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p-Energy forms on fractals: recent progress

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Abstract. In this article, we survey recent progress on self-similar p-energy forms on self-similar fractals, where $p \in (1, \infty)$. While for $p = 2$ the notion of such forms coincides with that of self-similar Dirichlet forms and there have been plenty of studies on them since the late 1980s, studies on the case of $p \in (1, \infty) \setminus \{2\}$ was initiated much later in 2004 by Herman, Peirone and Strichartz [*Potential Anal.* **20** (2004), 125–148] and Strichartz and Wong [*Nonlinearity* **17** (2004), 595–616] and no essential progress on this case had been made since then until a few years ago. The recent progress by Kigami, Shimizu, Cao–Gu–Qiu and Murugan–Shimizu has established the existence of such p-energy forms on general post-critically finite (p.-c.f.) self-similar sets and on large classes of low-dimensional infinitely ramified self-similar sets, and the authors have proved further detailed properties of these forms and associated p-harmonic functions, mainly for p.-c.f. self-similar sets. This article is devoted to a review of these results, focusing on the most recent developments by the authors and illustrating them in the simplest non-trivial setting of the two-dimensional standard Sierpiński gasket.

Keywords: (Two-dimensional standard) Sierpiński gasket, post-critically finite (p.-c.f.) self-similar set, p-resistance form, generalized contraction property, p-harmonic function, strong comparison principle, p-energy measure

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1 Introduction

In this article, we survey recent progress on *self-similar p-energy forms* on self-similar fractals for $p \in (1, \infty)$. Namely, we are concerned with a functional $\mathcal{E}_p \colon \mathcal{F}_p \to [0, \infty)$ defined on a linear space $\mathcal{F}_p$ of $\mathbb{R}$-valued functions over a fractal K which is self-similar with respect to a finite family of continuous injections $\{F_i\}_{i \in S}$, such that $\mathcal{E}_p^{1/p}$ is a seminorm on $\mathcal{F}_p$ and (at least) the following hold:

$$u^+ \wedge 1 \in \mathcal{F}_p \quad \text{and} \quad \mathcal{E}_p(u^+ \wedge 1) \leq \mathcal{E}_p(u) \quad \text{for any } u \in \mathcal{F}_p, \tag{1.1}$$

$$\{u \circ F_i\}_{i \in S} \subset \mathcal{F}_p \quad \text{and} \quad \mathcal{E}_p(u) = \sum_{i \in S} \rho_{p,i} \mathcal{E}_p(u \circ F_i) \quad \text{for any } u \in \mathcal{F}_p \tag{1.2}$$

for some $(\rho_{p,i})_{i \in S} \in (0, \infty)^S$. We usually assume also that $\mathcal{F}_p$ is complete under a suitable norm involving $\mathcal{E}_p$ and that $\mathcal{E}_p$ satisfies some L^p-type convexity and smoothness inequalities like p-Clarkson's ones. Such p-energy forms $(\mathcal{E}_p, \mathcal{F}_p)$ could be considered as natural analogs of the p-th power of the canonical seminorm $(\int_{\mathbb{R}^n} |\nabla u|^p \, dx)^{1/p}$ on the first-order L^p-Sobolev space $W^{1,p}(\mathbb{R}^n)$ over $\mathbb{R}^n$.

The case of $p = 2$, where such forms are nothing but self-similar Dirichlet forms on self-similar fractals, has been extensively studied since the late 1980s (see, e.g., the textbooks [2,21,29], a recent study [17, Section 6] and references therein). On the other hand, there had been no study on the case of $p \in (1, \infty) \setminus \{2\}$ until the first construction of such forms on a class of post-critically finite (p.-c.f.) self-similar sets by Herman, Peirone and Strichartz [12] in 2004 and a subsequent study of the standard p-Laplacian on the two-dimensional standard Sierpiński gasket (Figure 1 below) by Strichartz and Wong [30] in 2004, and no essential progress on this case had been made since then until a few years ago.

The situation changed in 2021, when Kigami [23] and Shimizu [28] started the construction of self-similar p-energy forms on much larger classes of self-similar fractals including the Sierpiński carpet and various other infinitely ramified self-similar sets, with the motivation toward better understanding of the Ahlfors regular conformal dimension and related conformal-geometric properties of the Sierpiński carpet; see also [17, (1.6) and Problem 7.7] for some backgrounds on this motivation. Soon after the first appearance of the works [23,28] on arXiv, Cao, Gu and Qiu [7] gave a comprehensive study of the case of p.-c.f. self-similar sets and in particular extended the construction of self-similar p-energy forms as in [12] to general p.-c.f. self-similar sets. The constructions in [23,28] needed to assume that p is greater than the Ahlfors-regular conformal dimension of the fractal K, in order for the domain $\mathcal{F}_p$ to be included in the space $C(K)$ of continuous functions on K (see also [6] in this connection), but Murugan and Shimizu [26] have recently removed this restriction on p in the case of the Sierpiński carpet. Another important aspect of the works [28,26] is that they have constructed the associated *p-energy measure* $\mu^p_{\langle u \rangle}$ of each $u \in \mathcal{F}_p$, i.e., the unique Borel measure $\mu^p_{\langle u \rangle}$ on the fractal K such that

$$\mu^p_{\langle u \rangle}(F_w(K)) = \rho_{p,w} \mathcal{E}_p(u \circ F_w) \qquad \text{for any } w \in \bigcup_{n \in \mathbb{N} \cup \{0\}} S^n, \tag{1.3}$$

where $F_w := F_{w_1} \circ \cdots \circ F_{w_n}$ and $\rho_{p,w} := \rho_{p,w_1} \cdots \rho_{p,w_n}$ for $w = w_1 \ldots w_n \in S^n$ ($F_w := \mathrm{id}_K$ and $\rho_{p,w} := 1$ if $n = 0$), and proved some fundamental properties of this family of measures, including the following chain rule:

$$d\mu^p_{\langle \Phi(u) \rangle} = |\Phi'(u)|^p \, d\mu^p_{\langle u \rangle} \quad \text{for any } u \in \mathcal{F}_p \cap C(K) \text{ and any } \Phi \in C^1(\mathbb{R}). \tag{1.4}$$

While these results have established the existence of self-similar p-energy forms on very large classes of self-similar sets, detailed properties of such forms are yet to be studied. In fact, we have recently made essential progress in analyzing p-harmonic functions and p-energy measures $\mu^p_{\langle u \rangle}$ further, mainly in the setting of p.-c.f. self-similar sets. Our main results can be summarized as follows; see [21, Chapter 1] for the definition of the notion of p.-c.f. self-similar structure.

(1) ([18]) The constructions of a self-similar p-energy form as in the main results of [12,23,28,7,26] can be modified so as to guarantee that the resulting p-energy form $(\mathcal{E}_p, \mathcal{F}_p)$ satisfies additionally the following properties:

  (1-1) (Generalized contraction property) If $q \in (0, p]$, $r \in [p, \infty]$, $m, n \in \mathbb{N}$ and $T = (T_1, \ldots, T_n) \colon \mathbb{R}^m \to \mathbb{R}^n$ satisfies $\|T(x) - T(y)\|_{\ell^r} \leq \|x - y\|_{\ell^q}$ for any $x, y \in \mathbb{R}^m$ and $T(0) = 0$, then for any $u = (u_1, \ldots, u_m) \in \mathcal{F}_p^m$, $T(u) = (T_1(u), \ldots, T_n(u)) \in \mathcal{F}_p^n$ and

$$\|(\mathcal{E}_p(T_k(u))^{1/p})_{k=1}^n\|_{\ell^r} \leq \|(\mathcal{E}_p(u_j)^{1/p})_{j=1}^m\|_{\ell^q}. \tag{1.5}$$

  (1-2) (Differentiability) $\mathcal{E}_p$ is Fréchet differentiable with locally α_p-Hölder continuous Fréchet derivative with respect to a natural norm on $\mathcal{F}_p$, where $\alpha_p := \frac{1}{p} \wedge \frac{p-1}{p}$.

(2) Assume that $\mathcal{L} := (K, S, \{F_i\}_{i \in S})$ is a p.-c.f. self-similar structure with K connected and containing at least two elements, and that $(\mathcal{E}_p, \mathcal{F}_p)$ is a self-similar p-energy form over $\mathcal{L}$ satisfying (1-1). Then the following hold:

  (2-1) (Strong comparison principle; [19]) Let $U \subsetneq K$ be a connected open subset of K. If $u, v \in C(\overline{U}^K)$ are harmonic on U with respect to $\mathcal{E}_p$ and $u(x) \leq v(x)$ for any $x \in \partial_K U$, then either $u(x) < v(x)$ for any $x \in U$ or $u(x) = v(x)$ for any $x \in \overline{U}^K$.

  (2-2) (Uniqueness under good symmetry; [19]) If $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ has certain good geometric symmetry and $(\mathcal{E}_p, \mathcal{F}_p)$ is invariant under this symmetry of $\mathcal{L}$, then any other such p-energy form over K satisfying (1.2) with the same $(\rho_{p,i})_{i \in S}$ as $(\mathcal{E}_p, \mathcal{F}_p)$ has domain $\mathcal{F}_p$ and is a constant multiple of $\mathcal{E}_p$.

  (2-3) (Singularity of p- and q-energy measures under very good symmetry; [20]) If $\mathcal{L} = (K, S, \{F_i\}_{i \in S})$ has certain very good geometric symmetry, $p, q \in (1, \infty)$, $p \neq q$, and $(\mathcal{E}_p, \mathcal{F}_p)$ and $(\mathcal{E}_q, \mathcal{F}_q)$ are invariant under this symmetry of $\mathcal{L}$, then the p-energy measure $\mu^p_{\langle u \rangle}$ of any $u \in \mathcal{F}_p$ and the q-energy measure $\mu^q_{\langle v \rangle}$ of any $v \in \mathcal{F}_q$ are mutually singular.

The purpose of this article is to give some concise descriptions of how these results are related to each other and how they can be obtained, by illustrating them in the simplest non-trivial setting of the (two-dimensional standard)

Sierpiński gasket (Figure 1 below), even for which all the results in (1) and (2) above are new. In fact, Strichartz and Wong [30] supposed without proofs some conditions implied by (1-1), (1-2) and (2-1) on the canonical p-energy form on the Sierpiński gasket to derive some properties of it. Our above results (1) and (2-1) ensure that a p-energy form on the Sierpiński gasket satisfying those conditions exists and thereby turns out to have the properties derived in [30]; see also Subsection 3.3 below, where we provide a self-contained treatment of principal results in [30, Section 5].

The rest of this paper is organized as follows. First in Section 2, we give a brief summary of the theory of p-resistance forms planned to appear in [18]. In Section 3, after introducing the (two-dimensional standard) Sierpiński gasket in Subsection 3.1, we survey the construction and basic properties of the canonical p-resistance form on the Sierpiński gasket in Subsection 3.2, describing the modification obtained in [18] of the original construction due to [12,7]. In Subsection 3.3, we present some details of the strong comparison principle and its application to uniqueness of a self-similar p-resistance form planned to be presented in [19]. Finally in Section 4, keeping the setting of the Sierpiński gasket, we introduce the p-energy measures, mention their basic properties, and prove that the p-energy measures and the q-energy measures are mutually singular for any $p, q \in (1, \infty)$ with $p \neq q$, whose proof in the greater generality described in (2-3) above is planned to appear in [20].

Notation. In this paper, we adopt the following notation and conventions.

(1) The symbols $\subset$ and $\supset$ for set inclusion *allow* the case of the equality.

(2) $\mathbb{N} := \{n \in \mathbb{Z} \mid n > 0\}$, i.e., $0 \notin \mathbb{N}$.

(3) The cardinality (the number of elements) of a set A is denoted by $\#A$.

(4) We set $\sup \emptyset := 0$, set $a \vee b := \max\{a, b\}$, $a \wedge b := \min\{a, b\}$, $a^+ := a \vee 0$, $a^- := -(a \wedge 0)$ and $\mathrm{sgn}(a) := |a|^{-1}a$ ($\mathrm{sgn}(0) := 0$) for $a, b \in \mathbb{R}$, and use the same notation also for $\mathbb{R}$-valued functions and equivalence classes of them. All numerical functions in this paper are assumed to be $[-\infty, \infty]$-valued.

(5) Let $n \in \mathbb{N}$. For $x = (x_k)_{k=1}^n \in \mathbb{R}^n$, we set $\|x\|_{\ell^p} := (\sum_{k=1}^n |x_k|^p)^{1/p}$ for $p \in (0, \infty)$, $\|x\|_{\ell^\infty} := \max_{1 \le k \le n} |x_k|$ and $|x| := \|x\|_{\ell^2}$. For $\Phi \colon \mathbb{R}^n \to \mathbb{R}$ which is differentiable on $\mathbb{R}^n$ and for $k \in \{1, \dots, n\}$, its first-order partial derivative in the k-th coordinate is denoted by $\partial_k \Phi$, and we set $C^1(\mathbb{R}^n) := \{\Phi \mid \Phi \colon \mathbb{R}^n \to \mathbb{R},\ \Phi$ is differentiable on $\mathbb{R}^n$, $\partial_1 \Phi, \dots, \partial_n \Phi$ are continuous on $\mathbb{R}^n\}$.

(6) Let K be a non-empty set. We define $\mathrm{id}_K \colon K \to K$ by $\mathrm{id}_K(x) := x$, $\mathbb{1}_A = \mathbb{1}_A^K \in \mathbb{R}^K$ for $A \subset K$ by $\mathbb{1}_A(x) := \mathbb{1}_A^K(x) := \left\{\begin{smallmatrix} 1 \text{ if } x \in A, \\ 0 \text{ if } x \notin A, \end{smallmatrix}\right.$ set $\mathbb{1}_x := \mathbb{1}_x^K := \mathbb{1}_{\{x\}}$ for $x \in K$, $\|u\|_{\sup} := \|u\|_{\sup, K} := \sup_{x \in K} |u(x)|$ for $u \in \mathbb{R}^K$, and $\mathrm{osc}_K[u] := \sup_{x, y \in K} |u(x) - u(y)| = \sup_{x \in K} u(x) - \inf_{x \in K} u(x)$ for $u \in \mathbb{R}^K$ with $\|u\|_{\sup} < \infty$.

(7) Let K be a topological space. The Borel σ-algebra of K is denoted by $\mathcal{B}(K)$. The closure and boundary of $A \subset K$ in K are denoted by $\overline{A}^K$ and $\partial_K A$, respectively. We set $C(K) := \{u \in \mathbb{R}^K \mid u$ is continuous$\}$ and $\mathrm{supp}_K[u] := \overline{K \setminus u^{-1}(0)}^K$ for $u \in C(K)$.

(8) Let $(K, \mathcal{B})$ be a measurable space and let μ, ν be measures on $(K, \mathcal{B})$. We write $\nu \ll \mu$ and $\nu \perp \mu$ to mean that ν is absolutely continuous and singular, respectively, with respect to μ.

2 p-Resistance forms: the definition and basic properties

As treated in [2,21,29], the theory of self-similar Dirichlet forms on p.-c.f. self-similar sets has been developed on the basis of that of resistance forms. Likewise, the construction and basic properties of self-similar p-energy forms on p.-c.f. self-similar sets can be presented most efficiently by referring to some general facts for certain natural L^p-analogs of resistance forms which we call *p-resistance forms*. For this reason, we start our discussion with introducing the notion of such forms and stating some basic properties of them. The details of the results in this section will appear in [18].

Let $p \in (1, \infty)$, which we fix throughout this section.

Definition 2.1 (p-Resistance forms). Let K be a non-empty set. The pair $(\mathcal{E}_p, \mathcal{F}_p)$ of a linear subspace $\mathcal{F}_p$ of $\mathbb{R}^K$ and a functional $\mathcal{E}_p \colon \mathcal{F}_p \to [0, \infty)$ is said to be a *p-resistance form* on K if and only if the following five conditions are satisfied:

$(\mathrm{RF1})_p$ $\mathcal{E}_p^{1/p}$ is a seminorm on $\mathcal{F}_p$ and $\{u \in \mathcal{F}_p \mid \mathcal{E}_p(u) = 0\} = \mathbb{R}\mathbb{1}_K$.

$(\mathrm{RF2})_p$ The quotient normed space $(\mathcal{F}_p/\mathbb{R}\mathbb{1}_K, \mathcal{E}_p^{1/p})$ is a Banach space.

$(\mathrm{RF3})_p$ For any $x, y \in K$ with $x \neq y$ there exists $u \in \mathcal{F}_p$ such that $u(x) \neq u(y)$.

$(\mathrm{RF4})_p$ $R_{\mathcal{E}_p}(x, y) := \sup\left\{ \dfrac{|u(x) - u(y)|^p}{\mathcal{E}_p(u)} \;\middle|\; u \in \mathcal{F}_p \backslash \mathbb{R}\mathbb{1}_K \right\} < \infty$ for any $x, y \in K$.

