is devoted to giving our elementary self-contained proof of the main theorem (Theorem 2.7), which is an adaptation of, and has been inspired by, an elementary proof of the counterpart of Theorem 2.7 for Sierpiński gaskets presented in [23, Proof of Proposition 5.3, Second paragraph]. We start with basic definitions and some simple lemmas.

Definition 3.1. We set $W_m := S^m = \{w_1 \dots w_m \mid w_i \in S \text{ for } i \in \{1, \dots, m\}\}$ for $m \in \mathbb{N}$ and $W_* := \bigcup_{m=1}^\infty W_m$. For each $w = w_1 \dots w_m \in W_*$, the unique $m \in \mathbb{N}$ with $w \in W_m$ is denoted by $|w|$, and we set $F_w := F_{w_1} \circ \cdots \circ F_{w_m}$, $K_w := F_w(K)$ and $q^w = (q_k^w)_{k=1}^d := F_w(\mathbf{0}_d)$.

Lemma 3.2. *If $w, v \in W_*$, $|w| = |v|$ and $w \ne v$, then $\mu(F_w(K \setminus (0,1)^d)) = 0 = \mu(K_w \cap K_v)$.*

Proof. This follows easily from (GSC1) and the fact that μ is a Borel probability measure on K satisfying $\mu(K_w) = (\#S)^{-|w|}$ for any $w \in W_*$. $\square$

LEMMA 3.3. *Let $w \in W_*$. Then for any Borel measurable function $u\colon K \to [-\infty, \infty]$, $\int_K |u \circ F_w|\,d\mu = (\#S)^{|w|} \int_{K_w} |u|\,d\mu$ and $\int_{K_w} |u \circ F_w^{-1}|\,d\mu = (\#S)^{-|w|} \int_K |u|\,d\mu$. In particular, bounded linear operators $F_w^*, (F_w)_*\colon L^2(K,\mu) \to L^2(K,\mu)$ can be defined by setting*

$$F_w^* u := u \circ F_w \qquad \text{and} \qquad (F_w)_* u := \begin{cases} u \circ F_w^{-1} & \text{on } K_w, \\ 0 & \text{on } K \setminus K_w \end{cases} \tag{3.1}$$

for each $u \in L^2(K,\mu)$. Moreover, $u \circ F_w \in \mathcal{F}$ and (2.2) holds for any $u \in \mathcal{F}$.

Proof. The former assertions are immediate from $\mu = (\#S)^{|w|} \mu \circ F_w$. For the latter ones, let $u \in \mathcal{F}$. Since $\mathcal{F} \cap \mathcal{C}(K)$ is dense in the Hilbert space $(\mathcal{F}, \mathcal{E}_1 = \mathcal{E} + \langle \cdot, \cdot \rangle)$ by the regularity of $(\mathcal{E}, \mathcal{F})$, we can choose $\{u_n\}_{n=1}^\infty \subset \mathcal{F} \cap \mathcal{C}(K)$ so that $\lim_{n\to\infty} \mathcal{E}_1(u - u_n, u - u_n) = 0$, and then $\{u_n \circ F_w\}_{n=1}^\infty$ is a Cauchy sequence in $(\mathcal{F}, \mathcal{E}_1)$ with $\lim_{n\to\infty} \|u \circ F_w - u_n \circ F_w\|_2 = 0$ by (GSCDF2) and (GSCDF3) and therefore has to converge to $u \circ F_w$ in norm in $(\mathcal{F}, \mathcal{E}_1)$. Thus $u \circ F_w \in \mathcal{F}$, and (2.2) for u follows by letting $n \to \infty$ in (2.2) for $u_n \in \mathcal{F} \cap \mathcal{C}(K)$. $\square$

LEMMA 3.4. *$\mathcal{E}(\mathbf{1}_K, v) = 0$ for any $v \in \mathcal{F}$.*

Proof. This is immediate from the Cauchy–Schwarz inequality for $\mathcal{E}$ and $\mathcal{E}(\mathbf{1}_K, \mathbf{1}_K) = 0$. $\square$

DEFINITION 3.5. Let U be a non-empty open subset of K.

(1) Equipping $\mathcal{F}$ with the inner product $\mathcal{E}_1 = \mathcal{E} + \langle \cdot, \cdot \rangle$, we define

$$\mathcal{C}_U := \{u \in \mathcal{F} \cap \mathcal{C}(K) \mid \operatorname{supp}_K[u] \subset U\} \qquad \text{and} \qquad \mathcal{F}_U := \overline{\mathcal{C}_U}^{\mathcal{F}}, \tag{3.2}$$

which are linear subspaces of $\mathcal{F}$, and for each $u \in \mathcal{F}$ we also set $u + \mathcal{C}_U := \{u + v \mid v \in \mathcal{C}_U\}$ and $u + \mathcal{F}_U := \{u + v \mid v \in \mathcal{F}_U\}$, so that $\overline{u + \mathcal{C}_U}^{\mathcal{F}} = u + \mathcal{F}_U$.

