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Heat kernel asymptotics for the measurable Riemannian structure on the Sierpinski gasket

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Abstract For the measurable Riemannian structure on the Sierpinski gasket introduced by Kigami, various short time asymptotics of the associated heat kernel are established, including Varadhan's asymptotic relation, some sharp one-dimensional asymptotics at vertices, and a non-integer-dimensional on-diagonal behavior at almost every point. Moreover, it is also proved that the asymptotic order of the eigenvalues of the corresponding Laplacian is given by the Hausdorff and box-counting dimensions of the space.

Keywords Sierpinski gasket · Kusuoka measure · Riemannian structure · geodesic metric · heat kernel · short time asymptotics

Mathematics Subject Classification (2000) Primary 28A80 · 60J35; Secondary 31C05 · 31C25 · 53C23

Contents

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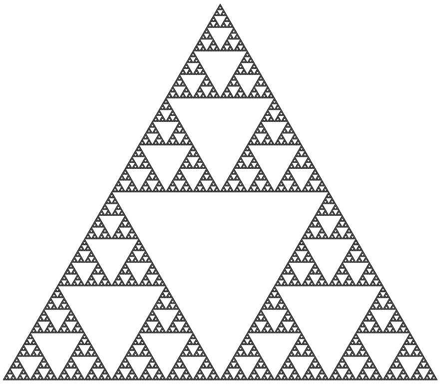

Figure 1 *Sierpinski gasket*

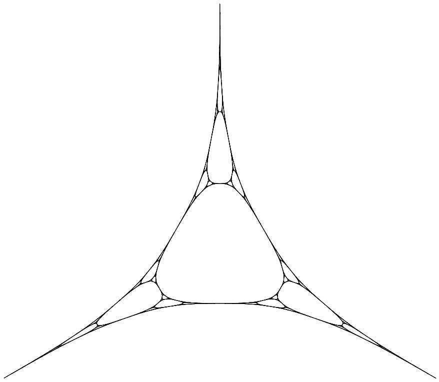

Figure 2 *Harmonic Sierpinski gasket*

1 Introduction

Recently there have been attempts to develop a theory of "manifold-like" analysis and geometry on fractals. As a prototype of such a theory, based on Kusuoka's construction in [29] of "weak gradients" for Dirichlet forms on fractals, Kigami [22,25] has introduced a measure-theoretic *"Riemannian structure"* on the Sierpinski gasket (Figure 1). He has further proved in [25] that the associated heat kernel satisfies the two-sided *Gaussian* bound in terms of the natural geodesic metric, unlike typical fractal diffusions treated e.g. in [5, 27, 12, 2, 3] for whose transition densities (heat kernels) the two-sided *sub-Gaussian* bounds hold. The purpose of this paper is to analyze this "Riemannian structure" on the Sierpinski gasket more in detail. We are particularly interested in short time asymptotic behaviors of the heat kernel, and our results include *"manifold-like"* ones as well as *"fractal-like"* ones.

Let us describe briefly our framework of the "Riemannian structure" on the Sierpinski gasket. Let K be the Sierpinski gasket constructed from an equilateral triangle in $\mathbb{R}^2$ with vertices q_1, q_2, q_3, and set $V_0 := \{q_1, q_2, q_3\}$. As studied in [1,23,34], a standard Dirichlet form $(\mathcal{E}, \mathcal{F})$ is defined on K, where the domain $\mathcal{F}$ is in fact a dense subalgebra of $C(K)$. By choosing $h_1, h_2 \in \mathcal{F}$ so that $2\mathcal{E}(h_i, h_j) = \delta_{ij}$ and they are harmonic on $K \setminus V_0$, we have a "harmonic map" $\Phi : K \to \mathbb{R}^2$ given by $\Phi(x) := (h_1(x), h_2(x))$. Φ is injective by [22, Theorem 3.6] and hence a homeomorphism from K onto its image $K_{\mathcal{H}} := \Phi(K)$, which is called the *harmonic Sierpinski gasket* (Figure 2). Moreover, Φ admits an associated $\mathcal{E}$-energy measure μ on K, called the *Kusuoka measure on the Sierpinski gasket* after [29].

By [29, §1] and [22, §3 and §4] (see Proposition 2.15 and Theorem 2.16 below), we can associate with the Dirichlet space $(K, \mu, \mathcal{E}, \mathcal{F})$ a *"one-dimensional tangent bundle with a Riemannian metric (Riemannian structure)"* on K inherited from $\mathbb{R}^2$ through the embedding Φ, where μ plays the role of the *"Riemannian volume measure"*. The heat kernel $p_\mu(t, x, y)$ of this Dirichlet space, which is the jointly continuous integral kernel of the associated Markovian semigroup on $L^2(K, \mu)$, is the main subject of our study.

Note that the "Riemannian structure" on K is different in several respects from usual Riemannian structures on manifolds; the notion of the *"tangent space $T_x K$ at x"*, which is a one-dimensional subspace of $\mathbb{R}^2$, makes sense only for μ-a.e. $x \in K$, and $T_x K$ depends discontinuously on $x \in K$. (In fact, the set of points where the tangent space cannot be defined is dense in K; see [22, Theorem B.5-(1)].) Therefore the associated heat kernel

$p_\mu(t,x,y)$ is expected to behave differently from those on Riemannian manifolds, and this is the case for the asymptotics of $p_\mu(t,x,x)$ as $t \downarrow 0$, as described in Theorem 1.3 below.

Now we outline the main results of this paper. Following [25, Theorem 5.1], we define the *harmonic geodesic metric* $\rho_{\mathcal{H}}$ *on* K by

$$\rho_{\mathcal{H}}(x,y) := \inf\{\ell(\Phi\circ\gamma) \mid \gamma : [0,1] \to K,\ \gamma \text{ is continuous},\ \gamma(0)=x,\ \gamma(1)=y\} \quad (1.1)$$

for $x, y \in K$, where $\ell(\Phi\circ\gamma)$ is the length of $\Phi\circ\gamma : [0,1] \to \mathbb{R}^2$ with respect to the Euclidean metric. Then $\rho_{\mathcal{H}}$ is a metric on K compatible with the original topology of K, and the first main result of this paper is the following characterization of the metric $\rho_{\mathcal{H}}$.

Theorem 1.1. *For any* $x, y \in K$,

$$\rho_{\mathcal{H}}(x,y) = \sup\{u(x)-u(y) \mid u \in \mathcal{F},\ |\widetilde{\nabla} u| \le 1\ \mu\text{-a.e.}\}, \quad (1.2)$$

where $\widetilde{\nabla} u$ *denotes the* "gradient vector field" *of* u*; see Theorem* 2.17 *below.*

It is not difficult to prove the equality analogous to (1.2) for Riemannian manifolds, whereas in the present case (1.2) is not straightforward and its proof, which is given in Section 4, is an important step of this paper. By virtue of Theorem 1.1, the general results of Sturm [35,36] and Ramírez [32] apply to the present case to yield the following off-diagonal Gaussian behaviors of $p_\mu(t,x,y)$ in terms of $\rho_{\mathcal{H}}$. For $(r,x) \in (0,\infty)\times K$ we set $B_r(x,\rho_{\mathcal{H}}) := \{y \in K \mid \rho_{\mathcal{H}}(x,y) < r\}$.

Corollary 1.2. (1) *There exist* $c_{\mathrm{L}}, c_{\mathrm{U}} \in (0,\infty)$ *such that for any* $(t,x,y) \in (0,\infty)\times K\times K$,

$$c_{\mathrm{L}} \frac{\exp\bigl(-\frac{\rho_{\mathcal{H}}(x,y)^2}{c_{\mathrm{L}} t}\bigr)}{\mu\bigl(B_{\sqrt{t}}(x,\rho_{\mathcal{H}})\bigr)} \le p_\mu(t,x,y) \le c_{\mathrm{U}} \frac{\bigl(1+\frac{\rho_{\mathcal{H}}(x,y)^2}{t}\bigr)^{\frac{\log_5 15}{2}} \exp\bigl(-\frac{\rho_{\mathcal{H}}(x,y)^2}{2t}\bigr)}{\sqrt{\mu\bigl(B_{\sqrt{t}}(x,\rho_{\mathcal{H}})\bigr)\mu\bigl(B_{\sqrt{t}}(y,\rho_{\mathcal{H}})\bigr)}}. \quad (1.3)$$

(2) *For any* $x, y \in K$,

$$\lim_{t\downarrow 0} 2t \log p_\mu(t,x,y) = -\rho_{\mathcal{H}}(x,y)^2. \quad (1.4)$$

For the heat kernels on Riemannian manifolds, the asymptotic behavior of exactly the same form as (1.4), called *Varadhan's asymptotic relation*, is well-known and has been obtained by Varadhan [38] (see also Norris [31]). Also the two-sided Gaussian heat kernel bound like (1.3) is known to hold for Riemannian manifolds which are either compact or complete with non-negative Ricci curvature; see [8,15,30,33,35,36] and references therein.

We remark that Kigami [25, Theorem 6.3] has already obtained a two-sided Gaussian bound for $p_\mu(t,x,y)$ similar to (1.3) where the upper bound involves $\exp\bigl(-\frac{\rho_{\mathcal{H}}(x,y)^2}{Ct}\bigr)$ with some constant $C \in (2,\infty)$ instead of $\exp\bigl(-\frac{\rho_{\mathcal{H}}(x,y)^2}{2t}\bigr)$. Here we can conclude a better Gaussian upper bound as in (1.3) by virtue of Theorem 1.1 and Sturm's results [35,36].

Note that Corollary 1.2 is in sharp contrast with the behaviors of the transition density $p(t,x,y)$ of the Brownian motion on the Sierpinski gasket K; $p(t,x,y)$ is nothing but the heat kernel associated with the Dirichlet space $(K,\nu,\mathcal{E},\mathcal{F})$ where ν is the $\log_2 3$-dimensional Hausdorff measure on K with respect to the Euclidean metric, and by [5, Theorem 1.5] we have the following *sub-Gaussian* bound

$$\frac{c_{1.1}}{t^{d_f/d_w}} \exp\biggl(-\Bigl(\frac{|x-y|^{d_w}}{c_{1.1}t}\Bigr)^{\frac{1}{d_w-1}}\biggr) \le p(t,x,y) \le \frac{c_{1.2}}{t^{d_f/d_w}} \exp\biggl(-\Bigl(\frac{|x-y|^{d_w}}{c_{1.2}t}\Bigr)^{\frac{1}{d_w-1}}\biggr),$$

where $d_f := \log_2 3$ and $d_w := \log_2 5 > \mathbf{2}$. Furthermore by [28, Theorem 1.2-a)], for any distinct $x, y \in K$, *the limit* $\lim_{t\downarrow 0} t^{\frac{1}{d_w-1}} \log p(t,x,y)$ *does not exist.*

Corollary 1.2 concerns the off-diagonal Gaussian behaviors of $p_\mu(t,x,y)$. On the other hand, for its on-diagonal behaviors we will establish the following statements, which include both "manifold-like" and "fractal-like" asymptotics.

Theorem 1.3. (1) *For any* $x \in V_0$ (recall $V_0 = \{q_1, q_2, q_3\}$), *it holds that*

$$p_\mu(t,x,x) = \frac{1}{\sqrt{2\pi t}}\bigl(2 + O(t^{\log_{5/3} 3})\bigr) \qquad \text{as } t \downarrow 0. \tag{1.5}$$

(2) *There exists a constant* $d_{\mathrm{S}}^{\mathrm{loc}} \in (1, 2\log_{25/3} 5]$ (note $2\log_{25/3} 5 = 1.5181\ldots$) *such that*

$$\lim_{t\downarrow 0} \frac{2\log p_\mu(t,x,x)}{-\log t} = d_{\mathrm{S}}^{\mathrm{loc}} \qquad \mu\text{-a.e. } x \in K. \tag{1.6}$$

(3) $\dim_{\mathrm{H}}(K, \rho_{\mathcal{H}}) = \dim_{\mathrm{B}}(K, \rho_{\mathcal{H}}) \in [d_{\mathrm{S}}^{\mathrm{loc}}, 2\log_{25/3} 5]$, *where* $\dim_{\mathrm{H}}$ *and* $\dim_{\mathrm{B}}$ *denote Hausdorff and box-counting dimensions, respectively. Moreover, set* $d_{\mathrm{S}} := \dim_{\mathrm{H}}(K, \rho_{\mathcal{H}})$, *let* $\{\lambda_n^\mu\}_{n\in\mathbb{N}}$ *be the eigenvalues of the Laplacian associated with* $(K, \mu, \mathcal{E}, \mathcal{F})$ *and let* $\mathcal{N}_\mu(s) := \#\{n \in \mathbb{N} \mid \lambda_n^\mu \le s\}$ *and* $\mathcal{Z}_\mu(t) := \sum_{n\in\mathbb{N}} e^{-t\lambda_n^\mu} \bigl(= \int_K p_\mu(t,x,x)d\mu(x)\bigr)$ *for* $s, t \in (0,\infty)$. *Then there exist* $c_{1.3}, c_{1.4} \in (0,\infty)$ *such that for any* $s \in [1,\infty)$ *and any* $t \in (0,1]$,

$$c_{1.3} s^{d_{\mathrm{S}}/2} \le \mathcal{N}_\mu(s) \le c_{1.4} s^{d_{\mathrm{S}}/2} \quad \text{and} \quad c_{1.3} t^{-d_{\mathrm{S}}/2} \le \mathcal{Z}_\mu(t) \le c_{1.4} t^{-d_{\mathrm{S}}/2}. \tag{1.7}$$

(1.5) is "manifold-like" and reflects our intuition on the picture of $K_{\mathcal{H}}$ (Figure 2) that, near $\Phi(x)$, $K_{\mathcal{H}}$ looks very much like its *"tangent line at* $\Phi(x)$*"*. In fact, for each $x \in V_*$ (i.e. a vertex x *of any level*), we prove a more detailed one-dimensional asymptotic behavior of $p_\mu(t,x,y)$ when $t \in (0,\infty)$ is small and $y \in K$ is close to x, as well as the existence of the limit $\lim_{r\downarrow 0} \mu(B_r(x,\rho_{\mathcal{H}}))/r \in (0,\infty)$. On the other hand, according to (1.6) and (1.7), p_μ exhibits *non-integer-dimensional* behaviors at μ-a.e. point in the short time limit, thereby reflecting the fractal nature of the space.

Lastly let us give a few remarks on the framework. One may expect that the main results of this paper can be generalized to the case of other self-similar fractals like ones in Figure 3, but such generalizations are not straightforward and the actual situation is quite subtle, as suggested by the following facts.

First, our proof of Theorem 1.1 utilizes a complete knowledge about the structure of geodesics due to [25, Section 5] (see Proposition 3.15 below), where the *two-dimensionality of the space* has played an essential role. Therefore some additional task should be necessary to verify Theorem 1.1 even in the (probably simplest) case of the d-dimensional (level-2) Sierpinski gasket with $d \ge 3$, although most of our main results will be valid also for them. Secondary, in another simple case, the case of the two-dimensional level-l Sierpinski gasket with $l \ge 3$ (see Figure 3), we can show that the "Riemannian volume measure" is *not volume doubling* with respect to the harmonic geodesic metric, based on the denseness of vertices from which the space spreads away in three directions. Hence by [24, Theorem 3.2.3], *even the on-diagonal upper bound* $p_\mu(t,x,x) \le c_{\mathrm{U}}/\mu\bigl(B_{\sqrt{t}}(x,\rho_{\mathcal{H}})\bigr)$ *is false there*, whereas Theorem 1.1 and part of Theorem 1.3 are still expected to be true. Finally, for most other typical fractals, such as pentagasket and snowflake in Figure 3, non-constant harmonic functions can be constant on non-empty open subsets and, as a consequence, harmonic maps into finite dimensional spaces and their associated energy measures cannot be used to introduce a "Riemannian structure". Thus it is already a highly non-trivial problem how we should introduce "Riemannian structures" on such fractals.

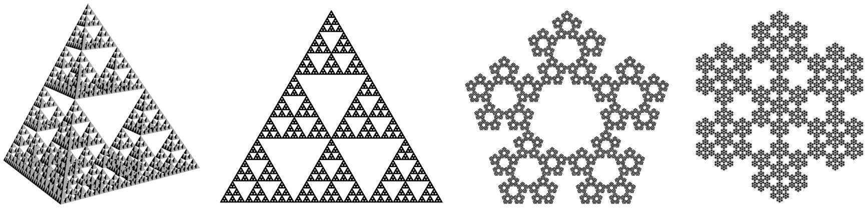

Figure 3 *From the left, three-dimensional (level-2) Sierpinski gasket, two-dimensional level-3 Sierpinski gasket, pentagasket and snowflake.*

In view of these observations, it seems reasonable for this present moment to content ourselves with the case of the two-dimensional Sierpinski gasket only. We leave possible extensions of our main results to other fractals for future studies.

The organization of this article is as follows. In Section 2, we collect basic facts concerning the standard Dirichlet form and the measurable Riemannian structure on the Sierpinski gasket. In Section 3 we briefly recall the results of [25] on the volume doubling property of the Kusuoka measure and basics on the harmonic geodesic metric, with slight improvements. Based on these preparations, we give the proofs of our main results in the subsequent sections; Theorem 1.1 and consequently Corollary 1.2 are proved in Section 4, and (1), (2) and (3) of Theorem 1.3 together with some more detailed results are treated respectively in Sections 5, 6 and 7.

Notation. In this paper, we adopt the following notations and conventions.
(1) $\mathbb{N} = \{1, 2, 3, \dots\}$, i.e. $0 \notin \mathbb{N}$.
(2) The cardinality (the number of all the elements) of a set A is denoted by $\#A$.
(3) We set $\sup \emptyset := 0$ and $\inf \emptyset := \infty$. We write $a \vee b := \max\{a, b\}$, $a \wedge b := \min\{a, b\}$, $a^+ := a \vee 0$ and $a^- := -(a \wedge 0)$ for $a, b \in [-\infty, \infty]$. We use the same notations also for functions. All functions treated in this paper are assumed to be $[-\infty, \infty]$-valued.
(4) Let $N \in \mathbb{N}$. The Euclidean inner product and norm on $\mathbb{R}^N$ are denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ respectively. For $\gamma : [a, b] \to \mathbb{R}^N$ continuous, where $a, b \in \mathbb{R}$, $a \leq b$, let $\ell(\gamma)$ be its length with respect to $|\cdot|$. We set $\mathcal{L}(\mathbb{R}^N) := \{T \mid T : \mathbb{R}^N \to \mathbb{R}^N,\ T \text{ is linear}\}$, and for $T \in \mathcal{L}(\mathbb{R}^N)$ let $\det T$ be its determinant, and T^* its adjoint and $\|T\|$ its *Hilbert-Schmidt* norm with respect to $\langle \cdot, \cdot \rangle$.
(5) Let E be a topological space. The Borel σ-field of E is denoted by $\mathcal{B}(E)$. We set $C(E) := \{f \mid f : E \to \mathbb{R},\ f \text{ is continuous}\}$ and $\|f\|_\infty := \sup_{x \in E} |f(x)|$, $f \in C(E)$.
(6) Let (E, ρ) be a metric space. We set $B_r(x, \rho) := \{y \in E \mid \rho(x, y) < r\}$ for $(r, x) \in (0, \infty) \times E$ and $\operatorname{diam}(A, \rho) := \sup_{x, y \in A} \rho(x, y)$ for $A \subset E$. Also for $f : E \to \mathbb{R}$ we set $\mathbf{Lip}_\rho f := \sup_{x, y \in E,\, x \neq y} |f(x) - f(y)|/\rho(x, y)$.

2 Measurable Riemannian structure on the Sierpinski gasket

In this section, we briefly recall basic facts concerning the measurable Riemannian structure on the Sierpinski gasket, including the definitions of the standard Dirichlet form (resistance form) and the harmonic Sierpinski gasket, which is the geometric realization of the measurable Riemannian structure. We follow mainly [25] for the presentation of this section, but we sometimes refer to also [17,22,23,26,29] for related facts. See [37] for possible generalizations to other finitely ramified fractals.

Definition 2.1 (Sierpinski gasket)**.** Let $V_0 = \{q_1, q_2, q_3\} \subset \mathbb{R}^2$ be the set of the three vertices of an equilateral triangle, set $S := \{1, 2, 3\}$, and for $i \in S$ define $F_i : \mathbb{R}^2 \to \mathbb{R}^2$ by $F_i(x) := (x + q_i)/2$. The *Sierpinski gasket* (Figure 1) is defined as the *self-similar set associated with* $\{F_i\}_{i\in S}$, i.e. the unique non-empty compact subset K of $\mathbb{R}^2$ that satisfies $K = \bigcup_{i\in S} F_i(K)$. We also define V_m for $m \in \mathbb{N}$ inductively by $V_m := \bigcup_{i\in S} F_i(V_{m-1})$ and set $V_* := \bigcup_{m\in\mathbb{N}} V_m$.

Note that $V_{m-1} \subset V_m$ for any $m \in \mathbb{N}$. K is always regarded as equipped with the relative topology inherited from $\mathbb{R}^2$, and V_* is dense in K in this topology. Hereafter we always regard F_i for each $i \in S$ as a continuous map from K to itself.

Definition 2.2. (1) Let $W_0 := \{\emptyset\}$, where $\emptyset$ is an element called the *empty word*, let $W_m := S^m = \{w_1 \dots w_m \mid w_i \in S \text{ for } i \in \{1, \dots, m\}\}$ for $m \in \mathbb{N}$ and $W_* := \bigcup_{m\in\mathbb{N}\cup\{0\}} W_m$. For $w \in W_*$, the unique $m \in \mathbb{N} \cup \{0\}$ with $w \in W_m$ is denoted by $|w|$ and called the *length of* w. Also for $i \in S$ and $n \in \mathbb{N} \cup \{0\}$ we write $i^n := i \dots i \in W_n$.
(2) We set $\Sigma := S^{\mathbb{N}} = \{\omega_1\omega_2\omega_3 \dots \mid \omega_i \in S \text{ for } i \in \mathbb{N}\}$, and define the *shift map* $\sigma : \Sigma \to \Sigma$ by $\sigma(\omega_1\omega_2\omega_3 \dots) := \omega_2\omega_3\omega_4 \dots$. Also for $i \in S$ we define $\sigma_i : \Sigma \to \Sigma$ by $\sigma_i(\omega_1\omega_2\omega_3 \dots) := i\omega_1\omega_2\omega_3 \dots$ and set $i^\infty := iii \dots \in \Sigma$. For $\omega = \omega_1\omega_2\omega_3 \dots \in \Sigma$ and $m \in \mathbb{N} \cup \{0\}$, we write $[\omega]_m := \omega_1 \dots \omega_m \in W_m$.
(3) For $w = w_1 \dots w_m \in W_*$, we set $F_w := F_{w_1} \circ \dots \circ F_{w_m}$ ($F_\emptyset := id_K$), $K_w := F_w(K)$, $\sigma_w := \sigma_{w_1} \circ \dots \circ \sigma_{w_m}$ ($\sigma_\emptyset := id_\Sigma$) and $\Sigma_w := \sigma_w(\Sigma)$.

Associated with the triple $(K, S, \{F_i\}_{i\in S})$ is a natural projection $\pi : \Sigma \to K$ given by the following proposition, which is used to describe the topological structure of K.

Proposition 2.3. *There exists a unique continuous surjective map* $\pi : \Sigma \to K$ *such that* $F_i \circ \pi = \pi \circ \sigma_i$ *for any* $i \in S$*, and it satisfies* $\{\pi(\omega)\} = \bigcap_{m\in\mathbb{N}} K_{[\omega]_m}$ *for any* $\omega \in \Sigma$*. Moreover,* $\#\pi^{-1}(x) = 1$ *for* $x \in K \setminus V_*$*,* $\pi^{-1}(q_i) = \{i^\infty\}$ *for* $i \in S$*, and for* $m \in \mathbb{N}$ *and each* $x \in V_m \setminus V_{m-1}$ *there exist* $w \in W_{m-1}$ *and* $i, j \in S$ *with* $i \neq j$ *such that* $\pi^{-1}(x) = \{wij^\infty, wji^\infty\}$.

Recall the following basic fact ([23, Proposition 1.3.5-(2)]) which we will use below without further notice: if $w, v \in W_*$ and $\Sigma_w \cap \Sigma_v = \emptyset$ then $K_w \cap K_v = F_w(V_0) \cap F_v(V_0)$.

As studied in [1,23,34], a standard Dirichlet form (or resistance form, strictly speaking) $(\mathcal{E}, \mathcal{F})$ is defined on the Sierpinski gasket K as follows.

Definition 2.4. Let $m \in \mathbb{N} \cup \{0\}$. We define a non-negative definite symmetric bilinear form $\mathcal{E}_m : \mathbb{R}^{V_m} \times \mathbb{R}^{V_m} \to \mathbb{R}$ on V_m by

$$\mathcal{E}_m(u, v) := \frac{1}{4} \cdot \frac{1}{2}\Big(\frac{5}{3}\Big)^m \sum_{x,y\in V_m,\, x\overset{m}{\sim}y} (u(x) - u(y))(v(x) - v(y)), \tag{2.1}$$

where, for $x, y \in V_m$, we write $x \overset{m}{\sim} y$ if and only if $x, y \in F_w(V_0)$ for some $w \in W_m$ and $x \neq y$.

The usual definition of $\mathcal{E}_m$ does not contain the factor $1/4$ so that each edge in the graph $(V_m, \overset{m}{\sim})$ has resistance $(3/5)^m$. Here it has been added for simplicity of the subsequent arguments; see Definition 2.11. It is easily shown that, for any function $u : K \to \mathbb{R}$, $\{\mathcal{E}_m(u|_{V_m}, u|_{V_m})\}_{m\in\mathbb{N}\cup\{0\}}$ is non-decreasing and hence has the limit in $[0, \infty]$. Then we have the following theorem; see [23, Chapter 2] and [26, Part 1] for the definition and basic properties of resistance forms.