$(\mathrm{RF5})_p$ (Generalized contraction property) If $q \in (0, p]$, $r \in [p, \infty]$, $m, n \in \mathbb{N}$ and $T = (T_1, \ldots, T_n) \colon \mathbb{R}^m \to \mathbb{R}^n$ satisfies $\|T(x) - T(y)\|_{\ell^r} \leq \|x - y\|_{\ell^q}$ for any $x, y \in \mathbb{R}^m$ and $T(0) = 0$, then for any $u = (u_1, \ldots, u_m) \in \mathcal{F}_p^m$, $T(u) = (T_1(u), \ldots, T_n(u)) \in \mathcal{F}_p^n$ and

$$\|(\mathcal{E}_p(T_k(u))^{1/p})_{k=1}^n\|_{\ell^r} \leq \|(\mathcal{E}_p(u_j)^{1/p})_{j=1}^m\|_{\ell^q}. \tag{2.1}$$

The generalized contraction property $(\mathrm{RF5})_p$ is arguably the strongest possible form of contraction properties of energy functionals, including as the special case with $p = 2$, $q = 1$ and $n = 1$ the so-called normal contractivity of symmetric Dirichlet forms (see, e.g., [25, Theorem 4.12] or [4, Proposition I.3.3.1]). As discussed in Example 2.2-(2) and Remark 2.3 below, if $\mathcal{E}_2 \colon \mathcal{F}_2 \to [0, \infty)$ is a quadratic form on a linear space $\mathcal{F}_2$ of (equivalence classes of) $\mathbb{R}$-valued functions and has some suitable completeness property, then $(\mathrm{RF5})_2$ is equivalent to (1.1). On the other hand, for $p \neq 2$, at the moment we do not know any characterization of $(\mathrm{RF5})_p$ by simpler conditions, but it turns out that we can still modify the existing constructions of self-similar p-energy forms in [12,23,28,7,26] so as to get ones satisfying $(\mathrm{RF5})_p$. In this sense we miss essentially no examples by requiring our p-energy forms to satisfy the strong condition $(\mathrm{RF5})_p$.

Example 2.2. (1) Let V be a non-empty finite set. Note that in this case $(\mathcal{E}_p, \mathcal{F}_p)$ is a p-resistance form on V if and only if $\mathcal{F}_p = \mathbb{R}^V$ and $\mathcal{E}_p \colon \mathbb{R}^V \to [0, \infty)$ satisfies (RF1)$_p$ and (RF5)$_p$; indeed, since (RF5)$_p$ implies (1.1), (RF3)$_p$ is equivalent to $\mathcal{F}_p = \mathbb{R}^V$ under (RF1)$_p$ and (RF5)$_p$ by [22, Proposition 3.2], and (RF2)$_p$ and (RF4)$_p$ are easily implied by (RF1)$_p$ and $\dim \mathcal{F}_p/\mathbb{R}\mathbb{1}_V < \infty$. On the basis of this observation, for p-resistance forms $(\mathcal{E}_p, \mathcal{F}_p)$ on finite sets, we refer only to $\mathcal{E}_p$ and say simply that $\mathcal{E}_p$ is a p-resistance form on the set. Now, consider any functional $\mathcal{E}_p \colon \mathbb{R}^V \to [0, \infty)$ of the form

$$\mathcal{E}_p(u) = \frac{1}{2} \sum_{x,y \in V} L_{xy} |u(x) - u(y)|^p \tag{2.2}$$

for some $L = (L_{xy})_{x,y \in V} \in [0, \infty)^{V \times V}$ such that $L_{xy} = L_{yx}$ for any $x, y \in V$. It is obvious that $\mathcal{E}_p$ satisfies (RF1)$_p$ if and only if the graph (V, E_L) is connected, where $E_L := \{\{x, y\} \mid x, y \in V,\, x \neq y,\, L_{xy} > 0\}$. It is also easy to see, by using the reverse Minkowski inequality for $\|\cdot\|_{\ell^{p/r}}$ (see, e.g., [1, Theorem 2.13]) and Minkowski's inequality for $\|\cdot\|_{\ell^{p/q}}$, that $\mathcal{E}_p$ satisfies (RF5)$_p$. It thus follows that $\mathcal{E}_p$ is a p-resistance form on V if and only if (V, E_L) is connected.

Note that, while any 2-resistance form on V is of the form (2.2) with $p = 2$, *the counterpart of this fact for $p \neq 2$ is NOT true* unless $\#V \leq 2$. As we will see in Section 3 and summarize in Remark 3.25, the analysis of self-similar p-energy forms on p.-c.f. self-similar sets inevitably involves p-energy forms on finite sets which may not be of the form (2.2), and therefore the validity of contraction properties like (RF5)$_p$ for such forms becomes non-trivial.

(2) Recall from [21, Definition 2.3.1] and [22, Definition 3.1] that a *resistance form* on a non-empty set K is defined as the pair $(\mathcal{E}, \mathcal{F})$ of a linear subspace $\mathcal{F}$ of $\mathbb{R}^K$ and a non-negative definite symmetric bilinear form $\mathcal{E} \colon \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ such that the associated quadratic form $(\mathcal{E}_2, \mathcal{F})$ given by $\mathcal{E}_2(u) := \mathcal{E}(u, u)$ satisfies (RF1)$_2$, (RF2)$_2$, (RF3)$_2$, (RF4)$_2$ and (1.1). As our terminology indicates, resistance forms on K can be canonically identified with 2-resistance forms on K as defined in Definition 2.1, through the correspondence $(\mathcal{E}, \mathcal{F}) \mapsto (\mathcal{E}_2, \mathcal{F})$. Indeed, if $(\mathcal{E}, \mathcal{F})$ is a resistance form on K, then $(\mathcal{E}_2, \mathcal{F})$ satisfies (RF5)$_2$ since, by [15, Corollary 2.37] (see also [21, Theorems 2.3.6, 2.3.7 and Lemma 2.3.8]), $\mathcal{E}_2$ can be expressed as a certain kind of supremum over resistance forms on the non-empty finite subsets of K, which are necessarily of the form (2.2) with $p = 2$ and hence satisfy (RF5)$_2$ by (1) above. Conversely, if $(\mathcal{E}_2, \mathcal{F}_2)$ is a 2-resistance form on K, then two simple applications of (RF5)$_2$ show (1.1) with $p = 2$ and that $\mathcal{E}_2(u + v) + \mathcal{E}_2(u - v) = 2(\mathcal{E}_2(u) + \mathcal{E}_2(v))$ for any $u, v \in \mathcal{F}_2$, so that $\mathcal{E} \colon \mathcal{F}_2 \times \mathcal{F}_2 \to \mathbb{R}$ defined by $\mathcal{E}(u, v) := \frac{1}{4}(\mathcal{E}_2(u + v) - \mathcal{E}_2(u - v))$ is bilinear and symmetric, satisfies $\mathcal{E}(u, u) = \mathcal{E}_2(u) \geq 0$ for any $u \in \mathcal{F}_2$ and thus gives a resistance form $(\mathcal{E}, \mathcal{F}_2)$ on K.

Remark 2.3. Similarly, *any symmetric Dirichlet form satisfies* (RF5)$_2$, which does not seem to have ever been stated in the literature. Indeed, this claim follows by using the argument in Example 2.2-(1) based on the reverse Minkowski inequality for $\|\cdot\|_{\ell^{2/r}}$ and Minkowski's inequality for $\|\cdot\|_{\ell^{2/q}}$ to modify [25, Proof of Theorem 4.12] in the right manner.

Another application of Minkowski's inequality for $\|\cdot\|_{\ell^{r/p}}$ and the reverse Minkowski inequality for $\|\cdot\|_{\ell^{q/p}}$ also shows the following lemma.

Lemma 2.4. *Let K be a non-empty set, let $\mathcal{F}_p$ be a linear subspace of $\mathbb{R}^K$, let $\mathcal{E}_{p,1}, \mathcal{E}_{p,2} \colon \mathcal{F}_p \to [0,\infty)$ and $a_1, a_2 \in [0,\infty)$.*

(1) *If $\mathcal{E}_{p,1}^{1/p}, \mathcal{E}_{p,2}^{1/p}$ are seminorms on $\mathcal{F}_p$, then so is $(a_1\mathcal{E}_{p,1} + a_2\mathcal{E}_{p,2})^{1/p}$.*
(2) *If $(\mathcal{E}_{p,1}, \mathcal{F}_p), (\mathcal{E}_{p,2}, \mathcal{F}_p)$ satisfy (RF5)$_p$, then so does $(a_1\mathcal{E}_{p,1} + a_2\mathcal{E}_{p,2}, \mathcal{F}_p)$.*

Throughout the rest of this section, we assume that K is a non-empty set and that $(\mathcal{E}_p, \mathcal{F}_p)$ is a p-resistance form on K. The following inequality, which is immediate from (RF4)$_p$, is of fundamental importance in this setting.

Proposition 2.5. *For any $u \in \mathcal{F}_p$ and any $x, y \in K$,*

$$|u(x) - u(y)|^p \leq R_{\mathcal{E}_p}(x,y)\mathcal{E}_p(u). \tag{2.3}$$

(RF5)$_p$ transfers various inequalities satisfied by L^p-norms to $\mathcal{E}_p^{1/p}$ and allows us thereby to analyze p-energy forms more deeply. We collect some important special cases of (RF5)$_p$ in the following proposition; see [5, Theorem 4.7] and the references therein for some background for (4) and, e.g., [1, Lemma 2.37 and Theorem 2.38] for (5).

Proposition 2.6. *Let $u, v \in \mathcal{F}_p$, and let $\varphi \in C(\mathbb{R})$ satisfy $|\varphi(t) - \varphi(s)| \leq |t-s|$ for any $s, t \in \mathbb{R}$.*

(1) *$\varphi(u) \in \mathcal{F}_p$ and $\mathcal{E}_p(\varphi(u)) \leq \mathcal{E}_p(u)$. In particular, $f \in \mathcal{F}_p$ and $\mathcal{E}_p(f) \leq \mathcal{E}_p(u)$ for any $f \in \{u^+ \wedge 1, |u|, u^+, u^-\}$, and $u - \varphi(u-v), v + \varphi(u-v), u \wedge v, u \vee v \in \mathcal{F}_p$.*
(2) *If $\|u\|_{\sup} \vee \|v\|_{\sup} < \infty$, then $uv \in \mathcal{F}_p$ and $\mathcal{E}_p(uv)^{1/p} \leq \|v\|_{\sup}\mathcal{E}_p(u)^{1/p} + \|u\|_{\sup}\mathcal{E}_p(v)^{1/p}$.*
(3) (Strong subadditivity) *$\mathcal{E}_p(u \wedge v) + \mathcal{E}_p(u \vee v) \leq \mathcal{E}_p(u) + \mathcal{E}_p(v)$.*
(4) *If φ is non-decreasing, then $\mathcal{E}_p(u - \varphi(u-v)) + \mathcal{E}_p(v + \varphi(u-v)) \leq \mathcal{E}_p(u) + \mathcal{E}_p(v)$.*
(5) (p-Clarkson's inequalities) *If $p \leq 2$, then*

$$\mathcal{E}_p(u+v) + \mathcal{E}_p(u-v) \geq 2\bigl(\mathcal{E}_p(u)^{1/(p-1)} + \mathcal{E}_p(v)^{1/(p-1)}\bigr)^{p-1}, \tag{2.4}$$
$$\mathcal{E}_p(u+v) + \mathcal{E}_p(u-v) \leq 2(\mathcal{E}_p(u) + \mathcal{E}_p(v)). \tag{2.5}$$

If $p \geq 2$, then

$$\mathcal{E}_p(u+v) + \mathcal{E}_p(u-v) \leq 2\bigl(\mathcal{E}_p(u)^{1/(p-1)} + \mathcal{E}_p(v)^{1/(p-1)}\bigr)^{p-1}, \tag{2.6}$$
$$\mathcal{E}_p(u+v) + \mathcal{E}_p(u-v) \geq 2(\mathcal{E}_p(u) + \mathcal{E}_p(v)). \tag{2.7}$$

p-Clarkson's inequalities (2.5) and (2.6), together with (RF1)$_p$, the convexity of $|\cdot|^p$, (2.4) and (2.7), imply the following properties.

Theorem 2.7. *The quotient norm $\mathcal{E}_p\colon \mathcal{F}_p/\mathbb{R}\mathbb{1}_K \to \mathbb{R}$ is Fréchet differentiable on $(\mathcal{F}_p/\mathbb{R}\mathbb{1}_K, \mathcal{E}_p^{1/p})$. In particular, for any $u, v \in \mathcal{F}_p$,*

$$\text{the derivative} \quad \mathcal{E}_p(u;v) := \frac{1}{p}\frac{d}{dt}\mathcal{E}_p(u+tv)\bigg|_{t=0} \in \mathbb{R} \quad \text{exists}, \tag{2.8}$$

$\mathcal{E}_p(u;\cdot)\colon \mathcal{F}_p \to \mathbb{R}$ is linear, $\mathcal{E}_p(u;u) = \mathcal{E}_p(u)$ and $\mathcal{E}_p(u;\mathbb{1}_K) = 0$. Moreover, for any $u, u_1, u_2, v \in \mathcal{F}_p$ and any $a \in \mathbb{R}$, the following hold:

$$\mathbb{R} \ni t \mapsto \mathcal{E}_p(u+tv;v) \in \mathbb{R} \text{ is strictly increasing if and only if } v \notin \mathbb{R}\mathbb{1}_K. \tag{2.9}$$

$$\mathcal{E}_p(au;v) = \operatorname{sgn}(a)|a|^{p-1}\mathcal{E}_p(u;v), \qquad \mathcal{E}_p(u+a\mathbb{1}_K;v) = \mathcal{E}_p(u;v). \tag{2.10}$$

$$|\mathcal{E}_p(u;v)| \le \mathcal{E}_p(u)^{(p-1)/p}\mathcal{E}_p(v)^{1/p}. \tag{2.11}$$

$$|\mathcal{E}_p(u_1;v) - \mathcal{E}_p(u_2;v)| \le c_p(\mathcal{E}_p(u_1) \vee \mathcal{E}_p(u_2))^{(p-1-\alpha_p)/p}\mathcal{E}_p(u_1-u_2)^{\alpha_p/p}\mathcal{E}_p(v)^{1/p} \tag{2.12}$$

for $\alpha_p := \frac{1}{p} \wedge \frac{p-1}{p}$ and some $c_p \in (0,\infty)$ determined solely and explicitly by p.

Theorem 2.7 plays fundamental roles in establishing fine properties of harmonic functions with respect to p-resistance forms. Below we summarize some important consequences of it. First, the following proposition states that the variational and distributional formulations of the notion of harmonicity of functions coincide for p-resistance forms.

Proposition 2.8. *Let $h \in \mathcal{F}_p$ and $B \subset K$. Then the following two conditions are equivalent:*

(1) $\mathcal{E}_p(h) = \inf\{\mathcal{E}_p(u) \mid u \in \mathcal{F}_p,\, u|_B = h|_B\}$.
(2) $\mathcal{E}_p(h;v) = 0$ *for any $v \in \mathcal{F}_p$ with $v|_B = 0$.*

Definition 2.9 ($\mathcal{E}_p$-harmonic functions). Let $B \subset K$. We say that $h \in \mathcal{F}_p$ is *$\mathcal{E}_p$-harmonic* on $K \setminus B$ if and only if h satisfies either (and hence both) of (1) and (2) in Proposition 2.8. We set $\mathcal{H}_{\mathcal{E}_p,B} := \{h \in \mathcal{F}_p \mid h \text{ is } \mathcal{E}_p\text{-harmonic on } K \setminus B\}$.

$\mathcal{E}_p$-harmonic functions with given boundary values uniquely exist, and their energies under $\mathcal{E}_p$ define a new p-resistance form on the boundary set, as follows.

Theorem 2.10. *Let $B \subset K$ be non-empty, set $\mathcal{F}_p|_B := \{u|_B \mid u \in \mathcal{F}_p\}$, and define $\mathcal{E}_p|_B\colon \mathcal{F}_p|_B \to [0,\infty)$ by*

$$\mathcal{E}_p|_B(u) := \inf\{\mathcal{E}_p(v) \mid v \in \mathcal{F}_p,\, v|_B = u\}, \quad u \in \mathcal{F}_p|_B. \tag{2.13}$$

Then $(\mathcal{E}_p|_B, \mathcal{F}_p|_B)$ is a p-resistance form on B and $R_{\mathcal{E}_p|_B} = R_{\mathcal{E}_p}|_{B\times B}$ (recall (RF4)$_p$)*. Moreover, for any $u \in \mathcal{F}_p|_B$ there exists a unique $h_B^{\mathcal{E}_p}[u] \in \mathcal{F}_p$ such that $h_B^{\mathcal{E}_p}[u]\big|_B = u$ and $\mathcal{E}_p\big(h_B^{\mathcal{E}_p}[u]\big) = \mathcal{E}_p|_B(u)$, so that $\mathcal{H}_{\mathcal{E}_p,B} = h_B^{\mathcal{E}_p}(\mathcal{F}_p|_B)$, and*

$$h_B^{\mathcal{E}_p}[au+b\mathbb{1}_B] = ah_B^{\mathcal{E}_p}[u] + b\mathbb{1}_K \quad \text{for any } u \in \mathcal{F}_p|_B \text{ and any } a, b \in \mathbb{R}, \tag{2.14}$$

$$\mathcal{E}_p|_B(u|_B;v|_B) = \mathcal{E}_p(u;v) \quad \text{for any } u \in \mathcal{H}_{\mathcal{E}_p,B} \text{ and any } v \in \mathcal{F}_p, \tag{2.15}$$

where $\mathcal{E}_p|_B(u;v) := \frac{1}{p}\frac{d}{dt}\mathcal{E}_p|_B(u+tv)|_{t=0}$ for $u, v \in \mathcal{F}_p|_B$ (recall (2.8))*.*

Remark 2.11. In contrast to (2.14), *the map* $h_B^{\mathcal{E}_p}\colon \mathcal{F}_p|_B \to \mathcal{F}_p$ *does NOT satisfy either* $h_B^{\mathcal{E}_p}[u+v] \leq h_B^{\mathcal{E}_p}[u] + h_B^{\mathcal{E}_p}[v]$ *for any* $u, v \in \mathcal{F}_p|_B$ *or* $h_B^{\mathcal{E}_p}[u+v] \geq h_B^{\mathcal{E}_p}[u] + h_B^{\mathcal{E}_p}[v]$ *for any* $u, v \in \mathcal{F}_p|_B$ *in general*, unless $p = 2$ or $\#B \leq 2$.