(2) A function $h \in \mathcal{F}$ is said to be *$\mathcal{E}$-harmonic* on U if and only if either of the following two conditions, which are easily seen to be equivalent to each other, holds:

$$\mathcal{E}(h,h) = \inf\{\mathcal{E}(u,u) \mid u \in h + \mathcal{F}_U\}, \tag{3.3}$$

$$\mathcal{E}(h,v) = 0 \quad \text{for any } v \in \mathcal{C}_U, \text{ or equivalently, for any } v \in \mathcal{F}_U, \tag{3.4}$$

where the equivalence stated in (3.4) is immediate from (3.2).

DEFINITION 3.6.

(1) We set $V_0^\varepsilon := K \cap (\{\varepsilon\} \times \mathbb{R}^{d-1})$ for each $\varepsilon \in \{0,1\}$ and $U_0 := K \setminus (V_0^0 \cup V_0^1)$.

(2) We fix an arbitrary $\varphi_0 \in \mathcal{C}_{K \setminus V_0^0}$ with $\operatorname{supp}_K[\mathbf{1}_K - \varphi_0] \subset K \setminus V_0^1$, which exists by [15, Exercise 1.4.1]; note that $\varphi_0 + \mathcal{C}_{U_0}, \varphi_0 + \mathcal{F}_{U_0}$ are independent of a particular choice of φ_0.

(3) We define $g_\varepsilon \in \mathcal{G}_0$ by $g_\varepsilon := \tau_\varepsilon|_K$ for each $\varepsilon = (\varepsilon_k)_{k=1}^d \in \{0,1\}^d$, where $\tau_\varepsilon\colon \mathbb{R}^d \to \mathbb{R}^d$ is given by $\tau_\varepsilon((x_k)_{k=1}^d) := (\varepsilon_k + (1 - 2\varepsilon_k)x_k)_{k=1}^d$, and define a subgroup $\mathcal{G}_1$ of $\mathcal{G}_0$ by

$$\mathcal{G}_1 := \{g_\varepsilon \mid \varepsilon \in \{0\} \times \{0,1\}^{d-1}\}. \tag{3.5}$$

Now we proceed to the core part of the proof of Theorem 2.7. It is divided into three propositions, proving respectively the existence of a good sequence $\{u_n\}_{n=1}^\infty \subset \mathcal{F} \cap \mathcal{C}(K)$ converging in norm in $(\mathcal{F}, \mathcal{E}_1)$ to $h_0 \in \varphi_0 + \mathcal{F}_{U_0}$ which is $\mathcal{E}$-harmonic on U_0 (Proposition 3.7), $\mathcal{E}(h_0, h_0) > 0$ (Proposition 3.10) and the *non-*$\mathcal{E}$-harmonicity on U_0 of $h_2 := \sum_{w \in W_2} (F_w)_*(l^{-2} h_0 + q_1^w \mathbf{1}_K) \in h_0 + \mathcal{F}_{U_0}$ (Proposition 3.11); see Figures 2 and 3 below for an illustration of h_0 and h_2. Then Theorem 2.7 will follow from $\mathcal{E}(h_0, h_0) < \mathcal{E}(h_2, h_2)$ and (2.2) for $u \in \mathcal{F}$. While the existence of such h_0 is implied by [15, Theorems 7.2.1, 4.6.5, 1.5.2-(iii), A.2.6-(i), 4.1.3, 4.2.1-(ii) and Corollary 2.3.1], that of $\{u_n\}_{n=1}^{\infty} \subset \mathcal{F} \cap \mathcal{C}(K)$ as in the following proposition cannot be obtained directly from the theory of regular symmetric Dirichlet forms in [15, 11].

Proposition 3.7. *There exist $h_0 \in \mathcal{F}$ and $\{u_n\}_{n=1}^{\infty} \subset \mathcal{F} \cap \mathcal{C}(K)$ satisfying the following:*

(1) *h_0 is $\mathcal{E}$-harmonic on U_0 and $h_0 \in \varphi_0 + \mathcal{F}_{U_0}$. In particular, $h_0 + \mathcal{F}_{U_0} = \varphi_0 + \mathcal{F}_{U_0}$.*
(2) *For each $n \in \mathbb{N}$, $u_n \circ g = u_n$ for any $g \in \mathcal{G}_1$ and $u_n \in \varphi_0 + \mathcal{C}_{U_0}$.*
(3) *$\lim_{n\to\infty} \mathcal{E}_1(h_0 - u_n, h_0 - u_n) = 0$.*

Proof. Recalling (3.3), for each $\alpha \in [0, \infty)$ we set

$$a_\alpha := \inf\{\mathcal{E}(u,u) + \alpha\|u\|_2^2 \mid u \in \varphi_0 + \mathcal{C}_{U_0}\} = \inf\{\mathcal{E}(u,u) + \alpha\|u\|_2^2 \mid u \in \varphi_0 + \mathcal{F}_{U_0}\}, \tag{3.6}$$

where the latter equality in (3.6) is immediate from $\overline{\varphi_0 + \mathcal{C}_{U_0}}^{\mathcal{F}} = \varphi_0 + \mathcal{F}_{U_0}$. Then for any $\alpha \in [0, \infty)$ and any $u \in \varphi_0 + \mathcal{C}_{U_0}$, by the unit contraction operating on $(\mathcal{E}, \mathcal{F})$ (see [15, Section 1.1 and Theorem 1.4.1]) we have $u^+ \wedge 1 \in \varphi_0 + \mathcal{C}_{U_0}$ and

$$\mathcal{E}(u,u) \geq \mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) \geq \mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) + \alpha\|u^+ \wedge 1\|_2^2 - \alpha \geq a_\alpha - \alpha,$$

and hence $a_0 \geq a_\alpha - \alpha$, so that for each $n \in \mathbb{N}$ we can take $v_n \in \varphi_0 + \mathcal{C}_{U_0}$ such that

$$\mathcal{E}(v_n^+ \wedge 1, v_n^+ \wedge 1) \leq \mathcal{E}(v_n, v_n) + n^{-1}\|v_n\|_2^2 < a_{n^{-1}} + n^{-1} \leq a_0 + 2n^{-1}. \tag{3.7}$$