Theorem 2.5. *Define $\mathcal{F} \subset C(K)$ by $\mathcal{F} := \{u \in C(K) \mid \lim_{m\to\infty} \mathcal{E}_m(u|_{V_m}, u|_{V_m}) < \infty\}$ and $\mathcal{E} : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$ by $\mathcal{E}(u, v) := \lim_{m\to\infty} \mathcal{E}_m(u|_{V_m}, v|_{V_m}) (\in \mathbb{R})$ for $u, v \in \mathcal{F}$. Then $(\mathcal{E}, \mathcal{F})$ is a resistance form on K whose resistance metric $R_{\mathcal{E}} = R_{\mathcal{E}}(x, y) : K \times K \to [0, \infty)$ is compatible with the original topology of K. Moreover, for any $u, v \in \mathcal{F}$,*

$$u \circ F_i \in \mathcal{F} \text{ for any } i \in S \qquad \text{and} \qquad \mathcal{E}(u, v) = \frac{5}{3} \sum_{i \in S} \mathcal{E}(u \circ F_i, v \circ F_i). \tag{2.2}$$

$(\mathcal{E}, \mathcal{F})$ is called the *standard resistance form on the Sierpinski gasket*. Furthermore [26, Corollary 6.4, Theorems 9.4, 9.9 and 10.4], (2.2), $\mathcal{E}(\mathbf{1}, \mathbf{1}) = 0$ and [24, Theorem A.4] imply the following theorem. See [13, Section 1.1] for the notions of regular Dirichlet forms and their strong locality, and see [13, Section 2.1] and [26, Definition 9.8] for the definition of their associated capacity.

Theorem 2.6. *Let ν be a finite Borel measure on K* with full support, *i.e. such that $\nu(U) > 0$ for any non-empty open subset U of K. Then $(\mathcal{E}, \mathcal{F})$ is a strong local regular Dirichlet form on $L^2(K, \nu)$ whose associated capacity* Cap_ν *satisfies* $\inf_{x\in K} \mathrm{Cap}_\nu(\{x\}) > 0$. *Moreover, its associated Markovian semigroup $\{T_t^\nu\}_{t\in(0,\infty)}$ on $L^2(K, \nu)$ admits a unique* continuous integral kernel p_ν, *i.e. a continuous function $p_\nu = p_\nu(t, x, y) : (0, \infty) \times K \times K \to (0, \infty)$ such that for each $f \in L^2(K, \nu)$ and $t \in (0, \infty)$,*

$$T_t^\nu f = \int_K p_\nu(t, \cdot, y) f(y) d\nu(y) \qquad \nu\text{-a.e.} \tag{2.3}$$

In the situation of Theorem 2.6, ν is called the *reference measure of the Dirichlet space* $(K, \nu, \mathcal{E}, \mathcal{F})$, and p_ν is called the *heat kernel associated with* $(K, \nu, \mathcal{E}, \mathcal{F})$; see [26, Theorem 10.4] for basic properties of p_ν.

Since we have a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ with state space K, by [13, pp. 110–111] we can define $\mathcal{E}$-energy measures as in the following definition.

Definition 2.7. The *$\mathcal{E}$-energy measure of $u \in \mathcal{F}$* is defined as the unique Borel measure $\mu_{\langle u\rangle}$ on K such that

$$\int_K f d\mu_{\langle u\rangle} = 2\mathcal{E}(uf, u) - \mathcal{E}(u^2, f) \qquad \text{for any } f \in \mathcal{F}. \tag{2.4}$$

We also define $\lambda_{\langle u\rangle}$ to be the unique positive Borel measure on Σ that satisfies $\lambda_{\langle u\rangle}(\Sigma_w) = 2(5/3)^{|w|}\mathcal{E}(u \circ F_w, u \circ F_w)$ for any $w \in W_*$, which exists by (2.2) and the Kolmogorov extension theorem. For $u, v \in \mathcal{F}$ we set $\mu_{\langle u,v\rangle} := (\mu_{\langle u+v\rangle} - \mu_{\langle u-v\rangle})/4$ and $\lambda_{\langle u,v\rangle} := (\lambda_{\langle u+v\rangle} - \lambda_{\langle u-v\rangle})/4$, so that they are finite Borel signed measures on K and on Σ respectively and are symmetric and bilinear in $(u, v) \in \mathcal{F} \times \mathcal{F}$.

Let $u \in \mathcal{F}$. According to [6, Proof of Theorem I.7.1.1], the strong locality of $(\mathcal{E}, \mathcal{F})$ implies that the image measure $\mu_{\langle u\rangle} \circ u^{-1}$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. In particular, $\mu_{\langle u\rangle}(\{x\}) = 0$ for any $x \in K$. We also easily see the following proposition by using (2.2) and (2.4). Note that $\pi(A) \in \mathcal{B}(K)$ for $A \in \mathcal{B}(\Sigma)$ by Proposition 2.3.

Proposition 2.8. *$\lambda_{\langle u,v\rangle} = \mu_{\langle u,v\rangle} \circ \pi$ and $\lambda_{\langle u,v\rangle} \circ \pi^{-1} = \mu_{\langle u,v\rangle}$ for any $u, v \in \mathcal{F}$.*

The definition of the measurable Riemannian structure on the Sierpinski gasket involves certain harmonic functions. In the present setting, harmonic functions are formulated as follows.

Definition 2.9. (1) We define $\mathcal{F}_U := \{u \in \mathcal{F} \mid u|_{K\setminus U} = 0\}$ for each open subset U of K.
(2) Let F be a closed subset of K. Then $h \in \mathcal{F}$ is called *F-harmonic* if and only if

$$\mathcal{E}(h,h) = \inf_{u\in\mathcal{F},\, u|_F = h|_F} \mathcal{E}(u,u) \quad \text{or equivalently,} \quad \mathcal{E}(h,u) = 0, \quad {}^\forall u \in \mathcal{F}_{K\setminus F}. \tag{2.5}$$

We set $\mathcal{H}_F := \{h \in \mathcal{F} \mid h \text{ is } F\text{-harmonic}\}$, which is a linear subspace of $\mathcal{F}$, and $\mathcal{H}_m := \mathcal{H}_{V_m}$, $m \in \mathbb{N}\cup\{0\}$. Note that for $u \in \mathcal{F}$, $u \in \mathcal{H}_m$ if and only if $\mathcal{E}(u,u) = \mathcal{E}_m(u|_{V_m}, u|_{V_m})$, which holds if and only if $u \circ F_w \in \mathcal{H}_0$ for any $w \in W_m$ by (2.2).

The following proposition easily follows from [26, Lemma 8.2].

Proposition 2.10. *Let F be a non-empty closed subset of K.*
(1) *Let $u \in \mathcal{F}$. Then there exists a unique $h \in \mathcal{H}_F$ such that $h|_F = u|_F$.*
(2) *Let $h \in \mathcal{H}_F$. Then* $\min_F h \le h(x) \le \max_F h$ *for any $x \in K$.*

Now we define a *"harmonic embedding"* Φ of K into $\mathbb{R}^2$, through which we will regard K as a kind of *"Riemannian submanifold in $\mathbb{R}^2$"* to obtain its measurable Riemannian structure. We also introduce a measure μ which is the $\mathcal{E}$-energy measure of the "embedding" Φ and will play the role of the *"Riemannian volume measure"*. Recall $V_0 = \{q_1, q_2, q_3\}$, and see [23, Section 3.2] and Proposition 2.12 below for basic properties of V_0-harmonic functions.

Definition 2.11. (0) Let $i \in S$, and let $j,k \in S$ be such that $j \equiv i+1 \bmod 3$ and $k \equiv i+2 \bmod 3$. We define $h^i_1, h^i_2 \in \mathcal{F}$ to be the V_0-harmonic functions satisfying $h^i_1(q_i) = h^i_2(q_i) = 0$, $h^i_1(q_j) = h^i_1(q_k) = 1$ and $-h^i_2(q_j) = h^i_2(q_k) = 1/\sqrt{3}$, so that $2\mathcal{E}(h^i_1, h^i_1) = 2\mathcal{E}(h^i_2, h^i_2) = 1$ (recall the factor $1/4$ in (2.1)), $\mathcal{E}(h^i_1, h^i_2) = 0$, $h^i_1 \circ F_i = (3/5)h^i_1$ and $h^i_2 \circ F_i = (1/5)h^i_2$.
(1) We set $h_1 := h^1_1$ and $h_2 := h^1_2$, and define $\Phi : K \to \mathbb{R}^2$ and $K_{\mathcal{H}}$ by

$$\Phi(x) := (h_1(x), h_2(x)), \quad x \in K \qquad \text{and} \qquad K_{\mathcal{H}} := \Phi(K). \tag{2.6}$$

$K_{\mathcal{H}}$ is called the *harmonic Sierpinski gasket* (Figure 2). We also set $\hat{q}_i := \Phi(q_i)$ for $i \in S$, so that $\{\hat{q}_1, \hat{q}_2, \hat{q}_3\} = \Phi(V_0)$ is the set of vertices of an equilateral triangle.
(2) We define finite Borel measures μ on K and λ on Σ by respectively

$$\mu := \mu_{\langle h_1\rangle} + \mu_{\langle h_2\rangle} \qquad \text{and} \qquad \lambda := \lambda_{\langle h_1\rangle} + \lambda_{\langle h_2\rangle}, \tag{2.7}$$

so that $\lambda = \mu \circ \pi$ and $\lambda \circ \pi^{-1} = \mu$ by Proposition 2.8. We call μ the *Kusuoka measure on the Sierpinski gasket.*

Notation. In what follows h^i_1, h^i_2, h_1, h_2 always denote the V_0-harmonic functions given in Definition 2.11. We often regard $\{h^i_1, h^i_2\}$ as forming an orthonormal basis of $(\mathcal{H}_0/\mathbb{R}\mathbf{1}, 2\mathcal{E})$. Moreover, we set

$$\|u\|_{\mathcal{E}} := \sqrt{2\mathcal{E}(u,u)}, \quad u \in \mathcal{F} \qquad \text{and} \qquad \mathcal{S}_{\mathcal{H}_0} := \{h \in \mathcal{H}_0 \mid \|h\|_{\mathcal{E}} = 1\}. \tag{2.8}$$

The following proposition provides an alternative geometric definition of $K_{\mathcal{H}}$, and essentially as its corollary we also see the injectivity of Φ (Theorem 2.13), Proposition 2.14 below and that $\mu_{\langle h\rangle}$ has full support for any $h \in \mathcal{S}_{\mathcal{H}_0}$.

Proposition 2.12 ([22, §3]). *For $i \in S$, define $T_i \in \mathcal{L}(\mathbb{R}^2)$ and $H_i : \mathbb{R}^2 \to \mathbb{R}^2$ by*

$$T_i\big(a(\hat{q}_j + \hat{q}_k - 2\hat{q}_i) + b(\hat{q}_k - \hat{q}_j)\big) := \frac{3}{5}a(\hat{q}_j + \hat{q}_k - 2\hat{q}_i) + \frac{1}{5}b(\hat{q}_k - \hat{q}_j), \quad a, b \in \mathbb{R},$$

where $\{i, j, k\} = S$, and $H_i(x) = \hat{q}_i + T_i(x - \hat{q}_i)$, $x \in \mathbb{R}^2$. Also for $w = w_1 \dots w_m \in W_*$ let $T_w := T_{w_1} \cdots T_{w_m}$ ($T_\emptyset := id_{\mathbb{R}^2}$), which we regard as its matrix representation through the standard basis of $\mathbb{R}^2$. Then we have the following statements:
(i) $T_1 = \begin{pmatrix} 3/5 & 0 \\ 0 & 1/5 \end{pmatrix}$, $T_2 = \begin{pmatrix} 3/10 & -\sqrt{3}/10 \\ -\sqrt{3}/10 & 1/2 \end{pmatrix}$, $T_3 = \begin{pmatrix} 3/10 & \sqrt{3}/10 \\ \sqrt{3}/10 & 1/2 \end{pmatrix}$.
(ii) *For each $w \in W_*$, $T_w^* := (T_w)^*$ is equal to the matrix representation of the linear map $F_w^* : \mathcal{H}_0/\mathbb{R}\mathbf{1} \to \mathcal{H}_0/\mathbb{R}\mathbf{1}$, $F_w^* h := h \circ F_w$ by the basis $\{h_1, h_2\}$ of $\mathcal{H}_0/\mathbb{R}\mathbf{1}$.*
(iii) *$H_i \circ \Phi = \Phi \circ F_i$ and hence $H_i \circ (\Phi \circ \pi) = (\Phi \circ \pi) \circ \sigma_i$ for any $i \in S$. In particular, $K_\mathcal{H} = \bigcup_{i \in S} H_i(K_\mathcal{H})$, i.e. $K_\mathcal{H}$ is the self-similar set associated with $\{H_i\}_{i \in S}$.*

Theorem 2.13 ([22, Theorem 3.6]). *The map $\Phi : K \to K_\mathcal{H}$ is a homeomorphism.*

Proposition 2.14. *$\mu(K_w) = \lambda(\Sigma_w) = (5/3)^{|w|}\|T_w\|^2$ for any $w \in W_*$. Moreover, it holds that $\lambda \circ \sigma^{-1} = \lambda$.*

Kusuoka [29, Example 1] has proved that λ *is ergodic with respect to the shift map* σ, i.e. $\lambda(A)\lambda(\Sigma \setminus A) = 0$ for any $A \in \mathcal{B}(\Sigma)$ with $\sigma^{-1}(A) = A$, and that it is singular with respect to the Bernoulli measure on Σ with weight $(1/3, 1/3, 1/3)$. The ergodicity of λ plays an essential role in Section 6, where we provide an alternative simple proof of it.

Now we introduce the measurable Riemannian structure on K, which is formulated as a matrix-valued Borel measurable map Z on K, as follows.

Proposition 2.15 ([29, §1], [22, Proposition B.2]). *Define $\Sigma_Z \in \mathcal{B}(\Sigma)$ and $K_Z \in \mathcal{B}(K)$ by*

$$\Sigma_Z := \left\{\omega \in \Sigma \;\middle|\; Z_\Sigma(\omega) := \lim_{m \to \infty} \frac{T_{[\omega]_m} T_{[\omega]_m}^*}{\|T_{[\omega]_m}\|^2} \text{ exists in } \mathcal{L}(\mathbb{R}^2)\right\}, \quad K_Z := \pi(\Sigma_Z). \tag{2.9}$$

Then $Z_\Sigma(\omega)$ is an orthogonal projection of rank 1 for any $\omega \in \Sigma_Z$, $\lambda(\Sigma \setminus \Sigma_Z) = \mu(K \setminus K_Z) = 0$, $\pi^{-1}(V_*) \subset \Sigma_Z$ and $Z_\Sigma(\omega) = Z_\Sigma(\tau)$ for $\omega, \tau \in \pi^{-1}(x)$, $x \in V_* \setminus V_0$. Hence (by Proposition 2.3) setting $Z_x := Z(x) := Z_\Sigma(\omega)$, $\omega \in \pi^{-1}(x)$ for $x \in K_Z$ and $Z_x := Z(x) := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ for $x \in K \setminus K_Z$ gives a well-defined Borel measurable map $Z : K \to \mathcal{L}(\mathbb{R}^2)$.

Theorem 2.16 ([22, §4]). *Set $C^1(K) := \{v \circ \Phi \mid v \in C^1(\mathbb{R}^2)\}$. Then for each $u \in C^1(K)$, $\nabla u := (\nabla v) \circ \Phi$ is independent of a particular choice of $v \in C^1(\mathbb{R}^2)$ satisfying $u = v \circ \Phi$. Moreover, $C^1(K) \subset \mathcal{F}$, $C^1(K)/\mathbb{R}\mathbf{1}$ is dense in $(\mathcal{F}/\mathbb{R}\mathbf{1}, \mathcal{E})$, and for any $u, v \in C^1(K)$,*

$$d\mu_{\langle u,v \rangle} = \langle Z\nabla u, Z\nabla v \rangle d\mu \quad \text{and} \quad \mathcal{E}(u, v) = \frac{1}{2}\int_K \langle Z\nabla u, Z\nabla v \rangle d\mu. \tag{2.10}$$

In view of Theorem 2.16, especially (2.10), we may regard Z as defining a *"one-dimensional tangent space of K at x together with a metric"* for μ-a.e. $x \in K$ in a measurable way, with μ considered as the associated *"Riemannian volume measure"* and $Z\nabla u$ as the *"gradient vector field"* of $u \in C^1(K)$. Then the Dirichlet space associated with this *"Riemannian structure"* is $(K, \mu, \mathcal{E}, \mathcal{F})$. The main subject of the present paper is detailed asymptotic analysis of *this* Dirichlet space, especially its associated heat kernel p_μ.

As a matter of fact, any $u \in \mathcal{F}$ admits a natural *"gradient vector field"* $\widetilde{\nabla} u$, thereby (2.10) extended to functions in $\mathcal{F}$, as in the following theorem whose essential part is due to Hino [17, Theorem 5.4].

Theorem 2.17. *Let $h \in \mathcal{S}_{\mathcal{H}_0}$. Then for any $u \in \mathcal{F}$ we have the following statements:*
(1) *For μ-a.e. $x \in K$, there exists $\widetilde{\nabla} u(x) \in \operatorname{Im} Z_x$ such that for any $\omega \in \pi^{-1}(x)$,*

$$\sup_{y \in K_{[\omega]_m}} \left| u(y) - u(x) - \langle \widetilde{\nabla} u(x), \Phi(y) - \Phi(x) \rangle \right| = o(\|T_{[\omega]_m}\|) \quad \text{as} \quad m \to \infty. \tag{2.11}$$

Such $\widetilde{\nabla} u(x) \in \operatorname{Im} Z_x$ as in (2.11) *is unique for each $x \in K_Z$, and $d\mu_{\langle u \rangle} = |\widetilde{\nabla} u|^2 d\mu$.*
(2) *For $\mu_{\langle h \rangle}$-a.e. $x \in K$, there exists $\frac{du}{dh}(x) \in \mathbb{R}$ such that for any $\omega \in \pi^{-1}(x)$,*

$$\sup_{y \in K_{[\omega]_m}} \left| u(y) - u(x) - \frac{du}{dh}(x)(h(y) - h(x)) \right| = o(\|h \circ F_{[\omega]_m}\|_{\mathcal{E}}) \quad \text{as} \quad m \to \infty. \tag{2.12}$$

Such $\frac{du}{dh}(x) \in \mathbb{R}$ as in (2.12) *is unique for each $x \in K$, and $d\mu_{\langle u \rangle} = \left(\frac{du}{dh}\right)^2 d\mu_{\langle h \rangle}$.*

We need the following definition and lemma for the proof of Theorem 2.17. Recall that the map $Z : K \to \mathcal{L}(\mathbb{R}^2)$ satisfies $Z^2 = Z^* = Z$, $\det Z = 0$ and $\operatorname{tr} Z = 1$.

Definition 2.18. Let $Z^{i,j} := \langle e_i, Z e_j \rangle$ for $i, j \in \{1,2\}$, where $e_1 := (1,0)$ and $e_2 := (0,1)$. We define $\zeta = (\zeta^1, \zeta^2) : K \to \mathbb{R}^2$ by

$$\zeta := \left(\sqrt{Z^{1,1}}, Z^{1,2}/\sqrt{Z^{1,1}}\right) \quad \text{if } Z^{1,1} \neq 0, \qquad \text{otherwise} \quad \zeta := (0,1), \tag{2.13}$$

so that $Z^{i,j} = \zeta^i \zeta^j$ for $i, j \in \{1,2\}$, $|\zeta| = 1$ and $\zeta(x) \in \operatorname{Im} Z_x$ for any $x \in K$. Also for each $x \in K$, we write $\zeta_x = (\zeta^1_x, \zeta^2_x)$ for $\zeta(x) = (\zeta^1(x), \zeta^2(x))$ and define $h_x, h_x^\perp$ by

$$\begin{aligned} h_x &:= \zeta^1_x (h_1 - h_1(x)\mathbf{1}) + \zeta^2_x (h_2 - h_2(x)\mathbf{1}), \\ h_x^\perp &:= -\zeta^2_x (h_1 - h_1(x)\mathbf{1}) + \zeta^1_x (h_2 - h_2(x)\mathbf{1}), \end{aligned} \tag{2.14}$$

so that $h_x, h_x^\perp \in \mathcal{S}_{\mathcal{H}_0}$, $\mathcal{E}(h_x, h_x^\perp) = 0$ and $h_x(x) = h_x^\perp(x) = 0$.

Lemma 2.19. *Let $x \in K_Z$ and $\omega \in \pi^{-1}(x)$. Then*

$$\lim_{m \to \infty} \frac{\|h_x \circ F_{[\omega]_m}\|_{\mathcal{E}}}{\|T_{[\omega]_m}\|} = 1 \quad \text{and} \quad \lim_{m \to \infty} \frac{\|h_x^\perp \circ F_{[\omega]_m}\|_{\mathcal{E}}}{\|T_{[\omega]_m}\|} = 0. \tag{2.15}$$

Proof. This is immediate from a direct calculation using Proposition 2.12-(ii), (2.9) and (2.14). □

Proof of Theorem 2.17. By [17, Theorem 5.6], $\mu_{\langle v \rangle}$ is absolutely continuous with respect to both μ and $\mu_{\langle h \rangle}$ for any $v \in \mathcal{F}$. Moreover, by [23, Theorem 3.2.5] and a direct calculation we have

$$\|h \circ F_w\|_{\mathcal{E}} \leq \max_{K_w} h - \min_{K_w} h \leq \frac{2}{\sqrt{3}} \|h \circ F_w\|_{\mathcal{E}} \qquad \text{for any } w \in W_*. \tag{2.16}$$

Therefore an application of [17, Theorem 5.4] to h and u yields (2). Thanks to (2.14), (2.15) and (2.16), (1) follows by applying (2) to $h = h_1$ and setting $\widetilde{\nabla} u(x) := \frac{du}{dh_1}(x) \zeta^1_x \zeta_x$; note that μ and $\mu_{\langle h_1 \rangle}$ are mutually absolutely continuous and that $(\zeta^1)^2 = |Ze_1|^2 = d\mu_{\langle h_1 \rangle}/d\mu$ μ-a.e. □

Remark 2.20. *The "gradient vector field"* $\widetilde{\nabla}u$ *in Theorem* 2.17-(1) *coincides with the "weak gradient"* $Y(\cdot\,;u)$ *defined by Kusuoka* [29, *Lemma* 5.1] (see also [25, Definition 4.11]). Indeed, noting that we can naturally define ∇u on $K \setminus V_m$ for $m \in \mathbb{N}$ and $u \in \mathcal{H}_m$ in the same way as in Theorem 2.16, from (2.15) and (2.16) we can easily verify $\widetilde{\nabla}u(x) = Z_x \nabla u(x)$ for $x \in K_Z$ if $u \in C^1(K)$ and for $x \in K_Z \setminus V_m$ if $m \in \mathbb{N}$ and $u \in \mathcal{H}_m$. Let $u \in \mathcal{F}$, and for each $m \in \mathbb{N}$ let $u_m \in \mathcal{H}_m$ be such that $u_m|_{V_m} = u|_{V_m}$. Then by Theorem 2.17-(1) and [23, Lemma 3.2.17],

$$\int_K |\widetilde{\nabla}u - Z\nabla u_m|^2 d\mu = \int_K |\widetilde{\nabla}(u - u_m)|^2 d\mu = \|u - u_m\|_{\mathcal{E}}^2 \xrightarrow{m\to\infty} 0,$$

whereas $Y(\cdot\,;u)$ is defined as the $L^2(K,\mu)$-limit of $\{Z\nabla u_m\}_{m\in\mathbb{N}}$ in [29]. Thus $\widetilde{\nabla}u = Y(\cdot\,;u)$ μ-a.e.

3 Geometry under the measurable Riemannian structure

This section is devoted to preparing preliminary facts required for the subsequent arguments. First we introduce basic notions and results concerning the description of geometry of K, following [24]. Then we treat the volume doubling property of energy measures, construction of geodesic metrics and weak Poincaré inequality. For the Dirichlet space $(K, \mu, \mathcal{E}, \mathcal{F})$, which corresponds to the measurable Riemannian structure on K, essential parts of the results of this section are already established in Kigami [25]. Here we slightly improve his results, and prove the same results also for the Dirichlet space $(K, \mu_{\langle h \rangle}, \mathcal{E}, \mathcal{F})$, $h \in \mathcal{S}_{\mathcal{H}_0}$. The extensions to $(K, \mu_{\langle h \rangle}, \mathcal{E}, \mathcal{F})$ are of independent interest and will play central roles in Sections 4 and 5.

Definition 3.1. (1) Let $w, v \in W_*$, $w = w_1 \dots w_m$, $v = v_1 \dots v_n$. We define $wv \in W_*$ by $wv := w_1 \dots w_m v_1 \dots v_n$ ($w\emptyset := w$, $\emptyset v := v$). We also define $w^1 \dots w^k$ for $k \geq 3$ and $w^1, \dots, w^k \in W_*$ inductively by $w^1 \dots w^k := (w^1 \dots w^{k-1})w^k$. We write $w \leq v$ if and only if $w = v\tau$ for some $\tau \in W_*$. Note that $\Sigma_w \cap \Sigma_v = \emptyset$ if and only if neither $w \leq v$ nor $v \leq w$.
(2) Let Λ be a finite subset of W_*. We call Λ a *partition of* Σ if and only if $\Sigma_w \cap \Sigma_v = \emptyset$ for any $w, v \in \Lambda$ with $w \neq v$ and $\Sigma = \bigcup_{w \in \Lambda} \Sigma_w$.
(3) Let Λ_1 and Λ_2 be two partitions of Σ. Then we say that Λ_1 is a *refinement of* Λ_2, and write $\Lambda_1 \leq \Lambda_2$, if and only if for each $w^1 \in \Lambda_1$ there exists $w^2 \in \Lambda_2$ such that $w^1 \leq w^2$.

Suppose $\Lambda_1 \leq \Lambda_2$. Then we have a natural surjection $\Lambda_1 \to \Lambda_2$ by which $w^1 \in \Lambda_1$ is mapped to the unique $w^2 \in \Lambda_2$ such that $w^1 \leq w^2$. In particular, $\#\Lambda_1 \geq \#\Lambda_2$.

Definition 3.2. (1) A family $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ of partitions of Σ is called a *scale on* Σ if and only if $\mathcal{S}$ satisfies the following three properties:
(S1) $\Lambda_1 = W_0\ (= \{\emptyset\})$. $\Lambda_{s_1} \leq \Lambda_{s_2}$ for any $s_1, s_2 \in (0,1]$ with $s_1 \leq s_2$.
(S2) $\min\{|w| \mid w \in \Lambda_s\} \to \infty$ as $s \downarrow 0$.
(Sr) For each $s \in (0,1)$ there exists $\varepsilon \in (0, 1-s]$ such that $\Lambda_{s'} = \Lambda_s$ for any $s' \in (s, s+\varepsilon)$.
(2) A function $l : W_* \to (0,1]$ is called a *gauge function on* W_* if and only if $l(wi) \leq l(w)$ for any $(w,i) \in W_* \times S$ and $\lim_{m\to\infty} \max\{l(w) \mid w \in W_m\} = 0$.

There is a natural one-to-one correspondence between scales on Σ and gauge functions on W_*, as in the following proposition. See [24, Section 1.1] for a proof.