Definition 2.12 (Trace of p-resistance form). Let $B \subset K$ be non-empty. The p-resistance form $(\mathcal{E}_p|_B, \mathcal{F}_p|_B)$ on B as defined in Theorem 2.10 is called the *trace* of $(\mathcal{E}_p, \mathcal{F}_p)$ to B.

Proposition 2.13. *Let* $A, B \subset K$ *satisfy* $\emptyset \neq A \subset B$. *Then* $(\mathcal{E}_p|_B|_A, \mathcal{F}_p|_B|_A) = (\mathcal{E}_p|_A, \mathcal{F}_p|_A)$ *and* $h_B^{\mathcal{E}_p} \circ h_A^{\mathcal{E}_p|_B} = h_A^{\mathcal{E}_p}$. *In particular,* $h_A^{\mathcal{E}_p|_B}[u] = h_A^{\mathcal{E}_p}[u]\big|_B$ *for any* $u \in \mathcal{F}_p|_A$.

As an easy consequence of the strong subadditivity of $\mathcal{E}_p$ (Proposition 2.6-(3)), we have the following natural analog of weak maximum principle.

Proposition 2.14 (Weak comparison principle). *Let* $B \subset K$ *be non-empty, and let* $u, v \in \mathcal{H}_{\mathcal{E}_p,B}$ *satisfy* $u(x) \leq v(x)$ *for any* $x \in B$. *Then* $u(x) \leq v(x)$ *for any* $x \in K$.

The strong subadditivity of $\mathcal{E}_p$ (Proposition 2.6-(3)) also implies the following important property of the derivative of $\mathcal{E}_p$.

Proposition 2.15. *Let* $u_1, u_2, v \in \mathcal{F}_p$ *satisfy* $((u_2 - u_1) \wedge v)(x) = 0$ *for any* $x \in K$. *Then* $\mathcal{E}_p(u_1; v) \geq \mathcal{E}_p(u_2; v)$.

Combining (2.10), Proposition 2.8, Theorem 2.10, Propositions 2.13, 2.14 and 2.15, we can prove the following Hölder continuity estimate for $\mathcal{E}_p$-harmonic functions, which plays central roles in our analysis of p-harmonic functions and p-energy measures for self-similar p-energy forms on self-similar sets in [18,19,20].

Theorem 2.16. *Let* $B \subset K$ *be non-empty and set* $B^{\mathcal{F}_p} := \bigcap_{u \in \mathcal{F}_p,\, u|_B = 0} u^{-1}(0)$. *Let* $x \in K \setminus B^{\mathcal{F}_p}$ *and set* $R_{\mathcal{E}_p}(x, B) := \mathcal{E}_p|_{B \cup \{x\}}\bigl(\mathbb{1}_x^{B \cup \{x\}}\bigr)^{-1}$. *Then for any* $y \in K$,

$$h_{B \cup \{x\}}^{\mathcal{E}_p}\bigl[\mathbb{1}_x^{B \cup \{x\}}\bigr](y) \leq \frac{R_{\mathcal{E}_p}(x, y)^{1/(p-1)}}{R_{\mathcal{E}_p}(x, B)^{1/(p-1)}}. \tag{2.16}$$

Moreover, for any $h \in \mathcal{H}_{\mathcal{E}_p,B}$ *with* $\|h\|_{\sup,B} < \infty$ *and any* $y \in K$,

$$|h(x) - h(y)| \leq \frac{R_{\mathcal{E}_p}(x, y)^{1/(p-1)}}{R_{\mathcal{E}_p}(x, B)^{1/(p-1)}} \operatorname{osc}_B[h]. \tag{2.17}$$

The special case of (2.16) with $B = \{z\}$ for $z \in K \setminus \{x\}$, the same inequality with x and z interchanged, and the equality $h_{\{x,z\}}^{\mathcal{E}_p}\bigl[\mathbb{1}_z^{\{x,z\}}\bigr] = \mathbb{1}_K - h_{\{x,z\}}^{\mathcal{E}_p}\bigl[\mathbb{1}_x^{\{x,z\}}\bigr]$ implied by (2.14) together yield the triangle inequality for $R_p^{1/(p-1)}$ and thereby show the following corollary.

Corollary 2.17. $R_p^{1/(p-1)}\colon K \times K \to [0, \infty)$ *is a metric on* K.

We conclude this section by presenting an expression of $(\mathcal{E}_p, \mathcal{F}_p)$ as the "inductive limit" of its traces $\{\mathcal{E}_p|_V\}_{V\subset K,\,1\leq\#V<\infty}$ to finite subsets, which is a straightforward extension of the counterpart for resistance forms given in [15, Corollary 2.37] (see also [15, Proof of Theorem 2.36-(2)] and [21, Theorems 2.3.6, 2.3.7 and Lemma 2.3.8]), and a few applications of it to convergence in the semi-norm $\mathcal{E}_p^{1/p}$.

Theorem 2.18. *It holds that*

$$\mathcal{F}_p = \{u \in \mathbb{R}^K \mid \sup_{V\subset K,\,1\leq\#V<\infty} \mathcal{E}_p|_V(u|_V) < \infty\}, \tag{2.18}$$

$$\mathcal{E}_p(u) = \sup_{V\subset K,\,1\leq\#V<\infty} \mathcal{E}_p|_V(u|_V) \quad \text{for any } u \in \mathcal{F}_p. \tag{2.19}$$

Theorem 2.18 easily implies the following simple characterization of the convergence in $\mathcal{F}_p$ with respect to $\mathcal{E}_p^{1/p}$.

Proposition 2.19. *Let $u \in \mathcal{F}_p$ and $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_p$.*

(1) *Assume that $\lim_{n\to\infty}(u_n(x) - u_n(y)) = u(x) - u(y)$ for any $x, y \in K$. Then $\mathcal{E}_p(u) \leq \liminf_{n\to\infty} \mathcal{E}_p(u_n)$.*

(2) *$\lim_{n\to\infty} \mathcal{E}_p(u - u_n) = 0$ if and only if $\limsup_{n\to\infty} \mathcal{E}_p(u_n) \leq \mathcal{E}_p(u)$ and $\lim_{n\to\infty}(u_n(x) - u_n(y)) = u(x) - u(y)$ for any $x, y \in K$.*

Proposition 2.19-(2) further yields the following useful approximation results.

Proposition 2.20. (1) *Let $\{\varphi_n\}_{n\in\mathbb{N}} \subset C(\mathbb{R})$ satisfy $|\varphi_n(t) - \varphi_n(s)| \leq |t - s|$ for any $n \in \mathbb{N}$ and any $s, t \in \mathbb{R}$ and $\lim_{n\to\infty} \varphi_n(t) = t$ for any $t \in \mathbb{R}$. Then $\{\varphi_n(u)\}_{n\in\mathbb{N}} \subset \mathcal{F}_p$ and $\lim_{n\to\infty} \mathcal{E}_p(u - \varphi_n(u)) = 0$ for any $u \in \mathcal{F}_p$.*

(2) *Let $u \in \mathcal{F}_p$, $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_p$ and $\varphi \in C(\mathbb{R})$ satisfy $\lim_{n\to\infty} \mathcal{E}_p(u - u_n) = 0$, $\lim_{n\to\infty} u_n(x) = u(x)$ for some $x \in K$, $|\varphi(t) - \varphi(s)| \leq |t - s|$ for any $s, t \in \mathbb{R}$ and $\varphi(u) = u$. Then $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subset \mathcal{F}_p$ and $\lim_{n\to\infty} \mathcal{E}_p(u - \varphi(u_n)) = 0$.*

Remark 2.21. Typical choices of $\{\varphi_n\}_{n\in\mathbb{N}} \subset C(\mathbb{R})$ in Proposition 2.20-(1) are $\varphi_n(t) = (-n) \vee (t \wedge n)$ and $\varphi_n(t) = t - (-\frac{1}{n}) \vee (t \wedge \frac{1}{n})$. A typical use of Proposition 2.20-(2) is to obtain a sequence of I-valued functions converging to u in $\mathcal{E}_p^{1/p}$ when $I \subset \mathbb{R}$ is a closed interval and $u \in \mathcal{F}_p$ is I-valued, by considering $\varphi \in C(\mathbb{R})$ given by $\varphi(t) := (\inf I) \vee (t \wedge \sup I)$.

3 Canonical p-resistance form on the Sierpiński gasket

In this section, we introduce the (two-dimensional standard) Sierpiński gasket and discuss the construction, the uniqueness and some basic properties of the canonical p-resistance form on it.

3.1 Sierpiński gasket

To start with, the Sierpiński gasket is defined as follows.

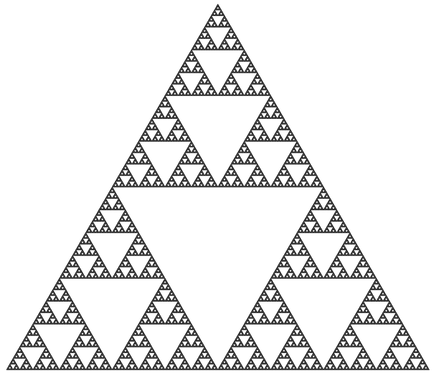

Figure 1. The (two-dimensional standard) Sierpiński gasket

Definition 3.1 (Sierpiński gasket). Set $S := \{1, 2, 3\}$, and let $V_0 := \{q_i \mid i \in S\} \subset \mathbb{R}^2$ be the set of the vertices of an equilateral triangle $\triangle \subset \mathbb{R}^2$, so that $\triangle$ is the convex hull of V_0 in $\mathbb{R}^2$. We further define $f_i \colon \mathbb{R}^2 \to \mathbb{R}^2$ by $f_i(x) := q_i + \frac{1}{2}(x - q_i)$ for each $i \in S$, let K be the *self-similar set* associated with $\{f_i\}_{i \in S}$, i.e., the unique non-empty compact subset of $\mathbb{R}^2$ such that $K = \bigcup_{i \in S} f_i(K)$, which exists and satisfies $K \subsetneq \triangle$ thanks to $\bigcup_{i \in S} f_i(\triangle) \subsetneq \triangle$ by [21, Theorem 1.1.4], and set $F_i := f_i|_K$ for each $i \in S$. The set K is called the *(two-dimensional standard) Sierpiński gasket* (see Figure 1 above). We equip K with the Euclidean metric $d \colon K \times K \to [0, \infty)$ given by $d(x, y) := |x - y|$, and the group of isometries of (K, d) is denoted by $\mathcal{G}_{\mathrm{sym}}$.

Throughout the rest of this paper, we fix the setting of Definition 3.1, and use the following notation, which is standard in studying self-similar sets.

Definition 3.2. (1) We set $W_0 := \{\emptyset\}$, where $\emptyset$ is an element called the *empty word*, $W_n := S^n = \{w_1 \dots w_n \mid w_i \in S \text{ for } i \in \{1, \dots, n\}\}$ for each $n \in \mathbb{N}$, and $W_* := \bigcup_{n \in \mathbb{N} \cup \{0\}} W_n$. For $w \in W_*$, the unique $n \in \mathbb{N} \cup \{0\}$ with $w \in W_n$ is denoted by $|w|$ and called the *length* of w.

(2) For $w = w_1 \dots w_{|w|}, v = v_1 \dots v_{|v|} \in W_*$, we define $wv \in W_*$ by $wv := w_1 \dots w_{|w|} v_1 \dots v_{|v|}$ ($w\emptyset := w$, $\emptyset v := v$). We also define $w^{(1)} \dots w^{(k)} \in W_*$ for $k \in \mathbb{N} \setminus \{1, 2\}$ and $w^{(1)}, \dots, w^{(k)} \in W_*$ inductively by $w^{(1)} \dots w^{(k)} := (w^{(1)} \dots w^{(k-1)}) w^{(k)}$. For $w \in W_*$ and $n \in \mathbb{N} \cup \{0\}$ we set $w^n := w \dots w \in W_{n|w|}$.

(3) We set $F_w := F_{w_1} \circ \cdots \circ F_{w_n}$ ($F_\emptyset := \mathrm{id}_K$) and $K_w := F_w(K)$ for each $w = w_1 \dots w_n \in W_*$, $V_n := \bigcup_{w \in W_n} F_w(V_0)$ for each $n \in \mathbb{N}$, and $V_* := \bigcup_{n \in \mathbb{N} \cup \{0\}} V_n$, so that $V_{n-1} \subsetneq V_n$ for any $n \in \mathbb{N}$ and V_* is dense in K.

The properties stated in the following proposition are crucial in developing the theory of self-similar energy forms on the Sierpiński gasket K.

Proposition 3.3 (Cf. [21, Proposition 1.3.5-(2) and Example 1.3.15]). *Let $w = w_1 \dots w_{|w|}, v = v_1 \dots v_{|v|} \in W_* \setminus \{\emptyset\}$ satisfy $w_k \neq v_k$ for some $k \in \{1, \dots, |w| \wedge |v|\}$.*

(1) $K_w \cap K_v = F_w(V_0) \cap F_v(V_0)$.

(2) $K_w \cap K_v \neq \emptyset$ *if and only if there exist* $\tau \in W_*$, $n, m \in \mathbb{N} \cup \{0\}$ *and* $i, j \in S$ *with* $i \neq j$ *such that* $w = \tau i j^n$ *and* $v = \tau j i^m$, *in which case* $K_w \cap K_v = \{F_\tau(q_{ij})\}$, *where* $q_{ij} := F_i(q_j) = F_j(q_i)$.

The following proposition can be easily verified by showing, for any $g \in \mathcal{G}_{\mathrm{sym}}$, first that $g(V_0) = V_0$ and then by an induction on n that $f^{-1} \circ g|_{V_n} = \mathrm{id}_{V_n}$ for any $n \in \mathbb{N}$ for the isometry f of $\mathbb{R}^2$ with $f|_{V_0} = g|_{V_0}$.

Proposition 3.4. $\mathcal{G}_{\mathrm{sym}} = \{f|_K \mid f$ *is an isometry of* $\mathbb{R}^2$, $f(V_0) = V_0\}$, *and the group* $\mathcal{G}_{\mathrm{sym}}$ *is generated by* $\{g_{xy}|_K \mid x, y \in V_0, x \neq y\}$, *where* $g_{xy} \colon \mathbb{R}^2 \to \mathbb{R}^2$ *denotes the reflection in the line* $\{z \in \mathbb{R}^2 \mid |z - x| = |z - y|\}$.

3.2 Construction of the canonical p-resistance form

Now we let $p \in (1, \infty)$ and fix it throughout the rest of this section. The following lemma is immediate from Proposition 3.3-(1), Lemma 2.4 and the characterization of p-resistance forms on finite sets given at the beginning of Example 2.2-(1).

Lemma 3.5. *Let* $\mathcal{E}_p^{(0)}$ *be a* p*-resistance form on* V_0, $\rho_p \in (0, \infty)$, $n \in \mathbb{N} \cup \{0\}$, *and define* $\mathcal{R}_{\rho_p,n}(\mathcal{E}_p^{(0)}) \colon \mathbb{R}^{V_n} \to [0, \infty)$ *by*

$$\mathcal{R}_{\rho_p,n}(\mathcal{E}_p^{(0)})(u) := \sum_{w \in W_n} \rho_p^n \mathcal{E}_p^{(0)}(u \circ F_w|_{V_0}), \qquad u \in \mathbb{R}^{V_n}. \tag{3.1}$$

Then $\mathcal{R}_{\rho_p,n}(\mathcal{E}_p^{(0)})$ *is a* p*-resistance form on* V_n.

As presented in [2, Sections 6 and 7], [21, Chapter 3] and [29, Chapter 1 and Section 4.2], the problem of constructing a self-similar Dirichlet form on a given p.-c.f. self-similar set is reduced to finding $\rho_2 \in (0, \infty)$ and a resistance form $\mathcal{E}_2^{(0)}$ on V_0 satisfying $\mathcal{R}_{\rho_2,1}(\mathcal{E}_2^{(0)})\big|_{V_0} = \mathcal{E}_2^{(0)}$ (or a generalization of it where $\rho_2 = \rho_{2,i}$ is allowed to depend on $i \in S$ as in (1.2)). The same is true also for p-energy forms with general p, and such a construction was achieved first by Herman, Peirone and Strichartz [12], for the Sierpiński gasket in [12, Corollary 3.7] and for a class of p.-c.f. self-similar sets in [12, Theorem 5.8]. Recently Cao, Gu and Qiu [7] have extended these results in [12] to much wider classes of p.-c.f. self-similar sets and developed a theory of the construction of p-energy forms on general p.-c.f. self-similar sets, extending many of the results in [21, Chapter 3] to general p. (See also [23, Section 4.6], where Kigami has constructed self-similar p-energy forms on a large class of p.-c.f. self-similar sets, as a special case of his general result [23, Theorem 4.6] whose proof is based on a different method.)