Recalling (GSCDF1), now for each $n \in \mathbb{N}$ we can define $u_n \in \mathcal{F} \cap \mathcal{C}(K)$ with the properties in (2) by $u_n := (\#\mathcal{G}_1)^{-1} \sum_{g \in \mathcal{G}_1} (v_n^+ \wedge 1) \circ g$ and see from the triangle inequality for $\mathcal{F} \ni u \mapsto \mathcal{E}(u,u)^{1/2}$, $\mathcal{E}((v_n^+ \wedge 1) \circ g, (v_n^+ \wedge 1) \circ g) = \mathcal{E}(v_n^+ \wedge 1, v_n^+ \wedge 1)$ for $g \in \mathcal{G}_1$ and (3.7) that

$$\mathcal{E}(u_n, u_n) \leq \mathcal{E}(v_n^+ \wedge 1, v_n^+ \wedge 1) < a_0 + 2n^{-1}. \tag{3.8}$$

Further, since $\|u_n\|_2 \leq 1$ by $0 \leq u_n \leq 1$ for any $n \in \mathbb{N}$, the Banach–Saks theorem [11, Theorem A.4.1-(i)] yields $h_0 \in L^2(K, \mu)$ and a strictly increasing sequence $\{j_k\}_{k=1}^{\infty} \subset \mathbb{N}$ such that the Cesàro mean sequence $\{\overline{u}_n\}_{n=1}^{\infty} \subset \mathcal{F} \cap \mathcal{C}(K)$ of $\{u_{j_k}\}_{k=1}^{\infty}$ given by $\overline{u}_n := n^{-1}\sum_{k=1}^{n} u_{j_k}$ satisfies $\lim_{n\to\infty} \|h_0 - \overline{u}_n\|_2 = 0$. Then (2) obviously holds for $\{\overline{u}_n\}_{n=1}^{\infty}$, and it follows from (3.6) and (3.8) that $\lim_{n\to\infty} \mathcal{E}(\overline{u}_n, \overline{u}_n) = a_0$ and that for any $n, k \in \mathbb{N}$,

$$\begin{aligned}\mathcal{E}(\overline{u}_n - \overline{u}_k, \overline{u}_n - \overline{u}_k) &= 2\mathcal{E}(\overline{u}_n, \overline{u}_n) + 2\mathcal{E}(\overline{u}_k, \overline{u}_k) - 4\mathcal{E}((\overline{u}_n + \overline{u}_k)/2, (\overline{u}_n + \overline{u}_k)/2)\\ &\leq 2\mathcal{E}(\overline{u}_n, \overline{u}_n) + 2\mathcal{E}(\overline{u}_k, \overline{u}_k) - 4a_0 \xrightarrow{n \wedge k \to \infty} 0,\end{aligned}$$

which together with $\lim_{n\to\infty} \|h_0 - \overline{u}_n\|_2 = 0$ and the completeness of $(\mathcal{F}, \mathcal{E}_1)$ implies that $h_0 \in \mathcal{F}$ and $\lim_{n\to\infty} \mathcal{E}_1(h_0 - \overline{u}_n, h_0 - \overline{u}_n) = 0$. Thus $h_0 \in \overline{\varphi_0 + \mathcal{C}_{U_0}}^{\mathcal{F}} = \varphi_0 + \mathcal{F}_{U_0} = h_0 + \mathcal{F}_{U_0}$ by $\{\overline{u}_n\}_{n=1}^{\infty} \subset \varphi_0 + \mathcal{C}_{U_0}$, $\mathcal{E}(h_0, h_0) = \lim_{n\to\infty} \mathcal{E}(\overline{u}_n, \overline{u}_n) = a_0$, and therefore h_0 is $\mathcal{E}$-harmonic on U_0 in view of (3.6) and (3.3), completing the proof. □

We need the following two lemmas for the remaining two propositions and their proofs.