Proposition 3.3. (1) *Let l be a gauge function on W_*. For each $s \in (0,1]$, define*

$$\Lambda_s(l) := \{w \mid w = w_1 \dots w_m \in W_*,\ l(w_1 \dots w_{m-1}) > s \geq l(w)\} \qquad (3.1)$$

where $l(w_1 \dots w_{m-1}) := 2$ when $w = \emptyset$. Then the collection $\mathcal{S}(l) := \{\Lambda_s(l)\}_{s\in(0,1]}$ is a scale on Σ. We call $\mathcal{S}(l)$ the scale induced by the gauge function l.
(2) *Let $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ be a scale on Σ. Then there exists a unique gauge function $l_{\mathcal{S}}$ on W_* such that $\mathcal{S} = \mathcal{S}(l_{\mathcal{S}})$. We call $l_{\mathcal{S}}$ the* gauge function of the scale $\mathcal{S}$.

Definition 3.4. Let $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ be a scale on Σ. For $s \in (0,1]$ and $x \in K$, we define

$$K_s(x,\mathcal{S}) := \bigcup_{w\in\Lambda_s,\, x\in K_w} K_w, \qquad U_s(x,\mathcal{S}) := \bigcup_{w\in\Lambda_s,\, K_w\cap K_s(x,\mathcal{S})\neq\emptyset} K_w. \qquad (3.2)$$

Clearly, $K_s(x,\mathcal{S})$ and $U_s(x,\mathcal{S})$ are decreasing as s decreases and $\{K_s(x,\mathcal{S})\}_{s\in(0,1]}$ and $\{U_s(x,\mathcal{S})\}_{s\in(0,1]}$ are fundamental systems of neighborhoods of x in K.

Proposition 2.3 easily yields the following lemma.

Lemma 3.5. *Let $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ be a scale on Σ and let $s \in (0,1]$, $x \in K$ and $w \in \Lambda_s$. Then $\#\{v \in \Lambda_s \mid K_v \cap K_s(x,\mathcal{S}) \neq \emptyset\} \leq 6$ and $\#\{v \in \Lambda_s \mid K_w \cap K_v \neq \emptyset\} \leq 4$.*

Definition 3.6. Let $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ be a scale on Σ.
(1) A function $\varphi : W_* \to [0,\infty)$ is called *gentle with respect to* $\mathcal{S}$ if and only if there exists $c_{\mathrm{gen}} \in (0,\infty)$ such that $\varphi(w) \leq c_{\mathrm{gen}}\varphi(v)$ whenever $w, v \in \Lambda_s$ for some $s \in (0,1]$ and $K_w \cap K_v \neq \emptyset$. We say that a finite Borel measure ν on K is *gentle with respect to* $\mathcal{S}$ if and only if the function $W_* \ni w \mapsto \nu(K_w)$ is gentle with respect to $\mathcal{S}$.
(2) A metric ρ on K is called *adapted to* $\mathcal{S}$ if and only if there exist $\beta_1, \beta_2 \in (0,\infty)$ such that

$$B_{\beta_1 s}(x,\rho) \subset U_s(x,\mathcal{S}) \subset B_{\beta_2 s}(x,\rho), \qquad (s,x) \in (0,1] \times K. \qquad (3.3)$$

Lemma 3.7. *Let $\mathcal{S} = \{\Lambda_s\}_{s\in(0,1]}$ be a scale on Σ with gauge function l and let ρ be a metric on K adapted to $\mathcal{S}$. Then ρ is compatible with the original topology of K, and* $\mathrm{diam}(K_w,\rho) \leq \beta_2 l(w)$ *for any $w \in W_*$, where $\beta_2 \in (0,\infty)$ is as in* (3.3).

Proof. The first assertion is clear. Let $w \in W_*$, $x, y \in K_w$ and $s := l(w)$. Then $w \leq v$ for a unique $v \in \Lambda_s$, and $K_w \subset K_v \subset U_s(x,\mathcal{S}) \subset B_{\beta_2 s}(x,\rho)$ by (3.3). Thus $\rho(x,y) < \beta_2 s = \beta_2 l(w)$. □

Now we discuss the volume doubling property of μ and $\mu_{\langle h\rangle}$, $h \in \mathcal{S}_{\mathcal{H}_0}$. First we state their volume doubling property in terms of certain scales, to which the corresponding geodesic metrics are shown to be adapted later in this section.

Definition 3.8. (1) We define $\mathcal{S}^{\mathcal{H}} = \{\Lambda^{\mathcal{H}}_s\}_{s\in(0,1]}$ to be the scale on Σ induced by the gauge function $l_{\mathcal{H}} : W_* \to (0,1]$, $l_{\mathcal{H}}(w) := \|T_w\| \wedge 1 = \sqrt{(3/5)^{|w|}\mu(K_w)} \wedge 1$.
(2) Let $h \in \mathcal{S}_{\mathcal{H}_0}$. We define $\mathcal{S}^h = \{\Lambda^h_s\}_{s\in(0,1]}$ to be the scale on Σ induced by the gauge function $l_h : W_* \to (0,1]$, $l_h(w) := \|h \circ F_w\|_{\mathcal{E}} = \sqrt{(3/5)^{|w|}\mu_{\langle h\rangle}(K_w)}$.

Lemma 3.9 (cf. [25, Lemma 3.5 and Proof of Theorem 3.2])**.** *Let $h \in \mathcal{S}_{\mathcal{H}_0}$.*
(1) *For any $(w,i) \in W_* \times S$,*

$$\frac{1}{15}\mu(K_w) \leq \mu(K_{wi}) \leq \frac{3}{5}\mu(K_w), \qquad \frac{1}{5}\|T_w\| \leq \|T_{wi}\| \leq \frac{3}{5}\|T_w\|, \qquad (3.4)$$

$$\frac{1}{15}\mu_{\langle h\rangle}(K_w) \leq \mu_{\langle h\rangle}(K_{wi}) \leq \frac{3}{5}\mu_{\langle h\rangle}(K_w), \qquad \frac{1}{5}l_h(w) \leq l_h(wi) \leq \frac{3}{5}l_h(w). \qquad (3.5)$$

(2) *If* $w, v \in W_*$ *satisfies* $|w| = |v|$ *and* $K_w \cap K_v \neq \emptyset$ *then*

$$\mu_{\langle h \rangle}(K_w) \leq 9\mu_{\langle h \rangle}(K_v), \quad l_h(w) \leq 3l_h(v) \quad \text{and} \quad l_{\mathcal{H}}(w) \leq 3l_{\mathcal{H}}(v). \tag{3.6}$$

Proof. (1) By considering $\|h \circ F_w\|_{\mathcal{E}}^{-1} h \circ F_w$ and $\emptyset$ instead of h and w respectively, a direct calculation easily yields (3.5), from which (3.4) is immediate.
(2) This is proved in essentially the same way as [25, Proof of Lemma 3.5]. □

Proposition 3.10 (cf. [25, Theorem 6.2]). (1) *There exists* $c_{\mathrm{G}} \in (0, \infty)$ *such that for any* $g, h \in \mathcal{S}_{\mathcal{H}_0}$, $\mu_{\langle g \rangle}$ *is gentle with respect to both* $\mathcal{S}^{\mathcal{H}}$ *and* $\mathcal{S}^h$ *with constant* $c_{\mathrm{gen}} = c_{\mathrm{G}}$, *i.e.* $\mu_{\langle g \rangle}(K_w) \leq c_{\mathrm{G}}\mu_{\langle g \rangle}(K_v)$ *whenever either* $w, v \in \Lambda_s^{\mathcal{H}}$ *or* $w, v \in \Lambda_s^h$ *for some* $s \in (0, 1]$ *and* $K_w \cap K_v \neq \emptyset$.
(2) *Let* $\kappa := \log_5 15$ *and* $\hat{\kappa} := \log_{5/3} 15$. *Then there exists* $c_{\mathrm{v}} \in (0, \infty)$ *such that for any* $g, h \in \mathcal{S}_{\mathcal{H}_0}$, $x \in K$ *and* $s, t \in (0, 1]$ *with* $s \leq t$,

$$\frac{\mu(U_t(x, \mathcal{S}^{\mathcal{H}}))}{\mu(U_s(x, \mathcal{S}^{\mathcal{H}}))} \leq c_{\mathrm{v}}\Big(\frac{t}{s}\Big)^{\kappa}, \qquad \frac{\mu_{\langle h \rangle}(U_t(x, \mathcal{S}^h))}{\mu_{\langle h \rangle}(U_s(x, \mathcal{S}^h))} \leq c_{\mathrm{v}}\Big(\frac{t}{s}\Big)^{\kappa}, \tag{3.7}$$

$$\frac{\mu_{\langle g \rangle}(U_t(x, \mathcal{S}^{\mathcal{H}}))}{\mu_{\langle g \rangle}(U_s(x, \mathcal{S}^{\mathcal{H}}))} \leq c_{\mathrm{v}}\Big(\frac{t}{s}\Big)^{\hat{\kappa}}, \qquad \frac{\mu_{\langle g \rangle}(U_t(x, \mathcal{S}^h))}{\mu_{\langle g \rangle}(U_s(x, \mathcal{S}^h))} \leq c_{\mathrm{v}}\Big(\frac{t}{s}\Big)^{\hat{\kappa}}. \tag{3.8}$$

Proof. (1) This is proved in exactly the same way as [25, Proof of Theorem 6.2]. Here [25, Proof of Theorem 1.4.3] together with (3.4), (3.5) and (3.6) easily shows that the constant $c_{\mathrm{G}} \in (0, \infty)$ can be chosen independently of g, h.
(2) We essentially follow [24, Proof of Theorem 1.3.5], but slightly more detailed arguments are required to deduce the explicit constants κ and $\hat{\kappa}$. Let $g \in \mathcal{S}_{\mathcal{H}_0}$, $x \in K$ and $\omega \in \pi^{-1}(x)$. For each $s \in (0, 1)$, let $n(s)$ be the unique $n \in \mathbb{N} \cup \{0\}$ satisfying $[\omega]_n \in \Lambda_s^{\mathcal{H}}$, so that $s/5 \leq \big\|T_{[\omega]_{n(s)}}\big\| \leq s$ by (3.4). Then (1) and Lemma 3.5 easily imply that for any $s \in (0, 1)$,

$$1 \leq \frac{\mu(U_s(x, \mathcal{S}^{\mathcal{H}}))}{\mu\big(K_{[\omega]_{n(s)}}\big)} \leq 6c_{\mathrm{G}}^2 \quad \text{and} \quad 1 \leq \frac{\mu_{\langle g \rangle}(U_s(x, \mathcal{S}^{\mathcal{H}}))}{\mu_{\langle g \rangle}\big(K_{[\omega]_{n(s)}}\big)} \leq 6c_{\mathrm{G}}^2. \tag{3.9}$$

Let $s, t \in (0, 1)$, $s \leq t$. Then $n(s) \geq n(t)$, and (3.4) yields

$$\frac{1}{5}\Big(\frac{1}{5}\Big)^{n(s)-n(t)} \leq \frac{\big\|T_{[\omega]_{n(s)}}\big\|}{5\big\|T_{[\omega]_{n(t)}}\big\|} \leq \frac{s}{t} \leq \frac{5\big\|T_{[\omega]_{n(s)}}\big\|}{\big\|T_{[\omega]_{n(t)}}\big\|} \leq 5\Big(\frac{3}{5}\Big)^{n(s)-n(t)}. \tag{3.10}$$

Now from (3.9) and (3.10) we conclude that

$$6c_{\mathrm{G}}^2\frac{\mu(U_s(x, \mathcal{S}^{\mathcal{H}}))}{\mu(U_t(x, \mathcal{S}^{\mathcal{H}}))} \geq \frac{\mu\big(K_{[\omega]_{n(s)}}\big)}{\mu\big(K_{[\omega]_{n(t)}}\big)} \geq \Big(\frac{t}{5s}\Big)^{\log_5(5/3)}\frac{s^2}{25t^2} = \frac{3}{125}\Big(\frac{s}{t}\Big)^{\kappa}$$

and, using also (3.5), that

$$6c_{\mathrm{G}}^2\frac{\mu_{\langle g \rangle}(U_s(x, \mathcal{S}^{\mathcal{H}}))}{\mu_{\langle g \rangle}(U_t(x, \mathcal{S}^{\mathcal{H}}))} \geq \frac{\mu_{\langle g \rangle}\big(K_{[\omega]_{n(s)}}\big)}{\mu_{\langle g \rangle}\big(K_{[\omega]_{n(t)}}\big)} \geq \Big(\frac{1}{15}\Big)^{n(s)-n(t)} \geq 5^{-\hat{\kappa}}\Big(\frac{s}{t}\Big)^{\hat{\kappa}},$$

proving the assertions for $\mathcal{S}^{\mathcal{H}}$; the case with $t = 1$ follows since $U_{4/5}(x, \mathcal{S}^{\mathcal{H}}) = K$. In view of (3.5), exactly the same proof applies to the assertions for $\mathcal{S}^h$ as well. □

Remark 3.11. *The powers* κ *in* (3.7) *and* $\hat{\kappa}$ *in* (3.8) *are best possible.* Indeed, for $n \in \mathbb{N}$, since $T^*_{1^n} = \begin{pmatrix} (3/5)^n & 0 \\ 0 & (1/5)^n \end{pmatrix}$, $T^*_{1^n 32^n} = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \begin{pmatrix} 0 & (\sqrt{3}/5)^{2n+1} \\ -(\sqrt{3}/5)^{2n+1} & -2(1/5)^{2n+1} \end{pmatrix}$, we easily see $5^n \leq t_n/s_n$ and hence $\mu(U_{s_n}(x_n, \mathcal{S}^{\mathcal{H}}))/\mu(U_{t_n}(x_n, \mathcal{S}^{\mathcal{H}})) \leq 10c_{\mathrm{G}}(s_n/t_n)^{\kappa}$ by (3.9), where $x_n := \pi(1^n 32^\infty)$, $s_n := \|T_{1^n 32^n}\|$ and $t_n := \|T_{1^n}\|$. Similar calculations work with $\mathcal{S}^h$ and $\mu_{\langle h \rangle}$ for each $h \in \mathcal{S}_{\mathcal{H}_0}$. For the first part of (3.8) it suffices to choose $g := h_2$, $x := q_1$, $s := \|T_{1 2^n}\|$ and $t := \|T_{1^n}\|$ to let $n \to \infty$, and similarly for the latter of (3.8) for each $h \in \mathcal{S}_{\mathcal{H}_0}$.

Next we define the corresponding geodesic metrics on K and state their basic properties.

Definition 3.12. Let $h \in \mathcal{S}_{\mathcal{H}_0}$. We define the *harmonic geodesic metric* $\rho_{\mathcal{H}}$ *on* K and the *h-geodesic metric* ρ_h *on* K by respectively

$$\rho_{\mathcal{H}}(x, y) := \inf\{\ell_{\mathcal{H}}(\gamma) \mid \gamma : [0,1] \to K,\ \gamma \text{ is continuous},\ \gamma(0) = x,\ \gamma(1) = y\},$$
$$\rho_h(x, y) := \inf\{\ell_h(\gamma) \mid \gamma : [0,1] \to K,\ \gamma \text{ is continuous},\ \gamma(0) = x,\ \gamma(1) = y\}$$

for $x, y \in K$, where we set $\ell_{\mathcal{H}}(\gamma) := \ell(\Phi \circ \gamma)$ and $\ell_h(\gamma) := \ell(h \circ \gamma)$ for a continuous map $\gamma : [a, b] \to K$, $a, b \in \mathbb{R}$, $a \leq b$.

Definition 3.13. (1) Let $m \in \mathbb{N} \cup \{0\}$ and let $x, y \in V_m$ satisfy $x \overset{m}{\sim} y$, where $\overset{m}{\sim}$ is as in Definition 2.4. Let $w(x, y)$ be the unique $w \in W_m$ such that $x, y \in F_w(V_0)$, and let $\overline{xy}$ ($\subset K_{w(x,y)}$) denote the line segment from x to y which is also regarded as the map $[0, 1] \ni t \mapsto x + t(y - x)$. Note that $\overline{xy} \subset K_Z$ by [25, Theorem 5.4].
(2) Let $m \in \mathbb{N} \cup \{0\}$. A sequence $\Gamma = \{x_k\}_{k=0}^N \subset V_m$, where $N \in \mathbb{N}$, is called an *m-walk* if and only if $x_{k-1} \overset{m}{\sim} x_k$ for $k \in \{1, \ldots, N\}$ and $w(x_{k-1}, x_k) \neq w(x_k, x_{k+1})$ for $k \in \{1, \ldots, N-1\}$. For such Γ we define continuous maps $\overline{\Gamma} : [0, N] \to K$ and $\widehat{\Gamma} : [0, \ell_{\mathcal{H}}(\overline{\Gamma})] \to K$ by

$$\overline{\Gamma}(t) := x_{k-1} + (t - k + 1)(x_k - x_{k-1}), \qquad t \in [k-1, k],\ \ k \in \{1, \ldots, N\},$$

and $\widehat{\Gamma} := \overline{\Gamma} \circ \varphi_\Gamma^{-1}$, where φ_Γ is the homeomorphism $\varphi_\Gamma : [0, N] \to [0, \ell_{\mathcal{H}}(\overline{\Gamma})]$, $\varphi_\Gamma(t) := \ell_{\mathcal{H}}(\overline{\Gamma}|_{[0,t]})$; note that $\ell_{\mathcal{H}}(\overline{\Gamma}) < \infty$ and $\widehat{\Gamma}([0, \ell_{\mathcal{H}}(\overline{\Gamma})]) \subset K_Z$ by [25, Theorem 5.4].
(3) Let $\gamma : [a, b] \to K$ be continuous, $a, b \in \mathbb{R}$, $a < b$. γ is called a *harmonic m-geodesic*, where $m \in \mathbb{N} \cup \{0\}$, if and only if $\gamma(t) = \widehat{\Gamma}\big(\ell_{\mathcal{H}}(\overline{\Gamma})\frac{t-a}{b-a}\big)$, $t \in [a, b]$ for some m-walk Γ. γ is called a *harmonic geodesic* if and only if there exist $n \in \mathbb{N} \cup \{0\}$ and sequences $\{a_m\}_{m \geq n}, \{b_m\}_{m \geq n} \subset [a, b]$ with $\lim_{m\to\infty} a_m = a$ and $\lim_{m\to\infty} b_m = b$ such that $a_{m+1} \leq a_m < b_m \leq b_{m+1}$ and $\gamma|_{[a_m, b_m]}$ is a harmonic m-geodesic for each $m \geq n$.

Proposition 3.14 ([25, Theorem 5.4])**.** *If* $m \in \mathbb{N} \cup \{0\}$ *and* Γ *is an m-walk, then* $\Phi \circ \widehat{\Gamma}$ *is* C^1, *and* $(\Phi \circ \widehat{\Gamma})'(t) \in \mathrm{Im}\, Z_{\widehat{\Gamma}(t)}$ *and* $|(\Phi \circ \widehat{\Gamma})'(t)| = 1$ *for any* $t \in [0, \ell_{\mathcal{H}}(\overline{\Gamma})]$.

For the harmonic geodesic metric $\rho_{\mathcal{H}}$ we have the following proposition due to Kigami [25]; it is not explicitly stated in [25, Theorem 5.1], but is actually shown in the proof there, that we can take harmonic geodesics as shortest paths for the length $\ell_{\mathcal{H}}(\cdot)$. This fact plays a crucial role in the proof of Proposition 4.10 below.

Proposition 3.15 ([25, Theorems 5.1 and 5.11])**.** (1) $\rho_{\mathcal{H}}$ *is a metric on* K *satisfying*

$$B_{\sqrt{2}s/50}(x, \rho_{\mathcal{H}}) \subset U_s(x, \mathcal{S}^{\mathcal{H}}) \subset B_{10s}(x, \rho_{\mathcal{H}}), \qquad (s, x) \in (0, 1] \times K. \tag{3.11}$$

(2) *For each* $x, y \in K$ *with* $x \neq y$, *there exists a harmonic geodesic* $\gamma_{xy} : [0, 1] \to K$ *such that* $\gamma_{xy}(0) = x$, $\gamma_{xy}(1) = y$ *and* $\rho_{\mathcal{H}}(x, y) = \ell_{\mathcal{H}}(\gamma_{xy})$, *and in particular* $\rho_{\mathcal{H}}(\gamma_{xy}(s), \gamma_{xy}(t)) = \ell_{\mathcal{H}}(\gamma_{xy}|_{[s,t]}) = (t - s)\rho_{\mathcal{H}}(x, y)$ *for any* $s, t \in [0, 1]$ *with* $s \leq t$. *Moreover, if* $m \in \mathbb{N} \cup \{0\}$ *and* $x, y \in V_m$ *then we can take a harmonic m-geodesic as* γ_{xy}.

In fact, similar assertions are valid also for ρ_h, as follows.

Proposition 3.16. *Let $h \in \mathcal{S}_{\mathcal{H}_0}$.*
(1) *ρ_h is a metric on K satisfying*

$$B_{s/25}(x,\rho_h) \subset U_s(x,\mathcal{S}^h) \subset B_{7s}(x,\rho_h), \qquad (s,x) \in (0,1] \times K. \tag{3.12}$$

(2) *For each $x, y \in K$ with $x \neq y$, there exists a harmonic geodesic $\gamma^h_{xy} : [0,1] \to K$ such that $\gamma^h_{xy}(0) = x$, $\gamma^h_{xy}(1) = y$ and $\rho_h(x,y) = \ell_h(\gamma^h_{xy})$. In particular, if we define $\varphi^h_{xy} : [0,1] \to [0,1]$ to be the inverse of* $[0,1] \ni t \mapsto \ell_h(\gamma^h_{xy}|_{[0,t]})/\rho_h(x,y)$, *then* $\rho_h(\gamma^h_{xy} \circ \varphi^h_{xy}(s), \gamma^h_{xy} \circ \varphi^h_{xy}(t)) = \ell_h(\gamma^h_{xy} \circ \varphi^h_{xy}|_{[s,t]}) = (t-s)\rho_h(x,y)$ *for any $s, t \in [0,1]$ with $s \leq t$. Moreover, if $m \in \mathbb{N} \cup \{0\}$ and $x, y \in V_m$ then we can take a harmonic m-geodesic as γ^h_{xy}.*

Remark 3.17. If $\gamma : [0,1] \to K$ is a harmonic geodesic and $h \in \mathcal{S}_{\mathcal{H}_0}$, then by [25, Theorem 5.4] (see also (3.15) below), the set $\{t \in (0,1) \mid (h \circ \gamma)'(t) = 0\}$ is discrete and hence $[0,1] \ni t \mapsto \ell_h(\gamma|_{[0,t]})$ is strictly increasing. Therefore φ^h_{xy} as above does exist as a homeomorphism.

We need the following lemma for the proof of Proposition 3.16.

Lemma 3.18 (cf. [25, Lemma 5.6])**.** *Set* $\mathrm{Osc}_A f := \sup_A f - \inf_A f$ *for $f \in C(K)$ and $A \subset K$, $A \neq \emptyset$. Let $h \in \mathcal{S}_{\mathcal{H}_0}$, $w \in W_*$ and $x, y \in F_w(V_0)$, $x \neq y$. Then*

$$\ell_h(\overline{xy}) = \inf\{\ell_h(\gamma) \mid \gamma : [0,1] \to K_w,\ \gamma \textit{ is continuous},\ \gamma(0) = x,\ \gamma(1) = y\}, \tag{3.13}$$

$$\frac{1}{5}\operatorname*{Osc}_{K_w} h \leq \ell_h(\overline{xy}) \leq 2\operatorname*{Osc}_{K_w} h \quad \textit{and} \quad \frac{l_h(w)}{5} \leq \ell_h(\overline{xy}) \leq \frac{4}{\sqrt{3}} l_h(w). \tag{3.14}$$

Proof. It is easy to see that we may assume $w = \emptyset$ without loss of generality by considering $\|h \circ F_w\|_{\mathcal{E}}^{-1} h \circ F_w, \emptyset, F_w^{-1}(x)$ and $F_w^{-1}(y)$ instead of h, w, x and y. Then by the symmetry of K and $(\mathcal{E}, \mathcal{F})$ we may further assume that $x = q_2$ and $y = q_3$.

Let $I := [-1/\sqrt{3}, 1/\sqrt{3}]$. By [25, Theorem 5.4], $\Phi(\overline{q_2 q_3}) = \{(\varphi(t), t) \mid t \in I\}$ for some $\varphi \in C^1(I)$ and it possesses the following properties: $\varphi(-t) = \varphi(t)$ for $t \in I$, φ' is strictly increasing, $\varphi'(\pm 1/\sqrt{3}) = \pm 1/\sqrt{3}$, and $K_{\mathcal{H}} \subset \{(s,t) \in \mathbb{R}^2 \mid s \leq \varphi(t)\}$, i.e. $h_1 \leq \varphi \circ h_2$. We set $\gamma_{23}(t) := \Phi^{-1}(\varphi(t), t)$, $t \in I$. Choose $a, b, c \in \mathbb{R}$ so that $h = ah_1 + bh_2 + c\mathbf{1}$. Then $h \circ \gamma_{23}(t) = a\varphi(t) + bt + c$ for $t \in I$ and $(h \circ \gamma_{23})' = a\varphi' + b$. Since $a^2 + b^2 = \|h\|_{\mathcal{E}}^2 = 1 \neq 0$ it follows that

$$\text{either} \quad (h \circ \gamma_{23})'(t) \neq 0 \text{ for any } t \in I \quad \text{or} \quad (h \circ \gamma_{23})'(t_0) = 0 \text{ for a unique } t_0 \in I, \tag{3.15}$$

from which and $h_1 \leq \varphi \circ h_2$ we can easily verify (3.13) and $\ell_h(\overline{q_2 q_3}) \leq 2\,\mathrm{Osc}_K h$.

To complete the proof of (3.14), let $q_{23} := F_2(q_3) = F_3(q_2)$, so that $5h(q_{23}) = h(q_1) + 2h(q_2) + 2h(q_3)$ by $h \in \mathcal{H}_0$ and [23, Example 3.2.6]. Since either of $h(q_2)$ and $h(q_3)$ is equal to either $\max_{V_0} h$ or $\min_{V_0} h$, we see that

$$\begin{aligned} 5\ell_h(\overline{q_2 q_3}) &\geq 5|h(q_2) - h(q_{23})| + 5|h(q_{23}) - h(q_3)| \\ &= |h(q_1) + 2h(q_3) - 3h(q_2)| + |h(q_1) + 2h(q_2) - 3h(q_3)| \geq \operatorname*{Osc}_{V_0} h = \operatorname*{Osc}_{K} h, \end{aligned}$$

proving the former assertion of (3.14) which and (2.16) yield the latter. □

Proof of Proposition 3.16. This is proved in exactly the same way as [25, Proofs of Theorems 5.1 and 5.11] by using Lemma 3.18 instead of [25, Lemma 5.6]. □

By virtue of Propositions 3.10, 3.15 and 3.16, now we arrive at the following theorem, which improves and generalizes [25, Theorem 6.2] and will be used to deduce the remainder estimates in Theorem 5.8 below.