A serious problem with all these results is that the constructed p-energy forms are explicitly claimed to satisfy only (1.1), which is far from strong enough for further detailed analysis of them. Our contribution in this context is that we have identified (RF5)$_p$ as the right property to assume for p-energy forms and have verified that the constructions in [12,7,23] can be modified or seen to yield p-energy forms satisfying (RF5)$_p$. The details of this result will appear in [18]. In the present setting of the Sierpiński gasket, a version of it can be stated as follows. Recall Theorem 2.10 and Definition 2.12 for traces of p-resistance forms.

Theorem 3.6. *There exists a unique $\rho_p \in (0,\infty)$ such that $\mathcal{R}_{\rho_p,1}(\mathcal{E}_p^{(0)})|_{V_0} = \mathcal{E}_p^{(0)}$ for some p-resistance form $\mathcal{E}_p^{(0)}$ on V_0. Moreover, $\mathcal{E}_p^{(0)}$ can be chosen so as to be $\mathcal{G}_{\mathrm{sym}}$-invariant, i.e., satisfy $\mathcal{E}_p^{(0)}(u \circ g|_{V_0}) = \mathcal{E}_p^{(0)}(u)$ for any $u \in \mathbb{R}^{V_0}$ and any $g \in \mathcal{G}_{\mathrm{sym}}$.*

Sketch of the proof. The uniqueness of ρ_p is immediate from $\#V_0 < \infty$, (RF1)$_p$ for $\mathcal{E}_p^{(0)}$ and the fact that $\mathcal{R}_{\rho_p,n}(\mathcal{E}_p^{(0)})|_{V_n} = \mathcal{E}_p^{(0)}$ for any $n \in \mathbb{N}$ by Proposition 3.3-(1), $\mathcal{R}_{\rho_p,1}(\mathcal{E}_p^{(0)})|_{V_0} = \mathcal{E}_p^{(0)}$ and Proposition 2.13. For existence, by [7, Theorems 6.3 and 4.2] there exist $\rho_p \in (0,\infty)$ and $\mathcal{E}_p^{(0)} \in \mathcal{Q}_p(V_0)$ with the desired properties except (RF5)$_p$, where $\mathcal{Q}_p(V_0)$ is as defined in [7, Definition 2.8], but we see from the definition of $\mathcal{Q}_p(V_0)$, Example 2.2-(1) and Theorem 2.10 that any $E \in \mathcal{Q}_p(V_0)$ in fact satisfies (RF5)$_p$ and in particular $\mathcal{E}_p^{(0)}$ does, completing the proof. □

Throughout the rest of this paper, we let ρ_p denote the unique element of $(0,\infty)$ as in Theorem 3.6 and fix a $\mathcal{G}_{\mathrm{sym}}$-invariant p-resistance form $\mathcal{E}_p^{(0)}$ on V_0 satisfying $\mathcal{R}_{\rho_p,1}(\mathcal{E}_p^{(0)})|_{V_0} = \mathcal{E}_p^{(0)}$. As stated in Theorem 3.22 below, $\mathcal{E}_p^{(0)}$ will turn out to be unique up to constant multiples.

The following proposition is an easy consequence of Proposition 3.3-(1), $\mathcal{R}_{\rho_p,1}(\mathcal{E}_p^{(0)})|_{V_0} = \mathcal{E}_p^{(0)}$ and Proposition 2.13.

Proposition 3.7. *$\mathcal{R}_{\rho_p,n+m}(\mathcal{E}_p^{(0)})|_{V_n} = \mathcal{R}_{\rho_p,n}(\mathcal{E}_p^{(0)})$ for any $n, m \in \mathbb{N} \cup \{0\}$.*

As a reflection of the fact that the Sierpiński gasket is a p.-c.f. self-similar set, we also have the following important feature of ρ_p.

Proposition 3.8 ([12, Lemma 3.8 and Theorem 5.9], [7, Lemma 5.4]). *$\rho_p > 1$.*

Based on Propositions 3.7 and 3.8, the standard machinery for constructing the "inductive limit" of p-energy forms as presented in [7, Proposition 5.3], which is an adaptation of the relevant pieces of the theory of resistance forms due to [21, Sections 2.2, 2.3 and 3.3], gives the following result.

Definition 3.9. We set $\mathcal{E}_p^{(n)} := \mathcal{R}_{\rho_p,n}(\mathcal{E}_p^{(0)})$ for each $n \in \mathbb{N}$, and define a linear subspace $\mathcal{F}_{p,*}$ of $\mathbb{R}^{V_*}$, one $\mathcal{F}_p$ of $C(K)$ and a functional $\mathcal{E}_p \colon \mathcal{F}_p \to [0,\infty)$ by

$$\mathcal{F}_p := \left\{ u \in C(K) \;\middle|\; \lim_{n\to\infty} \mathcal{E}_p^{(n)}(u|_{V_n}) < \infty \right\}, \tag{3.2}$$

$$\mathcal{E}_p(u) := \lim_{n\to\infty} \mathcal{E}_p^{(n)}(u|_{V_n}) \in [0,\infty), \quad u \in \mathcal{F}_p; \tag{3.3}$$

note that $\{\mathcal{E}_p^{(n)}(u|_{V_n})\}_{n\in\mathbb{N}\cup\{0\}} \subset [0,\infty)$ is non-decreasing by Proposition 3.7 and hence has a limit in $[0,\infty]$ for any $u \in \mathbb{R}^{V_*}$.

Theorem 3.10 (Cf. [7, Proposition 5.3]). *$(\mathcal{E}_p, \mathcal{F}_p)$ is a p-resistance form on K with $\mathcal{E}_p|_{V_n} = \mathcal{E}_p^{(n)}$ for any $n \in \mathbb{N}\cup\{0\}$ and with the metric $R_{\mathcal{E}_p}^{1/(p-1)}$* (recall Corollary 2.17) *compatible with the original (Euclidean) topology of K, and satisfies the following* self-similarity*:*

(SSE1) *$\mathcal{F}_p = \{u \in C(K) \mid u \circ F_i \in \mathcal{F}_p$ for any $i \in S\}$.*

(SSE2) *$\mathcal{E}_p(u) = \sum_{i\in S} \rho_p \mathcal{E}_p(u \circ F_i)$ for any $u \in \mathcal{F}_p$.*

Moreover, $(\mathcal{E}_p, \mathcal{F}_p)$ is $\mathcal{G}_{\mathrm{sym}}$-invariant*, i.e., $u \circ g \in \mathcal{F}_p$ and $\mathcal{E}_p(u \circ g) = \mathcal{E}_p(u)$ for any $u \in \mathcal{F}_p$ and any $g \in \mathcal{G}_{\mathrm{sym}}$.*

Below are some basic properties of $(\mathcal{E}_p, \mathcal{F}_p)$. First, the following simple lemma underlies the monotonicity of $\rho_p^{1/(p-1)}$ as a function of p (see Theorem 4.9 below), which is a key observation in our study of p-energy measures in [20].

Lemma 3.11. *For any $w \in W_*$ and any $x, y \in K$,*

$$R_{\mathcal{E}_p}(F_w(x), F_w(y)) \leq \rho_p^{-|w|} R_{\mathcal{E}_p}(x, y). \tag{3.4}$$

Proof. This is immediate from (SSE1), (SSE2) and (RF4)$_p$. □

From the above construction of $(\mathcal{E}_p, \mathcal{F}_p)$, we easily obtain the following characterizations of $\mathcal{E}_p$-harmonic functions on $K \setminus V_n$ for $n \in \mathbb{N}\cup\{0\}$. Recall Definition 2.9 and Theorem 2.10.

Proposition 3.12. *Let $n \in \mathbb{N}\cup\{0\}$. Then for each $h \in C(K)$, the following three conditions are equivalent to each other:*

(1) *$h \in \mathcal{H}_{\mathcal{E}_p, V_n}$.*

(2) *$h \circ F_w \in \mathcal{H}_{\mathcal{E}_p, V_0}$ for any $w \in W_n$.*

(3) *For any $m \in \mathbb{N}$ with $m > n$ and any $x \in V_m \setminus V_n$,*

$$\sum_{w\in W_m,\, x\in F_w(V_0)} \rho_p^m \mathcal{E}_p^{(0)}\bigl(h \circ F_w|_{V_0}; \mathbb{1}_{F_w^{-1}(x)}\bigr) = 0. \tag{3.5}$$

The implication from (1) to (2) in Proposition 3.12 enables us to conclude the following localized version of the weak comparison principle (Proposition 2.14).

Proposition 3.13 (A localized weak comparison principle). *Let $n \in \mathbb{N}\cup\{0\}$, $w \in W_n$, and let $u, v \in \mathcal{H}_{\mathcal{E}_p, V_n}$ satisfy $u(x) \leq v(x)$ for any $x \in F_w(V_0)$. Then $u(x) \leq v(x)$ for any $x \in K_w$.*

The following proposition is useful in reducing the proof of a statement for general $u \in \mathcal{F}_p$ to the case of $u \in \bigcup_{n\in\mathbb{N}\cup\{0\}} \mathcal{H}_{\mathcal{E}_p, V_n}$.

Proposition 3.14. *Let $u \in \mathcal{F}_p$ and set $u_n := h_{V_n}^{\mathcal{E}_p}[u|_{V_n}]$ for each $n \in \mathbb{N}\cup\{0\}$. Then $\lim_{n\to\infty} \mathcal{E}_p(u - u_n, u - u_n) = 0$.*

Proof. This is immediate from (2.4) and (2.7) with $\frac{u+u_n}{2}, \frac{u-u_n}{2}$ in place of u, v, $\mathcal{E}_p\bigl(\frac{u+u_n}{2}\bigr) \geq \mathcal{E}_p(u_n) = \mathcal{E}_p^{(n)}(u|_{V_n})$ for $n \in \mathbb{N} \cup \{0\}$ and (3.3). $\square$

We collect some important consequences of Theorem 3.10 in relation to the topology of K in the following theorem.

Theorem 3.15 (Cf. [7, Theorem 5.2]).

(1) *$(\mathcal{E}_p, \mathcal{F}_p)$ is* regular, *i.e., $\mathcal{F}_p$ is a dense subalgebra of $(C(K), \|\cdot\|_{\sup})$.*

(2) *$(\mathcal{E}_p, \mathcal{F}_p)$ is* strongly local, *i.e., $\mathcal{E}_p(u_1; v) = \mathcal{E}_p(u_2; v)$ for any $u_1, u_2, v \in \mathcal{F}_p$ that satisfy $(u_1(x) - u_2(x) - a)(v(x) - b) = 0$ for any $x \in K$ for some $a, b \in \mathbb{R}$.*

Proof. (1) This follows from the compactness of K, $(\mathrm{RF1})_p$, $(\mathrm{RF3})_p$, Proposition 2.6-(2) and the Stone–Weierstrass theorem (see, e.g., [9, Theorem 2.4.11]).

(2) Under the stronger assumption $\mathrm{supp}_K[u_1 - u_2 - a\mathbb{1}_K] \cap \mathrm{supp}_K[v - b\mathbb{1}_K] = \emptyset$ this is immediate from the compactness of K, (SSE1), (SSE2) and (2.8), and the general case follows by applying this special case with v replaced by $\varphi_n(v - b\mathbb{1}_K)$ for $\varphi_n \in C(\mathbb{R})$ given by $\varphi_n(t) = t - (-\frac{1}{n}) \vee (t \wedge \frac{1}{n})$ and using Proposition 2.20-(1) to let $n \to \infty$. $\square$

The following proposition is used to prove the weak comparison principle for arbitrary open subsets of K (Proposition 3.17), which is an intermediate step of the proof of the strong one (Theorem 3.18 below). Detailed proofs of Propositions 3.16 and 3.17 will be given in [18] in the setting of a local p-resistance form.

Proposition 3.16. *Let U be an open subset of K and let $u \in \mathcal{F}_p$ satisfy $u(x) = 0$ for any $x \in \partial_K U$. Then $u\mathbb{1}_U \in \mathcal{F}_p$.*

Sketch of the proof. Let $n \in \mathbb{N}$, define $\varphi_n \in C(\mathbb{R})$ by $\varphi_n(t) = t - (-\frac{1}{n}) \vee (t \wedge \frac{1}{n})$ and set $A_n := U \cap \mathrm{supp}_K[\varphi_n(u)]$. Since $A_n = \overline{U}^K \cap \mathrm{supp}_K[\varphi_n(u)]$ by $u|_{\partial_K U} = 0$, A_n is a compact subset of U, hence by Theorem 3.15-(1) and Proposition 2.6-(1) we can take $v_n \in \mathcal{F}_p$ such that $\mathbb{1}_{A_n}(x) \leq v_n(x) \leq \mathbb{1}_U(x)$ for any $x \in K$, and then $\varphi_n(u)\mathbb{1}_U = \varphi_n(u)v_n \in \mathcal{F}_p$ by Proposition 2.6-(2). Now it is not difficult to see that $\{\varphi_n(u)\mathbb{1}_U\}_{n\in\mathbb{N}}$ gives a Cauchy sequence in $(\mathcal{F}_p/\mathbb{R}\mathbb{1}_K, \mathcal{E}_p^{1/p})$ by Theorem 3.15-(2) and Proposition 2.20-(1) and thus converges in norm in $(\mathcal{F}_p/\mathbb{R}\mathbb{1}_K, \mathcal{E}_p^{1/p})$ to its pointwise limit $u\mathbb{1}_U$ by $(\mathrm{RF2})_p$ and Proposition 2.5, whence $u\mathbb{1}_U \in \mathcal{F}_p$. $\square$

Proposition 3.17 (Weak comparison principle). *Let $U \subsetneq K$ be an open subset of K and let $u, v \in \mathcal{H}_{\mathcal{E}_p, K\setminus U}$ satisfy $u(x) \leq v(x)$ for any $x \in \partial_K U$. Then $u(x) \leq v(x)$ for any $x \in U$.*

Sketch of the proof. Setting $f := u - (u-v)^+\mathbb{1}_{K\setminus U}$ and $g := v + (u-v)^+\mathbb{1}_{K\setminus U}$, we have $f, g \in \mathcal{F}_p$ by Proposition 2.6-(1), $u|_{\partial_K U} \leq v|_{\partial_K U}$ and Proposition 3.16, $f, g \in \mathcal{H}_{\mathcal{E}_p, K\setminus U}$ by Theorem 3.15-(2) and $u, v \in \mathcal{H}_{\mathcal{E}_p, K\setminus U}$, $f(x) = (u \wedge v)(x) \leq (u \vee v)(x) = g(x)$ for any $x \in K \setminus U$, and thus $u(x) = f(x) \leq g(x) = v(x)$ for any $x \in U$ by Proposition 2.14. $\square$

3.3 Strong comparison principle and the uniqueness of the form

While the weak comparison principle as in Propositions 3.13 and 3.17 above is already good enough in many applications, for detailed analysis of $\mathcal{E}_p$-harmonic functions it is very important to have the strong comparison principle as stated in Theorem 3.18 below. In fact, it can be proved for general p.-c.f. self-similar sets by combining Proposition 2.15 with the self-similarity of $(\mathcal{E}_p, \mathcal{F}_p)$ and $\#V_0 < \infty$, and allows us to apply a nonlinear version of Perron–Frobenius theory presented in [24, Chapters 5 and 6] and thereby to conclude the uniqueness of $\mathcal{E}_p^{(0)}$ under the assumption of some good geometric symmetry of the self-similar set by following [27, Proof of Theorem 2.6]. The details of these results will be provided in [19]. In this subsection, we give their precise statements and a brief sketch of the proofs of them in the current setting of the Sierpiński gasket. First, the strong comparison principle can be stated as follows.

Theorem 3.18 (Strong comparison principle). *Let $U \subsetneq K$ be a connected open subset of K and let $u, v \in \mathcal{H}_{\mathcal{E}_p, K\setminus U}$ satisfy $u(x) \leq v(x)$ for any $x \in \partial_K U$. Then either $u(x) < v(x)$ for any $x \in U$ or $u(x) = v(x)$ for any $x \in \overline{U}^K$.*

Sketch of the proof. For $x \in K$ and $m \in \mathbb{N}$ we set $K_{m,x} := \bigcup_{w \in W_m,\, x \in K_w} K_w$ and $U_{m,x} := \bigcup_{y \in K_{m,x} \cap V_m} K_{m,y}$. First, we see from (SSE1), (SSE2), (2.8), (2.15) and $\mathcal{E}_p|_{V_0} = \mathcal{E}_p^{(0)}$ from Theorem 3.10 that for any $h \in \mathcal{H}_{\mathcal{E}_p, K\setminus U}$,

$$h \circ F_w \in \mathcal{H}_{\mathcal{E}_p, V_0} \quad \text{for any } w \in W_* \text{ with } K_w \subset U, \text{ and} \tag{3.6}$$

$$(3.5) \text{ holds for any } m \in \mathbb{N} \text{ and any } x \in V_m \text{ with } K_{m,x} \subset U. \tag{3.7}$$

Let $q \in U \cap V_*$, let $m \in \mathbb{N}$ satisfy $q \in V_m$ and $U_{m,q} \subset U$, let $w \in W_m$ satisfy $q \in K_w$, and assume that $u(q) = v(q)$. We claim that $u \circ F_w = v \circ F_w$. Suppose to the contrary that $B := \{y \in V_0 \mid u \circ F_w(y) < v \circ F_w(y)\} \neq \emptyset$. Then setting $a := \min_{y \in B}(v \circ F_w(y) - u \circ F_w(y))$ and $f := u \circ F_w|_{V_0} + a\mathbb{1}_B$, for any $y \in V_0 \setminus B$ we would be able to see from (3.7) with $x = F_w(y)$ and Proposition 2.15 that

$$\mathcal{E}_p^{(0)}(u \circ F_w|_{V_0}; \mathbb{1}_y) = \mathcal{E}_p^{(0)}(f; \mathbb{1}_y) = \mathcal{E}_p^{(0)}(v \circ F_w|_{V_0}; \mathbb{1}_y), \tag{3.8}$$

whereas by $F_w^{-1}(q) \in V_0 \setminus B \neq \emptyset \neq B$, (2.9) and $a > 0$ we would also have

$$\mathcal{E}_p^{(0)}(u \circ F_w|_{V_0}; \mathbb{1}_B) < \mathcal{E}_p^{(0)}(f; \mathbb{1}_B). \tag{3.9}$$

The conjunction of (3.8) and (3.9) would contradict $\mathcal{E}_p^{(0)}(u \circ F_w|_{V_0}; \mathbb{1}_{V_0}) = 0 = \mathcal{E}_p^{(0)}(f; \mathbb{1}_{V_0})$, proving $u \circ F_w|_{V_0} = v \circ F_w|_{V_0}$ and hence $u \circ F_w = v \circ F_w$ by (3.6).