Lemma 3.8. *Let $h_0 \in \mathcal{F}$ be as in Proposition 3.7, let $m \in \mathbb{N}$ and define $h_m \in L^2(K, \mu)$ by*

$$h_m := \sum_{w \in W_m} (F_w)_*(l^{-m} h_0 + q_1^w \mathbf{1}_K) \tag{3.9}$$

(see Figure 3 below for an illustration of (3.9)). *Then $h_m \in h_0 + \mathcal{F}_{U_0}$.*

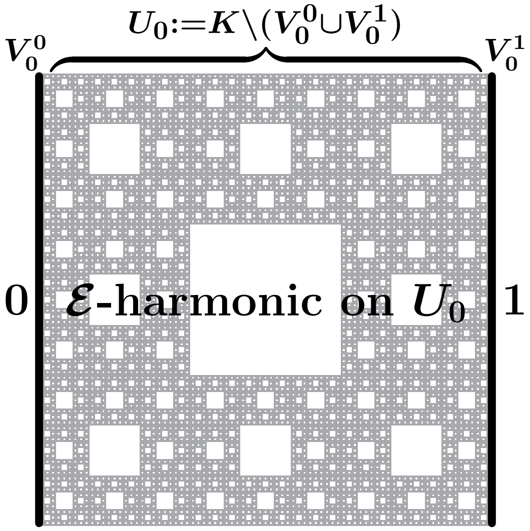

FIGURE 2. *The choice of* $h_0 \in \mathcal{F}$

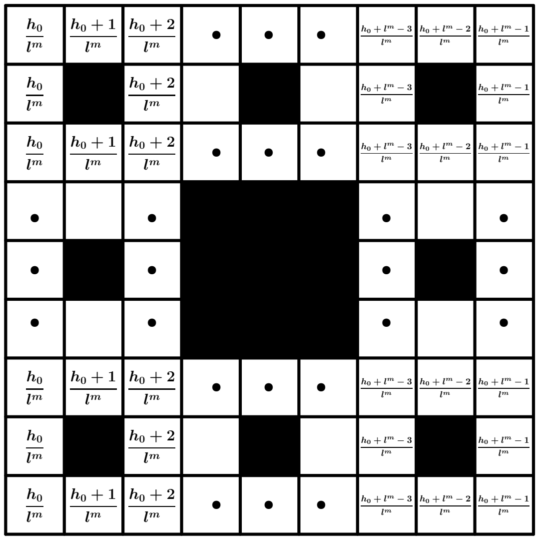

FIGURE 3. *The construction of* $h_m \in h_0 + \mathcal{F}_{U_0}$

Proof. Let $\{u_n\}_{n=1}^{\infty} \subset \mathcal{F} \cap \mathcal{C}(K)$ be as in Proposition 3.7. For each $n \in \mathbb{N}$, since $u_n \circ g = u_n$ for any $g \in \mathcal{G}_1$, $u_n \in \varphi_0 + \mathcal{C}_{U_0}$ and hence $u_n|_{V_0^0 \cup V_0^1} = \varphi_0|_{V_0^0 \cup V_0^1} = \mathbf{1}_{V_0^1}$ by Proposition 3.7-(2), we can define $u_{m,n} \in \mathcal{C}(K)$ by setting $u_{m,n}|_{K_w} := (l^{-m} u_n + q_1^w \mathbf{1}_K) \circ F_w^{-1}$ for each $w \in W_m$, so that $u_{m,n} \circ F_w = l^{-m} u_n + q_1^w \mathbf{1}_K \in \mathcal{F}$ by $\mathbf{1}_K \in \mathcal{F}$ and thus $u_{m,n} \in \varphi_0 + \mathcal{C}_{U_0}$ by (GSCDF2) and $u_n \in \varphi_0 + \mathcal{C}_{U_0}$. Then we see from (GSCDF3), Lemmas 3.2, 3.3 and Proposition 3.7-(3) that $\{u_{m,n}\}_{n=1}^{\infty}$ is a Cauchy sequence in the Hilbert space $(\mathcal{F}, \mathcal{E}_1)$ with $\lim_{n\to\infty} \|h_m - u_{m,n}\|_2 = 0$ and therefore has to converge to h_m in norm in $(\mathcal{F}, \mathcal{E}_1)$, whence $h_m \in \overline{\varphi_0 + \mathcal{C}_{U_0}}^{\mathcal{F}} = \varphi_0 + \mathcal{F}_{U_0} = h_0 + \mathcal{F}_{U_0}$ by $\{u_{m,n}\}_{n=1}^{\infty} \subset \varphi_0 + \mathcal{C}_{U_0}$ and Proposition 3.7-(1). $\square$

LEMMA 3.9. *Let $k \in \{1, 2, \ldots, d\}$ and define $f_k \in \mathcal{C}(\mathbb{R}^d)$ by $f_k((x_j)_{j=1}^d) := x_k$. Then either $f_k|_K \in \mathcal{F}$ and $\mathcal{E}(f_k|_K, f_k|_K) > 0$ or $f_k|_K \notin \mathcal{F}$.*