Theorem 3.19. *Let* $\kappa := \log_5 15$ *and* $\hat{\kappa} := \log_{5/3} 15$*, as in Proposition* 3.10-(2)*. Then there exists* $c_{\mathrm{V}} \in (0,\infty)$ *such that for any* $g, h \in \mathcal{S}_{\mathcal{H}_0}$*,* $x, y \in K$ *and* $r, R \in (0,\infty)$ *with* $r \leq R$*,*

$$\frac{\mu(B_R(x,\rho_{\mathcal{H}}))}{\mu(B_r(y,\rho_{\mathcal{H}}))} \leq c_{\mathrm{V}}\Bigl(\frac{R+\rho_{\mathcal{H}}(x,y)}{r}\Bigr)^{\kappa}, \quad \frac{\mu_{\langle h\rangle}(B_R(x,\rho_h))}{\mu_{\langle h\rangle}(B_r(y,\rho_h))} \leq c_{\mathrm{V}}\Bigl(\frac{R+\rho_h(x,y)}{r}\Bigr)^{\kappa}, \tag{3.16}$$

$$\frac{\mu_{\langle g\rangle}(B_R(x,\rho_{\mathcal{H}}))}{\mu_{\langle g\rangle}(B_r(y,\rho_{\mathcal{H}}))} \leq c_{\mathrm{V}}\Bigl(\frac{R+\rho_{\mathcal{H}}(x,y)}{r}\Bigr)^{\hat{\kappa}}, \quad \frac{\mu_{\langle g\rangle}(B_R(x,\rho_h))}{\mu_{\langle g\rangle}(B_r(y,\rho_h))} \leq c_{\mathrm{V}}\Bigl(\frac{R+\rho_h(x,y)}{r}\Bigr)^{\hat{\kappa}}. \tag{3.17}$$

Proof. Since $B_R(x,\rho) \subset B_{R+\rho(x,y)}(y,\rho)$ for $\rho = \rho_{\mathcal{H}}, \rho_h$, it suffice to prove the assertions when $x = y$. (3.7), (3.8), (3.11) and (3.12) easily yield (3.16) and (3.17) for $R \leq \sqrt{2}/50$, and then the case of $R \geq \sqrt{2}/50$ is easily proved by using (3.4), (3.5), (3.11) and (3.12). □

Finally we prove the weak Poincaré inequality for $(K, \mu, \mathcal{E}, \mathcal{F})$ and $(K, \mu_{\langle h\rangle}, \mathcal{E}, \mathcal{F})$, $h \in \mathcal{S}_{\mathcal{H}_0}$.

Proposition 3.20. *Let* $c_{\mathrm{G}} \in (0,\infty)$ *be as in Proposition* 3.10-(1) *and* $c_{\mathrm{P}} := 3^4 10^6 c_{\mathrm{G}}^4$*. Let* $h \in \mathcal{S}_{\mathcal{H}_0}$ *and let* (ν,ρ) *denote any one of* $(\mu, \rho_{\mathcal{H}})$ *and* $(\mu_{\langle h\rangle}, \rho_h)$*. Then*

$$\int_{B_r(x,\rho)} \bigl|u - \overline{u}_{r,x}^{\nu,\rho}\bigr|^2 d\nu \leq c_{\mathrm{P}} r^2 \mu_{\langle u\rangle}\bigl(B_{250\sqrt{2}r}(x,\rho)\bigr), \qquad u \in \mathcal{F} \tag{3.18}$$

for any $(r,x) \in (0,\infty) \times K$*, where* $\overline{u}_{r,x}^{\nu,\rho} := \nu(B_r(x,\rho))^{-1} \int_{B_r(x,\rho)} u\, d\nu$*.*

Proof. Let $u \in \mathcal{F}$. Recall that $R_{\mathcal{E}}$ denotes the resistance metric on K associated with $(\mathcal{E},\mathcal{F})$. Since $\mathrm{diam}(K, R_{\mathcal{E}}) \leq 6$ which easily follows by using [23, Lemma 3.3.5], for any $w \in W_*$ and any $y, z \in K_w$ we have

$$|u(y) - u(z)|^2 \leq R_{\mathcal{E}}\bigl(F_w^{-1}(y), F_w^{-1}(z)\bigr)\mathcal{E}(u \circ F_w, u \circ F_w) \leq 3\Bigl(\frac{3}{5}\Bigr)^{|w|}\mu_{\langle u\rangle}(K_w). \tag{3.19}$$

Also for $s \in (0,1)$ and $w, v \in \Lambda_s^{\mathcal{H}}$ with $K_w \cap K_v \neq \emptyset$, (3.4) and Proposition 3.10-(1) yield

$$\frac{s^2}{25}\Bigl(\frac{5}{3}\Bigr)^{|w|} \leq \mu(K_w) \leq c_{\mathrm{G}}\mu(K_v) \leq c_{\mathrm{G}}\Bigl(\frac{5}{3}\Bigr)^{|v|} s^2, \quad \text{thus} \quad \Bigl(\frac{3}{5}\Bigr)^{|v|} \leq 25 c_{\mathrm{G}}\Bigl(\frac{3}{5}\Bigr)^{|w|}. \tag{3.20}$$

Let $(r,x) \in (0,\infty) \times K$. Suppose $r < \sqrt{2}/50$ and take $w \in \Lambda_{25\sqrt{2}r}^{\mathcal{H}}$ such that $x \in K_w$. Then by considering $U_{25\sqrt{2}r}(x, \mathcal{S}^{\mathcal{H}})$, from (3.11), (3.19) and (3.20) we easily see that

$$|u(y) - u(z)| \leq 60\sqrt{3}c_{\mathrm{G}}\sqrt{\Bigl(\frac{3}{5}\Bigr)^{|w|}\mu_{\langle u\rangle}\bigl(B_{250\sqrt{2}r}(x,\rho_{\mathcal{H}})\bigr)}, \quad y, z \in B_r(x,\rho_{\mathcal{H}}). \tag{3.21}$$

Now since $\mu(B_r(x,\rho_{\mathcal{H}})) \leq \mu\bigl(U_{25\sqrt{2}r}(x,\mathcal{S}^{\mathcal{H}})\bigr) \leq 6c_{\mathrm{G}}^2\mu(K_w)$ by (3.11) and (3.9), and $(3/5)^{|w|}\mu(K_w) = \|T_w\|^2 \leq 1250r^2$ by $w \in \Lambda_{25\sqrt{2}r}^{\mathcal{H}}$, (3.18) for $(\mu, \rho_{\mathcal{H}})$ immediately follows by integrating (3.21) in z under $\mu|_{B_r(x,\rho_{\mathcal{H}})}$ and then in y after taking the square. The case of $r \geq \sqrt{2}/50$ can be verified in a similar way by using (3.19) with $w = \emptyset$ since $B_{250\sqrt{2}r}(x,\rho_{\mathcal{H}}) = K$ by (3.11), and exactly the same proof applies to the case of $(\mu_{\langle h\rangle}, \rho_h)$ as well by virtue of (3.5), Proposition 3.10-(1) and (3.12). □

Notation. In the rest of this paper, we will use the constants $\kappa = \log_5 15$, $\hat{\kappa} = \log_{5/3} 15$, c_{G} and c_{V} appearing in Proposition 3.10 and Theorem 3.19 without further notice. In particular, for $g, h \in \mathcal{S}_{\mathcal{H}_0}$, $\mu_{\langle g \rangle}$ is gentle with respect to both $\mathcal{S}^{\mathcal{H}}$ and $\mathcal{S}^h$ with $c_{\mathrm{gen}} = c_{\mathrm{G}}$. Also *in what follows, for $a, b \in [0, \infty)$ we write $a \lesssim b$ if and only if $a \leq cb$ for some constant $c \in (0, \infty)$ determined solely by $\kappa, \hat{\kappa}, c_{\mathrm{G}}, c_{\mathrm{V}}$, and write $a \asymp b$ if and only if both $a \lesssim b$ and $b \lesssim a$ hold.*

4 Off-diagonal Gaussian heat kernel behavior

The main purpose of this section is further analysis of the geodesic metrics $\rho_{\mathcal{H}}$ and ρ_h, $h \in \mathcal{S}_{\mathcal{H}_0}$, and as a consequence we will get the two-sided Gaussian bound and Varadhan's asymptotic relation for the heat kernels p_μ and $p_{\mu_{\langle h \rangle}}$.

Let us start this section with the following standard definition.

Definition 4.1. Let ν be a finite Borel measure on K with full support. We define

$$\rho_\nu(x, y) = \sup\{u(x) - u(y) \mid u \in \mathcal{F},\ \mu_{\langle u \rangle} \leq \nu\}, \qquad x, y \in K. \tag{4.1}$$

Clearly, $\rho_\nu(x, y) = \rho_\nu(y, x) \in [0, \infty)$, $\rho_\nu(x, x) = 0$ and $\rho_\nu(x, y) \leq \rho_\nu(x, z) + \rho_\nu(z, y)$ for any $x, y, z \in K$; in fact, $\rho_\nu(x, y)^2 \leq \nu(K) R_{\mathcal{E}}(x, y)/2$. ρ_ν is called the *intrinsic metric of the Dirichlet space* $(K, \nu, \mathcal{E}, \mathcal{F})$ or simply the ν*-intrinsic metric on* K.

The notion of the intrinsic metric of a strong local Dirichlet space appears in many places such as [35,36,32,18]. The results there suggest that the intrinsic metric is the most *"natural"* metric for a given strong local Dirichlet space; for example, according to Ramírez [32] and Hino and Ramírez [18], Varadhan's asymptotic relation like (1.4) is true for a large class of strong local Dirichlet spaces as long as the metric in the right-hand side is replaced by the intrinsic metric.

Then a problem arises as to how the intrinsic metric is characterized for concrete examples. For the canonical Dirichlet space associated with a smooth Riemannian manifold M, it is not difficult to see that the intrinsic metric is equal to the geodesic metric on M; see [31] and references therein for related results on Riemannian manifolds. The same assertion is in fact true also for our Dirichlet spaces $(K, \mu, \mathcal{E}, \mathcal{F})$ and $(K, \mu_{\langle h \rangle}, \mathcal{E}, \mathcal{F})$, $h \in \mathcal{S}_{\mathcal{H}_0}$, which is the main theorem of this section:

Theorem 4.2. (1) *$\rho_{\mathcal{H}} = \rho_\mu$. Moreover, $\rho_{\mathcal{H}}(x, \cdot) \in \mathcal{F}$ and $\mu_{\langle \rho_{\mathcal{H}}(x, \cdot) \rangle} = \mu$ for any $x \in K$.*
(2) *Let $h \in \mathcal{S}_{\mathcal{H}_0}$. Then $\rho_h = \rho_{\mu_{\langle h \rangle}}$. Moreover, $\rho_h(x, \cdot) \in \mathcal{F}$ and $\mu_{\langle \rho_h(x, \cdot) \rangle} = \mu_{\langle h \rangle}$ for any $x \in K$.*

Then based on Theorem 3.19 and Proposition 3.20, the general results of Sturm [35,36] and Ramírez [32] imply the following Gaussian bounds and Varadhan's asymptotic relation.

Corollary 4.3. *Let $h \in \mathcal{S}_{\mathcal{H}_0}$ and let (ν, ρ) denote any one of $(\mu, \rho_{\mathcal{H}})$ and $(\mu_{\langle h \rangle}, \rho_h)$. Let $n \in \mathbb{N}$. Then for any $(t, x, y) \in (0, \infty) \times K \times K$,*

$$c_{\mathrm{L}} \frac{\exp\bigl(-\frac{\rho(x,y)^2}{c_{\mathrm{L}} t}\bigr)}{\nu\bigl(B_{\sqrt{t}}(x, \rho)\bigr)} \leq p_\nu(t, x, y) \leq c_{\mathrm{U}} \frac{\bigl(1 + \frac{\rho(x,y)^2}{t}\bigr)^{\kappa/2} \exp\bigl(-\frac{\rho(x,y)^2}{2t}\bigr)}{\sqrt{\nu\bigl(B_{\sqrt{t}}(x, \rho)\bigr)\nu\bigl(B_{\sqrt{t}}(y, \rho)\bigr)}}, \tag{4.2}$$

$$\bigl|\partial_t^n p_\nu(t, x, y)\bigr| \leq c_{\mathrm{U}}(n) \frac{\bigl(1 + \frac{\rho(x,y)^2}{t}\bigr)^{\kappa/2+n} \exp\bigl(-\frac{\rho(x,y)^2}{2t}\bigr)}{t^n \sqrt{\nu\bigl(B_{\sqrt{t}}(x, \rho)\bigr)\nu\bigl(B_{\sqrt{t}}(y, \rho)\bigr)}}, \tag{4.3}$$

where $c_{\mathrm{L}}, c_{\mathrm{U}} \in (0, \infty)$ are determined solely by $\kappa, c_{\mathrm{G}}, c_{\mathrm{V}}$ and $c_{\mathrm{U}}(n) \in (0, \infty)$ by $n, \kappa, c_{\mathrm{G}}, c_{\mathrm{V}}$.

Proof. Note that $\partial_t^n p_\nu$ exists and is continuous on $(0,\infty)\times K\times K$ by [8, Proof of Theorem 2.1.4]. On the basis of $\rho=\rho_\nu$, (3.16) and (3.18), [36, Corollary 4.10] yields the lower bound in (4.2), and [36, Theorem 2.6] and [35, Corollary 2.7] imply the other assertions. □

Corollary 4.4. *Let $h\in\mathcal{S}_{\mathcal{H}_0}$ and let (ν,ρ) denote any one of $(\mu,\rho_{\mathcal{H}})$ and $(\mu_{\langle h\rangle},\rho_h)$. Then*

$$\lim_{t\downarrow 0} 2t\log p_\nu(t,x,y) = -\rho(x,y)^2, \qquad x,y\in K. \tag{4.4}$$

Proof. (4.2) and (3.16) yield $\limsup_{t\downarrow 0} 2t\log p_\nu(t,x,y)\le -\rho(x,y)^2$. We can also easily show $\liminf_{t\downarrow 0} 2t\log p_\nu(t,x,y)\ge -\rho(x,y)^2$ in exactly the same way as [32, Proof of Theorem 4.1] by using $\rho=\rho_\nu$ and the lower bound in (4.2), since [32, Theorem 1.1] (or [18, Theorem 1.1]) applies to the present situation by the strong locality of $(\mathcal{E},\mathcal{F})$ and [13, Theorem 3.2.2]. □

The rest of this section is devoted to the proof of Theorem 4.2. Unlike the case of Riemannian manifolds, this result is not straightforward and requires a long complicated proof, mainly due to the geometric singularity of the space. The proof relies heavily on Theorem 2.17, Propositions 3.15 and 3.16 and the ideas in [20].

Lemma 4.5. (1) *If $u\in C(K)$ and $\mathbf{Lip}_{\rho_{\mathcal{H}}} u\le 1$ then $u\in\mathcal{F}$ and $\mu_{\langle u\rangle}\le\mu$. Moreover, $\rho_{\mathcal{H}}(x,\cdot)\in\mathcal{F}$ and $\mu_{\langle\rho_{\mathcal{H}}(x,\cdot)\rangle}=\mu$ for any $x\in K$.*
(2) *Let $h\in\mathcal{S}_{\mathcal{H}_0}$. If $u\in C(K)$ and $\mathbf{Lip}_{\rho_h} u\le 1$ then $u\in\mathcal{F}$ and $\mu_{\langle u\rangle}\le\mu_{\langle h\rangle}$. Moreover, $\rho_h(x,\cdot)\in\mathcal{F}$ and $\mu_{\langle\rho_h(x,\cdot)\rangle}=\mu_{\langle h\rangle}$ for any $x\in K$.*

Proof. (1) We fix $x\in K$ throughout this proof. Let $u\in C(K)$ satisfy $\mathbf{Lip}_{\rho_{\mathcal{H}}} u\le 1$. Since $|u(y)-u(z)|\le\rho_{\mathcal{H}}(y,z)\le\ell_{\mathcal{H}}(\overline{yz})\le(4\sqrt{6}/3)\|T_w\|$ for $w\in W_*$ and $y,z\in F_w(V_0)$ with $y\neq z$ by (3.14), from (2.1) we see that for $m\in\mathbb{N}\cup\{0\}$,

$$\mathcal{E}_m(u|_{V_m},u|_{V_m})\le\frac{1}{8}\Big(\frac{5}{3}\Big)^m\sum_{w\in W_m}\sum_{y,z\in F_w(V_0),\,y\neq z}\frac{32}{3}\|T_w\|^2=\sum_{w\in W_m}8\mu(K_w)=16,$$

i.e. $u\in\mathcal{F}$ and $\mathcal{E}(u,u)\le 16$. Recalling Theorem 2.17, let $y\in K_Z\setminus V_*$, $y\neq x$ and suppose that $\widetilde{\nabla}u(y)\in\operatorname{Im}Z_y$ as in (2.11) exists. We show that $|\widetilde{\nabla}u(y)|\le 1$, from which $\mu_{\langle u\rangle}\le\mu$ follows since $d\mu_{\langle u\rangle}=|\widetilde{\nabla}u|^2d\mu$. Let $\omega\in\pi^{-1}(y)$, and set

$$R_y(z):=u(z)-u(y)-\langle\widetilde{\nabla}u(y),\Phi(z)-\Phi(y)\rangle, \qquad z\in K. \tag{4.5}$$

By Proposition 3.15, there exists a harmonic geodesic $\gamma:[0,1]\to K$ such that $\gamma(0)=x$, $\gamma(1)=y$ and $\rho_{\mathcal{H}}(\gamma(s),\gamma(t))=|s-t|\rho_{\mathcal{H}}(x,y)$ for any $s,t\in[0,1]$.

Let $m\in\mathbb{N}$ satisfy $x\notin K_{[\omega]_m}$. Set $a:=\sup\{t\in[0,1]\mid\gamma(t)\notin K_{[\omega]_m}\}$, so that $a\in(0,1)$, $\gamma(a)\in F_{[\omega]_m}(V_0)$ and $\gamma([a,1])\subset K_{[\omega]_m}$. Choose $i\in S$ so that $\gamma(a)=F_{[\omega]_m}(q_i)$, and let $n:=\min\{k\in\mathbb{N}\mid k>m,\ \omega_k\neq i\}-1$, $w:=[\omega]_n$ and $j:=\omega_{n+1}$. Then $n\ge m$, $i\neq j$, $\gamma(a)=F_{wi}(q_i)$ and $y\in K_{wj}\setminus V_*$. Further set $b:=\inf\{t\in[a,1]\mid\gamma(t)\notin K_{wi}\}$, so that $b\in(a,1)$, $\gamma(b)\in F_{wi}(V_0)$ and $\gamma([a,b])\subset K_{wi}$. Now by [25, Lemma 5.6], these facts together with $\rho_{\mathcal{H}}(\gamma(a),\gamma(b))=\ell_{\mathcal{H}}(\gamma|_{[a,b]})$ imply that $\ell_{\mathcal{H}}(\gamma|_{[a,b]})=\ell_{\mathcal{H}}(\widehat{z_az_b})$, where $z_a:=\gamma(a)$, $z_b:=\gamma(b)$ and $\widehat{z_az_b}:[a,b]\to K$ denotes the harmonic $(n+1)$-geodesic determined by the $(n+1)$-walk $\{z_a,z_b\}$. Therefore if we define $\gamma_0:[0,b]\to K$ by $\gamma_0|_{[0,a]}:=\gamma|_{[0,a]}$ and $\gamma_0|_{[a,b]}:=\widehat{z_az_b}$, then it is continuous, $\gamma_0|_{(0,b]}$ is C^1 with $|\gamma_0'(t)|=\rho_{\mathcal{H}}(x,y)$ for $t\in(0,b]$, and $\ell_{\mathcal{H}}(\gamma_0)=\ell_{\mathcal{H}}(\gamma|_{[0,b]})=\rho_{\mathcal{H}}(x,z_b)=\rho_{\mathcal{H}}(x,\gamma_0(b))$. Hence

$$\rho_{\mathcal{H}}(\gamma_0(s),\gamma_0(t))=\ell_{\mathcal{H}}(\gamma_0|_{[s,t]})=(t-s)\rho_{\mathcal{H}}(x,y) \quad \text{for } s,t\in[0,b],\ s\le t. \tag{4.6}$$

Since $(h_y \circ \widehat{z_a z_b})'(t) = 0$ for at most one $t \in [a, b]$ by (3.15), we can take $c, d \in [a, b]$ so that $d - c \geq (b - a)/2$ and $h_y \circ \widehat{z_a z_b}|_{[c,d]}$ is strictly monotone. Then letting $z_c := \widehat{z_a z_b}(c) = \gamma_0(c)$ and $z_d := \widehat{z_a z_b}(d) = \gamma_0(d)$ and using (3.14), (4.6) and (3.4), we have $\ell_{h_y^\perp}(\widehat{z_a z_b}) \leq (4/\sqrt{3}) l_{h_y^\perp}(wi) \leq 3 l_{h_y^\perp}(w)$, $\ell_{h_y}(\widehat{z_a z_b}|_{[c,d]}) = |h_y(z_c) - h_y(z_d)|$,

$$\begin{aligned} \rho_{\mathcal{H}}(z_c, z_d) = \ell_{\mathcal{H}}(\widehat{z_a z_b}|_{[c,d]}) &\leq \ell_{h_y}(\widehat{z_a z_b}|_{[c,d]}) + \ell_{h_y^\perp}(\widehat{z_a z_b}) \\ &\leq |h_y(z_c) - h_y(z_d)| + 3 l_{h_y^\perp}(w), \end{aligned} \tag{4.7}$$

$$\rho_{\mathcal{H}}(z_c, z_d) = (d - c)\rho_{\mathcal{H}}(x, y) \geq \frac{b - a}{2}\rho_{\mathcal{H}}(x, y) = \frac{\ell_{\mathcal{H}}(\widehat{z_a z_b})}{2} \geq \frac{\|T_{wi}\|}{10\sqrt{2}} \geq \frac{\|T_w\|}{100}. \tag{4.8}$$

Now let $c_{u,y} \in \mathbb{R}$ be such that $\widetilde{\nabla} u(y) = c_{u,y}\zeta_y$. Then since $\langle \widetilde{\nabla} u(y), \Phi(\cdot) - \Phi(y)\rangle = c_{u,y} h_y$ by (2.14), (4.7) and (4.5) yield

$$\begin{aligned} &|c_{u,y}|\rho_{\mathcal{H}}(z_c, z_d) \leq |c_{u,y}(h_y(z_c) - h_y(z_d))| + 3|c_{u,y}| l_{h_y^\perp}(w) \\ &\leq \left|\langle \widetilde{\nabla} u(y), \Phi(z_c) - \Phi(z_d)\rangle + R_y(z_c) - R_y(z_d)\right| + 2\sup_{K_w}|R_y| + 3|c_{u,y}| l_{h_y^\perp}(w) \\ &= |u(z_c) - u(z_d)| + 2\sup_{K_w}|R_y| + 3|c_{u,y}| l_{h_y^\perp}(w) \\ &\leq \rho_{\mathcal{H}}(z_c, z_d) + 2\sup_{K_w}|R_y| + 3|c_{u,y}| l_{h_y^\perp}(w). \end{aligned} \tag{4.9}$$

Recalling $w = [\omega]_n$ and $n \geq m$, we divide (4.9) by $\rho_{\mathcal{H}}(z_c, z_d)$ and use (4.8) to get

$$|c_{u,y}| \leq 1 + 100 \cdot \frac{2\sup_{z \in K_{[\omega]_n}} |R_y| + 3|c_{u,y}| \|h_y^\perp \circ F_{[\omega]_n}\|_{\mathcal{E}}}{\|T_{[\omega]_n}\|} \xrightarrow{m \to \infty, n \to \infty} 1$$

by virtue of (2.11) and Lemma 2.19, proving $|\widetilde{\nabla} u(y)| \leq 1$. Finally, noting that $\mathbf{Lip}_{\rho_{\mathcal{H}}} \rho_{\mathcal{H}}^x \leq 1$, where $\rho_{\mathcal{H}}^x := \rho_{\mathcal{H}}(x, \cdot)$, we let $u := \rho_{\mathcal{H}}^x$ in the above argument and use (4.6) to obtain

$$\begin{aligned} \rho_{\mathcal{H}}(z_c, z_d) = \rho_{\mathcal{H}}^x(z_d) - \rho_{\mathcal{H}}^x(z_c) &= \langle \widetilde{\nabla} \rho_{\mathcal{H}}^x(y), \Phi(z_d) - \Phi(z_d)\rangle + R_y(z_c) - R_y(z_c) \\ &= c_{\rho_{\mathcal{H}}^x, y}(h_y(z_d) - h_y(z_c)) + R_y(z_c) - R_y(z_c) \\ &\leq |c_{\rho_{\mathcal{H}}^x, y}|\rho_{\mathcal{H}}(z_c, z_d) + 2\sup_{K_{[\omega]_n}}|R_y|, \end{aligned}$$

from which we conclude that $1 \leq |c_{\rho_{\mathcal{H}}^x, y}| = |\widetilde{\nabla} \rho_{\mathcal{H}}^x(y)| (\leq 1)$ by using (4.8) and (2.11) to let $m \to \infty, n \to \infty$. Thus $1 = |\widetilde{\nabla} \rho_{\mathcal{H}}^x|^2 = d\mu_{\langle \rho_{\mathcal{H}}^x\rangle}/d\mu$ μ-a.e., that is, $\mu_{\langle \rho_{\mathcal{H}}^x\rangle} = \mu$.
(2) This is proved in exactly the same way as above by using Theorem 2.17-(2), (3.5), Proposition 3.16 and Lemma 3.18. □

Lemma 4.6. *$\rho_{\mathcal{H}} \leq \rho_\mu \leq 9\rho_{\mathcal{H}}$, and $\rho_h \leq \rho_{\mu_{\langle h\rangle}} \leq 6\rho_h$ for any $h \in \mathcal{S}_{\mathcal{H}_0}$.*

Proof. Let $x, y \in K$. Since $\rho_{\mathcal{H}}(x, \cdot) \in \mathcal{F}$ and $\mu_{\langle \rho_{\mathcal{H}}(x,\cdot)\rangle} = \mu$ by Lemma 4.5-(1), we have $\rho_{\mathcal{H}}(x, y) = \rho_{\mathcal{H}}(x, y) - \rho_{\mathcal{H}}(x, x) \leq \rho_\mu(x, y)$. Next for the proof of $\rho_\mu \leq 9\rho_{\mathcal{H}}$ let $u \in \mathcal{F}$ satisfy $\mu_{\langle u\rangle} \leq \mu$. It suffices to show that $|u(x) - u(y)| \leq 9\rho_{\mathcal{H}}(x, y)$ when $x, y \in V_m$ for some $m \in \mathbb{N}$ and $x \neq y$ since $u \in C(K)$ and V_* is dense in K. For any $w \in W_*$, from $\mu_{\langle u\rangle}(K_w) \leq \mu(K_w)$ we easily see $\|u \circ F_w\|_{\mathcal{E}} \leq \|T_w\|$ and therefore

$$\|T_w\| \geq \|u \circ F_w\|_{\mathcal{E}} \geq \sqrt{2\mathcal{E}_0(u \circ F_w|_{V_0}, u \circ F_w|_{V_0})} \geq \frac{\sqrt{3}}{2} \operatorname*{Osc}_{F_w(V_0)} u. \tag{4.10}$$

By Proposition 3.15, there exists an m-walk $\{x_k\}_{k=0}^N \subset V_m$ such that $x_0 = x, x_N = y$ and $\rho_{\mathcal{H}}(x,y) = \sum_{k=1}^N \ell_{\mathcal{H}}(\overline{x_{k-1}x_k})$. Then (3.14) and (4.10) yield (recall Definition 3.13-(1))

$$\sum_{k=1}^N \ell_{\mathcal{H}}(\overline{x_{k-1}x_k}) \geq \sum_{k=1}^N \frac{\|T_{w(x_{k-1},x_k)}\|}{5\sqrt{2}} \geq \sum_{k=1}^N \frac{|u(x_{k-1}) - u(x_k)|}{9} \geq \frac{|u(x) - u(y)|}{9}$$

and hence $|u(x) - u(y)| \leq 9\rho_{\mathcal{H}}(x,y)$. Exactly the same argument using Lemma 4.5-(2), Proposition 3.16 and (3.14) shows the other assertion, completing the proof. □

We need the following two lemmas for the next proposition (Proposition 4.9). The first lemma is elementary and easily follows from [26, Theorems 10.3 and 10.4], whereas *the latter plays a central role in the proof of Proposition* 4.9.