Finally, if $x \in U \setminus V_*$ and $u(x) = v(x)$, then choosing $m \in \mathbb{N}$ and $w \in W_m$ so that $x \in K_w$ and $\bigcup_{q \in F_w(V_0)} U_{m,q} \subset U$, we have $u(q) = v(q)$ for some $q \in F_w(V_0)$ by (3.6), (2.14), Proposition 2.14 and $u(x) = v(x)$, and therefore $u \circ F_w = v \circ F_w$ by the previous paragraph. It follows that $\{x \in U \mid u(x) = v(x)\}$ is both open and closed in U and thus either $\emptyset$ or U by the connectedness of U. □

Theorem 3.18 has an important application to identifying the principal term of the asymptotic behavior of $h \in \mathcal{H}_{\mathcal{E}_p,V_0}$ at each point of $V_0 = \{q_i \mid i \in S\}$ as proved in Theorem 3.20 below. For this purpose we need the following proposition, which indicates what the asymptotic decay rate here should be. In fact, part of Proposition 3.19-(1),(2),(3) and of Theorem 3.20 was obtained in [30, Section 5] under the suppositions of (2.8), the strict convexity of $(\mathcal{F}_p/\mathbb{R}\mathbb{1}_K, \mathcal{E}_p^{1/p})$ implied by (2.4) and (2.7), and some special cases of Theorem 3.18 and Proposition 2.14, but we provide here self-contained proofs of them for the reader's convenience.

Proposition 3.19. *Let $i \in S$, $j \in S \cap \{i-2, i+1\}$ and $k \in S \cap \{i-1, i+2\}$.*

(1) $\mathcal{E}_p^{(0)}(h|_{V_0}; \mathbb{1}_{q_i}) = \rho_p^n \mathcal{E}_p^{(0)}(h \circ F_{i^n}|_{V_0}; \mathbb{1}_{q_i})$ *for any* $h \in \mathcal{H}_{\mathcal{E}_p,V_0}$ *and any* $n \in \mathbb{N}$.

(2) $\mathbb{R} \ni t \mapsto \mathcal{E}_p^{(0)}(a\mathbb{1}_{q_j} + t\mathbb{1}_{q_k}; \mathbb{1}_{q_i}) \in \mathbb{R}$ *and* $\mathbb{R} \ni t \mapsto \mathcal{E}_p^{(0)}(t\mathbb{1}_{q_j} - a\mathbb{1}_{q_k}; \mathbb{1}_{q_i}) \in \mathbb{R}$ *are strictly decreasing and* $\mathcal{E}_p^{(0)}(a\mathbb{1}_{q_j} - a\mathbb{1}_{q_k}; \mathbb{1}_{q_i}) = 0$ *for any* $a \in \mathbb{R}$.

(3) *Define* $h_{p,i} := h_{V_0}^{\mathcal{E}_p}[\mathbb{1}_{q_j} + \mathbb{1}_{q_k}]$ *and* $\check{h}_{p,i} := h_{V_0}^{\mathcal{E}_p}[\mathbb{1}_{q_k} - \mathbb{1}_{q_j}]$. *Then* $h_{p,i} \circ F_i = \rho_p^{-1/(p-1)} h_{p,i}$, *and there exists* $\lambda_p \in (0, \rho_p^{-1/(p-1)})$ *such that* $\check{h}_{p,i} \circ F_i = \lambda_p \check{h}_{p,i}$.

(4) *Let* $h \in \mathcal{H}_{\mathcal{E}_p,V_0}$. *Then* $\operatorname{osc}_K[h \circ F_i] = \rho_p^{-1/(p-1)} \operatorname{osc}_K[h]$ *if* $h(q_j) = h(q_k)$, *and* $0 < \operatorname{osc}_K[h \circ F_i] < \rho_p^{-1/(p-1)} \operatorname{osc}_K[h]$ *if* $h(q_j) \neq h(q_k)$.

(5) $h \circ F_i \in \mathcal{H}_{\mathcal{E}_p,V_0} \setminus \mathbb{R}\mathbb{1}_K$ *for any* $h \in \mathcal{H}_{\mathcal{E}_p,V_0} \setminus \mathbb{R}\mathbb{1}_K$.

Proof. (1) This is the counterpart of [21, Corollary 3.2.2] and can be proved in the same way. Indeed, since $h|_{V_n} \in \mathcal{H}_{\mathcal{E}_p^{(n)},V_0}$ by $\mathcal{E}_p|_{V_n} = \mathcal{E}_p^{(n)}$ from Theorem 3.10 and Proposition 2.13, $\mathcal{E}_p^{(0)}(h|_{V_0}; \mathbb{1}_{q_i}) = \mathcal{E}_p^{(n)}(h|_{V_n}; \mathbb{1}_{q_i}^{V_n})$ by $\mathcal{E}_p^{(0)} = \mathcal{E}_p^{(n)}|_{V_0}$ from Proposition 3.7 and (2.15), and the assertion follows from this last equality, the definition (3.1) of $\mathcal{E}_p^{(n)} = \mathcal{R}_{\rho_p,n}(\mathcal{E}_p^{(0)})$ and (2.8).

(2) For the first claim, since $h_{V_0}^{\mathcal{E}_p}[t\mathbb{1}_{q_j} + s\mathbb{1}_{q_k}](F_i(q_j))$ and $h_{V_0}^{\mathcal{E}_p}[t\mathbb{1}_{q_j} + s\mathbb{1}_{q_k}](F_i(q_k))$ are strictly increasing in each of $t, s \in \mathbb{R}$ by Theorem 3.18, in view of (1) with $n = 1$ it suffices to show that $\mathcal{E}_p^{(0)}(u + t\mathbb{1}_{q_j} + s\mathbb{1}_{q_k}; \mathbb{1}_{q_i}) < \mathcal{E}_p^{(0)}(u; \mathbb{1}_{q_i})$ for any $u \in \mathbb{R}^{V_0}$ and any $t, s \in (0, \infty)$, which in turn follows from Proposition 2.15 and the fact that $\mathcal{E}_p^{(0)}(u + t\mathbb{1}_{q_j} + t\mathbb{1}_{q_k}; \mathbb{1}_{q_i}) = \mathcal{E}_p^{(0)}(u - t\mathbb{1}_{q_i}; \mathbb{1}_{q_i}) < \mathcal{E}_p^{(0)}(u; \mathbb{1}_{q_i})$ for any $t \in (0, \infty)$ by (2.10) and (2.9). The latter assertion is immediate from the invariance of $\mathcal{E}_p^{(0)}$ under $g_{q_j q_k}|_{V_0}$ (recall Proposition 3.4) and (2.10).

(3) The invariance of $\mathcal{E}_p$ under $g_{q_j q_k}|_K$ yields $h_{p,i} \circ g_{q_j q_k}|_K = h_{p,i}$ and $\check{h}_{p,i} \circ g_{q_j q_k}|_K = -\check{h}_{p,i}$, which together with $h_{p,i} \circ F_i, \check{h}_{p,i} \circ F_i \in \mathcal{H}_{\mathcal{E}_p,V_0}$ from Proposition 3.12 implies that $h_{p,i} \circ F_i = \kappa_p h_{p,i}$ and $\check{h}_{p,i} \circ F_i = \lambda_p \check{h}_{p,i}$ for some $\kappa_p, \lambda_p \in \mathbb{R}$. The equality $\kappa_p = \rho_p^{-1/(p-1)}$ is the counterpart of [21, Lemma A.1.5] and follows in the same way from (1) with $n = 1$, (2.10) and the fact that $\mathcal{E}_p^{(0)}(\mathbb{1}_{q_j} + \mathbb{1}_{q_k}; \mathbb{1}_{q_i}) < 0$ by (2) (or (2.10)). Finally, $\lambda_p = \check{h}_{p,i}(F_i(q_k)) < h_{p,i}(F_i(q_k)) = \kappa_p = \rho_p^{-1/(p-1)}$ and $\lambda_p = \check{h}_{p,i}(F_i(q_k)) > 0$ by Theorem 3.18 with, respectively, $U = K \setminus V_0$ and $U = \{x \in K \setminus V_0 \mid |x - q_j| > |x - q_k|\}$, which satisfies $\check{h}_{p,i}|_{\partial_K U} = \mathbb{1}_{q_k}^{\partial_K U}$ by $\check{h}_{p,i} \circ g_{q_j q_k}|_K = -\check{h}_{p,i}$.

(4) By replacing h with $h - h(q_i)\mathbb{1}_K$ on the basis of (2.14) we may assume that $h(q_i) = 0$, and then this is immediate from (2.14), (3) and Theorem 3.18.

(5) This is immediate from Proposition 3.12 and (4). $\square$

Theorem 3.20 (Perron–Frobenius type theorem). *Let $i \in S$, $h \in \mathcal{H}_{\mathcal{E}_p,V_0}$ and set $c_{p,i}(h) := -\operatorname{sgn}\bigl(\mathcal{E}_p^{(0)}(h|_{V_0}; \mathbb{1}_{q_i})\bigr)\bigl|\mathcal{E}_p^{(0)}(h|_{V_0}; \mathbb{1}_{q_i})/\mathcal{E}_p^{(0)}(h_{p,i}|_{V_0}; \mathbb{1}_{q_i})\bigr|^{1/(p-1)}$. Then in the norm topologies of both $(C(K), \|\cdot\|_{\sup})$ and $(\mathcal{F}_p/\mathbb{R}\mathbb{1}_K, \mathcal{E}_p^{1/p})$,*

$$\lim_{n\to\infty} \rho_p^{n/(p-1)}(h \circ F_{i^n} - h(q_i)\mathbb{1}_K) = c_{p,i}(h)h_{p,i}. \tag{3.10}$$

Proof. Let $j, k \in S$ be as in Proposition 3.19, and set $h_n := \rho_p^{n/(p-1)}(h \circ F_{i^n} - h(q_i)\mathbb{1}_K)$, $a_n := h_n(q_j)$ and $b_n := h_n(q_k)$ for $n \in \mathbb{N} \cup \{0\}$. Replacing h with $-h$ on the basis of (2.14) if $a_0 > b_0$, we may and do assume that $a_0 \le b_0$. Then noting that $\{h_n\}_{n\in\mathbb{N}\cup\{0\}} \subset \mathcal{H}_{\mathcal{E}_p,V_0}$ by Proposition 3.12 and (2.14), we see by an induction on n using (2.14), Proposition 3.19-(3) and Proposition 3.17 that $a_{n-1} \le a_n \le b_n \le b_{n-1}$ for any $n \in \mathbb{N}$, where $a_n \le b_n$ follows from Proposition 3.17 with $U = \{x \in K \setminus V_0 \mid |x - q_j| > |x - q_k|\}$, $u = h \circ F_{i^{n-1}} \circ g_{q_j q_k}|_K$ and $v = h \circ F_{i^{n-1}}$. Thus $a := \lim_{n\to\infty} a_n \in \mathbb{R}$ and $b := \lim_{n\to\infty} b_n \in \mathbb{R}$ exist, which means that $\lim_{n\to\infty}\|h_\infty - h_n\|_{\sup} = 0$ for $h_\infty := h^{\mathcal{E}_p}_{V_0}[a\mathbb{1}_{q_j} + b\mathbb{1}_{q_k}]$ since $\|u - v\|_{\sup} = \|u|_{V_0} - v|_{V_0}\|_{\sup,V_0}$ for any $u, v \in \mathcal{H}_{\mathcal{E}_p,V_0}$ by (2.14) and Proposition 2.14. Then letting $n \to \infty$ in the obvious equality $h_{n+1} = \rho_p^{1/(p-1)} h_n \circ F_i$ shows that $h_\infty = \rho_p^{1/(p-1)} h_\infty \circ F_i$, which cannot hold under $a \ne b$ by (2.14), Proposition 3.19-(3) and Theorem 3.18 and thus implies that $a = b$. Therefore $h_\infty = ah_{p,i}$ by (2.14), and we further see from Proposition 3.19-(1), (2.10), $\#V_0 < \infty$ and the continuity of $\mathcal{E}_p^{(0)}(\cdot; \mathbb{1}_{q_i})$ on $\mathbb{R}^{V_0}/\mathbb{R}\mathbb{1}_{V_0}$ from (2.12) that

$$\begin{aligned}\mathcal{E}_p^{(0)}(h|_{V_0}; \mathbb{1}_{q_i}) &= \lim_{n\to\infty} \mathcal{E}_p^{(0)}(h_n|_{V_0}; \mathbb{1}_{q_i}) = \mathcal{E}_p^{(0)}(h_\infty|_{V_0}; \mathbb{1}_{q_i}) \\ &= \mathcal{E}_p^{(0)}(ah_{p,i}|_{V_0}; \mathbb{1}_{q_i}) = \operatorname{sgn}(a)|a|^{p-1}\mathcal{E}_p^{(0)}(h_{p,i}|_{V_0}; \mathbb{1}_{q_i}),\end{aligned} \tag{3.11}$$

so that $a = c_{p,i}(h)$ since $\mathcal{E}_p^{(0)}(h_{p,i}|_{V_0}; \mathbb{1}_{q_i}) < 0$ by Proposition 3.19-(2) (or (2.10)). Finally, by $h_\infty \in \mathcal{H}_{\mathcal{E}_p,V_0}$, $\{h_n\}_{n\in\mathbb{N}\cup\{0\}} \subset \mathcal{H}_{\mathcal{E}_p,V_0}$, $\mathcal{E}_p|_{V_0} = \mathcal{E}_p^{(0)}$ from Theorem 3.10, (RF1)$_p$, $\#V_0 < \infty$ and $\lim_{n\to\infty}\|h_\infty|_{V_0} - h_n|_{V_0}\|_{\sup,V_0} = 0$ we have

$$\mathcal{E}_p(h_n) = \mathcal{E}_p^{(0)}(h_n|_{V_0}) \xrightarrow{n\to\infty} \mathcal{E}_p^{(0)}(h_\infty|_{V_0}) = \mathcal{E}_p(h_\infty), \tag{3.12}$$

which together with $\lim_{n\to\infty}\|h_\infty - h_n\|_{\sup} = 0$ and Proposition 2.19-(2) yields $\lim_{n\to\infty} \mathcal{E}_p(h_\infty - h_n) = 0$, completing the proof. $\square$

Remark 3.21. The above proof of Theorem 3.20 and that of the existence of $h_{p,i} \in \mathcal{H}_{\mathcal{E}_p,V_0}$ with $h_{p,i}(q_i) = 0$, $h_{p,i}(q_j) \wedge h_{p,i}(q_k) > 0$ and $h_{p,i} \circ F_i = \rho_p^{-1/(p-1)} h_{p,i}$ in Proposition 3.19-(3) rely heavily on the invariance of $\mathcal{E}_p$ with respect to $g_{q_j q_k}|_K$ and the fact that $K_i \cap (K_j \cup K_k) = \{F_i(q_j), F_i(q_k)\}$ consists of two points and is invariant with respect to $g_{q_j q_k}|_K$. For more general p.-c.f. self-similar sets, such very nice geometric features are usually only partially available or not available, and some alternative arguments are needed to establish the same kind of results. Fortunately, there is a well-established theory on Perron–Frobenius type theorems for *nonlinear* 1-homogeneous order-preserving maps presented in [24], and our strong comparison principle as in Theorem 3.18 is strong enough to let us apply the most relevant results [24, Corollary 5.4.2 and Theorem 6.5.1] there. The statements obtained in this way are similar to Proposition 3.19-(3) and Theorem 3.20, but considerably weaker in the sense that the counterpart of the eigenfunction $h_{p,i}$ is never explicit and that the convergence result as in (3.10) requires the additional assumption that $h(x) - h(q_i) \geq 0$ for any $x \in V_0 \setminus \{q_i\}$. It is not clear to the authors how negative $(h - h(q_i)\mathbb{1}_K)|_{V_0 \setminus \{q_i\}}$ is allowed to be in order for the convergence as in (3.10) to remain holding.