Proof. Suppose to the contrary that $f_k|_K \in \mathcal{F}$ and $\mathcal{E}(f_k|_K, f_k|_K) = 0$. Then $f|_K \in \mathcal{F}$ and $\mathcal{E}(f|_K, f|_K) = 0$ for any $f \in \{\mathbf{1}_{\mathbb{R}^d}, f_1, f_2, \ldots, f_d\}$ by $\mathbf{1}_K \in \mathcal{F}$, $\mathcal{E}(\mathbf{1}_K, \mathbf{1}_K) = 0$ and (GSCDF1), and hence also for any polynomial $f \in \mathcal{C}(\mathbb{R}^d)$ since for any $u, v \in \mathcal{F} \cap \mathcal{C}(K)$ we have $uv \in \mathcal{F} \cap \mathcal{C}(K)$ and $\mathcal{E}(uv, uv)^{1/2} \leq \|u\|_{\sup} \mathcal{E}(v, v)^{1/2} + \|v\|_{\sup} \mathcal{E}(u, u)^{1/2}$ by [15, Theorem 1.4.2-(ii)]. On the other hand, $\mathcal{E}(u, u) > 0$ for some $u \in \mathcal{F} \cap \mathcal{C}(K)$ by the existence of such $u \in \mathcal{F}$ and the denseness of $\mathcal{F} \cap \mathcal{C}(K)$ in $(\mathcal{F}, \mathcal{E}_1)$, the Stone–Weierstrass theorem [13, Theorem 2.4.11] implies that $\lim_{n\to\infty} \|u - f_n|_K\|_{\sup} = 0$ for some sequence $\{f_n\}_{n=1}^{\infty} \subset \mathcal{C}(\mathbb{R}^d)$ of polynomials, then $\lim_{n\to\infty} \|u - f_n|_K\|_2 = 0$ and $\lim_{n \wedge k \to \infty} \mathcal{E}_1(f_n|_K - f_k|_K, f_n|_K - f_k|_K) = 0$. Thus $\lim_{n\to\infty} \mathcal{E}_1(u - f_n|_K, u - f_n|_K) = 0$ by the completeness of $(\mathcal{F}, \mathcal{E}_1)$ and therefore $0 < \mathcal{E}(u, u) = \lim_{n\to\infty} \mathcal{E}(f_n|_K, f_n|_K) = 0$, which is a contradiction and completes the proof. $\square$

PROPOSITION 3.10. *Let $h_0 \in \mathcal{F}$ be as in Proposition 3.7. Then $\mathcal{E}(h_0, h_0) > 0$.*

Proof. Let $f_1 \in \mathcal{C}(\mathbb{R}^d)$ be as in Lemma 3.9 with $k = 1$ and for each $m \in \mathbb{N}$ let $h_m \in \mathcal{F}$ be as in Lemma 3.8, so that by (3.9), Lemmas 3.2 and 3.3 we have $\|f_1|_K - h_m\|_2 = l^{-m} \|f_1|_K - h_0\|_2$ and $h_m \circ F_w = l^{-m} h_0 + q_1^w \mathbf{1}_K$ μ-a.e. for any $w \in W_m$ and hence (2.2) for $u \in \mathcal{F}$ from Lemma

3.3 and Lemma 3.4 together yield

$$\mathcal{E}(h_m,h_m)=\sum_{w\in W_m}\frac{1}{r^m}\mathcal{E}(l^{-m}h_0+q_1^w\mathbf{1}_K,l^{-m}h_0+q_1^w\mathbf{1}_K)=\Big(\frac{\#S}{r}l^{-2}\Big)^m\mathcal{E}(h_0,h_0). \tag{3.10}$$

Now if $\mathcal{E}(h_0,h_0)=0$, then $\mathcal{E}(h_m,h_m)=0$ by (3.10) for any $m\in\mathbb{N}$, thus $\{h_m\}_{m=1}^\infty$ would be a Cauchy sequence in the Hilbert space $(\mathcal{F},\mathcal{E}_1)$ with $\lim_{m\to\infty}\|f_1|_K-h_m\|_2=0$ and therefore convergent to $f_1|_K$ in norm in $(\mathcal{F},\mathcal{E}_1)$, hence $f_1|_K\in\mathcal{F}$ and $\mathcal{E}(f_1|_K,f_1|_K)=\lim_{m\to\infty}\mathcal{E}(h_m,h_m)=0$, which contradicts Lemma 3.9 and completes the proof. □

It is the proof of the following proposition that requires our standing assumption that $S\neq\{0,1,\dots,l-1\}^d$, which excludes the case of $K=[0,1]^d$ from the present framework.

Proposition 3.11. *Let $h_2\in\mathcal{F}$ be as in Lemma 3.8 with $m=2$. Then h_2 is not $\mathcal{E}$-harmonic on U_0.*

Proof. We claim that, if h_2 were $\mathcal{E}$-harmonic on U_0, then $h_0\in\mathcal{F}$ as in Proposition 3.7 would turn out to be $\mathcal{E}$-harmonic on $K\setminus V_0^0$, which would imply that $\mathcal{E}(h_0,h_0)=\lim_{n\to\infty}\mathcal{E}(h_0,u_n)=0$ for $\{u_n\}_{n=1}^\infty\subset\varphi_0+\mathcal{C}_{U_0}\subset\mathcal{C}_{K\setminus V_0^0}$ as in Proposition 3.7 by (3.4), a contradiction to Proposition 3.10 and will thereby prove that h_2 is not $\mathcal{E}$-harmonic on U_0.