Lemma 4.7. *Let ν be a finite Borel measure on K with full support, let U be a non-empty open subset of K and set $\nu|_U := \nu|_{\mathcal{B}(U)}$ and $\mathcal{E}^U := \mathcal{E}|_{\mathcal{F}_U \times \mathcal{F}_U}$. Then $(\mathcal{E}^U, \mathcal{F}_U)$ is a strong local regular Dirichlet form on $L^2(U, \nu|_U)$ whose associated Markovian semigroup admits a unique continuous integral kernel $p_\nu^U = p_\nu^U(t,x,y) : (0,\infty) \times U \times U \to [0,\infty)$, and p_ν^U is extended to a continuous function on $(0,\infty) \times K \times K$ by setting $p_\nu^U := 0$ on $(0,\infty) \times (K \times K \setminus U \times U)$. p_ν^U is called the* heat kernel associated with $(U, \nu|_U, \mathcal{E}^U, \mathcal{F}_U)$.

Lemma 4.8. $\limsup_{t\downarrow 0} 2t \log p_{\mu_{\langle h\rangle}}(t,x,y) \leq -\rho_{\mu_{\langle h\rangle}}(x,y)^2$ *for any $x, y \in K$, $h \in \mathcal{S}_{\mathcal{H}_0}$.*

Proof. Let $h \in \mathcal{S}_{\mathcal{H}_0}$. By Lemma 4.6 and (3.12), $\rho_{\mu_{\langle h\rangle}}$ is a metric on K adapted to $\mathcal{S}^h$. Then $(\mu_{\langle h\rangle}, \rho_{\mu_{\langle h\rangle}})$ has the volume doubling property similar to (3.16). Moreover, for $(\nu,\rho) = (\mu_{\langle h\rangle}, \rho_{\mu_{\langle h\rangle}})$, the proof of Proposition 3.20 still works and hence (3.18) holds with the constants $3^4 10^6$ and $250\sqrt{2}$ suitably replaced. Now the assertion follows from [36, Theorem 2.6] and [35, Corollary 2.7]. □

Proposition 4.9. *Let $h \in \mathcal{S}_{\mathcal{H}_0}$, $i \in S$, $b \in (h(q_i),\infty)$ and set $a := h(q_i)$. Suppose that the connected component U of $h^{-1}((-\infty,b))$ with $q_i \in U$ satisfies $U \cap V_0 = \{q_i\}$. Let $p_{[a,b)} = p_{[a,b)}(t,x,y) : (0,\infty) \times [a,b] \times [a,b] \to [0,\infty)$ be the heat kernel for $\frac{1}{2}\frac{d^2}{dx^2}$ on $[a,b]$ with Neumann (reflecting) boundary condition at a and Dirichlet (absorbing) boundary condition at b. Then*

$$\mu_{\langle h\rangle} \circ (h|_U)^{-1} = 2\mathcal{E}(h, h_1^i)\mathbf{1}_{[a,b]}dx \qquad (dx \text{ is the Lebesgue measure on } \mathbb{R}), \tag{4.11}$$

$$p_{\mu_{\langle h\rangle}}^U(t,q_i,x) = (2\mathcal{E}(h,h_1^i))^{-1} p_{[a,b)}(t,a,h(x)), \qquad (t,x) \in (0,\infty) \times \overline{U}, \tag{4.12}$$

$$\rho_h(q_i,x) = \rho_{\mu_{\langle h\rangle}}(q_i,x) = h(x) - a, \qquad x \in \overline{U}. \tag{4.13}$$

Proof. Let $h_b := h\mathbf{1}_U + b\mathbf{1}_{K\setminus U}$. We show $h_b \in \mathcal{H}_{\{q_i\}\cup(K\setminus U)}$. Note that, by [13, Problem 1.4.1] and the locality of $(\mathcal{E},\mathcal{F})$, given open subsets U_1, U_2 of K with $U_1 \cap U_2 = \emptyset$ we can verify $\mathcal{F}_{U_1\cup U_2} = \mathcal{F}_{U_1} \oplus \mathcal{F}_{U_2}$ and $\mathcal{E}(u_1,u_2) = 0$ for $u_i \in \mathcal{F}_{U_i}$, $i = 1,2$. Set $\widehat{U} := h^{-1}((-\infty,b))\setminus U$. Since $U, \widehat{U}$ are open in K and $(b\mathbf{1}-h)^+ \in \mathcal{F}_{U\cup\widehat{U}}$, $(b\mathbf{1}-h)^+\mathbf{1}_U \in \mathcal{F}_U$ and $h_b = b\mathbf{1}-(b\mathbf{1}-h)^+\mathbf{1}_U \in \mathcal{F}$. By $\partial U \subset \overline{U}\setminus(U\cup\widehat{U}) \subset h^{-1}(b)$, $h = b$ on ∂U, $h_b - h = (b\mathbf{1}-h)\mathbf{1}_{K\setminus U} \in \mathcal{F}_{K\setminus\overline{U}}$ and therefore $\mathcal{E}(h_b,u) = \mathcal{E}(h_b - h, u) = \mathcal{E}((b\mathbf{1}-h)\mathbf{1}_{K\setminus U}, u) = 0$ for $u \in \mathcal{F}_{U\setminus\{q_i\}}$, proving the claim.

Proposition 2.10-(2) yields $a \leq h_b \leq b$. Moreover, we have $h_b^{-1}(a) = \{q_i\}$. Indeed, choose $n \in \mathbb{N}$ so that $K_{i^{n-1}} \subset U$. Then $h_b \circ F_{i^{n-1}} = h \circ F_{i^{n-1}} \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$ by $h \in \mathcal{H}_0 \setminus \mathbb{R}\mathbf{1}$ and hence $h_b > a$ on $K_{i^n} \setminus \{q_i\}$ by the strong maximum principle [23, Theorem 3.2.14]. Set $c := \min_{F_{i^n}(V_0)\setminus\{q_i\}} h_b$ and $g := h_b\mathbf{1}_{K\setminus K_{i^n}} + (h_b \vee c)\mathbf{1}_{K_{i^n}}$. Then $g \in \mathcal{H}_{K_{i^n}\cup(K\setminus U)}$, and Proposition 2.10-(2) implies that $h_b(x) = g(x) \geq c > a$ for $x \in K \setminus K_{i^n}$. Thus $h_b^{-1}(a) = \{q_i\}$.

By [11, Proposition 2.9] (see also [20, Corollary 2.11]), $\mu_{\langle h_b \rangle} \circ h_b^{-1} = \delta \mathbf{1}_{[a,b]} dx$ for some $\delta \in (0,\infty)$, and $\mu_{\langle h_b \rangle}(K \setminus U) = \mu_{\langle h_b \rangle}(h_b^{-1}(b)) = 0$. Since $\mu_{\langle h \rangle}|_U = \mu_{\langle h_b \rangle}|_U$ by [13, Corollary 3.2.1] (or by Theorem 2.17) and $\mu_{\langle h_b \rangle}(K \setminus U) = 0$, we have $\mu_{\langle h \rangle} \circ (h|_U)^{-1} = \mu_{\langle h_b \rangle} \circ (h_b|_U)^{-1} = \mu_{\langle h_b \rangle} \circ h_b^{-1} = \delta \mathbf{1}_{[a,b]} dx$. Take $a_h, b_h \in \mathbb{R}$ such that $h = a_h h_1^i + b_h h_2^i + a\mathbf{1}$. Let $n \in \mathbb{N}$ satisfy $K_{i^{n-1}} \subset U$. Then $2\mathcal{E}(h, h_1^i) = a_h > 0$ since $h > a$ on $K_{i^n} \setminus \{q_i\}$, and the argument in the previous paragraph together with Proposition 2.10-(2) also yields

$$(h|_U)^{-1}\bigl([a, a + (\tfrac{3}{5})^n a_h - (\tfrac{1}{5})^n \tfrac{|b_h|}{\sqrt{3}}]\bigr) \subset K_{i^n} \subset (h|_U)^{-1}\bigl([a, a + (\tfrac{3}{5})^n a_h + (\tfrac{1}{5})^n \tfrac{|b_h|}{\sqrt{3}}]\bigr). \tag{4.14}$$

Taking the values of $\mu_{\langle h \rangle}$ on each side of (4.14) yields $|a_h^2 + 9^{-n} b_h^2 - \delta a_h| \leq 3^{-n} \delta |b_h| / \sqrt{3}$, and letting $n \to \infty$ results in $\delta a_h = a_h^2$. Thus $\delta = a_h = 2\mathcal{E}(h, h_1^i)$, proving (4.11).

We could give a probabilistic proof of (4.12) based on [20, Theorem 3.6], as in [20, Proof of Theorem 4.1], but we provide an alternative analytic proof here. For $n \in \mathbb{N}$ let $\varphi_n(x) := \bigl(\frac{2}{b-a}\bigr)^{1/2} \cos\bigl(\frac{2n-1}{2}\pi \frac{x-a}{b-a}\bigr)$ and $\lambda_n := \frac{\pi^2}{8}\bigl(\frac{2n-1}{b-a}\bigr)^2$, so that $-\frac{1}{2}\varphi_n'' = \lambda_n \varphi_n$, $\varphi_n'(a) = \varphi_n(b) = 0$ and therefore $\int_a^b p_{[a,b)}(t, \cdot, y) \varphi_n(y) dy = e^{-\lambda_n t} \varphi_n$ for $t \in (0,\infty)$. Then $\{\varphi_n\}_{n \in \mathbb{N}}$ is a complete orthonormal system of $L^2([a,b], dx)$. On the other hand, let $\Delta_{h,U}$ be the non-positive self-adjoint operator of the Dirichlet space $(U, \mu_{\langle h \rangle}|_U, \mathcal{E}^U, \mathcal{F}_U)$ with domain $\mathcal{D}[\Delta_{h,U}]$. Then $\varphi_n(h_b) \in \mathcal{D}[\Delta_{h,U}]$ and $\Delta_{h,U}[\varphi_n(h_b)] = \frac{1}{2}\varphi_n''(h_b) = -\lambda_n \varphi_n(h_b)$ by [20, Theorem 2.12-(2)] and hence

$$\int_U p^U_{\mu_{\langle h \rangle}}(t, \cdot, y) \varphi_n(h_b) d\mu_{\langle h \rangle}(y) = e^{-\lambda_n t} \varphi_n(h_b), \qquad t \in (0,\infty). \tag{4.15}$$

Let $f \in L^2([a,b], dx)$ and $a_n := \int_a^b f \varphi_n dx$, $n \in \mathbb{N}$. Then $f = \sum_{n \in \mathbb{N}} a_n \varphi_n$ in $L^2([a,b], dx)$ and hence $f(h_b)\mathbf{1}_U \in L^2(U, \mu_{\langle h \rangle}|_U)$ and $f(h_b)\mathbf{1}_U = \sum_{n \in \mathbb{N}} a_n \varphi_n(h_b)$ in $L^2(U, \mu_{\langle h \rangle}|_U)$ by (4.11). Therefore for $(t,x) \in (0,\infty) \times K$, (4.15) yields

$$\begin{aligned}\int_U p^U_{\mu_{\langle h \rangle}}(t, y, x) f(h_b(y)) d\mu_{\langle h \rangle}(y) &= \sum_{n \in \mathbb{N}} a_n e^{-\lambda_n t} \varphi_n(h_b(x)) \\ &= \int_a^b p_{[a,b)}(t, y, h_b(x)) f(y) dy. \end{aligned} \tag{4.16}$$

Now (4.12) follows by letting $s \in (a,b)$, $f := (s-a)^{-1}\mathbf{1}_{[a,s]}$ in (4.16) and $s \downarrow a$ since $h_b^{-1}(a) = \{q_i\}$ and $\mu_{\langle h \rangle}\bigl(h_b^{-1}([a,s])\bigr) = 2\mathcal{E}(h, h_1^i)(s-a)$. Finally, since $p^U_{\mu_{\langle h \rangle}} \leq p_{\mu_{\langle h \rangle}}$ by [24, (C.2)], we see from Lemmas 4.6 and 4.8 and a direct calculation using [21, Proposition 2.8.10] that for $x \in U$,

$$\begin{aligned}(h(x)-a)^2 &= -\lim_{t \downarrow 0} 2t \log p_{[a,b)}(t, a, h(x)) = -\lim_{t \downarrow 0} 2t \log p^U_{\mu_{\langle h \rangle}}(t, q_i, x) \\ &\geq -\limsup_{t \downarrow 0} 2t \log p_{\mu_{\langle h \rangle}}(t, q_i, x) \geq \rho_{\mu_{\langle h \rangle}}(q_i, x)^2 \geq \rho_h(q_i, x)^2 \geq (h(x)-a)^2,\end{aligned}$$

proving (4.13) for $x \in U$, and hence also for $x \in \overline{U}$. □

Proposition 4.10. (1) $\{u \in \mathcal{F} \mid \mu_{\langle u \rangle} \leq \mu\} = \{u \in C(K) \mid \mathbf{Lip}_{\rho_{\mathcal{H}}} u \leq 1\}$.
(2) *Let* $h \in \mathcal{S}_{\mathcal{H}_0}$. *Then* $\{u \in \mathcal{F} \mid \mu_{\langle u \rangle} \leq \mu_{\langle h \rangle}\} = \{u \in C(K) \mid \mathbf{Lip}_{\rho_h} u \leq 1\}$.

Proof. (1) Let $u \in \mathcal{F}$ satisfy $\mu_{\langle u \rangle} \leq \mu$, let $l \in \mathbb{N}$ and $x, y \in V_l$, $x \neq y$. It suffices to show $|u(x) - u(y)| \leq \rho_{\mathcal{H}}(x, y)$, since V_* is dense in K and we already have Lemma 4.5. We follow [7, Proof of Proposition 1.11]. Note that $\mathbf{Lip}_{\rho_{\mathcal{H}}} u \leq 9 < \infty$ by Lemma 4.6. By Proposition 3.15, we can choose a harmonic l-geodesic $\gamma : [0, 1] \to K$ arising from an l-walk $\Gamma = \{z_k\}_{k=0}^N$ so that $\gamma(0) = x$, $\gamma(1) = y$ and $\rho_{\mathcal{H}}(\gamma(s), \gamma(t)) = |s - t| \rho_{\mathcal{H}}(x, y)$ for any $s, t \in [0, 1]$. Set $\psi := u \circ \gamma$. Then we have $|\psi(s) - \psi(t)| \leq (\mathbf{Lip}_{\rho_{\mathcal{H}}} u)|s - t| \rho_{\mathcal{H}}(x, y)$ for $s, t \in [0, 1]$ and hence ψ is absolutely continuous. In particular, $\psi'(t)$ exists for dt-a.e. $t \in [0, 1]$, $\psi' \in L^1([0, 1], dt)$ and $\psi(t) = \int_0^t \psi'(s) ds$, $t \in [0, 1]$. Thus it suffices to prove that $|\psi'(t)| \leq \rho_{\mathcal{H}}(x, y)$ for dt-a.e. $t \in [0, 1]$.

Let $t \in [0, 1]$ and suppose $\psi'(t)$ exists. We may assume that $\gamma(t) \notin V_*$ since $\gamma^{-1}(V_*)$ is countable. Let $z := \gamma(t)$ and $\omega \in \pi^{-1}(z)$. Choose $k \in \{1, \ldots, N\}$ and $i, j \in S$ so that $z \in \overline{z_{k-1} z_k}$, $z_{k-1} = F_w(q_i)$ and $z_k = F_w(q_j)$, where $w := w(z_{k-1}, z_k)$. For $m \geq |w|$ we set

$$u_m := \frac{u \circ F_{[\omega]_m}}{\| T_{[\omega]_m} \|}, \qquad h_m := \frac{h_z \circ F_{[\omega]_m}}{\| T_{[\omega]_m} \|}, \qquad h_m^{\perp} := \frac{h_z^{\perp} \circ F_{[\omega]_m}}{\| T_{[\omega]_m} \|}. \tag{4.17}$$

Then $\| u_m \|_{\mathcal{E}} \leq 1$ by $\mu_{\langle u \rangle} \leq \mu$, and Lemma 2.19 yields $1 \geq \| h_m \|_{\mathcal{E}} \to 1$ and $\| h_m^{\perp} \|_{\mathcal{E}} \to 0$ as $m \to \infty$ since $z \in \overline{z_{k-1} z_k} \subset K_Z$. Choosing subsequences $\{u_{m_n}\}_{n \in \mathbb{N}}$ and $\{h_{m_n}\}_{n \in \mathbb{N}}$, we have $u_{m_n} \to v$ weakly in $(\mathcal{F}/\mathbb{R}\mathbf{1}, \mathcal{E})$ and $\| h_{m_n} - g \|_{\mathcal{E}} \to 0$ as $n \to \infty$ for some $v \in \mathcal{F}$ and $g \in \mathcal{S}_{\mathcal{H}_0}$ with $v(z) = g(z) = 0$. We further define

$$v_n := u_{m_n} - u_{m_n}(z)\mathbf{1}, \quad g_n := h_{m_n} - h_{m_n}(z)\mathbf{1}, \quad g_n^{\perp} := h_{m_n}^{\perp} - h_{m_n}^{\perp}(z)\mathbf{1}. \tag{4.18}$$

We have $\lim_{n \to \infty} \| g_n - g \|_{\infty} = 0$ and $v_n(p) = v_n(p) - v_n(z) \to v(p) - v(z) = v(p)$ as $n \to \infty$ for any $p \in K$ since $\mathcal{F}/\mathbb{R}\mathbf{1} \ni f \mapsto f - f(z)\mathbf{1} \in C(K)$ is a well-defined bounded linear operator $(\mathcal{F}/\mathbb{R}\mathbf{1}, \mathcal{E}) \to (C(K), \| \cdot \|_{\infty})$ by Theorem 2.5.

We claim that $\mu_{\langle v \rangle} \leq \mu_{\langle g \rangle}$. Let $\tau \in W_*$. Since $F_\tau^* : f \mapsto f \circ F_\tau$ is a bounded linear operator on $(\mathcal{F}/\mathbb{R}\mathbf{1}, \mathcal{E})$ by (2.2), we have $\| g_n \circ F_\tau - g \circ F_\tau \|_{\mathcal{E}} \vee \| g_n^{\perp} \circ F_\tau \|_{\mathcal{E}} \to 0$ and $v_n \circ F_\tau \to v \circ F_\tau$ weakly in $(\mathcal{F}/\mathbb{R}\mathbf{1}, \mathcal{E})$ as $n \to \infty$. By $\mu_{\langle u \rangle} \leq \mu = \mu_{\langle h_z \rangle} + \mu_{\langle h_z^{\perp} \rangle}$ we see that $\| v_n \circ F_\tau \|_{\mathcal{E}}^2 \leq \| g_n \circ F_\tau \|_{\mathcal{E}}^2 + \| g_n^{\perp} \circ F_\tau \|_{\mathcal{E}}^2$, and letting $n \to \infty$ results in $\| v \circ F_\tau \|_{\mathcal{E}} \leq \liminf_{n \to \infty} \| v_n \circ F_\tau \|_{\mathcal{E}} \leq \| g \circ F_\tau \|_{\mathcal{E}}$, i.e. $\mu_{\langle v \rangle}(K_\tau) \leq \mu_{\langle g \rangle}(K_\tau)$. Thus the claim follows.

Note that either $g \notin \mathbb{R}h_2^i + \mathbb{R}\mathbf{1}$ or $g \notin \mathbb{R}h_2^j + \mathbb{R}\mathbf{1}$. Suppose $g \notin \mathbb{R}h_2^i + \mathbb{R}\mathbf{1}$; the proof for the other case is similar. Take $\zeta_g = (\zeta_g^1, \zeta_g^2) \in \mathbb{R}^2$ so that $g - \zeta_g^1 h_1^i - \zeta_g^2 h_2^i \in \mathbb{R}\mathbf{1}$. Then $\zeta_g^1 \neq 0$, and since $h_1^i \circ F_{i^M} = (3/5)^M h_1^i$ and $h_2^i \circ F_{i^M} = (1/5)^M h_2^i$ we can choose $M \in \mathbb{N}$ so that $\varepsilon g(q_i) < \min_{p \in V_0 \setminus \{q_i\}} \varepsilon g \circ F_{i^M}(p) =: b$, where $\varepsilon := \zeta_g^1 / |\zeta_g^1|$. Let U be the connected component of $(\varepsilon g)^{-1}((-\infty, b))$ with $q_i \in U$, and choose $q \in \overline{q_i q_j} \cap U \setminus \{q_i\}$. The definition of b implies $U \subset K_{i^M}$ and hence Proposition 4.9 together with $\mu_{\langle v \rangle} \leq \mu_{\langle g \rangle}$ shows that

$$|v(q_i) - v(q)| \leq \rho_{\mu_{\langle g \rangle}}(q_i, q) = |g(q_i) - g(q)| \neq 0. \tag{4.19}$$

Now noting that γ is injective and that $F_{[\omega]_{m_n}}(q_i), F_{[\omega]_{m_n}}(q) \in \overline{z_{k-1} z_k}$, we set $s_n := \gamma^{-1}\big(F_{[\omega]_{m_n}}(q_i)\big)$ and $t_n := \gamma^{-1}\big(F_{[\omega]_{m_n}}(q)\big)$ for $n \in \mathbb{N}$. Then $\lim_{n \to \infty} s_n = \lim_{n \to \infty} t_n = \gamma^{-1}(z) = t$, and Lemma 3.7 and (3.11) imply

$$(|s_n - t| \vee |t_n - t|) \rho_{\mathcal{H}}(x, y) = \rho_{\mathcal{H}}(\gamma(s_n), z) \vee \rho_{\mathcal{H}}(\gamma(t_n), z) \leq 10 \big\| T_{[\omega]_{m_n}} \big\|. \tag{4.20}$$

Let $a := g(q_i) - g(q)$. By $\lim_{n\to\infty}(g_n(q_i) - g_n(q)) = a \neq 0$ and (4.20), for sufficiently large $n \in \mathbb{N}$ we have $|g_n(q_i) - g_n(q)| \geq |a|/2$ and

$$|s_n - t_n| = \frac{\rho_{\mathcal{H}}(\gamma(s_n), \gamma(t_n))}{\rho_{\mathcal{H}}(x, y)} \geq \frac{\|T_{[\omega]_{m_n}}\| |g_n(q_i) - g_n(q)|}{\rho_{\mathcal{H}}(x, y)} \geq \frac{|a|}{20}(|s_n - t| \vee |t_n - t|), \tag{4.21}$$

from which $\lim_{n\to\infty} \frac{\psi(s_n)-\psi(t_n)}{s_n-t_n} = \psi'(t)$ easily follows. Then the first inequality in (4.21) and (4.19) together imply

$$\frac{|\psi'(t)|}{\rho_{\mathcal{H}}(x, y)} = \lim_{n\to\infty} \frac{|\psi(s_n) - \psi(t_n)|}{|s_n - t_n|\rho_{\mathcal{H}}(x, y)} \leq \lim_{n\to\infty} \left| \frac{v_n(q_i) - v_n(q)}{g_n(q_i) - g_n(q)} \right| = \left| \frac{v(q_i) - v(q)}{g(q_i) - g(q)} \right| \leq 1,$$

proving $|u(x) - u(y)| = |\psi(0) - \psi(1)| \leq \rho_{\mathcal{H}}(x, y)$ and $\mathbf{Lip}_{\rho_{\mathcal{H}}} u \leq 1$.
(2) Let $l \in \mathbb{N}$ and $x, y \in V_l$, $x \neq y$ and set $\gamma := \gamma^h_{xy} \circ \varphi^h_{xy}$, where γ^h_{xy} and φ^h_{xy} are as in Proposition 3.16 with γ^h_{xy} a harmonic l-geodesic. Then exactly the same proof as that of (1) still works with $u_m := \|h \circ F_{[\omega]_m}\|_{\mathcal{E}}^{-1} u \circ F_{[\omega]_m}$ and $h_m := \|h \circ F_{[\omega]_m}\|_{\mathcal{E}}^{-1} h \circ F_{[\omega]_m}$. □

Proof of Theorem 4.2. Let $h \in \mathcal{S}_{\mathcal{H}_0}$. By Lemmas 4.5 and 4.6, it only remains to show that $\rho_\mu \leq \rho_{\mathcal{H}}$ and $\rho_{\mu_{\langle h \rangle}} \leq \rho_h$, which are immediate from Proposition 4.10. □

5 One-dimensional asymptotics at vertices

In this section, we prove sharp *"one-dimensional"* asymptotic behaviors of $\mu(B_r(x, \rho_{\mathcal{H}}))$ and $p_\mu(t, x, y)$ for $x \in V_*$, which reflect our observation that, near $\Phi(x)$, the harmonic Sierpinski gasket $K_{\mathcal{H}}$ (Figure 2) looks very much like its *"tangent line at* $\Phi(x)$*"*. We treat the results for $\mu(B_r(x, \rho_{\mathcal{H}}))$ and $p_\mu(t, x, y)$ respectively in Subsections 5.1 and 5.2. Then Subsection 5.3 presents an application of the result for p_μ to moments of displacement of the corresponding diffusion.