Theorem 3.20 allows us to adapt the argument in [27, Remark 2.7] to conclude the uniqueness of a p-resistance form $\tilde{\mathcal{E}}_p^{(0)}$ satisfying $\mathcal{R}_{\rho_p,1}(\tilde{\mathcal{E}}_p^{(0)})\big|_{V_0} = \tilde{\mathcal{E}}_p^{(0)}$. Note that, as presented in [27, Remark 2.7 and Corollary 5.7], *the assumption of the $\mathcal{G}_{\mathrm{sym}}$-invariance of $\tilde{\mathcal{E}}_p^{(0)}$ is NOT needed here* thanks to the $\mathcal{G}_{\mathrm{sym}}$-invariance of $\mathcal{E}_p^{(0)}$, $\#V_0 \geq 3$ and the fact that $\{g|_{V_0} \mid g \in \mathcal{G}_{\mathrm{sym}}\} = \{g \mid g \colon V_0 \to V_0,\ g \text{ is bijective}\}$.

Theorem 3.22. *Let $\tilde{\mathcal{E}}_p^{(0)}$ be a p-resistance form on V_0 with the property that $\mathcal{R}_{\rho_p,1}(\tilde{\mathcal{E}}_p^{(0)})\big|_{V_0} = \tilde{\mathcal{E}}_p^{(0)}$. Then $\tilde{\mathcal{E}}_p^{(0)} = c\mathcal{E}_p^{(0)}$ for some $c \in (0, \infty)$.*

Sketch of the proof. Given the Perron–Frobenius type theorem (Theorem 3.20) above, this is proved in exactly the same way as [27, Proof of Theorem 2.6 and Remark 2.7]. □

We can also translate Theorem 3.22 into the uniqueness of a p-resistance form on K with the properties (SSE1) and (SSE2) in Theorem 3.10, as follows.

Theorem 3.23. *Let $(\tilde{\mathcal{E}}_p, \tilde{\mathcal{F}}_p)$ be a p-resistance form on K satisfying (SSE1) and (SSE2). Then $\tilde{\mathcal{F}}_p = \mathcal{F}_p$ and $\tilde{\mathcal{E}}_p = c\mathcal{E}_p$ for some $c \in (0, \infty)$.*

Sketch of the proof. Set $\tilde{\mathcal{E}}_p^{(0)} := \tilde{\mathcal{E}}_p|_{V_0}$. Then we can see from (SSE1) and (SSE2) that $\tilde{\mathcal{E}}_p|_{V_n} = \mathcal{R}_{\rho_p,n}(\tilde{\mathcal{E}}_p^{(0)})$ for any $n \in \mathbb{N}$, hence $\mathcal{R}_{\rho_p,1}(\tilde{\mathcal{E}}_p^{(0)})\big|_{V_0} = \tilde{\mathcal{E}}_p|_{V_1}|_{V_0} = \tilde{\mathcal{E}}_p^{(0)}$ by Proposition 2.13, and thus $\tilde{\mathcal{E}}_p^{(0)} = c\mathcal{E}_p^{(0)}$ for some $c \in (0, \infty)$ by Theorem 3.22. Now it is not difficult to show that $\tilde{\mathcal{F}}_p = \mathcal{F}_p$ and $\tilde{\mathcal{E}}_p = c\mathcal{E}_p$, by the same argument as the proof of Theorem 2.18 (see also [15, Proof of Theorem 2.36-(2)] and [21, Proof of Lemma 2.3.8]). □

Definition 3.24 (Canonical p-resistance form). In view of its uniqueness obtained in Theorem 3.23, we call $(\mathcal{E}_p, \mathcal{F}_p)$ as defined in Definition 3.9 the *canonical p-resistance form* on the Sierpiński gasket K.

Remark 3.25. Here are a couple of remarks on our choice of the framework of p-resistance forms in relation to the existence and detailed properties of $\mathcal{E}_p^{(0)}$.

(1) As mentioned in the above sketch of the proof of Theorem 3.23, the existence of a p-resistance form $\mathcal{E}_p^{(0)}$ on V_0 with the property that $\mathcal{R}_{\rho_p,1}(\mathcal{E}_p^{(0)})\big|_{V_0} = \mathcal{E}_p^{(0)}$

for some $\rho_p \in (0, \infty)$ is necessary for that of a p-resistance form $(\mathcal{E}_p, \mathcal{F}_p)$ on K with the self-similarity (SSE1) and (SSE2). In this sense, the construction of $(\mathcal{E}_p, \mathcal{F}_p)$ based on such $\mathcal{E}_p^{(0)}$ as presented in Subsection 3.2 does not put any restriction on the class of resulting p-resistance forms on K as far as self-similar ones are concerned. On the other hand, any possible proof of the existence of such $\mathcal{E}_p^{(0)}$ would inevitably involve *the operation of taking traces to subsets, which does NOT preserve the class of p-energy forms on finite sets of the type* (2.2). This is why we are forced to consider some larger class of p-energy forms, and that of p-resistance forms as formulated in Definition 2.1 seems to be the right one for unifying the existing studies of self-similar p-energy forms on self-similar sets in [12,30,23,28,7].

(2) Observe the central roles played by the functional $\mathcal{E}_p^{(0)}(\cdot;\cdot)$ in the above proofs of Theorem 3.18, Proposition 3.19 and Theorem 3.20. These proofs are made possible by combining some of the basic properties of p-resistance forms given in Section 2 with the fact that $\mathcal{E}_p^{(0)}$ coincides with the trace $\mathcal{E}_p|_{V_0}$ of $(\mathcal{E}_p, \mathcal{F}_p)$ to V_0 and provides a useful discrete characterization (3.5) of the $\mathcal{E}_p$-harmonicity. This observation suggests that it is important to guarantee nice properties of $\mathcal{E}_p^{(0)}$ for further detailed analysis of the limit p-energy form $(\mathcal{E}_p, \mathcal{F}_p)$, and the framework of p-resistance forms is helpful for this purpose.

4 p-Energy measures and singularity among distinct p

In this last section, we introduce the p-energy measures on the Sierpiński gasket, present their basic properties, and give the proof that the p-energy measures and the q-energy measures are mutually singular for any $p, q \in (1, \infty)$ with $p \neq q$, which will be proved in [20] in a more general setting of self-similar p-energy forms on p.-c.f. self-similar sets with certain very good geometric symmetry.

Throughout this section, we continue to follow the notation in Section 3, and let p denote an arbitrary element of $(1, \infty)$ unless otherwise stated. First, the p-energy measure $\mu^p_{\langle u \rangle}$ of $u \in \mathcal{F}_p$ is defined as follows. Note that, as mentioned in [14, Proof of Lemma 4-(ii)], for $p = 2$ the following definition results in what is known as the $\mathcal{E}_2$-energy measure of $u \in \mathcal{F}_2$ in the theory of regular symmetric Dirichlet forms; see [11, (3.2.14)] for the definition of the latter.

Theorem 4.1. *Let $u \in \mathcal{F}_p$. Then there exists a unique Borel measure $\mu^p_{\langle u \rangle}$ on K such that $\mu^p_{\langle u \rangle}(K_w) = \rho_p^{|w|}\mathcal{E}_p(u \circ F_w)$ for any $w \in W_*$. Moreover, $\mu^p_{\langle u \rangle}(\{x\}) = 0$ for any $x \in K$.*

Sketch of the proof. The uniqueness of $\mu^p_{\langle u \rangle}$ is immediate from (SSE2) and the Dynkin class theorem (see, e.g., [10, Appendixes, Theorem 4.2]). To see its existence, we follow the construction in [13, Lemma 4.1] (see also [28, Section 7]). Namely, we consider the (unique, by the Dynkin class theorem) Borel measure $\mathfrak{m}^p_{\langle u \rangle}$ on $S^{\mathbb{N}}$ such that $\mathfrak{m}^p_{\langle u \rangle}(\{w\} \times S^{\mathbb{N} \cap (|w|, \infty)}) = \rho_p^{|w|}\mathcal{E}_p(u \circ F_w)$ for any $w \in W_*$, which exists by (SSE2) and Kolmogorov's extension theorem (see, e.g., [9, Theorem 12.1.2]), and define a Borel measure $\mu^p_{\langle u\rangle}$ on K by $\mu^p_{\langle u\rangle} := \mathfrak{m}^p_{\langle u\rangle} \circ \pi^{-1}$, where $\pi\colon S^{\mathbb{N}} \to K$ is the continuous surjection given by $\{\pi((\omega_n)_{n\in\mathbb{N}})\} := \bigcap_{n\in\mathbb{N}} K_{\omega_1\ldots\omega_n}$ (see, e.g., [21, Theorem 1.2.3]). Then since $\pi^{-1}(K_w \setminus V_*) = \bigl(\{w\} \times S^{\mathbb{N}\cap(|w|,\infty)}\bigr) \setminus \pi^{-1}(V_*)$ for any $w \in W_*$ by Proposition 3.3-(1) and $\sup_{x\in K} \#\pi^{-1}(x) = 2 < \infty$ by Proposition 3.3-(2) (see also [21, Proof of Lemma 4.2.3]), the desired properties of $\mu^p_{\langle u\rangle}$ will follow from the definition of $\mathfrak{m}^p_{\langle u\rangle}$ and the countability of V_* once we have shown that

$$\mathfrak{m}^p_{\langle u\rangle}(\{\omega\}) = 0 \qquad \text{for any } \omega \in S^{\mathbb{N}}. \tag{4.1}$$

Indeed, if $u \in \bigcup_{n\in\mathbb{N}\cup\{0\}} \mathcal{H}_{\mathcal{E}_p,V_n}$, then (4.1) is not difficult see by combining the implication from (1) to (2) in Proposition 3.12 with Proposition 3.19-(4) (or with (2.17), Lemma 3.11 and $\sup_{x,y\in K} R_{\mathcal{E}_p}(x,y)^{1/(p-1)}(x,y) < \infty$ from Theorem 3.10). Then (4.1) for $u \in \mathcal{F}_p$ follows from the inequalities $\mathfrak{m}^p_{\langle u\rangle}(\{\omega\})^{1/p} \le \mathfrak{m}^p_{\langle v\rangle}(\{\omega\})^{1/p} + \mathfrak{m}^p_{\langle u-v\rangle}(\{\omega\})^{1/p} \le \mathfrak{m}^p_{\langle v\rangle}(\{\omega\})^{1/p} + \mathcal{E}_p(u-v)^{1/p}$ for $u, v \in \mathcal{F}_p$ implied by (RF1)$_p$ and (SSE2), Proposition 3.14, and (4.1) for $u \in \bigcup_{n\in\mathbb{N}\cup\{0\}} \mathcal{H}_{\mathcal{E}_p,V_n}$. □

Definition 4.2 (p-Energy measure). For each $u \in \mathcal{F}_p$, the Borel measure $\mu^p_{\langle u\rangle}$ on K as in Theorem 4.1 is called the *$\mathcal{E}_p$-energy measure* of u, or the *canonical p-energy measure* of u on the Sierpiński gasket K.

We collect some basic properties of the p-energy measures in the following propositions and theorems. The details of these results will be presented in [18] under a more general setting of self-similar p-energy forms on self-similar sets.

Proposition 4.3. *If $f\colon K \to [0,\infty)$ is Borel measurable and $\|f\|_{\sup} < \infty$, then:*

(1) $\bigl(\int_K f\, d\mu^p_{\langle\cdot\rangle}\bigr)^{1/p}$ *is a seminorm on* $\mathcal{F}_p$ *and* $\int_K f\, d\mu^p_{\langle \mathbb{1}_K\rangle} = 0$*;*

(2) $\bigl(\int_K f\, d\mu^p_{\langle\cdot\rangle}, \mathcal{F}_p\bigr)$ *satisfies* (RF5)$_p$*;*

(3) *Proposition 2.6 with $\int_K f\, d\mu^p_{\langle\cdot\rangle}$ in place of $\mathcal{E}_p$ holds.*

Sketch of the proof. Since $\{A \times S^{\mathbb{N}\cap(n,\infty)} \mid n \in \mathbb{N},\, A \subset S^n\}$ is an algebra in $S^{\mathbb{N}}$ generating $\mathcal{B}(S^{\mathbb{N}})$, it is not difficult to see from Lemma 2.4 and the monotone class theorem (see, e.g., [9, Theorem 4.4.2]) that $\mathfrak{m}^p_{\langle\cdot\rangle}(A)$ has the properties stated in (1) and (2) for any $A \in \mathcal{B}(S^{\mathbb{N}})$, where $\mathfrak{m}^p_{\langle u\rangle}$ denotes the Borel measure on $S^{\mathbb{N}}$ defined in the sketch of the proof of Theorem 4.1 above. Then (1) and (2) are immediate by Lemma 2.4 and monotone convergence, and (3) follows from (2) and (1) in exactly the same way as Proposition 2.6. □

Proposition 4.4. *If $f\colon K \to [0,\infty)$ is Borel measurable and $\|f\|_{\sup} < \infty$, then $\int_K f\, d\mu^p_{\langle\cdot\rangle}\colon \mathcal{F}_p/\mathbb{R}\mathbb{1}_K \to \mathbb{R}$ is Fréchet differentiable on $(\mathcal{F}_p/\mathbb{R}\mathbb{1}_K, \mathcal{E}_p^{1/p})$ and has the same properties as those of $\mathcal{E}_p$ in Theorem 2.7 with "$v \notin \mathbb{R}\mathbb{1}_K$" in (2.9)* *replaced by "$\int_K f\,d\mu^p_{\langle v\rangle} > 0$" and with the same c_p, and for any $u, v \in \mathcal{F}_p$,*

$$\mathcal{B}(K) \ni A \mapsto \mu^p_{\langle u;v\rangle}(A) := \frac{1}{p}\frac{d}{dt}\mu^p_{\langle u+tv\rangle}(A)\bigg|_{t=0} \quad \textit{is a Borel signed measure on } K$$
$$\textit{and} \quad \int_K f\,d\mu^p_{\langle u;v\rangle} = \frac{1}{p}\frac{d}{dt}\int_K f\,d\mu^p_{\langle u+tv\rangle}\bigg|_{t=0}. \tag{4.2}$$

Further, for any $u, v \in \mathcal{F}_p$ and any Borel measurable functions $f, g\colon K \to [0,\infty]$,

$$\int_K fg\,d\bigl|\mu^p_{\langle u;v\rangle}\bigr| \le \left(\int_K f^{p/(p-1)}\,d\mu^p_{\langle u\rangle}\right)^{(p-1)/p}\left(\int_K g^p\,d\mu^p_{\langle v\rangle}\right)^{1/p}. \tag{4.3}$$

Sketch of the proof. The first part is proved in exactly the same way as Theorem 2.7 on the basis of Proposition 4.3-(1),(3). (4.2) follows by the finite additivity of $\frac{1}{p}\frac{d}{dt}\int_K f\,d\mu^p_{\langle u+tv\rangle}\big|_{t=0}$ in f, (2.11) for $\int_K f\,d\mu^p_{\langle\cdot\rangle}$ and dominated convergence. (4.3) can be shown first for $f = g = \mathbb{1}_A$ with $A \in \mathcal{B}(K)$ by the definition of $\bigl|\mu^p_{\langle u;v\rangle}\bigr|(A)$, (2.11) for $\int_K \mathbb{1}_B\,d\mu^p_{\langle\cdot\rangle} = \mu^p_{\langle\cdot\rangle}(B)$ with $B \in \mathcal{B}(K)$ and Hölder's inequality, and then for general f, g by Hölder's inequality and monotone convergence. □

Theorem 4.5 (Chain rule for p-energy measures). *Let $n \in \mathbb{N}$, $u \in \mathcal{F}_p$, $v = (v_1, \dots, v_n) \in \mathcal{F}_p^n$, $\Phi \in C^1(\mathbb{R})$ and $\Psi \in C^1(\mathbb{R}^n)$. Then $\Phi(u), \Psi(v) \in \mathcal{F}_p$ and*

$$d\mu^p_{\langle \Phi(u);\Psi(v)\rangle} = \sum_{k=1}^n \operatorname{sgn}(\Phi'(u))|\Phi'(u)|^{p-1}\partial_k\Psi(v)\,d\mu^p_{\langle u;v_k\rangle}. \tag{4.4}$$

Sketch of the proof. (RF5)$_p$ implies $\Phi(u), \Psi(v) \in \mathcal{F}_p$. (4.4) on K_w for $w \in W_*$ can be proved by approximating its right-hand side by a Riemann sum associated with the partition $\{K_{w\tau}\}_{\tau\in W_n}$ for $n \in \mathbb{N}$, estimating the difference between the term for $K_{w\tau}$ in the sum and $\mu^p_{\langle\Phi(u);\Psi(v)\rangle}(K_{w\tau}) = \rho_p^{|w\tau|}\mathcal{E}_p(\Phi(u\circ F_{w\tau}); \Psi(v\circ F_{w\tau}))$ through (2.10), (2.11), (2.12) and Proposition 2.6-(1), adding up the resulting bounds over $\tau \in W_n$ via Hölder's inequality, and then letting $n \to \infty$. Thus (4.4) follows by the Dynkin class theorem (see, e.g., [10, Appendixes, Theorem 4.2]). □

Theorem 4.6. *For any $u \in \mathcal{F}_p$, the Borel measure $\mu^p_{\langle u\rangle} \circ u^{-1}$ on $\mathbb{R}$ defined by $\mu^p_{\langle u\rangle} \circ u^{-1}(A) := \mu^p_{\langle u\rangle}(u^{-1}(A))$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.*

Proof. This is proved, on the basis of Theorem 4.5, in exactly the same way as [28, Proposition 7.6], which is a simple adaptation of [8, Theorem 4.3.8]. □

The following theorem is an improvement on Theorem 3.15-(2) and gives arguably the strongest possible form of the strong locality of $(\mathcal{E}_p, \mathcal{F}_p)$.