For each $\varepsilon=(\varepsilon_k)_{k=1}^d\in\{1\}\times\{0,1\}^{d-1}$, set $U^\varepsilon:=K\cap\prod_{k=1}^d(\varepsilon_k-1,\varepsilon_k+1)$, $K^\varepsilon:=K\cap\prod_{k=1}^d[\varepsilon_k-1/2,\varepsilon_k+1/2]$ and choose $\varphi_\varepsilon\in\mathcal{C}_{U^\varepsilon}$ so that $\varphi_\varepsilon|_{K^\varepsilon}=\mathbf{1}_{K^\varepsilon}$; such φ_ε exists by [15, Exercise 1.4.1]. Let $v\in\mathcal{C}_{K\setminus V_0^0}$ and, taking an enumeration $\{\varepsilon^{(k)}\}_{k=1}^{2^{d-1}}$ of $\{1\}\times\{0,1\}^{d-1}$ and recalling that $v_1v_2\in\mathcal{F}\cap\mathcal{C}(K)$ for any $v_1,v_2\in\mathcal{F}\cap\mathcal{C}(K)$ by [15, Theorem 1.4.2-(ii)], define $v_\varepsilon\in\mathcal{C}_{U^\varepsilon}$ for $\varepsilon\in\{1\}\times\{0,1\}^{d-1}$ by $v_{\varepsilon^{(1)}}:=v\varphi_{\varepsilon^{(1)}}$ and $v_{\varepsilon^{(k)}}:=v\varphi_{\varepsilon^{(k)}}\prod_{j=1}^{k-1}(\mathbf{1}_K-\varphi_{\varepsilon^{(j)}})$ for $k\in\{2,\dots,2^{d-1}\}$. Then $v-\sum_{\varepsilon\in\{1\}\times\{0,1\}^{d-1}}v_\varepsilon=v\prod_{\varepsilon\in\{1\}\times\{0,1\}^{d-1}}(\mathbf{1}_K-\varphi_\varepsilon)\in\mathcal{C}_{U_0}$, hence $\mathcal{E}(h_0,v)=\sum_{\varepsilon\in\{1\}\times\{0,1\}^{d-1}}\mathcal{E}(h_0,v_\varepsilon)$ by Proposition 3.7-(1) and (3.4), and therefore the desired $\mathcal{E}$-harmonicity of h_0 on $K\setminus V_0^0$, i.e., (3.4) with $h=h_0$ and $U=K\setminus V_0^0$, would be obtained by deducing that $\mathcal{E}(h_0,v_\varepsilon)=0$ for any $\varepsilon\in\{1\}\times\{0,1\}^{d-1}$.

To this end, set $\varepsilon^{(0)}:=(\mathbf{1}_{\{1\}}(k))_{k=1}^d$, take $i=(i_k)_{k=1}^d\in S$ with $i_1<l-1$ and $i+\varepsilon^{(0)}\notin S$, which exists by $\emptyset\neq S\subsetneq\{0,1,\dots,l-1\}^d$ and (GSC1), and let $\varepsilon=(\varepsilon_k)_{k=1}^d\in\{1\}\times\{0,1\}^{d-1}$. We will choose $i^\varepsilon\in S$ with $F_{ii^\varepsilon}(\varepsilon)\in F_i(K\cap(\{1\}\times(0,1)^{d-1}))$ and assemble $v_\varepsilon\circ g_w\circ F_w^{-1}$ with a suitable $g_w\in\mathcal{G}_1$ for $w\in W_2$ with $F_{ii^\varepsilon}(\varepsilon)\in K_w$ into a function $v_{\varepsilon,2}\in\mathcal{C}_{U_0}$. Specifically, set $i^{\varepsilon,\eta}:=\big((l-1)(\mathbf{1}_{\{1\}}(k)+1-\varepsilon_k)+(2\varepsilon_k-1)\eta_k\big)_{k=1}^d$ for each $\eta=(\eta_k)_{k=1}^d\in\{0\}\times\{0,1\}^{d-1}$ and $I^\varepsilon:=\{\eta\in\{0\}\times\{0,1\}^{d-1}\mid i^{\varepsilon,\eta}\in S\}$, so that $i^\varepsilon:=i^{\varepsilon,\mathbf{0}_d}\in S$ by (GSC4) and (GSC1) and hence $\mathbf{0}_d\in I^\varepsilon$. Thanks to $v_\varepsilon\in\mathcal{C}_{U^\varepsilon}$ and $i+\varepsilon^{(0)}\notin S$ we can define $v_{\varepsilon,2}\in\mathcal{C}(K)$ by setting

$$v_{\varepsilon,2}|_{K_w}:=\begin{cases}v_\varepsilon\circ g_\eta\circ F_w^{-1} & \text{if } \eta\in I^\varepsilon \text{ and } w=ii^{\varepsilon,\eta}\\ 0 & \text{if } w\notin\{ii^{\varepsilon,\eta}\mid\eta\in I^\varepsilon\}\end{cases}\qquad\text{for each } w\in W_2, \tag{3.11}$$

then $\operatorname{supp}_K[v_{\varepsilon,2}]\subset K_i\setminus V_0^0\subset U_0$ by (3.11) and $i_1<l-1$, $v_{\varepsilon,2}\circ F_w\in\mathcal{F}$ for any $w\in W_2$ by (3.11), $v_\varepsilon\in\mathcal{F}\cap\mathcal{C}(K)$ and (GSCDF1), thus $v_{\varepsilon,2}\in\mathcal{F}$ by (GSCDF2) and therefore $v_{\varepsilon,2}\in\mathcal{C}_{U_0}$. Moreover, recalling that $h_2\circ F_w=l^{-2}h_0+q_1^w\mathbf{1}_K$ μ-a.e. for any $w\in W_2$ by (3.9), Lemmas 3.2 and 3.3 and letting $\{u_n\}_{n=1}^\infty\subset\mathcal{F}\cap\mathcal{C}(K)$ be as in Proposition 3.7, we see from (2.2) for $u\in\mathcal{F}$ in Lemma 3.3, (3.11), Lemma 3.4, Proposition 3.7-(3), (GSCDF1) and Proposition 3.7-(2) that