The following definition is fundamental for the arguments in this section.

Definition 5.1. For each $x \in V_*$, we define $\xi_x, c_x, r_x \in (0, \infty)$ and $K^x \subset K$ as follows:
(i) If $x = q_i \in V_0$, $i \in S$, then we set $\xi_{q_i} := 1/2$, $c_{q_i} := 1$, $r_{q_i} := 1$ and $K^{q_i} := K_i$.
(ii) If $x \in V_* \setminus V_0$, let $w \in W_*$ and $i, j \in S$, $i \neq j$ be such that $\pi^{-1}(x) = \{wij^\infty, wji^\infty\}$ (recall Proposition 2.3) and $a^i_x, b^i_x, a^j_x, b^j_x \in \mathbb{R}$ such that $h_x \circ F_{wi} = a^i_x h^j_1 + b^i_x h^j_2$ and $h_x \circ F_{wj} = a^j_x h^i_1 + b^j_x h^i_2$ (recall $h_x(x) = 0$). Noting that $a^j_x = -a^i_x$ by the hamonicity of h_x at x (see [23, (3.2.1)]) and that $a^i_x \neq 0$ by Lemma 2.19 and $\inf_{n\in\mathbb{N}}(5/3)^n \|T_{wij^n}\| > 0$, we define

$$\begin{aligned} \xi_x &:= \left(\frac{5}{3}\right)^{|w|+1} |a^i_x|, & c_x &:= \frac{\mu_{\langle h^\perp_x \rangle}(K_{wi} \cup K_{wj})}{|a^i_x|^{\hat{\kappa}}}, \\ r_x &:= \frac{4}{3}\left(\frac{3}{5}\right)^{N_x} |a^i_x|, & K^x &:= K_{wij^{N_x}} \cup K_{wji^{N_x}}, \end{aligned} \tag{5.1}$$

where $N_x := 1 + \min\{n \in \mathbb{N} \cup \{0\} \mid (\sqrt{3}/6)3^n |a^i_x| \geq |b^i_x| \vee |b^j_x|\}$.

Remark 5.2. *We can write down* ξ_x, c_x, r_x *explicitly in terms of* T_w *in the situation of Definition* 5.1-(ii), since h_x and $h^\perp_x$ are given by (2.14) and

$$\zeta_x = \varepsilon_i |T_{wi}\zeta_{q_j}|^{-1} T_{wi}\zeta_{q_j} = \varepsilon_j |T_{wj}\zeta_{q_i}|^{-1} T_{wj}\zeta_{q_i} \tag{5.2}$$

for some $\varepsilon_i, \varepsilon_j \in \{-1, 1\}$ by Proposition 3.14.

5.1 Measures of geodesic balls

The following is the main theorem of this subsection.

Theorem 5.3. *Let $x \in V_*$ and $s \in (0, r_x]$. Then*

$$\lim_{r\downarrow 0} \frac{\mu(B_r(x,\rho_{\mathcal{H}}))}{r} = \frac{\mu_{\langle h_x\rangle}(B_s(x,\rho_{h_x}))}{s} = 2\xi_x. \tag{5.3}$$

The rest of this subsection is devoted to the proof of Theorem 5.3. We need the following proposition and lemmas, which will play essential roles also in Subsections 5.2 and 5.3 below.

Proposition 5.4 (cf. Proposition 4.9)**.** *Let $x \in V_* \setminus V_0$, and let U^x be the connected component of $h_x^{-1}((-r_x, r_x))$ containing x. Let $w \in W_*$, $i, j \in S$, $a_x^i, a_x^j \in \mathbb{R}$ and $N_x \in \mathbb{N}$ be as in Definition* 5.1*, and without loss of generality assume $a_x^j > 0$.*
(1) *$h_x < 0$ on $K_{wij^{N_x-1}} \setminus \{x\}$ and $h_x > 0$ on $K_{wji^{N_x-1}} \setminus \{x\}$. Moreover, $K^x \subset U^x \subset K_{wij^{N_x-1}} \cup K_{wji^{N_x-1}}$.*
(2) *For $b \in (0,\infty)$ let $p_b = p_b(t,y,z) : (0,\infty) \times [-b,b] \times [-b,b] \to [0,\infty)$ denote the heat kernel for $\frac{1}{2}\frac{d^2}{dy^2}$ on $[-b,b]$ with Dirichlet (absorbing) boundary condition at $-b$ and b. Then*

$$\mu_{\langle h_x\rangle} \circ (h_x|_{U^x})^{-1} = \xi_x \mathbf{1}_{[-r_x,r_x]}dy \qquad (dy \text{ is the Lebesgue measure on } \mathbb{R}), \tag{5.4}$$

$$p^{U^x}_{\mu_{\langle h_x\rangle}}(t,x,y) = \xi_x^{-1} p_{r_x}(t,0,h_x(y)), \qquad (t,y) \in (0,\infty) \times \overline{U^x}, \tag{5.5}$$

$$\rho_{h_x}(x,y) = |h_x(y)|, \qquad y \in \overline{U^x}. \tag{5.6}$$

(3) *$B_r(x,\rho_{h_x}) = U^x \cap h_x^{-1}((-r,r))$ and $\mu_{\langle h_x\rangle}(B_r(x,\rho_{h_x})) = 2\xi_x r$ for any $r \in (0, r_x]$.*
(4) *$B_{2r_{x,n}/3}(x,\rho_{h_x}) \subset K_{wij^n} \cup K_{wji^n} \subset B_{5r_{x,n}/6}(x,\rho_{h_x})$ for $n \in \mathbb{N}$, $n \geq N_x$, where $r_{x,n} := \frac{4}{3}\bigl(\frac{3}{5}\bigr)^n |a_x^i|$.*

Proof. (1) In view of the definition of N_x, a direct calculation together with the strong maximum principle [23, Theorem 3.2.14] easily shows the assertions.
(2) (5.4) and (5.6) follow by applying Proposition 4.9 with $h = \|h_x \circ F_v\|_{\mathcal{E}}^{-1} h_x \circ F_v$, $b = r_x/\|h_x \circ F_v\|$, $a = 0$ and $U = F_v^{-1}(U^x \cap K_v)$, where $v := wji^{N_x-1}$, and similarly on $K_{wij^{N_x-1}}$. Also the same proof as that of (4.12) shows that for any $f \in L^2([-r_x,r_x],dy)$ and any $(t,y) \in (0,\infty) \times \overline{U^x}$,

$$\int_{U^x} p^{U^x}_{\mu_{\langle h_x\rangle}}(t,z,y) f(h_x(z)) d\mu_{\langle h\rangle}(z) = \int_{-r_x}^{r_x} p_{r_x}(t,z,h_x(y)) f(z) dz, \tag{5.7}$$

from which (5.5) easily follows by virtue of $h_x^{-1}(0) \cap \overline{U^x} = \{x\}$ and (5.4).
(3) This is immediate from (5.4), (5.6) and the fact that $|h_x| = r_x$ on ∂U^x.
(4) Similarly to (4.14), using the definition of N_x and Proposition 2.10-(2) we have

$$(h_x|_{U^x})^{-1}\bigl((-\tfrac{2}{3}r_{x,n}, \tfrac{2}{3}r_{x,n})\bigr) \subset K_{wij^n} \cup K_{wji^n} \subset (h_x|_{U^x})^{-1}\bigl((-\tfrac{5}{6}r_{x,n}, \tfrac{5}{6}r_{x,n})\bigr)$$

for $n \geq N_x$, which and the first assertion of (3) immediately yield (4). □

Lemma 5.5. $\frac{1}{225} c_x r^{\hat{k}} \leq \mu_{\langle h_x^{\perp}\rangle}(B_r(x,\rho_{h_x})) \leq 225 c_x r^{\hat{k}}$ *for $x \in V_*$ and any $r \in (0, r_x]$.*

Proof. Suppose $x \in V_* \setminus V_0$. Let $w \in W_*$, $i, j \in S$, $a_x^i, a_x^j \in \mathbb{R}$ and $N_x \in \mathbb{N}$ be as in Definition 5.1 and set $a := |a_x^i| = |a_x^j|$. Since $\|h_x^\perp \circ F_{wij^n}\|_\mathcal{E} \vee \|h_x^\perp \circ F_{wji^n}\|_\mathcal{E} = o((3/5)^n)$ as $n \to \infty$ by Lemma 2.19, $h_x^\perp \circ F_{wi} \in \mathbb{R}h_2^j$, $h_x^\perp \circ F_{wj} \in \mathbb{R}h_2^i$ and hence $\mu_{\langle h_x^\perp \rangle}(K_n^x) = (1/15)^n a^{\hat{\kappa}} c_x$ for $n \in \mathbb{N} \cup \{0\}$, where $K_n^x := K_{wij^n} \cup K_{wji^n}$. For $n \geq N_x$, $K_n^x \subset B_{r_{x,n}}(x, \rho_{h_x}) \subset K_{n-1}^x$ by Proposition 5.4-(1), (4) and hence $15^{-1}(r_{x,n})^{\hat{\kappa}} c_x \leq \mu_{\langle h_x^\perp \rangle}(B_{r_{x,n}}(x, \rho_{h_x})) \leq 15(r_{x,n})^{\hat{\kappa}} c_x$. Now for each $r \in (0, r_x]$, $r_{x,n+1} < r \leq r_{x,n}$ for a unique $n \geq N_x$, and then $\mu_{\langle h_x^\perp \rangle}(B_r(x, \rho_{h_x})) \leq 15((5/3)r_{x,n+1})^{\hat{\kappa}} c_x \leq 15^2 c_x r^{\hat{\kappa}}$ and $\mu_{\langle h_x^\perp \rangle}(B_r(x, \rho_{h_x})) \geq 15^{-1}((3/5)r_{x,n})^{\hat{\kappa}} c_x \geq 15^{-2} c_x r^{\hat{\kappa}}$.

The assertion for $x \in V_0$ is proved in the same way by using Proposition 4.9. □

Lemma 5.6. *Let $x \in V_* \setminus V_0$, and let $w \in W_*$, $i, j \in S$ and $a_x^i \in \mathbb{R}$ be as in Definition* 5.1. *Then for any $y \in K$,*

$$\lim_{n \to \infty} \left(\frac{5}{3}\right)^n \rho_\mathcal{H}\bigl(x, F_{wij^n}(y)\bigr) = |a_x^i| h_1^j(y). \tag{5.8}$$

Proof. Let $y \in K$. $\rho_{h_1^j}(q_j, y) = h_1^j(y)$ by (4.13), and by Proposition 3.16 we can choose a harmonic geodesic $\gamma_y : [0,1] \to K$ so that $\gamma_y(0) = q_j$, $\gamma_y(1) = y$ and $\ell_{h_1^j}(\gamma_y) = \rho_{h_1^j}(q_j, y) = h_1^j(y)$. Since $h_x(x) = 0$, $F_{wij^n} \circ \gamma_y(0) = F_{wij^n}(q_j) = x$, and $h_x^\perp \circ F_{wi} = c_x^i h_2^j$ for some $c_x^i \in \mathbb{R}$ by the proof of Lemma 5.5,

$$\begin{aligned}
|a_x^i| h_1^j(y) - \frac{|b_x^i h_2^j(y)|}{3^n} &\leq \left(\frac{5}{3}\right)^n |h_x \circ F_{wij^n}(y)| \leq \left(\frac{5}{3}\right)^n \rho_\mathcal{H}\bigl(x, F_{wij^n}(y)\bigr) \\
&\leq \left(\frac{5}{3}\right)^n \ell_\mathcal{H}(F_{wij^n} \circ \gamma_y) \leq \left(\frac{5}{3}\right)^n \bigl(\ell_{h_x}(F_{wij^n} \circ \gamma_y) + \ell_{h_x^\perp}(F_{wij^n} \circ \gamma_y)\bigr) \\
&= \ell\bigl((a_x^i h_1^j + 3^{-n} b_x^i h_2^j) \circ \gamma_y\bigr) + \frac{|c_x^i|}{3^n} \ell(h_2^j \circ \gamma_y) \\
&\leq |a_x^i| \ell_{h_1^j}(\gamma_y) + \frac{|b_x^i| + |c_x^i|}{3^n} \ell_{h_2^j}(\gamma_y) = |a_x^i| h_1^j(y) + \frac{|b_x^i| + |c_x^i|}{3^n} \ell_{h_2^j}(\gamma_y),
\end{aligned} \tag{5.9}$$

where $b_x^i \in \mathbb{R}$ is as in Definition 5.1. Now letting $n \to \infty$ in (5.9) yields (5.8). □

Remark 5.7. *In* (5.9)*, the author does not have any idea how to estimate $\ell_{h_2^j}(\gamma_y)$ uniformly in y.* This is why no remainder estimate is given for the limits in (5.3) and (5.8), and in (5.42) below, neither.

Proof of Theorem 5.3. $\mu_{\langle h_x \rangle}(B_s(x, \rho_{h_x})) = 2\xi_x s$ follows from Propositions 4.9 and 5.4. Let $r \in (0, r_x]$. Since $B_r(x, \rho_\mathcal{H}) \subset B_r(x, \rho_{h_x})$ by $\rho_{h_x} \leq \rho_\mathcal{H}$, $\mu = \mu_{\langle h_x \rangle} + \mu_{\langle h_x^\perp \rangle}$ and Lemma 5.5 imply

$$\mu(B_r(x, \rho_\mathcal{H})) \leq \mu_{\langle h_x \rangle}(B_r(x, \rho_{h_x})) + \mu_{\langle h_x^\perp \rangle}(B_r(x, \rho_{h_x})) \leq 2\xi_x r + 225 c_x r^{\hat{\kappa}}. \tag{5.10}$$

In the rest of this proof we suppose $x \in V_* \setminus V_0$; the case of $x \in V_0$ is proved similarly and more easily. Let $w \in W_*$, $i, j \in S$ and $a_x^i, b_x^i, a_x^j, b_x^j \in \mathbb{R}$ be as in Definition 5.1. Let $r \in (0, |a_x^i|]$ and $n \in \mathbb{N}$. Since $\mu_{\langle u \rangle}|_{K_v} := \mu_{\langle u \rangle}|_{\mathcal{B}(K_v)} = (5/3)^{|v|} \mu_{\langle u \circ F_v \rangle} \circ F_v^{-1}$ for $u \in \mathcal{F}$ and $v \in W_*$,

$$\left(\frac{5}{3}\right)^n \mu\bigl(B_{(3/5)^n r}(x, \rho_\mathcal{H})\bigr) \geq \left(\frac{5}{3}\right)^n \int_{K_{wij^n} \cup K_{wji^n}} \mathbf{1}_{[0,(3/5)^n r)}(\rho_\mathcal{H}(x, y)) d\mu_{\langle h_x \rangle}(y)$$

$$= \left(\frac{5}{3}\right)^{2n+|w|+1} \sum_{(k,l)} \int_{K_{wkl^n}} \mathbf{1}_{[0,r)}\Bigl(\bigl(\tfrac{5}{3}\bigr)^n \rho_{\mathcal{H}}(x,y)\Bigr) d\bigl(\mu_{\langle h_x \circ F_{wkl^n}\rangle} \circ F_{wkl^n}^{-1}\bigr)(y)$$

$$= \left(\frac{5}{3}\right)^{|w|+1} \sum_{(k,l)} \int_{K} \mathbf{1}_{[0,r)}\Bigl(\bigl(\tfrac{5}{3}\bigr)^n \rho_{\mathcal{H}}\bigl(x, F_{wkl^n}(y)\bigr)\Bigr) d\mu_{\langle a_x^k h_1^l + 3^{-n} b_x^k h_2^l\rangle}(y)$$

$$\geq \xi_x |a_x^i| \sum_{(k,l)} \left(\int_{K} \mathbf{1}_{[0,r)}\Bigl(\bigl(\tfrac{5}{3}\bigr)^n \rho_{\mathcal{H}}\bigl(x, F_{wkl^n}(y)\bigr)\Bigr) d\mu_{\langle h_1^l\rangle}(y) - \frac{2|b_x^k|}{3^n |a_x^k|} \right), \tag{5.11}$$

where (k,l) runs over $\{(i,j),(j,i)\}$ and we used $\mu_{\langle h_1^l, h_2^l\rangle}(A)^2 \leq \mu_{\langle h_1^l\rangle}(A)\mu_{\langle h_2^l\rangle}(A) \leq 1$, $A \in \mathcal{B}(K)$. Then by using Lemma 5.6 and Fatou's lemma to let $n \to \infty$ in (5.11), together with (4.11) and (5.10), we get $\lim_{n\to\infty}\bigl((\frac{3}{5})^n r\bigr)^{-1}\mu\bigl(B_{(3/5)^n r}(x,\rho_{\mathcal{H}})\bigr) = 2\xi_x$, from which $\lim_{r\downarrow 0} r^{-1}\mu(B_r(x,\rho_{\mathcal{H}})) = 2\xi_x$ immediately follows since $(0,\infty) \ni r \mapsto \mu(B_r(x,\rho_{\mathcal{H}}))$ is non-decreasing. □

5.2 Heat kernel

The main result of this subsection is a short time asymptotic behavior of $p_\mu(t,x,y)$ for $x \in V_*$ and is stated in the following theorem, whose proof makes full use of Propositions 4.9 and 5.4 and Lemma 5.5. Recall Definition 5.1 and that $\rho_{h_x}(x,y) = |h_x(y)|$ for $x \in V_*$ and $y \in K^x$ by (4.13) and (5.6).

Theorem 5.8. *Let $\delta \in (0,1]$ and $x \in V_*$. Then there exists $c_{\mathrm{R}} \in (0,\infty)$ determined solely by $\kappa, \hat{\kappa}, c_{\mathrm{G}}, c_{\mathrm{V}}$ such that for any $(t,y) \in (0, r_x^2] \times K^x$,*

$$\left| p_\mu(t,x,y) - \frac{\exp\bigl(-\frac{h_x(y)^2}{2t}\bigr)}{\xi_x\sqrt{2\pi t}} \right| \leq \left(\frac{c_x}{\xi_x} t^{\frac{\hat{\kappa}-1}{2}} + \delta^{\kappa+1}\left(\delta \wedge \frac{(\frac{c_x}{\xi_x})^{\frac{2}{\kappa+1}} |h_x(y)|^{\frac{2(\kappa+\hat{\kappa})}{\kappa+1}}}{t} \right) + \delta^{\frac{15}{4}\kappa+\frac{\hat{\kappa}}{2}+2} \exp\left(-\frac{r_x^2}{6t}\right) \right) \frac{c_{\mathrm{R}}}{\delta^{\frac{15}{4}\kappa+\frac{\hat{\kappa}}{2}+2}} \frac{\exp\bigl(-\frac{h_x(y)^2}{2(1+\delta)t}\bigr)}{\xi_x\sqrt{2\pi t}}. \tag{5.12}$$

In particular, there exists $t_x \in (0, r_x^2]$ determined solely by $r_x, \frac{c_x}{\xi_x}, \hat{\kappa}$ such that

$$\left| p_\mu(t,x,y) - \frac{\exp\bigl(-\frac{h_x(y)^2}{2t}\bigr)}{\xi_x\sqrt{2\pi t}} \right| \leq c_{\mathrm{R}}^{x,\delta}\left(t^{\frac{\hat{\kappa}-1}{2}} + \delta^{\kappa}|h_x(y)|^{\frac{2(\hat{\kappa}-1)}{\kappa+1}} \right) \frac{\exp\bigl(-\frac{h_x(y)^2}{2(1+\delta)t}\bigr)}{\xi_x\sqrt{2\pi t}} \tag{5.13}$$

for any $(t,y) \in (0,t_x] \times K^x$, where $c_{\mathrm{R}}^{x,\delta} := 5c_{\mathrm{R}}\bigl(\frac{c_x}{\xi_x} \vee (\frac{c_x}{\xi_x})^{\frac{2}{\kappa+1}}\bigr)(2/\delta)^{\frac{15}{4}\kappa+\frac{\hat{\kappa}}{2}+2}$.

By virtue of Propositions 4.9 and 5.4 and Lemma 5.5, Theorem 5.8 follows from the following general remainder estimate.

Theorem 5.9. *Let $h, h^\perp \in \mathcal{S}_{\mathcal{H}_0}$ satisfy $\mathcal{E}(h,h^\perp) = 0$, and let $\delta \in (0,1]$. Then there exists $C_{\mathrm{R}} \in (0,\infty)$ determined solely by $\kappa, \hat{\kappa}, c_{\mathrm{G}}, c_{\mathrm{V}}$ such that for any $(t,x,y) \in (0,\infty) \times K \times K$,*

$$\begin{aligned} &|p_\mu(t,x,y) - p_{\mu_{\langle h\rangle}}(t,x,y)| \\ &\leq \left(\frac{1}{t}\int_0^t \frac{\mu_{\langle h^\perp\rangle}\bigl(B_{\sqrt{s}}(x,\rho_h)\bigr)}{\mu_{\langle h\rangle}\bigl(B_{\sqrt{s}}(x,\rho_h)\bigr)} ds + \frac{\delta^{\kappa+1}}{t}\int_0^{\delta t} \frac{\mu_{\langle h^\perp\rangle}\bigl(B_{\sqrt{s}}(y,\rho_{\mathcal{H}})\bigr)}{\mu\bigl(B_{\sqrt{s}}(y,\rho_{\mathcal{H}})\bigr)} ds \right. \\ &\quad \left. + \delta^{\frac{9}{4}\kappa+2}\frac{\mu_{\langle h^\perp\rangle}\bigl(B_{\sqrt{\delta t}}(y,\rho_{\mathcal{H}})\bigr)}{\mu\bigl(B_{\sqrt{\delta t}}(y,\rho_{\mathcal{H}})\bigr)} \right) \frac{C_{\mathrm{R}}}{\delta^{\frac{15}{4}\kappa+\frac{\hat{\kappa}}{2}+2}} \frac{\exp\bigl(-\frac{\rho_h(x,y)^2}{2(1+\delta)t}\bigr)}{\mu_{\langle h\rangle}\bigl(B_{\sqrt{t}}(x,\rho_h)\bigr)}. \end{aligned} \tag{5.14}$$

The proof of Theorem 5.9 is given later. First we prove Theorem 5.8 based on Theorem 5.9. For this purpose we need the following lemma.

Lemma 5.10. $\mu_{\langle h\rangle}(B_r(x,\rho_h)) \lesssim \mu(B_r(x,\rho_{\mathcal{H}}))$ *for* $h \in \mathcal{S}_{\mathcal{H}_0}$ *and any* $(r,x) \in (0,\infty)\times K$.