Theorem 4.7. *Let $u_1, u_2, v \in \mathcal{F}_p$ and $a, b \in \mathbb{R}$. Then $\mu^p_{\langle u_1;v\rangle}(A) = \mu^p_{\langle u_2;v\rangle}(A)$ for any $A \in \mathcal{B}(K)$ with $A \subset (u_1 - u_2)^{-1}(a) \cup v^{-1}(b)$.*

Proof. This is proved in the same way as [28, Theorem 7.7] by combining Theorem 4.6 with the finite additivity of $\mu^p_{\langle u_1;v\rangle}, \mu^p_{\langle u_2;v\rangle}$ and (2.12) for $\int_K \mathbb{1}_B \, d\mu^p_{\langle\cdot\rangle} = \mu^p_{\langle\cdot\rangle}(B)$ with $B \in \{A \setminus v^{-1}(b), A \cap v^{-1}(b)\}$ from Proposition 4.4. □

We now turn to our last topic, planned to be treated in [20], of the singularity of the p-energy measures $\mu^p_{\langle u\rangle}$ and the q-energy measures $\mu^q_{\langle v\rangle}$ for $p, q \in (1,\infty)$ with $p < q$. It turns out that this singularity is implied by the *strict* inequality $\rho_p^{1/(p-1)} < \rho_q^{1/(q-1)}$, which on the other hand may or may not hold depending on the precise geometric nature of the self-similar set and has been proved in [20] only under the assumption of certain very good geometric symmetry. In the present setting of the Sierpiński gasket, these results can be stated as follows.

Theorem 4.8. *Let $p, q \in (1,\infty)$ satisfy $p \neq q$. Then $\mu^p_{\langle u\rangle} \perp \mu^q_{\langle v\rangle}$ for any $u \in \mathcal{F}_p$ and any $v \in \mathcal{F}_q$.*

Theorem 4.9. *The function $(1,\infty) \ni p \mapsto \rho_p^{1/(p-1)}$ is strictly increasing.*

Remark 4.10. *Theorems 4.8 and 4.9 are NOT always true for more general p.-c.f. self-similar sets, even under the assumption of good geometric symmetry of the set.* Indeed, in the case of the Vicsek set (Figure 2 below), Baudoin and Chen have recently observed in [3, Definition 2.7, Theorem 3.1 and Remark 3.11] for any $p \in (1,\infty)$ that the counterpart of ρ_p as in Theorem 3.6 is given by 3^{p-1} and that the p-energy measures are absolutely continuous with respect to the length (one-dimensional Lebesgue) measure on the skeleton of the Vicsek set.

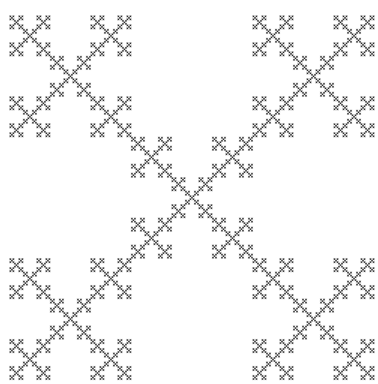

Figure 2. The Vicsek set

The rest of this section is devoted to the proofs of Theorems 4.8 and 4.9. We begin with the proof of Theorem 4.9, which goes as follows.

Proof of Theorem 4.9. Let $p, q \in (1,\infty)$ satisfy $p < q$. To highlight what causes the strict inequality $\rho_p^{1/(p-1)} < \rho_q^{1/(q-1)}$, we first give the proof of the non-strict one and then describe the necessary modifications to make it strict.

Recall that by Proposition 3.12 and Proposition 3.19-(4),

$$h \circ F_w \in \mathcal{H}_{\mathcal{E}_p,V_0} \qquad \text{and} \qquad \operatorname{osc}_K[h \circ F_w] \le \rho_p^{-|w|/(p-1)} \operatorname{osc}_K[h] \tag{4.5}$$

for any $h \in \mathcal{H}_{\mathcal{E}_p,V_0}$ and any $w \in W_*$. Note also that

$$\bigl(C_q^{-1}\mathcal{E}_q^{(0)}(u)\bigr)^{1/q} \le \operatorname{osc}_{V_0}[u] \le \bigl(C_p\mathcal{E}_p^{(0)}(u)\bigr)^{1/p} \qquad \text{for any } u \in \mathbb{R}^{V_0} \tag{4.6}$$

for some $C_q, C_p \in (0,\infty)$ by $\#V_0 < \infty$, $(\mathrm{RF1})_q$ for $\mathcal{E}_q^{(0)}$ and $(\mathrm{RF1})_p$ for $\mathcal{E}_p^{(0)}$. Recalling Theorem 2.10, choose any $h \in \mathcal{H}_{\mathcal{E}_p,V_0}$ satisfying $h|_{V_0} \notin \mathbb{R}\mathbb{1}_{V_0}$. Then for any $n \in \mathbb{N}$, it follows from $(\mathrm{RF1})_q$ for $\mathcal{E}_q^{(0)}$, Proposition 3.7, (3.1), (4.6), $q-p>0$, (4.5), $\mathcal{R}_{\rho_p,n}(\mathcal{E}_p^{(0)}) = \mathcal{E}_p^{(n)} = \mathcal{E}_p|_{V_n}$ from Theorem 3.10, and $h \in \mathcal{H}_{\mathcal{E}_p,V_0} \subset \mathcal{H}_{\mathcal{E}_p,V_n}$ that, with $C_{p,q,h} := C_qC_p(\operatorname{osc}_K[h])^{q-p} \in (0,\infty)$,

$$\begin{aligned}
0 < \mathcal{E}_q^{(0)}(h|_{V_0}) &\le \mathcal{R}_{\rho_q,n}(\mathcal{E}_q^{(0)})(h|_{V_n}) = \sum_{w \in W_n} \rho_q^n \mathcal{E}_q^{(0)}(h \circ F_w|_{V_0}) \\
&\le \sum_{w \in W_n} C_q\rho_q^n(\operatorname{osc}_{V_0}[h \circ F_w|_{V_0}])^q \\
&\le \sum_{w \in W_n} C_qC_p\rho_q^n\mathcal{E}_p^{(0)}(h \circ F_w|_{V_0})(\operatorname{osc}_{V_0}[h \circ F_w|_{V_0}])^{q-p} \\
&\le \sum_{w \in W_n} C_qC_p\rho_q^n\mathcal{E}_p^{(0)}(h \circ F_w|_{V_0})\rho_p^{-n(q-p)/(p-1)}(\operatorname{osc}_K[h])^{q-p} \qquad (4.7)\\
&= C_{p,q,h}\rho_q^n\rho_p^{-n(q-1)/(p-1)}\mathcal{E}_p^{(n)}(h|_{V_n}) = C_{p,q,h}\rho_q^n\rho_p^{-n(q-1)/(p-1)}\mathcal{E}_p(h),
\end{aligned}$$

whence $\rho_q^{1/(q-1)}/\rho_p^{1/(p-1)} \ge \bigl(C_{p,q,h}^{-1}\mathcal{E}_q^{(0)}(h|_{V_0})/\mathcal{E}_p(h)\bigr)^{1/(n(q-1))} \xrightarrow{n\to\infty} 1$, proving $\rho_p^{1/(p-1)} \le \rho_q^{1/(q-1)}$.

To achieve $\rho_p^{1/(p-1)} < \rho_q^{1/(q-1)}$, we improve the estimate in the line (4.7) by a compactness argument based on Proposition 3.19-(4),(5). Recall that $\#V_0 < \infty$ and that $\|u-v\|_{\sup} = \|u|_{V_0} - v|_{V_0}\|_{\sup,V_0}$ and $\operatorname{osc}_K[u] = \operatorname{osc}_{V_0}[u|_{V_0}]$ for any $u, v \in \mathcal{H}_{\mathcal{E}_p,V_0}$ by (2.14) and Proposition 2.14. In particular, for each $i \in S$ the map $\mathbb{R}^{V_0}/\mathbb{R}\mathbb{1}_{V_0} \ni u \mapsto h_{V_0}^{\mathcal{E}_p}[u] \circ F_i|_{V_0} \in \mathbb{R}^{V_0}/\mathbb{R}\mathbb{1}_{V_0}$ can be defined thanks to (2.14) and is continuous, and by considering the maximum of its post-composition by $\operatorname{osc}_{V_0}[\cdot]$ on the compact set $\{u \in \mathbb{R}^{V_0}/\mathbb{R}\mathbb{1}_{V_0} \mid \operatorname{osc}_{V_0}[u] = 1,\ \operatorname{osc}_{V_0\setminus\{q_i\}}[u|_{V_0\setminus\{q_i\}}] \ge \frac{1}{2}\}$, we see from Proposition 3.19-(4) and (2.14) that for some $\delta_p \in (0,1)$,

$$\operatorname{osc}_K[h \circ F_i] \le \delta_p\rho_p^{-1/(p-1)}\operatorname{osc}_K[h] \quad \text{for } \textit{some } i \in S \text{ for each } h \in \mathcal{H}_{\mathcal{E}_p,V_0}. \tag{4.8}$$

Similarly, since $(\mathcal{E}_p^{(0)})^{1/p}$ is a norm on $\mathbb{R}^{V_0}/\mathbb{R}\mathbb{1}_{V_0}$ by $(\mathrm{RF1})_p$, by considering the minimum of $\mathbb{R}^{V_0}/\mathbb{R}\mathbb{1}_{V_0} \ni u \mapsto \mathcal{E}_p^{(0)}\bigl(h_{V_0}^{\mathcal{E}_p}[u] \circ F_i|_{V_0}\bigr) \in [0,\infty)$ on $\{u \in \mathbb{R}^{V_0}/\mathbb{R}\mathbb{1}_{V_0} \mid \mathcal{E}_p^{(0)}(u) = 1\}$ for each $i \in S$, it follows from Proposition 3.19-(5) and (2.14) that

$$\rho_p\mathcal{E}_p^{(0)}(h \circ F_i|_{V_0}) \ge \varepsilon_p\mathcal{E}_p^{(0)}(h|_{V_0}) \quad \text{for any } i \in S \text{ and any } h \in \mathcal{H}_{\mathcal{E}_p,V_0} \tag{4.9}$$

for some $\varepsilon_p \in (0,1)$. Then for any $h \in \mathcal{H}_{\mathcal{E}_p,V_0}$, combining (4.5), (4.8) and (4.9) with the fact that $\sum_{i\in S} \rho_p \mathcal{E}_p^{(0)}(h \circ F_i|_{V_0}) = \mathcal{E}_p^{(1)}(h|_{V_1}) = \mathcal{E}_p(h) = \mathcal{E}_p^{(0)}(h|_{V_0})$ by (3.1) and $\mathcal{E}_p^{(n)} = \mathcal{E}_p|_{V_n}$ for $n \in \{0,1\}$ from Theorem 3.10, we obtain

$$\begin{aligned}\sum_{i\in S} &\rho_p \mathcal{E}_p^{(0)}(h \circ F_i|_{V_0})(\mathrm{osc}_{V_0}[h \circ F_i|_{V_0}])^{q-p} \\ &\leq (1 - \varepsilon_p(1 - \delta_p^{q-p}))\rho_p^{-(q-p)/(p-1)} \mathcal{E}_p^{(0)}(h|_{V_0})(\mathrm{osc}_{V_0}[h|_{V_0}])^{q-p}.\end{aligned} \tag{4.10}$$

Now letting $h \in \mathcal{H}_{\mathcal{E}_p,V_0} \setminus \mathbb{R}\mathbb{1}_K$ and recalling from (4.5) that $h \circ F_w \in \mathcal{H}_{\mathcal{E}_p,V_0}$ for any $w \in W_*$, we apply (4.10) in the line before (4.7) and then let $n \to \infty$ to get $\rho_q^{1/(q-1)}/\rho_p^{1/(p-1)} \geq (1 - \varepsilon_p(1 - \delta_p^{q-p}))^{-1/(q-1)} > 1$, completing the proof. $\square$

We next turn to the proof of Theorem 4.8, which is reduced to the case of $u \in \bigcup_{n\in\mathbb{N}\cup\{0\}} \mathcal{H}_{\mathcal{E}_p,V_n}$ and $v \in \bigcup_{n\in\mathbb{N}\cup\{0\}} \mathcal{H}_{\mathcal{E}_q,V_n}$ by Proposition 3.14 and the following lemma.

Lemma 4.11. *Let $p, q \in (1,\infty)$ satisfy $p \neq q$. If $u \in \mathcal{F}_p$, $\{u_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_p$, $v \in \mathcal{F}_q$ and $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_q$ satisfy $\lim_{n\to\infty} \mathcal{E}_p(u - u_n) = 0 = \lim_{n\to\infty} \mathcal{E}_q(v - v_n)$ and $\mu^p_{\langle u_l\rangle} \perp \mu^q_{\langle v_m\rangle}$ for any $l, m \in \mathbb{N}$, then $\mu^p_{\langle u\rangle} \perp \mu^q_{\langle v\rangle}$.*

Proof. We follow [16, Proof of Lemma 3.7-(b)]. First, for any $A \in \mathcal{B}(K)$, since $\mu^p_{\langle u-u_n\rangle}(A) \leq \mathcal{E}_p(u-u_n)$ and $\mu^q_{\langle v-v_n\rangle}(A) \leq \mathcal{E}_q(v-v_n)$ for any $n \in \mathbb{N}$, we see from Proposition 4.3-(1) for $\int_K \mathbb{1}_A \, d\mu^p_{\langle\cdot\rangle} = \mu^p_{\langle\cdot\rangle}(A)$ and $\int_K \mathbb{1}_A \, d\mu^q_{\langle\cdot\rangle} = \mu^q_{\langle\cdot\rangle}(A)$ that

$$\lim_{n\to\infty} \mu^p_{\langle u_n\rangle}(A) = \mu^p_{\langle u\rangle}(A) \qquad \text{and} \qquad \lim_{n\to\infty} \mu^q_{\langle v_n\rangle}(A) = \mu^q_{\langle v\rangle}(A). \tag{4.11}$$

For each $l, m \in \mathbb{N}$, by $\mu^p_{\langle u_l\rangle} \perp \mu^q_{\langle v_m\rangle}$ we can choose $A_{l,m} \in \mathcal{B}(K)$ so that $\mu^p_{\langle u_l\rangle}(A_{l,m}) = 0 = \mu^q_{\langle v_m\rangle}(K \setminus A_{l,m})$. Set $A_m := \bigcap_{l\in\mathbb{N}} A_{l,m}$ for each $m \in \mathbb{N}$ and $A := \bigcup_{m\in\mathbb{N}} A_m$. Then $\mu^p_{\langle u_l\rangle}(A_m) = 0 = \mu^q_{\langle v_m\rangle}(K \setminus A_m)$ for any $l, m \in \mathbb{N}$, hence $\mu^p_{\langle u\rangle}(A_m) = \lim_{l\to\infty} \mu^p_{\langle u_l\rangle}(A_m) = 0$ by (4.11) and $\mu^q_{\langle v_m\rangle}(K \setminus A) = 0$ for any $m \in \mathbb{N}$, and thus $\mu^p_{\langle u\rangle}(A) = 0$ and $\mu^q_{\langle v\rangle}(K \setminus A) = \lim_{m\to\infty} \mu^q_{\langle v_m\rangle}(K \setminus A) = 0$ by (4.11), proving $\mu^p_{\langle u\rangle} \perp \mu^q_{\langle v\rangle}$. $\square$

We prove that $\mu^p_{\langle u\rangle} \perp \mu^q_{\langle v\rangle}$ for $u \in \bigcup_{n\in\mathbb{N}\cup\{0\}} \mathcal{H}_{\mathcal{E}_p,V_n}$ and $v \in \bigcup_{n\in\mathbb{N}\cup\{0\}} \mathcal{H}_{\mathcal{E}_q,V_n}$, by combining Theorem 4.9 with the following lemma and theorem.