$$\begin{aligned}\mathcal{E}(h_2,v_{\varepsilon,2})&=\sum_{\eta\in I^\varepsilon}\frac{1}{r^2l^2}\mathcal{E}(h_0,v_\varepsilon\circ g_\eta)=\lim_{n\to\infty}\sum_{\eta\in I^\varepsilon}\frac{1}{r^2l^2}\mathcal{E}(u_n,v_\varepsilon\circ g_\eta)\\&=\lim_{n\to\infty}\sum_{\eta\in I^\varepsilon}\frac{1}{r^2l^2}\mathcal{E}(u_n\circ g_\eta,v_\varepsilon)=\lim_{n\to\infty}\frac{\#I^\varepsilon}{r^2l^2}\mathcal{E}(u_n,v_\varepsilon)=\frac{\#I^\varepsilon}{r^2l^2}\mathcal{E}(h_0,v_\varepsilon).\end{aligned}\tag{3.12}$$

Now, supposing that h_2 were $\mathcal{E}$-harmonic on U_0, from (3.12), $\#I^\varepsilon > 0$, $v_{\varepsilon,2} \in \mathcal{C}_{U_0}$ and (3.4) we would obtain $\mathcal{E}(h_0, v_\varepsilon) = r^2 l^2 (\#I^\varepsilon)^{-1} \mathcal{E}(h_2, v_{\varepsilon,2}) = 0$, which would imply a contradiction as explained in the last two paragraphs and thus completes the proof. $\square$

Proof of Theorem 2.7. Let $h_0 \in \mathcal{F}$ be as in Proposition 3.7 and let $h_2 \in h_0 + \mathcal{F}_{U_0}$ be as in Lemma 3.8 with $m = 2$, so that h_0 is $\mathcal{E}$-harmonic on U_0, $h_2 + \mathcal{F}_{U_0} = h_0 + \mathcal{F}_{U_0}$, h_2 is not $\mathcal{E}$-harmonic on U_0 by Proposition 3.11 and hence $\mathcal{E}(h_0, h_0) < \mathcal{E}(h_2, h_2)$ in view of (3.3). This strict inequality combined with (3.10) shows that

$$\mathcal{E}(h_0, h_0) < \mathcal{E}(h_2, h_2) = \Bigl(\frac{\#S}{r} l^{-2}\Bigr)^2 \mathcal{E}(h_0, h_0),$$

whence $l^2 < \#S/r$, namely $d_{\mathrm{w}} = \log_l(\#S/r) > 2$. $\square$

Remark 3.12. As an alternative to Proposition 3.11 and its proof above, we could have used the *strong maximum principle* for $\mathcal{E}$-harmonic functions to show that $h_1 \in \mathcal{F}$ as in Lemma 3.8 with $m = 1$ is not $\mathcal{E}$-harmonic on U_0, from which Theorem 2.7 follows in exactly the same way. Indeed, let $h_0 \in \mathcal{F}$ be as in Proposition 3.7, let $i = (i_k)_{k=1}^d \in S$ be as in the above proof of Proposition 3.11 and set $U_i := F_i(K \cap ((0,1] \times (0,1)^{d-1}))$, which is an open subset of U_0 by $i_1 < l - 1$ and $i + (\mathbf{1}_{\{1\}}(k))_{k=1}^d \not\in S$ and connected by (GSC4), (GSC1) and (GSC2). Then noting that $h_1|_{U_i} = l^{-1} h_0 \circ F_i^{-1}|_{U_i} + l^{-1} i_1 \mathbf{1}_{U_i}$ μ-a.e. by (3.9) and that $\mu(K_i \setminus U_i) = 0$ by Lemma 3.2, we easily see from Proposition 3.7 (and its proof), Lemma 3.3, Proposition 3.10 and $\mathcal{E}(\mathbf{1}_K, \mathbf{1}_K) = 0$ that $h_1 \le l^{-1}(i_1 + 1)$ μ-a.e. on U_i, "$h_1 = l^{-1}(i_1 + 1)$ on $U_i \cap F_i(V_0^1)$" and $h_1|_{U_i} \neq l^{-1}(i_1 + 1)\mathbf{1}_{U_i}$, so that h_1 cannot be $\mathcal{E}$-harmonic on U_i or on $U_0 \supset U_i$ by the strong maximum principle [12, Theorem 2.11] for $\mathcal{E}$-(sub)harmonic functions on U_i.

While this short "proof" nicely illustrates the heuristics behind the proof of Proposition 3.11 above, it is in fact highly non-trivial to justify our last application of the strong maximum principle [12, Theorem 2.11] because it requires the following set of conditions; see [15, Sections A.2, 1.4, 4.1, 4.2 and 1.6] for the definitions and further details of the notions involved. Note that (con) below also gives a precise formulation of the statement "$h_1 = l^{-1}(i_1 + 1)$ on $U_i \cap F_i(V_0^1)$" in the previous paragraph, which needs to be done since $\mu(U_i \cap F_i(V_0^1)) = 0$ by Lemma 3.2.