Proof. Let $h \in \mathcal{S}_{\mathcal{H}_0}$ and $(s,x) \in (0,1)\times K$. Let $w \in \Lambda_s^h$ satisfy $K_w \cap K_s(x,\mathcal{S}^h) \neq \emptyset$. Then $K_w \cap K_v \neq \emptyset$ for some $v \in \Lambda_s^h$ with $x \in K_v$, and $\tau \leq v$ for some $\tau \in \Lambda_s^{\mathcal{H}}$ with $x \in K_\tau$ by $l_h \leq l_{\mathcal{H}}$. Moreover $\|h\circ F_v\|_{\mathcal{E}} \leq s \leq 5\|T_\tau\|$ by (3.4), which and $|v| \leq |\tau|$ easily yield $\mu_{\langle h\rangle}(K_v) \leq 25\mu(K_\tau)$. Therefore using Proposition 3.10-(1) we see that $\mu_{\langle h\rangle}(K_w) \lesssim \mu_{\langle h\rangle}(K_v) \lesssim \mu(K_\tau) \leq \mu(U_s(x,\mathcal{S}^{\mathcal{H}}))$, which and Lemma 3.5 imply $\mu_{\langle h\rangle}(U_s(x,\mathcal{S}^h)) \lesssim \mu(U_s(x,\mathcal{S}^{\mathcal{H}}))$. Using this fact together with (3.16), (3.12) and (3.11), we conclude that

$$\begin{aligned}\mu_{\langle h\rangle}(B_{10s}(x,\rho_h)) &\lesssim \mu_{\langle h\rangle}(B_{s/25}(x,\rho_h)) \leq \mu_{\langle h\rangle}(U_s(x,\mathcal{S}^h))\\ &\lesssim \mu(U_s(x,\mathcal{S}^{\mathcal{H}})) \leq \mu(B_{10s}(x,\rho_{\mathcal{H}})).\end{aligned}$$

The case of $r \geq 10$ is clear since $B_{10}(x,\rho_{\mathcal{H}}) = B_{10}(x,\rho_h) = K$ by (3.11) and (3.12). □

Proof of Theorem 5.8 *under Theorem* 5.9. Let $\delta \in (0,1]$, $x \in V_*$ and $y \in K^x$. For $r \in (0,r_x]$, $B_r(y,\rho_{\mathcal{H}}) \subset B_r(y,\rho_{h_x})$ by $\rho_{h_x} \leq \rho_{\mathcal{H}}$, and then by Lemma 5.10, Theorem 3.19, Proposition 5.4 (Proposition 4.9 when $x \in V_0$) and Lemma 5.5 we have

$$\begin{aligned}\frac{\mu_{\langle h_x^\perp\rangle}(B_r(y,\rho_{\mathcal{H}}))}{\mu(B_r(y,\rho_{\mathcal{H}}))} &\lesssim \frac{\mu_{\langle h_x^\perp\rangle}(B_r(y,\rho_{h_x}))}{\mu_{\langle h_x\rangle}(B_r(y,\rho_{h_x}))} \lesssim \left(1+\frac{|h_x(y)|}{r}\right)^{\kappa+\hat\kappa}\frac{\mu_{\langle h_x^\perp\rangle}(B_r(x,\rho_{h_x}))}{\mu_{\langle h_x\rangle}(B_r(x,\rho_{h_x}))}\\ &\lesssim \left(1+\frac{|h_x(y)|^{\kappa+\hat\kappa}}{r^{\kappa+\hat\kappa}}\right)\frac{225c_x}{2\xi_x}r^{\hat\kappa-1} \lesssim \frac{c_x}{\xi_x}\left(r^{\hat\kappa-1}+\frac{|h_x(y)|^{\kappa+\hat\kappa}}{r^{\kappa+1}}\right).\end{aligned}\tag{5.15}$$

Let $t \in (0,r_x^2]$. Since $\mu_{\langle h_x^\perp\rangle} \leq \mu$ and $\kappa+1>2$, (5.15) yields

$$\begin{aligned}&\int_0^{\delta t}\frac{\mu_{\langle h_x^\perp\rangle}\big(B_{\sqrt{s}}(y,\rho_{\mathcal{H}})\big)}{\mu\big(B_{\sqrt{s}}(y,\rho_{\mathcal{H}})\big)}ds \lesssim \frac{c_x}{\xi_x}\int_0^{\delta t}s^{\frac{\hat\kappa-1}{2}}ds+\int_0^{\delta t}1\wedge\frac{c_x|h_x(y)|^{\kappa+\hat\kappa}}{\xi_x s^{\frac{\kappa+1}{2}}}ds\\ &\leq \begin{cases}\dfrac{c_x}{\xi_x}(\delta t)^{\frac{\hat\kappa+1}{2}}+D_x(y)+\displaystyle\int_{D_x(y)}^{\infty}\Big(\frac{D_x(y)}{s}\Big)^{\frac{\kappa+1}{2}}ds & \text{if } D_x(y)\leq\delta t\\ \dfrac{c_x}{\xi_x}(\delta t)^{\frac{\hat\kappa+1}{2}}+\delta t & \text{if } D_x(y)\geq\delta t\end{cases}\\ &\leq \frac{c_x}{\xi_x}(\delta t)^{\frac{\hat\kappa+1}{2}}+4t\Big(\delta\wedge\frac{D_x(y)}{t}\Big),\quad \text{where } D_x(y):=\Big(\frac{c_x}{\xi_x}\Big)^{\frac{2}{\kappa+1}}|h_x(y)|^{\frac{2(\kappa+\hat\kappa)}{\kappa+1}}.\end{aligned}\tag{5.16}$$

Similarly, by using (5.15) and $1\wedge s^{\frac{\kappa+1}{2}} \leq 1\wedge s$, $s\in[0,\infty)$, we see that

$$\frac{\mu_{\langle h_x^\perp\rangle}\big(B_{\sqrt{\delta t}}(y,\rho_{\mathcal{H}})\big)}{\mu\big(B_{\sqrt{\delta t}}(y,\rho_{\mathcal{H}})\big)} \lesssim \frac{c_x}{\xi_x}(\delta t)^{\frac{\hat\kappa-1}{2}}+\delta^{-1}\Big(\delta\wedge\frac{D_x(y)}{t}\Big).\tag{5.17}$$

Again by Proposition 5.4-(3) (Proposition 4.9 when $x \in V_0$) and Lemma 5.5, we also have

$$\int_0^t\frac{\mu_{\langle h_x^\perp\rangle}\big(B_{\sqrt{s}}(x,\rho_{h_x})\big)}{\mu\big(B_{\sqrt{s}}(x,\rho_{h_x})\big)}ds \leq \frac{225c_x}{2\xi_x}\int_0^t s^{\frac{\hat\kappa-1}{2}}ds = \frac{225}{\hat\kappa+1}\frac{c_x}{\xi_x}t^{\frac{\hat\kappa+1}{2}}.\tag{5.18}$$

On the other hand, let U^x be the connected component of $h_x^{-1}((-r_x, r_x))$ containing x and set $\psi_x(y,t) := 1 - \int_{U^x} p^{U^x}_{\mu_{\langle h_x\rangle}}(t,y,z)\,d\mu_{\langle h_x\rangle}(z)$. Then (4.16), (5.7) and a direct calculation using [21, Exercise 2.8.11] yield

$$0 \le \psi_x(y,t) = 1 - \int_{-r_x}^{r_x} p_{r_x}(t, h_x(y), z)\,dz \le 2\exp\biggl(-\frac{(r_x - |h_x(y)|)^2}{2t}\biggr). \tag{5.19}$$

By [16, Theorem 5.1] (or [14, Theorem 10.4]), (5.19), (3.16), (4.2) and $|h_x(y)| \le 5r_x/6$,

$$\begin{aligned}
0 &\le p_{\mu_{\langle h_x\rangle}}(t,x,y) - p^{U^x}_{\mu_{\langle h_x\rangle}}(t,x,y)\\
&\le \psi_x\bigl(x,\tfrac{t}{2}\bigr) \sup_{s\in[\frac{t}{2},t]} \sup_{w\in\partial U^x} p_{\mu_{\langle h_x\rangle}}(s,w,y) + \psi_x\bigl(y,\tfrac{t}{2}\bigr) \sup_{s\in[\frac{t}{2},t]} \sup_{z\in\partial U^x} p_{\mu_{\langle h_x\rangle}}(s,x,z)\\
&\lesssim \biggl(1+\frac{8r_x^2}{t}\biggr)^{3\kappa/4} \Biggl(\frac{\exp\bigl(-\frac{r_x^2}{t} - \frac{r_x^2}{72t}\bigr)}{\mu_{\langle h_x\rangle}\bigl(B_{\sqrt{t/2}}(y,\rho_{h_x})\bigr)} + \frac{\exp\bigl(-\frac{r_x^2}{36t} - \frac{r_x^2}{2t}\bigr)}{\mu_{\langle h_x\rangle}\bigl(B_{\sqrt{t/2}}(x,\rho_{h_x})\bigr)}\Biggr)\\
&\lesssim \biggl(1+\frac{8r_x^2}{t}\biggr)^{5\kappa/4} \exp\biggl(-\frac{13r_x^2}{72t}\biggr) \frac{\exp\bigl(-\frac{h_x(y)^2}{2t}\bigr)}{\mu_{\langle h_x\rangle}\bigl(B_{\sqrt{t}}(x,\rho_{h_x})\bigr)}\\
&\lesssim \exp\biggl(-\frac{r_x^2}{6t}\biggr) \frac{\exp\bigl(-\frac{h_x(y)^2}{2t}\bigr)}{2\xi_x\sqrt{t}}.
\end{aligned} \tag{5.20}$$

Also a direct calculation using [21, Proposition 2.8.10], $t \le r_x^2$ and $|h_x(y)| \le 5r_x/6$ yields

$$\begin{aligned}
0 &\le \frac{\exp\bigl(-\frac{h_x(y)^2}{2t}\bigr)}{\xi_x\sqrt{2\pi t}} - p^{U^x}_{\mu_{\langle h_x\rangle}}(t,x,y) = \frac{\exp\bigl(-\frac{h_x(y)^2}{2t}\bigr)}{\xi_x\sqrt{2\pi t}} - \xi_x^{-1} p_{r_x}(t,0,h_x(y))\\
&\le 3\exp\biggl(-\frac{2r_x(r_x - |h_x(y)|)}{t}\biggr)\frac{\exp\bigl(-\frac{h_x(y)^2}{2t}\bigr)}{\xi_x\sqrt{2\pi t}} \le 3\exp\biggl(-\frac{r_x^2}{3t}\biggr)\frac{\exp\bigl(-\frac{h_x(y)^2}{2t}\bigr)}{\xi_x\sqrt{2\pi t}}.
\end{aligned} \tag{5.21}$$

Now (5.12) is immediate from the inequality (5.14) with $h = h_x$ and $h^\perp = h_x^\perp$ and the estimates (5.16), (5.17), (5.18), (5.20) and (5.21). (5.13) follows by using $2se^{-s/(1+\delta/2)} \le 5\delta^{-1}e^{-s/(1+\delta)}$, $s := h_x(y)^2/2t$ to estimate the second term in (5.12), completing the proof of Theorem 5.8. □

The rest of this subsection is devoted to the proof of Theorem 5.9. We need to prepare several lemmas. The following lemma is immediate from (3.16) and Corollary 4.3; note that we have $(1+x)^\alpha e^{-x/\beta} \le (e^{-1}\alpha\beta)^\alpha e^{1/\beta}$ for $\alpha, \beta \in (0,\infty)$ and $x \in [-1,\infty)$.

Lemma 5.11. *Let $h \in \mathcal{S}_{\mathcal{H}_0}$. For $\delta \in (0,\infty)$ and $(t,x,y) \in (0,\infty)\times K \times K$, define*

$$\Psi_{\mathcal{H},\delta}(t,x,y) := \frac{\exp\bigl(-\frac{\rho_{\mathcal{H}}(x,y)^2}{2(1+\delta)t}\bigr)}{\mu\bigl(B_{\sqrt{t}}(x,\rho_{\mathcal{H}})\bigr)}, \qquad \Psi_{h,\delta}(t,x,y) := \frac{\exp\bigl(-\frac{\rho_h(x,y)^2}{2(1+\delta)t}\bigr)}{\mu_{\langle h\rangle}\bigl(B_{\sqrt{t}}(x,\rho_h)\bigr)}. \tag{5.22}$$

Then for each $n \in \mathbb{N}\cup\{0\}$ there exists $c_{\mathrm{hk}}(n) \in (0,\infty)$ determined solely by $n, \kappa, c_{\mathrm{G}}, c_{\mathrm{V}}$ such that for any $\delta \in (0,1]$ and any $(t,x,y) \in (0,\infty)\times K\times K$,

$$\bigl|\partial_t^n p_\mu(t,x,y)\bigr| \le \frac{c_{\mathrm{hk}}(n)}{\delta^{\frac{3}{4}\kappa+n}t^n}\Psi_{\mathcal{H},\delta}(t,x,y), \tag{5.23}$$

$$\bigl|\partial_t^n p_{\mu_{\langle h\rangle}}(t,x,y)\bigr| \le \frac{c_{\mathrm{hk}}(n)}{\delta^{\frac{3}{4}\kappa+n}t^n}\Psi_{h,\delta}(t,x,y). \tag{5.24}$$

Lemma 5.12. *Let* $\varepsilon, \delta \in (0,\infty)$, $\varepsilon < \delta$ *and set* $\theta(\varepsilon,\delta) := (\varepsilon\delta + 2\varepsilon + 1)/(\delta - \varepsilon)$. *Let* $h \in \mathcal{S}_{\mathcal{H}_0}$, $s,t \in (0,\infty)$ *and* $x,y,z \in K$. *Then*

$$\Psi_{\mathcal{H},\varepsilon}(s,x,z)\Psi_{h,\delta}(t,y,z) \le c_{\mathrm{V}}\Bigl(\frac{s+t}{t}\Bigr)^{\kappa/2}\Psi_{\mathcal{H},\theta(\varepsilon,\delta)}(s,x,z)\Psi_{h,\delta}(s+t,y,x), \quad (5.25)$$

$$\Psi_{h,\varepsilon}(s,x,z)\Psi_{h,\delta}(t,y,z) \le c_{\mathrm{V}}\Bigl(\frac{s+t}{t}\Bigr)^{\kappa/2}\Psi_{h,\theta(\varepsilon,\delta)}(s,x,z)\Psi_{h,\delta}(s+t,y,x). \quad (5.26)$$

Proof. Since $(1+\varepsilon)^{-1} = (1+\theta(\varepsilon,\delta))^{-1} + (1+\delta)^{-1}$ and $a^2/s + b^2/t \ge (a+b)^2/(s+t)$ for $a,b \in [0,\infty)$, a direct calculation using (3.16) and $\rho_h \le \rho_{\mathcal{H}}$ easily shows the assertion. □

Lemma 5.13. *Let* $g,h \in \mathcal{S}_{\mathcal{H}_0}$ *and* $\theta \in [1,\infty)$. *Then for any* $(t,x) \in (0,\infty)\times K$,

$$\int_K \Psi_\theta(t,x,y)d\nu(y) \lesssim \theta^{\kappa/2}, \qquad \int_K \Psi_\theta(t,x,y)d\mu_{\langle g\rangle}(y) \lesssim \theta^{\hat{\kappa}/2}\frac{\mu_{\langle g\rangle}\bigl(B_{\sqrt{t}}(x,\rho)\bigr)}{\nu\bigl(B_{\sqrt{t}}(x,\rho)\bigr)}, \quad (5.27)$$

where (ν,ρ,Ψ_θ) *denotes any one of* $(\mu,\rho_{\mathcal{H}},\Psi_{\mathcal{H},\theta})$ *and* $(\mu_{\langle h\rangle},\rho_h,\Psi_{h,\theta})$.

Proof. Let $(t,x) \in (0,\infty)\times K$ and $s := (1+\theta)t$. By (3.16) we see that

$$\begin{aligned}
&\int_K \Psi_\theta(t,x,y)d\nu(y)\\
&= \int_{B_{\sqrt{s}}(x,\rho)} \Psi_\theta(t,x,y)d\nu(y) + \sum_{n\in\mathbb{N}}\int_{B_{2^n\sqrt{s}}(x,\rho)\setminus B_{2^{n-1}\sqrt{s}}(x,\rho)} \Psi_\theta(t,x,y)d\nu(y)\\
&\le \nu\bigl(B_{\sqrt{t}}(x,\rho)\bigr)^{-1}\Bigl(\nu\bigl(B_{\sqrt{s}}(x,\rho)\bigr) + \sum_{n\in\mathbb{N}} e^{-4^n/8}\nu\bigl(B_{2^n\sqrt{s}}(x,\rho)\setminus B_{2^{n-1}\sqrt{s}}(x,\rho)\bigr)\Bigr)\\
&\le \nu\bigl(B_{\sqrt{t}}(x,\rho)\bigr)^{-1}\nu\bigl(B_{\sqrt{s}}(x,\rho)\bigr)c_{\mathrm{V}}\Bigl(1+\sum_{n\in\mathbb{N}}2^{\kappa n}e^{-4^n/8}\Bigr) \lesssim \theta^{\kappa/2}.
\end{aligned}$$

The latter assertion is proved in the same way by using (3.16) and (3.17). □

Next we introduce several probabilistic notions required for the proof of Theorem 5.9, which utilizes a time change argument on the diffusion. See [13, Part II and Section A.2] for details concerning diffusions associated with symmetric Dirichlet forms and their time changes by positive continuous additive functionals. Below $K_\partial := K \cup \{\partial\}$ denotes the one-point compactification of K and a function $f : K \to [-\infty,\infty]$ on K is always extended to K_∂ by setting $f(\partial) := 0$ when needed. Let $X = \bigl(\Omega, \mathcal{M}, \{X_t\}_{t\in[0,\infty]}, \{\mathbf{P}_x\}_{x\in K_\partial}\bigr)$ be a μ-symmetric diffusion on K with life time ζ^X and minimum completed admissible filtration $\mathcal{F}_* := \{\mathcal{F}_t\}_{t\in[0,\infty]}$ whose Dirichlet form on $L^2(K,\mu)$ is $(\mathcal{E},\mathcal{F})$; such X does exist by virtue of [13, Theorem 7.2.2]. Then $\mathbf{P}_x[X_t \in dy] = p_\mu(t,x,y)d\mu(y)$ for any $(t,x) \in (0,\infty)\times K$ by [26, Theorem 10.4], and $\mathbf{P}_x[\zeta^X = \infty] = 1$ for $x \in K$ since $\int_K p_\mu(t,x,y)d\mu(y) = 1$, $t \in (0,\infty)$. Expectation (i.e. integral on Ω) under the measure $\mathbf{P}_x$ is denoted by $\mathbf{E}_x[(\cdot)]$.

Take any $h, h^\perp \in \mathcal{S}_{\mathcal{H}_0}$ satisfying $\mathcal{E}(h,h^\perp) = 0$, so that $\mu = \mu_{\langle h\rangle} + \mu_{\langle h^\perp\rangle}$; we fix them in the rest of this subsection. Also fix a Borel measurable version of $d\mu_{\langle h\rangle}/d\mu$ satisfying $0 < (d\mu_{\langle h\rangle}/d\mu)(y) \le 1$ for any $y \in K$; such a version exists since $\mu_{\langle h\rangle} \le \mu$ and μ is absolutely continuous with respect to $\mu_{\langle h\rangle}$ by [17, Theorem 5.6]. We define

$$A_t := \int_0^t \frac{d\mu_{\langle h\rangle}}{d\mu}(X_s)ds, \qquad t \in [0,\infty], \quad (5.28)$$

so that $A = \{A_t\}_{t\in[0,\infty)}$ is the positive continuous additive functional of X with Revuz measure $\mu_{\langle h\rangle}$. For $t \in [0,\infty]$ we further define

$$\tau_t := \inf\{s \in [0,\infty) \mid A_s > t\}, \qquad Y_t := X_{\tau_t} \qquad \mathcal{G}_t := \mathcal{F}_{\tau_t}; \tag{5.29}$$

here τ_t is an $\mathcal{F}_*$-stopping time and hence $\mathcal{F}_{\tau_t}$ is defined as a sub-σ-field of $\mathcal{F}_\infty$. [13, Theorems A.2.12 and 6.2.1] imply that $Y := \bigl(\Omega, \mathcal{M}, \{Y_t\}_{t\in[0,\infty]}, \{\mathbf{P}_x\}_{x\in K_\partial}\bigr)$ is a $\mu_{\langle h\rangle}$-symmetric diffusion on K with life time A_∞ and admissible filtration $\mathcal{G}_* := \{\mathcal{G}_t\}_{t\in[0,\infty]}$ whose Dirichlet form on $L^2(K,\mu_{\langle h\rangle})$ is $(\mathcal{E},\mathcal{F})$. $\mathbf{P}_x[Y_t \in dy] = p_{\mu_{\langle h\rangle}}(t,x,y)d\mu_{\langle h\rangle}(y)$ for any $(t,x) \in (0,\infty)\times K$ by [26, Theorem 10.4] and hence $\mathbf{P}_x[A_\infty = \infty] = 1$, $x \in K$. For each $t \in [0,\infty)$, clearly $A_t \le t \le \tau_t$, and A_t is a $\mathcal{G}_*$-stopping time since $\{A_t > s\} = \{\tau_s < t\} \in \mathcal{F}_{\tau_s} = \mathcal{G}_s$, $s \in [0,\infty)$. On $\{\zeta^X = \infty\}$, $A_{(\cdot)}$ is strictly increasing and hence $\tau_{A_t} = t$ and $Y_{A_t} = X_t$ for any $t \in [0,\infty)$. For $x \in K$, since $\mathbf{P}_x[A_\infty = \infty] = 1$, a direct calculation shows that

$$\tau_t = \int_0^t \Bigl(\frac{d\mu_{\langle h\rangle}}{d\mu}(Y_s)\Bigr)^{-1} ds < \infty \qquad \text{for any } t \in [0,\infty),\ \mathbf{P}_x\text{-a.s.} \tag{5.30}$$

Lemma 5.14. *For any $\delta \in (0,1]$, $s,t \in (0,\infty)$ and $x,y \in K$,*

$$\frac{\mathbf{E}_x[(t-A_t)\Psi_{h,\delta}(s,y,X_t)]}{\Psi_{h,\delta}(s+t,y,x)} \lesssim \frac{\bigl(\frac{s+t}{s}\bigr)^{\kappa/2}}{\delta^{2\kappa+\hat{\kappa}/2}} \int_0^t \frac{\mu_{\langle h^\perp\rangle}\bigl(B_{\sqrt{u}}(x,\rho_{\mathcal{H}})\bigr)}{\mu\bigl(B_{\sqrt{u}}(x,\rho_{\mathcal{H}})\bigr)} du, \tag{5.31}$$

$$\frac{\mathbf{E}_x[(\tau_t-t)\Psi_{h,\delta}(s,y,Y_t)]}{\Psi_{h,\delta}(s+t,y,x)} \lesssim \frac{\bigl(\frac{s+t}{s}\bigr)^{\kappa/2}}{\delta^{2\kappa+\hat{\kappa}/2}} \int_0^t \frac{\mu_{\langle h^\perp\rangle}\bigl(B_{\sqrt{u}}(x,\rho_h)\bigr)}{\mu_{\langle h\rangle}\bigl(B_{\sqrt{u}}(x,\rho_h)\bigr)} du. \tag{5.32}$$

Proof. Let $\delta \in (0,1]$, $s,t \in (0,\infty)$ and $x,y \in K$. By (5.28) and the Markov property of X,

$$\begin{aligned}
&\mathbf{E}_x[(t-A_t)\Psi_{h,\delta}(s,y,X_t)] = \int_0^t \mathbf{E}_x\Bigl[\Bigl(\mathbf{1} - \frac{d\mu_{\langle h\rangle}}{d\mu}\Bigr)(X_u)\Psi_{h,\delta}(s,y,X_t)\Bigr]du \\
&= \int_0^t \int_K \int_K p_\mu(u,x,z)p_\mu(t-u,z,w)\Bigl(\mathbf{1} - \frac{d\mu_{\langle h\rangle}}{d\mu}\Bigr)(z)\Psi_{h,\delta}(s,y,w)d\mu(w)d\mu(z)du \\
&= \int_0^t \int_K \int_K p_\mu(u,x,z)p_\mu(t-u,z,w)\Psi_{h,\delta}(s,y,w)d\mu(w)d\mu_{\langle h^\perp\rangle}(z)du,
\end{aligned} \tag{5.33}$$

where we used $\mu = \mu_{\langle h\rangle} + \mu_{\langle h^\perp\rangle}$ in the last equality. Then (5.23) and (5.25) yield

$$\begin{aligned}
&(\delta/2)^{3\kappa/2} p_\mu(u,x,z)p_\mu(t-u,z,w)\Psi_{h,\delta}(s,y,w) \\
&\lesssim \Psi_{\mathcal{H},\delta/2}(u,x,z)\Psi_{\mathcal{H},\delta/2}(t-u,z,w)\Psi_{h,\delta}(s,y,w) \\
&\lesssim \Bigl(\frac{s+t-u}{s}\Bigr)^{\kappa/2} \Psi_{\mathcal{H},\delta/2}(u,x,z)\Psi_{\mathcal{H},\theta(\delta/2,\delta)}(t-u,z,w)\Psi_{h,\delta}(s+t-u,y,z) \\
&\lesssim \Bigl(\frac{s+t}{s}\Bigr)^{\kappa/2} \Psi_{\mathcal{H},\theta(\delta/2,\delta)}(u,x,z)\Psi_{\mathcal{H},\theta(\delta/2,\delta)}(t-u,z,w)\Psi_{h,\delta}(s+t,y,x).
\end{aligned} \tag{5.34}$$

Since $2/\delta \le \theta(\delta/2,\delta) \le 5/\delta$, from (5.33) and (5.34) we get (5.31) by using (5.27) to integrate (5.34) first by $d\mu(w)$ and then by $d\mu_{\langle h^\perp\rangle}(z)$. The same argument using (5.24), (5.26) and (5.27) easily shows (5.32) as well since similarly to (5.33) we have

$$\begin{aligned}
&\mathbf{E}_x[(\tau_t - t)\Psi_{h,\delta}(s,y,Y_t)] \\
&= \int_0^t \int_K \int_K p_{\mu_{\langle h\rangle}}(u,x,z)p_{\mu_{\langle h\rangle}}(t-u,z,w)\Psi_{h,\delta}(s,y,w)d\mu_{\langle h\rangle}(w)d\mu_{\langle h^\perp\rangle}(z)du
\end{aligned} \tag{5.35}$$

by virtue of (5.30), the Markov property of Y and $1/(d\mu_{\langle h\rangle}/d\mu) = d\mu/d\mu_{\langle h\rangle}$ $\mu_{\langle h\rangle}$-a.e. □

Lemma 5.15. *Let $s, t, a \in (0,\infty)$, $s < a < t$. Then for any $x, y \in K$,*

$$p_{\mu_{\langle h\rangle}}(t,x,y) = \mathbf{E}_y[p_{\mu_{\langle h\rangle}}(t-A_s, X_s, x)], \tag{5.36}$$

$$p_\mu(t,x,y) = \mathbf{E}_x[\mathbf{1}_{\{\tau_s\ge a\}} p_\mu(t-a, X_a, y) + \mathbf{1}_{\{\tau_s< a\}} p_\mu(t-\tau_s, Y_s, y)]. \tag{5.37}$$

Proof. Let $x, y \in K$. Since $A_s \le s < t$ and A_s is a $\mathcal{G}_*$-stopping time, the strong Markov property of Y together with [21, Corollary 2.6.18] implies that for any $r \in (0,\infty)$,

$$\begin{aligned}\int_{B_r(x,\rho_h)} p_{\mu_{\langle h\rangle}}(t,y,z)d\mu_{\langle h\rangle}(z) &= \mathbf{P}_y[Y_t \in B_r(x,\rho_h)] \\ &= \int_\Omega \mathbf{P}_{Y_{A_s(\omega)}(\omega)}[Y_{t-A_s(\omega)} \in B_r(x,\rho_h)]d\mathbf{P}_y(\omega) \\ &= \mathbf{E}_y\biggl[\int_{B_r(x,\rho_h)} p_{\mu_{\langle h\rangle}}(t-A_s, X_s, z)d\mu_{\langle h\rangle}(z)\biggr].\end{aligned} \tag{5.38}$$

Then noting that $0 < t-s \le t-A_s \le t$, we obtain (5.36) by dividing (5.38) by $\mu(B_r(x,\rho_h))$ and using the joint continuity of $p_{\mu_{\langle h\rangle}}$ to let $r \downarrow 0$. Similarly we can also show (5.37) based on the Markov property of X at time a and the strong Markov property of X at the $\mathcal{F}_*$-stopping time τ_s together with [21, Corollary 2.6.18]. □

Proof of Theorem 5.9. Let $\delta \in (0,1]$ and set $\varepsilon := \delta/4$, so that $(1+\varepsilon)^2 \le 1+\delta$. Let $(t,x,y) \in (0,\infty)\times K\times K$. From (5.36), (5.24), (3.16) and (5.31) we see that

$$\begin{aligned}&\left|p_{\mu_{\langle h\rangle}}(t,x,y) - \int_K p_\mu(\varepsilon t, y, z)p_{\mu_{\langle h\rangle}}((1-\varepsilon)t, z, x)d\mu(z)\right| \\ &= \left|\mathbf{E}_y[p_{\mu_{\langle h\rangle}}(t-A_{\varepsilon t}, X_{\varepsilon t}, x) - p_{\mu_{\langle h\rangle}}((1-\varepsilon)t, X_{\varepsilon t}, x)]\right| \\ &\le \mathbf{E}_y[(\varepsilon t - A_{\varepsilon t})\sup\nolimits_{u\in[(1-\varepsilon)t,t]}|\partial_u p_{\mu_{\langle h\rangle}}(u,x,X_{\varepsilon t})|] \\ &\lesssim \varepsilon^{-\frac{3}{4}\kappa-1}\mathbf{E}_y[(\varepsilon t - A_{\varepsilon t})\sup\nolimits_{u\in[(1-\varepsilon)t,t]} u^{-1}\Psi_{h,\varepsilon}(u,x,X_{\varepsilon t})] \\ &\lesssim \varepsilon^{-\frac{3}{4}\kappa-1}t^{-1}\mathbf{E}_y[(\varepsilon t - A_{\varepsilon t})\Psi_{h,\varepsilon}(t,x,X_{\varepsilon t})] \\ &\lesssim \frac{\Psi_{h,\varepsilon}((1+\varepsilon)t,x,y)}{\varepsilon^{\frac{11}{4}\kappa+\frac{\hat{\kappa}}{2}+1}t}\int_0^{\varepsilon t}\frac{\mu_{\langle h^\perp\rangle}\bigl(B_{\sqrt{u}}(y,\rho_{\mathcal{H}})\bigr)}{\mu\bigl(B_{\sqrt{u}}(y,\rho_{\mathcal{H}})\bigr)}du \\ &\lesssim \frac{\Psi_{h,\delta}(t,x,y)}{\delta^{\frac{11}{4}\kappa+\frac{\hat{\kappa}}{2}+1}t}\int_0^{\delta t}\frac{\mu_{\langle h^\perp\rangle}\bigl(B_{\sqrt{u}}(y,\rho_{\mathcal{H}})\bigr)}{\mu\bigl(B_{\sqrt{u}}(y,\rho_{\mathcal{H}})\bigr)}du.\end{aligned} \tag{5.39}$$