Lemma 4.12. *There exist $C_{p,1}, C_{p,2} \in (0,\infty)$ such that for any $h \in \mathcal{H}_{\mathcal{E}_p,V_0}$, any $w \in W_*$, any $i \in S$ with $h(q_i) \in \{\max_{x\in V_0} h(x), \min_{x\in V_0} h(x)\}$ and any $n \in \mathbb{N} \cup \{0\}$,*

$$\mathcal{E}_p(h \circ F_w) \leq C_{p,1}\rho_p^{-|w|p/(p-1)} \mathcal{E}_p(h), \tag{4.12}$$

$$\mathcal{E}_p(h \circ F_{i^n}) \geq C_{p,2}\rho_p^{-np/(p-1)} \mathcal{E}_p(h). \tag{4.13}$$

Proof. Recall that by $\#V_0 < \infty$ and (RF1)_p for $\mathcal{E}_p^{(0)}$ we have

$$C_p^{-1}(\mathrm{osc}_{V_0}[u])^p \leq \mathcal{E}_p^{(0)}(u) \leq C_p(\mathrm{osc}_{V_0}[u])^p \qquad \text{for any } u \in \mathbb{R}^{V_0} \tag{4.14}$$

for some $C_p \in (0,\infty)$, and that any $h \in \mathcal{H}_{\mathcal{E}_p,V_0}$ satisfies $\mathcal{E}_p(h) = \mathcal{E}_p^{(0)}(h|_{V_0})$ by $\mathcal{E}_p|_{V_0} = \mathcal{E}_p^{(0)}$ from Theorem 3.10, $h \circ F_w \in \mathcal{H}_{\mathcal{E}_p,V_0}$ for any $w \in W_*$ by Proposition 3.12, and $\mathrm{osc}_K[h] = \mathrm{osc}_{V_0}[h|_{V_0}]$ by (2.14) and Proposition 2.14. Therefore (4.12) holds with $C_{p,1} = C_p^2$ by (4.14) and (4.5), and we also see from (4.14) that (4.13) is implied by the existence of $C_{p,3} \in (0,\infty)$ such that for any $h \in \mathcal{H}_{\mathcal{E}_p,V_0}$, any $i \in S$ with $h(q_i) \in \{\max_{x\in V_0} h(x), \min_{x\in V_0} h(x)\}$ and any $n \in \mathbb{N}\cup\{0\}$,

$$\mathrm{osc}_K[h \circ F_{i^n}] \geq C_{p,3}\rho_p^{-n/(p-1)}\,\mathrm{osc}_K[h], \tag{4.15}$$

which we prove as follows. Thanks to (2.14) we may and do assume that $h(q_i) = 0 = \min_{x\in V_0} h(x)$ by replacing h with $h - h(q_i)\mathbb{1}_K$ or $h(q_i)\mathbb{1}_K - h$. Choose $j \in S \setminus \{i\}$ so that $h(q_j) = \mathrm{osc}_K[h]$ and set $C_{p,3} := \rho_p^{1/(p-1)} \min_{x\in V_0\setminus\{q_i\}} h_{V_0}^{\mathcal{E}_p}[\mathbb{1}_{q_j}] \circ F_i(x)$, which is independent of i, j by the $\mathcal{G}_{\mathrm{sym}}$-invariance of $(\mathcal{E}_p, \mathcal{F}_p)$ from Theorem 3.10. Then $C_{p,3} \in (0,1)$ by Theorem 3.18 and Proposition 3.19-(3), and it follows from Proposition 2.14, (2.14) and Proposition 3.19-(3) that for any $n \in \mathbb{N}$,

$$\begin{aligned}
\mathrm{osc}_K[h \circ F_{i^n}] &= \max_{x\in V_0} h(F_{i^n}(x)) \\
&\geq \max_{x\in V_0} h_{V_0}^{\mathcal{E}_p}\big[\mathrm{osc}_K[h]\mathbb{1}_{q_j}\big](F_{i^n}(x)) = \mathrm{osc}_K[h] \max_{x\in V_0} h_{V_0}^{\mathcal{E}_p}[\mathbb{1}_{q_j}] \circ F_i(F_{i^{n-1}}(x)) \\
&\geq \mathrm{osc}_K[h] \max_{x\in V_0} h_{V_0}^{\mathcal{E}_p}\big[C_{p,3}\rho_p^{-1/(p-1)}\mathbb{1}_{V_0\setminus\{q_i\}}\big](F_{i^{n-1}}(x)) = C_{p,3}\rho_p^{-n/(p-1)}\,\mathrm{osc}_K[h],
\end{aligned}$$

proving (4.15) and thereby (4.13). ◻

Theorem 4.13 ([13, Theorem 4.1]). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{\mathcal{F}_n\}_{n\in\mathbb{N}\cup\{0\}}$ a non-decreasing sequence of σ-algebras in Ω such that $\bigcup_{n\in\mathbb{N}\cup\{0\}} \mathcal{F}_n$ generates $\mathcal{F}$. Let $\widetilde{\mathbb{P}}$ be a probability measure on $(\Omega,\mathcal{F})$ such that $\widetilde{\mathbb{P}}|_{\mathcal{F}_n} \ll \mathbb{P}|_{\mathcal{F}_n}$ for any $n \in \mathbb{N}\cup\{0\}$, and for each $n \in \mathbb{N}$ define $\alpha_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P}|_{\mathcal{F}_n})$ by*

$$\alpha_n := \begin{cases} \dfrac{d(\widetilde{\mathbb{P}}|_{\mathcal{F}_n})/d(\mathbb{P}|_{\mathcal{F}_n})}{d(\widetilde{\mathbb{P}}|_{\mathcal{F}_{n-1}})/d(\mathbb{P}|_{\mathcal{F}_{n-1}})} & \text{on } \{d(\widetilde{\mathbb{P}}|_{\mathcal{F}_{n-1}})/d(\mathbb{P}|_{\mathcal{F}_{n-1}}) > 0\}, \\ 0 & \text{on } \{d(\widetilde{\mathbb{P}}|_{\mathcal{F}_{n-1}})/d(\mathbb{P}|_{\mathcal{F}_{n-1}}) = 0\}, \end{cases} \tag{4.16}$$

so that $\mathbb{E}[\sqrt{\alpha_n} \mid \mathcal{F}_{n-1}] \leq 1$ $\mathbb{P}|_{\mathcal{F}_{n-1}}$-a.s. by conditional Jensen's inequality, where $\mathbb{E}[\cdot \mid \mathcal{F}_{n-1}]$ denotes the conditional expectation given $\mathcal{F}_{n-1}$ with respect to $\mathbb{P}$. If

$$\sum_{n\in\mathbb{N}} (1 - \mathbb{E}[\sqrt{\alpha_n} \mid \mathcal{F}_{n-1}]) = \infty \qquad \mathbb{P}\text{-a.s.}, \tag{4.17}$$

then $\widetilde{\mathbb{P}} \perp \mathbb{P}$.

We will apply Theorem 4.13 under the setting of the following lemma.

Lemma 4.14. *Set $\Omega := K$, $\mathcal{F} := \mathcal{B}(K)$ and let $\mathbb{P}, \widetilde{\mathbb{P}}$ be probability measures on $(\Omega, \mathcal{F})$ such that $\mathbb{P}(K_w) > 0$ for any $w \in W_*$ and $\mathbb{P}(V_*) = \widetilde{\mathbb{P}}(V_*) = 0$. Let $N \in \mathbb{N}$, set $\mathcal{F}_n := \{A \cup \bigcup_{w\in\Lambda}(K_w \setminus V_{nN}) \mid \Lambda \subset W_{nN},\ A \subset V_{nN}\}$ for each $n \in \mathbb{N}\cup\{0\}$, so that $\{\mathcal{F}_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a non-decreasing sequence of σ-algebras in Ω by Proposition*

3.3-(1), $\bigcup_{n\in\mathbb{N}\cup\{0\}}\mathcal{F}_n$ *generates* $\mathcal{F}$, *and* $\widetilde{\mathbb{P}}|_{\mathcal{F}_n} \ll \mathbb{P}|_{\mathcal{F}_n}$ *for any* $n \in \mathbb{N}\cup\{0\}$*. Let* $n \in \mathbb{N}$ *and define* $\alpha_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P}|_{\mathcal{F}_n})$ *by* (4.16)*. Then for each* $w \in W_{(n-1)N}$*,*

$$\mathbb{E}[\sqrt{\alpha_n} \mid \mathcal{F}_{n-1}]|_{K_w\setminus V_{(n-1)N}} = \begin{cases} \sum_{\tau\in W_N} \sqrt{\frac{\widetilde{\mathbb{P}}(K_{w\tau})}{\widetilde{\mathbb{P}}(K_w)}}\sqrt{\frac{\mathbb{P}(K_{w\tau})}{\mathbb{P}(K_w)}} & \text{if } \widetilde{\mathbb{P}}(K_w) > 0, \\ 0 & \text{if } \widetilde{\mathbb{P}}(K_w) = 0. \end{cases} \tag{4.18}$$

Proof. This follows easily by direct calculations based on (4.16) and Proposition 3.3-(1). □

Proof of Theorem 4.8. We first prove $\mu^p_{\langle u\rangle} \perp \mu^q_{\langle v\rangle}$ for $u \in \mathcal{H}_{\mathcal{E}_p,V_0}$ and $v \in \mathcal{H}_{\mathcal{E}_q,V_0}$. Without loss of generality we may and do assume $p < q$, let $C_{p,2} \in (0,\infty)$ be as in (4.13), $C_{q,1} \in (0,\infty)$ be as in (4.12) with q in place of p, and choose $N \in \mathbb{N}$ so that $C_{q,1}\rho_q^{-N/(q-1)} \le \frac{1}{2}C_{p,2}\rho_p^{-N/(p-1)}$, which is possible since $\rho_q^{-1/(q-1)} < \rho_p^{-1/(p-1)}$ by Theorem 4.9. Noting that we clearly have $\mu^p_{\langle u\rangle} \perp \mu^q_{\langle v\rangle}$ if $\mu^p_{\langle u\rangle}(K)\mu^q_{\langle v\rangle}(K) = 0$, we assume that $\mu^p_{\langle u\rangle}(K)\mu^q_{\langle v\rangle}(K) > 0$, set $(\Omega, \mathcal{F}, \mathbb{P}) := \bigl(K, \mathcal{B}(K), \mu^p_{\langle u\rangle}(K)^{-1}\mu^p_{\langle u\rangle}\bigr)$, let $\{\mathcal{F}_n\}_{n\in\mathbb{N}\cup\{0\}}$ denote the non-decreasing sequence of σ-algebras in Ω with $\bigcup_{n\in\mathbb{N}\cup\{0\}}\mathcal{F}_n$ generating $\mathcal{F}$ as defined in Lemma 4.14, and set $\widetilde{\mathbb{P}} := \mu^q_{\langle v\rangle}(K)^{-1}\mu^q_{\langle v\rangle}$, so that $\mathbb{P}(K_w)\widetilde{\mathbb{P}}(K_w) > 0$ for any $w \in W_*$ by Proposition 3.19-(5), (RF1)$_p$ for $(\mathcal{E}_p, \mathcal{F}_p)$ and (RF1)$_q$ for $(\mathcal{E}_q, \mathcal{F}_q)$, and $\mathbb{P}(V_*) = 0 = \widetilde{\mathbb{P}}(V_*)$ by Theorem 4.1 and the countability of V_*. In particular, $\widetilde{\mathbb{P}}|_{\mathcal{F}_n} \ll \mathbb{P}|_{\mathcal{F}_n}$ for any $n \in \mathbb{N}\cup\{0\}$, and define $\alpha_n \in L^1(\Omega, \mathcal{F}_n, \mathbb{P}|_{\mathcal{F}_n})$ by (4.16) for each $n \in \mathbb{N}$. We claim that for any $n \in \mathbb{N}$,

$$\begin{aligned} &\|\mathbb{E}[\sqrt{\alpha_n} \mid \mathcal{F}_{n-1}]\|_{\sup,K\setminus V_{(n-1)N}} \\ &\le \max\bigl\{\sqrt{st} + \sqrt{(1-s)(1-t)} \bigm| s,t \in [0,1],\ 2s \le C_{p,2}\rho_p^{-N/(p-1)} \le t\bigr\} \\ &=: \delta_{p,N} \in (0,1). \end{aligned} \tag{4.19}$$

Indeed, let $n \in \mathbb{N}$ and $w \in W_{(n-1)N}$. Then since $\mathcal{E}_p(u\circ F_w)\mathcal{E}_q(v\circ F_w) > 0$ by $\mathbb{P}(K_w)\widetilde{\mathbb{P}}(K_w) > 0$ and $u\circ F_w \in \mathcal{H}_{\mathcal{E}_p,V_0}$ and $v\circ F_w \in \mathcal{H}_{\mathcal{E}_q,V_0}$ by Proposition 3.12, taking $i \in S$ such that $u\circ F_w(q_i) = \min_{x\in V_0} u\circ F_w(x)$, we see from (4.12) with q in place of p, the way above of having chosen N and (4.13) that

$$\begin{aligned} \frac{\widetilde{\mathbb{P}}(K_{wi^N})}{\widetilde{\mathbb{P}}(K_w)} &= \frac{\mu^q_{\langle v\rangle}(K_{wi^N})}{\mu^q_{\langle v\rangle}(K_w)} = \frac{\rho_q^N\mathcal{E}_q(v\circ F_{wi^N})}{\mathcal{E}_q(v\circ F_w)} \le C_{q,1}\rho_q^{-N/(q-1)} \le \frac{1}{2}C_{p,2}\rho_p^{-N/(p-1)}, \\ \frac{\mathbb{P}(K_{wi^N})}{\mathbb{P}(K_w)} &= \frac{\mu^p_{\langle u\rangle}(K_{wi^N})}{\mu^p_{\langle u\rangle}(K_w)} = \frac{\rho_p^N\mathcal{E}_p(u\circ F_{wi^N})}{\mathcal{E}_p(u\circ F_w)} \ge C_{p,2}\rho_p^{-N/(p-1)}, \end{aligned} \tag{4.20}$$

and hence from Lemma 4.14, the Cauchy–Schwarz inequality, $\widetilde{\mathbb{P}}(V_*) = 0 = \mathbb{P}(V_*)$, Proposition 3.3-(1) and (4.20) that

$$\begin{aligned}
&\mathbb{E}[\sqrt{\alpha_n} \mid \mathcal{F}_{n-1}]|_{K_w \setminus V_{(n-1)N}} \\
&\quad= \sqrt{\frac{\widetilde{\mathbb{P}}(K_{wi^N})}{\widetilde{\mathbb{P}}(K_w)}} \sqrt{\frac{\mathbb{P}(K_{wi^N})}{\mathbb{P}(K_w)}} + \sum_{\tau \in W_N \setminus \{i^N\}} \sqrt{\frac{\widetilde{\mathbb{P}}(K_{w\tau})}{\widetilde{\mathbb{P}}(K_w)}} \sqrt{\frac{\mathbb{P}(K_{w\tau})}{\mathbb{P}(K_w)}} \\
&\quad\leq \sqrt{\frac{\widetilde{\mathbb{P}}(K_{wi^N})}{\widetilde{\mathbb{P}}(K_w)}} \sqrt{\frac{\mathbb{P}(K_{wi^N})}{\mathbb{P}(K_w)}} + \sqrt{1 - \frac{\widetilde{\mathbb{P}}(K_{wi^N})}{\widetilde{\mathbb{P}}(K_w)}} \sqrt{1 - \frac{\mathbb{P}(K_{wi^N})}{\mathbb{P}(K_w)}} \leq \delta_{p,N},
\end{aligned} \tag{4.21}$$

proving (4.19). Thus $\sum_{n \in \mathbb{N}} (1 - \mathbb{E}[\sqrt{\alpha_n} \mid \mathcal{F}_{n-1}](x)) = \infty$ for any $x \in K \setminus V_*$ by (4.19) and in particular for $\mathbb{P}$-a.e. $x \in K = \Omega$ by $\mathbb{P}(V_*) = 0$, so that Theorem 4.13 is applicable and yields $\widetilde{\mathbb{P}} \perp \mathbb{P}$, namely $\mu^p_{\langle u \rangle} \perp \mu^q_{\langle v \rangle}$.

Next, let $u \in \bigcup_{n \in \mathbb{N} \cup \{0\}} \mathcal{H}_{\mathcal{E}_p, V_n}$ and $v \in \bigcup_{n \in \mathbb{N} \cup \{0\}} \mathcal{H}_{\mathcal{E}_q, V_n}$. Then choosing $n \in \mathbb{N} \cup \{0\}$ so that $u \in \mathcal{H}_{\mathcal{E}_p, V_n}$ and $v \in \mathcal{H}_{\mathcal{E}_q, V_n}$, for each $w \in W_n$ we have $\mu^p_{\langle u \rangle} \circ F_w = \rho_p^n \mu^p_{\langle u \circ F_w \rangle}$ and $\mu^q_{\langle v \rangle} \circ F_w = \rho_q^n \mu^q_{\langle v \circ F_w \rangle}$ by the uniqueness of $\mu^p_{\langle u \circ F_w \rangle}$ and $\mu^q_{\langle v \circ F_w \rangle}$ from Theorem 4.1, $u \circ F_w \in \mathcal{H}_{\mathcal{E}_p, V_0}$ and $v \circ F_w \in \mathcal{H}_{\mathcal{E}_q, V_0}$ by Proposition 3.12, and hence $\mu^p_{\langle u \rangle}(F_w(A_w)) = \rho_p^n \mu^p_{\langle u \circ F_w \rangle}(A_w) = 0$ and $\mu^q_{\langle v \rangle}(K_w \setminus F_w(A_w)) = \rho_q^n \mu^q_{\langle v \circ F_w \rangle}(K \setminus A_w) = 0$ for some $A_w \in \mathcal{B}(K)$ by the previous paragraph. Thus $\mu^p_{\langle u \rangle}\bigl(\bigcup_{w \in W_n} F_w(A_w)\bigr) = 0 = \mu^q_{\langle v \rangle}\bigl(K \setminus \bigcup_{w \in W_n} F_w(A_w)\bigr)$, proving $\mu^p_{\langle u \rangle} \perp \mu^q_{\langle v \rangle}$.

Finally, combining the result of the last paragraph with Proposition 3.14 and Lemma 4.11, we conclude that $\mu^p_{\langle u \rangle} \perp \mu^q_{\langle v \rangle}$ for any $u \in \mathcal{F}_p$ and any $v \in \mathcal{F}_q$. $\square$

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