(AC) There exists a μ-symmetric Hunt process $X = \bigl(\Omega, \mathcal{M}, \{X_t\}_{t\in[0,\infty]}, \{\mathbb{P}_x\}_{x\in K_\Delta}\bigr)$ on K such that its Dirichlet form on $L^2(K, \mu)$ is $(\mathcal{E}, \mathcal{F})$ and the Borel measure $\mathbb{P}_x[X_t \in \cdot]$ on K is *absolutely continuous with respect to* μ *for any* $(t, x) \in (0,\infty) \times K$, where $K_\Delta := K \cup \{\Delta\}$ denotes the one-point compactification of K.

(irr) $(\mathcal{E}|_{\mathcal{F}_{U_i} \times \mathcal{F}_{U_i}}, \mathcal{F}_{U_i})$ is *μ-irreducible*, i.e., $\mu(A)\mu(U_i \setminus A) = 0$ for any Borel subset A of U_i with the property that $u\mathbf{1}_A \in \mathcal{F}_{U_i}$ and $\mathcal{E}(u\mathbf{1}_A, u - u\mathbf{1}_A) = 0$ for any $u \in \mathcal{F}_{U_i}$.

(con) If h_1 were $\mathcal{E}$-harmonic on U_i, then there would exist a μ-version of $h_1|_{U_i}$ which would be nearly Borel measurable and finely continuous with respect to X as in (AC) and satisfy $h_1(x) = l^{-1}(i_1 + 1)$ for some $x \in U_i \cap F_i(V_0^1)$.

(Some other versions of the strong maximum principle for (sub)harmonic functions in the setting of a strongly local regular symmetric Dirichlet form have been proved in [29, 30, 31], but conditions very similar to (AC), (irr) and (con) are assumed also by them and therefore have to be verified anyway.)

It is possible to deduce (AC), (irr) and (con) from known properties of $(K, \rho, \mu, \mathcal{E}, \mathcal{F})$. Indeed, (AC) follows from [10, Chapter I, Theorem 9.4] and the fact that $(K, \rho, \mu, \mathcal{E}, \mathcal{F})$ has a version $p = p_t(x, y)$ of the heat kernel which is jointly continuous in $(t, x, y) \in (0,\infty) \times K \times K$ and satisfies (1.1) by [6, Theorem 4.30 and Remark 4.33] and [7, Theorem 3.1], and (irr) is implied by w-LLE$((\cdot)^{d_{\mathrm{w}}})$ from [7, Theorem 3.1], the connectedness of U_i and [15, Theorem 1.6.1]. For (con), let $\mathrm{Cap}_1^{\mathcal{E}}(A)$ denote the 1-capacity of $A \subset K$ with respect to $(K, \mu, \mathcal{E}, \mathcal{F})$ as defined in [15, (2.1.1), (2.1.2) and (2.1.3)], and suppose that h_1 were $\mathcal{E}$-harmonic on U_i. Then by w-LLE$((\cdot)^{d_{\mathrm{w}}})$

and [7, Corollary 4.2] there would exist a continuous μ-version of $h_1|_{U_i}$ and hence so would a continuous one of $h_0 = lh_1 \circ F_i - i_1 \mathbf{1}_K$ on $F_i^{-1}(U_i)$ by Lemma 3.3. Moreover, $\mathrm{Cap}_1^{\mathcal{E}}(V_0^1) > 0$ by Propositions 3.7-(1), 3.10, (3.3) and [15, Corollary 2.3.1], $\mathrm{Cap}_1^{\mathcal{E}}(F_i^{-1}(U_i) \cap V_0^1) > 0$ by (GSC4), (GSC1) and [19, Lemma 7.14], the continuous μ-version of h_0 on $F_i^{-1}(U_i)$ would satisfy $h_0 = 1$ on $F_i^{-1}(U_i) \cap V_0^1 \setminus N$ for some $N \subset K$ with $\mathrm{Cap}_1^{\mathcal{E}}(N) = 0$ by $h_0 \in \varphi_0 + \mathcal{F}_{U_0}$ and [15, Theorem 2.1.4-(i)], and $F_i^{-1}(U_i) \cap V_0^1 \setminus N \neq \emptyset$ by $\mathrm{Cap}_1^{\mathcal{E}}(F_i^{-1}(U_i) \cap V_0^1) > \mathrm{Cap}_1^{\mathcal{E}}(N)$. Thus $h_0(y) = 1$ for some $y \in F_i^{-1}(U_i) \cap V_0^1$ and $h_1(x) = l^{-1}(i_1 + 1)$ for $x := F_i(y) \in U_i \cap F_i(V_0^1)$, proving (con).

Note that the arguments in the last paragraph heavily rely on the demanding results in [6, 7], and also that the proofs of the strong maximum principle in [29, 30, 31, 12] make full use of the potential theory for regular symmetric Dirichlet forms in [15, 11]. In this sense, the above "short" proof of the non-$\mathcal{E}$-harmonicity of h_1 on U_0 based on the strong maximum principle is far from self-contained and much less elementary than the proof of Proposition 3.11.

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