Furthermore let $s := (1-\varepsilon)t$ and $a := (1-\varepsilon/2)t$. Since $\Psi_{\mathcal{H},\delta}(t,x,y) \lesssim \Psi_{h,\delta}(t,x,y)$ and $\mu_{\langle g\rangle}(B_r(x,\rho_{\mathcal{H}})) \le \mu_{\langle g\rangle}(B_r(x,\rho_h))$ by $\rho_h \le \rho_{\mathcal{H}}$ and Lemma 5.10, by using (5.37), $\{\tau_s \ge a\} = \{A_a \le s\}$, (5.23), (3.16), Lemmas 5.14 and 5.10 we obtain

$$\begin{aligned}&\left|p_\mu(t,x,y) - \int_K p_\mu(\varepsilon t,y,z)p_{\mu_{\langle h\rangle}}((1-\varepsilon)t,z,x)d\mu_{\langle h\rangle}(z)\right| \\ &= \left|\mathbf{E}_x[\mathbf{1}_{\{\tau_s\ge a\}}p_\mu(t-a,X_a,y) + \mathbf{1}_{\{\tau_s<a\}}p_\mu(t-\tau_s,Y_s,y) - p_\mu(t-s,Y_s,y)]\right| \\ &\le \mathbf{E}_x[\mathbf{1}_{\{\tau_s\ge a\}}p_\mu(t-a,y,X_a)] + \mathbf{E}_x[\mathbf{1}_{\{\tau_s\ge a\}}p_\mu(t-s,y,Y_s)] \\ &\qquad + \mathbf{E}_x[\mathbf{1}_{\{\tau_s<a\}}|p_\mu(t-\tau_s,y,Y_s) - p_\mu(t-s,y,Y_s)|]\end{aligned}$$

$$
\begin{aligned}
&\leq \mathbf{E}_x[\mathbf{1}_{\{a-A_a\geq a-s\}} p_\mu(t-a,y,X_a)] + \mathbf{E}_x[\mathbf{1}_{\{\tau_s-s\geq a-s\}} p_\mu(t-s,y,Y_s)] \\
&\qquad\qquad + \mathbf{E}_x[\mathbf{1}_{\{\tau_s<a\}}(\tau_s-s)\sup\nolimits_{u\in[t-a,t-s]}|\partial_u p_\mu(u,y,Y_s)|] \\
&\lesssim \varepsilon^{-3\kappa/4}(a-s)^{-1}\mathbf{E}_x[(a-A_a)\Psi_{\mathcal{H},\varepsilon}(t-a,y,X_a) + (\tau_s-s)\Psi_{\mathcal{H},\varepsilon}(t-s,y,Y_s)] \\
&\qquad\qquad + \varepsilon^{-\frac{3}{4}\kappa-1}\mathbf{E}_x[(\tau_s-s)\sup\nolimits_{u\in[t-a,t-s]} u^{-1}\Psi_{\mathcal{H},\varepsilon}(u,y,Y_s)] \\
&\lesssim \varepsilon^{-\frac{3}{4}\kappa-1}t^{-1}\mathbf{E}_x[(a-A_a)\Psi_{h,\varepsilon}(t-a,y,X_a) + (\tau_s-s)\Psi_{h,\varepsilon}(t-s,y,Y_s)] \\
&\qquad\qquad + \varepsilon^{-\frac{3}{4}\kappa-2}t^{-1}\mathbf{E}_x[(\tau_s-s)\Psi_{h,\varepsilon}(t-s,y,Y_s)] \\
&\lesssim \frac{\Psi_{h,\varepsilon}(t,y,x)}{\varepsilon^{\frac{13}{4}\kappa+\frac{\hat{\kappa}}{2}+2}t}\int_0^t \frac{\mu_{\langle h^\perp\rangle}\big(B_{\sqrt{u}}(x,\rho_h)\big)}{\mu_{\langle h\rangle}\big(B_{\sqrt{u}}(x,\rho_h)\big)}du \\
&\lesssim \frac{\Psi_{h,\delta}(t,x,y)}{\delta^{\frac{15}{4}\kappa+\frac{\hat{\kappa}}{2}+2}t}\int_0^t \frac{\mu_{\langle h^\perp\rangle}\big(B_{\sqrt{u}}(x,\rho_h)\big)}{\mu_{\langle h\rangle}\big(B_{\sqrt{u}}(x,\rho_h)\big)}du. \qquad (5.40)
\end{aligned}
$$

On the other hand, (5.23), (5.24), (5.25), (5.27), (3.16) and $1/\delta \leq \theta(\varepsilon,\delta) \leq 4/\delta$ together imply that

$$
\begin{aligned}
0 &\leq \int_K p_\mu(\varepsilon t,y,z)p_{\mu_{\langle h\rangle}}((1-\varepsilon)t,z,x)d\mu_{\langle h^\perp\rangle}(z) \\
&\lesssim (2/\delta)^{3\kappa/2}\int_K \Psi_{\mathcal{H},\varepsilon}(\varepsilon t,y,z)\Psi_{h,\delta}((1-\varepsilon)t,x,z)d\mu_{\langle h^\perp\rangle} \\
&\lesssim \delta^{-3\kappa/2}(1-\varepsilon)^{-\kappa/2}\int_K \Psi_{\mathcal{H},\theta(\varepsilon,\delta)}(\varepsilon t,y,z)\Psi_{h,\delta}(t,x,y)d\mu_{\langle h^\perp\rangle}(z) \\
&\lesssim \frac{\theta(\varepsilon,\delta)^{\hat{\kappa}/2}\Psi_{h,\delta}(t,x,y)}{\delta^{3\kappa/2}}\cdot\frac{\mu_{\langle h^\perp\rangle}\big(B_{\sqrt{\varepsilon t}}(y,\rho_{\mathcal{H}})\big)}{\mu\big(B_{\sqrt{\varepsilon t}}(y,\rho_{\mathcal{H}})\big)} \\
&\lesssim \frac{\Psi_{h,\delta}(t,x,y)}{\delta^{\frac{3}{2}\kappa+\frac{\hat{\kappa}}{2}}}\cdot\frac{\mu_{\langle h^\perp\rangle}\big(B_{\sqrt{\delta t}}(y,\rho_{\mathcal{H}})\big)}{\mu\big(B_{\sqrt{\delta t}}(y,\rho_{\mathcal{H}})\big)}. \qquad (5.41)
\end{aligned}
$$

Now Theorem 5.9 is immediate from (5.39), (5.40), (5.41) and $\mu = \mu_{\langle h\rangle} + \mu_{\langle h^\perp\rangle}$. □

5.3 Moments of displacement of the diffusion

The purpose of this subsection is to present an application of Theorem 5.8 to asymptotics of moments of displacement of the corresponding diffusion. The main result is the following.

Theorem 5.16. *Let $x \in V_*$ and $\alpha \in (-1,\infty)$. Then*

$$\lim_{t\downarrow 0}\frac{1}{t^{\alpha/2}}\int_K \rho_{\mathcal{H}}(x,y)^\alpha p_\mu(t,x,y)d\mu(y) = \int_{\mathbb{R}}|y|^\alpha\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy. \qquad (5.42)$$

Note that, if $X = \big(\Omega,\mathcal{M},\{X_t\}_{t\in[0,\infty]},\{\mathbf{P}_x\}_{x\in K_\partial}\big)$ is a μ-symmetric diffusion on K whose Dirichlet form on $L^2(K,\mu)$ is $(\mathcal{E},\mathcal{F})$, as in the previous subsection, then

$$\int_K \rho_{\mathcal{H}}(x,y)^\alpha p_\mu(t,x,y)d\mu(y) = \mathbf{E}_x[\rho_{\mathcal{H}}(x,X_t)^\alpha], \qquad (t,x)\in(0,\infty)\times K. \qquad (5.43)$$

(5.42) says that, in the short time limit, the moment $\mathbf{E}_x[\rho_{\mathcal{H}}(x,X_t)^\alpha]$ of displacement of X at $x \in V_*$ is asymptotically equal to that of one-dimensional Brownian motion.

Proof of Theorem 5.16. Since (5.42) for $\alpha = 0$ is trivial, we assume $\alpha \neq 0$. The following proof is based on the same idea as the proof of Theorem 5.3. It suffices to prove that

$$\lim_{n\to\infty}\bigl((\tfrac{3}{5})^{2n}t\bigr)^{-\alpha/2}I_{x,\alpha}\bigl((\tfrac{3}{5})^{2n}t\bigr) = \int_{\mathbb{R}}|y|^{\alpha}\frac{e^{-y^2/2}}{\sqrt{2\pi}}dy \tag{5.44}$$

for any $t \in (0,1)$, where $I_{x,\alpha}(t) := \int_K \rho_{\mathcal{H}}(x,y)^{\alpha}p_{\mu}\bigl(t,x,y\bigr)d\mu(y)$, $t \in (0,\infty)$; indeed, since $r/c_{x.1} \leq \mu(B_r(x,\rho_{\mathcal{H}})) \leq c_{x.2}r$ for any $r \in (0,1]$ for some $c_{x.1}, c_{x.2} \in (0,\infty)$ by (5.3), using (3.16) we have for any $t \in (0,1)$, similarly to the proof of Lemma 5.13,

$$\begin{aligned}
&\int_K \rho_{\mathcal{H}}(x,y)^{\alpha}\Psi_{\mathcal{H},1}(t,x,y)d\mu(y)\\
&= \sum_{n\in\mathbb{Z}}\int_{B_{2^n\sqrt{t}}(x,\rho_{\mathcal{H}})\setminus B_{2^{n-1}\sqrt{t}}(x,\rho_{\mathcal{H}})}\rho_{\mathcal{H}}(x,y)^{\alpha}\Psi_{\mathcal{H},1}(t,x,y)d\mu(y)\\
&\leq \sum_{n\in\mathbb{Z}}2^{|\alpha|}t^{\alpha/2}2^{\alpha n}e^{-4^{n-2}}\frac{\mu\bigl(B_{2^n\sqrt{t}}(x,\rho_{\mathcal{H}})\bigr)}{\mu\bigl(B_{\sqrt{t}}(x,\rho_{\mathcal{H}})\bigr)} \leq C_{x,\alpha}t^{\alpha/2},
\end{aligned} \tag{5.45}$$

where $C_{x,\alpha} := 2^{|\alpha|}\sum_{n\in\mathbb{N}\cup\{0\}}\bigl(c_{x.1}c_{x.2}2^{-(1+\alpha)n} + c_{\mathrm{V}}2^{(\kappa+\alpha)n}e^{-4^{n-2}}\bigr)$. Then by (5.23),

$$\left|\frac{dI_{x,\alpha}}{dt}(t)\right| = \left|\int_K \rho_{\mathcal{H}}(x,y)^{\alpha}\partial_t p_{\mu}(t,x,y)d\mu(y)\right| \leq c_{\mathrm{hk}}(1)C_{x,\alpha}t^{\alpha/2-1} \tag{5.46}$$

for $t \in (0,1)$, from which and (5.44) we can easily verify (5.42).

For the proof of (5.44), suppose $x \in V_* \setminus V_0$; the case of $x \in V_0$ is proved in the same way. Let $w \in W_*$, $i,j \in S$, $a_x^i, b_x^i, a_x^j, b_x^j \in \mathbb{R}$ and $N_x \in \mathbb{N}$ be as in Definition 5.1, and let $c_x^i, c_x^j \in \mathbb{R}$ be such that $h_x^{\perp}\circ F_{wi} = c_x^i h_2^j$ and $h_x^{\perp}\circ F_{wj} = c_x^j h_2^i$ (see the proof of Lemma 5.5). Let $t \in (0,1)$ and set $g_n^l(y) := (5/3)^n\rho_{\mathcal{H}}\bigl(x, F_{wkl^n}(y)\bigr)$ for $(k,l) \in \{(i,j),(j,i)\}$, $n \in \mathbb{N}$ and $y \in K$. Recalling $\mu_{\langle u\rangle}|_{K_v} = (5/3)^{|v|}\mu_{\langle u\circ F_v\rangle}\circ F_v^{-1}$, $u \in \mathcal{F}$, $v \in W_*$, similarly to (5.11) we have

$$\begin{aligned}
&\Bigl(\frac{5}{3}\Bigr)^{n\alpha}\int_{K_{wij^n}\cup K_{wji^n}}\rho_{\mathcal{H}}(x,y)^{\alpha}p_{\mu}\bigl((\tfrac{3}{5})^{2n}t,x,y\bigr)d\mu(y)\\
&= \Bigl(\frac{5}{3}\Bigr)^{|w|+1}\sum_{(k,l)\in\{(i,j),(j,i)\}}\int_K g_n^l(y)^{\alpha}\Bigl(\frac{3}{5}\Bigr)^n p_{\mu}\bigl((\tfrac{3}{5})^{2n}t,x,F_{wkl^n}(y)\bigr)\cdot\\
&\qquad d\Bigl(|a_x^i|^2\mu_{\langle h_1^l\rangle} + 2\cdot 3^{-n}a_x^k b_x^k\mu_{\langle h_1^l,h_2^l\rangle} + 9^{-n}(|b_x^k|^2+|c_x^k|^2)\mu_{\langle h_2^l\rangle}\Bigr)(y).
\end{aligned} \tag{5.47}$$

Let $(k,l) \in \{(i,j),(j,i)\}$ and $y \in K$. (5.13) immediately implies that

$$\lim_{n\to\infty}\Bigl(\frac{3}{5}\Bigr)^n p_{\mu}\bigl((\tfrac{3}{5})^{2n}t,x,F_{wkl^n}(y)\bigr) = \frac{\exp\bigl(-\frac{|a_x^i h_1^l(y)|^2}{2t}\bigr)}{\xi_x\sqrt{2\pi t}}. \tag{5.48}$$

Moreover for $n \in \mathbb{N}$ with $n \geq N_x$, [23, Theorem 3.2.5 and Example 3.2.6] easily yield $g_n^l(y) \geq (5/3)^n|h_x\circ F_{wkl^n}(y)| \geq |a_x^i|h_1^l(y)/2$ and therefore by (5.23),

$$\begin{aligned}
&g_n^l(y)^{\alpha}\Bigl(\frac{3}{5}\Bigr)^n p_{\mu}\bigl((\tfrac{3}{5})^{2n}t,x,F_{wkl^n}(y)\bigr)\\
&\lesssim c_{x.1}g_n^l(y)^{\alpha}\frac{\exp\bigl(-\frac{g_n^l(y)^2}{4t}\bigr)}{\sqrt{t}} \leq \begin{cases} c_{x.1}\alpha^{\alpha/2}t^{(\alpha-1)/2} & \text{if } \alpha \in (0,\infty),\\ c_{x.1}2^{-\alpha}|a_x^i|^{\alpha}t^{-1/2}h_1^l(y)^{\alpha} & \text{if } \alpha \in (-1,0).\end{cases}
\end{aligned} \tag{5.49}$$

Here $\int_K h_1^l(y)^\alpha d\mu(y) = \sum_{n\in\mathbb{N}} \int_{(h_1^l)^{-1}((2^{-n},2^{1-n}])} h_1^l(y)^\alpha d\mu(y) < \infty$ if $\alpha \in (-1,0)$, since $\mu_{\langle h_1^l\rangle} \circ (h_1^l)^{-1} = \mathbf{1}_{[0,1]}dy$ by (4.11) and $\mu_{\langle h_2^l\rangle}\bigl((h_1^l)^{-1}([0,2^{-n}])\bigr) \le 225\cdot 2^{-\hat{\kappa}n}$ for $n \in \mathbb{N}$ by (4.13) and Lemma 5.5. Thus by virtue of dominated convergence based on (5.49), from (5.47), (5.48), (5.8) and $\mu_{\langle h_1^l\rangle} \circ (h_1^l)^{-1} = \mathbf{1}_{[0,1]}dy$ we conclude that

$$\lim_{n\to\infty}\Bigl(\frac{5}{3}\Bigr)^{n\alpha}\int_{K_{wij^n}\cup K_{wji^n}} \rho_{\mathcal{H}}(x,y)^\alpha p_\mu\bigl((\tfrac{3}{5})^{2n}t,x,y\bigr)d\mu(y) = t^{\alpha/2}\int_{-|a_x^i|/\sqrt{t}}^{|a_x^i|/\sqrt{t}} |y|^\alpha \frac{e^{-y^2/2}}{\sqrt{2\pi}}dy; \tag{5.50}$$

note that $(\int_K f d\mu_{\langle u,v\rangle})^2 \le \int_K f d\mu_{\langle u\rangle}\int_K f d\mu_{\langle v\rangle}$ for $u,v \in \mathcal{F}$ and a bounded Borel measurable function $f : K \to [0,\infty)$.

On the other hand, let $n \in \mathbb{N}$, $n \ge N_x$ and define $\rho_{\mathcal{H}}^{x,n}(y) := (5/3)^n\rho_{\mathcal{H}}(x,y)$ for $y \in K$. Then Proposition 5.4-(4) yields $\rho_{\mathcal{H}}^{x,n}(y) \ge (5/3)^n\rho_{h_x}(x,y) \ge 8|a_x^i|/9$ for $y \in K \setminus (K_{wij^n} \cup K_{wji^n})$, and therefore by (5.23) with $\delta = 1/2$ and (5.27),

$$\begin{aligned}
&\Bigl(\frac{5}{3}\Bigr)^{n\alpha}\int_{K\setminus(K_{wij^n}\cup K_{wji^n})} \rho_{\mathcal{H}}(x,y)^\alpha p_\mu\bigl((\tfrac{3}{5})^{2n}t,x,y\bigr)d\mu(y)\\
&\lesssim \int_{K\setminus(K_{wij^n}\cup K_{wji^n})} \rho_{\mathcal{H}}^{x,n}(y)^\alpha \exp\biggl(-\frac{\rho_{\mathcal{H}}^{x,n}(y)^2}{12t}\biggr)\Psi_{\mathcal{H},1}\bigl((\tfrac{3}{5})^{2n}t,x,y\bigr)d\mu(y)\\
&\lesssim \bigl((50|\alpha|)^{\alpha/2} \vee (|a_x^i|/2)^\alpha\bigr)\exp\biggl(-\frac{|a_x^i|^2}{16t}\biggr).
\end{aligned} \tag{5.51}$$

Now (5.44) easily follows by substituting t by $(3/5)^{2N}t$ ($N \in \mathbb{N}$) in (5.50) and (5.51) and using them to let $n \to \infty$ first and then $N \to \infty$. Thus the proof of Theorem 5.16 is complete. □

6 On-diagonal asymptotics at almost every point

So far we have established Gaussian off-diagonal behaviors of the heat kernels as well as several one-dimensional asymptotics at each $x \in V_*$. In this and next sections, we will verify that $p_\mu(t,x,x)$ and $p_{\mu_{\langle h\rangle}}(t,x,x)$ for $h \in \mathcal{S}_{\mathcal{H}_0}$ exhibit *non-integer-dimensional* asymptotic behaviors as $t \downarrow 0$ for μ-a.e. $x \in K$.

The following is the main theorem of this section. Note that, for each $h \in \mathcal{S}_{\mathcal{H}_0}$, the term "*$\mu$-a.e.*" is a synonym for "*$\mu_{\langle h\rangle}$-a.e.*" since μ and $\mu_{\langle h\rangle}$ are mutually absolutely continuous by [17, Theorem 5.6]. Note also that $2\log_{25/3}5 = 1.5181\ldots < 2$.

Theorem 6.1. *There exists $d_{\mathrm{S}}^{\mathrm{loc}} \in (1, 2\log_{25/3}5]$ such that for each $h \in \mathcal{S}_{\mathcal{H}_0}$,*

$$\lim_{t\downarrow 0}\frac{2\log p_\mu(t,x,x)}{-\log t} = \lim_{t\downarrow 0}\frac{2\log p_{\mu_{\langle h\rangle}}(t,x,x)}{-\log t} = d_{\mathrm{S}}^{\mathrm{loc}} \qquad \mu\text{-a.e. } x \in K. \tag{6.1}$$

Remark 6.2. (1) We have a concrete expression for $d_{\mathrm{S}}^{\mathrm{loc}}$; see (6.10) and (6.12).
(2) In Theorem 7.2 below we will show that $d_{\mathrm{S}}^{\mathrm{loc}} \le \dim_{\mathrm{H}}(K,\rho_{\mathcal{H}})$, where $\dim_{\mathrm{H}}$ denotes Hausdorff dimension. Unfortunately, *the author has no idea whether $d_{\mathrm{S}}^{\mathrm{loc}} = \dim_{\mathrm{H}}(K,\rho_{\mathcal{H}})$ or not.*

The limit $\lim_{t\downarrow 0}\log p_\nu(t,x,x)/(-\log t)$, if exists, is often called the *local spectral dimension at x for the Dirichlet space* $(K,\nu,\mathcal{E},\mathcal{F})$. (6.1) says that the local spectral dimensions at x for $(K,\mu,\mathcal{E},\mathcal{F})$ and $(K,\mu_{\langle h\rangle},\mathcal{E},\mathcal{F})$ exist and are equal to a *non-integer* constant $d_{\mathrm{S}}^{\mathrm{loc}}$ for μ-a.e. $x\in K$.

One of the keys to Theorem 6.1 is the ergodicity of the Kusuoka measure μ (to be precise, of the measure $\lambda=\mu\circ\pi$) which has been obtained in [29, Example 1]. Unfortunately, however, the proof of this fact in [29] is indirect and complicated. We provide an alternative simple proof of it at the end of this section based on the self-similarity (2.2) of $(\mathcal{E},\mathcal{F})$.

Now we proceed to the proof of Theorem 6.1. We start with an easy lemma.

Lemma 6.3. *For any $\omega\in\Sigma$ and any $x\in\mathbb{R}^2\setminus\{0\}$,*

$$\log\frac{\sqrt{3}}{5}\le\liminf_{m\to\infty}\frac{\log\|T_{[\omega]_m}\|}{m}\le\limsup_{m\to\infty}\frac{\log\|T_{[\omega]_m}\|}{m}\le\log\frac{3}{5},\tag{6.2}$$

$$\log\frac{1}{5}\le\liminf_{m\to\infty}\frac{\log\bigl|T^*_{[\omega]_m}x\bigr|}{m}\le\limsup_{m\to\infty}\frac{\log\bigl|T^*_{[\omega]_m}x\bigr|}{m}\le\log\frac{3}{5}.\tag{6.3}$$

Proof. Since $\|A\|^2\ge 2|\det A|$ for any $A\in\mathcal{L}(\mathbb{R}^2)$, Proposition 2.12-(i) and (3.4) imply that

$$\sqrt{2}\bigl(\sqrt{3}/5\bigr)^{|w|}=\sqrt{2|\det T_w|}\le\|T_w\|\le(3/5)^{|w|}\|T_\emptyset\|=\sqrt{2}(3/5)^{|w|}\tag{6.4}$$

for any $w\in W_*$, which immediately yields (6.2). Similarly (6.3) follows by applying (3.5) to $h:=|x|^{-1}(x_1h_1+x_2h_2)$, where $x=(x_1,x_2)$. □

The following two propositions completely characterize when the local spectral dimensions at $\pi(\omega)$ exist for a given $\omega\in\Sigma$, in terms of the asymptotic behavior as $m\to\infty$ of the logarithms of the norms $\|T_{[\omega]_m}\|$ and $\bigl|T^*_{[\omega]_m}x\bigr|$, $x\in\mathbb{R}^2\setminus\{0\}$.

Proposition 6.4. *Let $\omega\in\Sigma$. Then it holds that*

$$\begin{aligned}\liminf_{t\downarrow 0}\frac{2\log p_\mu(t,\pi(\omega),\pi(\omega))}{-\log t}&=2+\frac{\log\frac{5}{3}}{\limsup_{m\to\infty}\frac{1}{m}\log\|T_{[\omega]_m}\|}\ge 1,\\ \limsup_{t\downarrow 0}\frac{2\log p_\mu(t,\pi(\omega),\pi(\omega))}{-\log t}&=2+\frac{\log\frac{5}{3}}{\liminf_{m\to\infty}\frac{1}{m}\log\|T_{[\omega]_m}\|}\le 2\log_{25/3}5.\end{aligned}\tag{6.5}$$

In particular, the limit $\lim_{t\downarrow 0}2\log p_\mu(t,\pi(\omega),\pi(\omega))/(-\log t)$ *exists if and only if so does* $\lim_{m\to\infty}\frac{1}{m}\log\|T_{[\omega]_m}\|$, *and if either of these two limits exists then*

$$\lim_{t\downarrow 0}\frac{2\log p_\mu(t,\pi(\omega),\pi(\omega))}{-\log t}=2+\frac{\log\frac{5}{3}}{\lim_{m\to\infty}\frac{1}{m}\log\|T_{[\omega]_m}\|}\in[1,2\log_{25/3}5].\tag{6.6}$$

Proof. Let $(s,x)\in(0,1]\times K$ and let $w\in\Lambda^{\mathcal{H}}_s$ satisfy $x\in K_w$. Then $\mu(U_s(x,\mathcal{S}^{\mathcal{H}}))\asymp\mu(K_w)$ by (3.9), and therefore (3.11) and (3.16) easily imply that

$$\mu(B_s(x,\rho_{\mathcal{H}}))\asymp\mu(U_s(x,\mathcal{S}^{\mathcal{H}}))\asymp\mu(K_w).\tag{6.7}$$

Let $\omega\in\Sigma$, and for each $t\in(0,1)$ let $m(t)$ be the unique $m\in\mathbb{N}$ satisfying $[\omega]_m\in\Lambda^{\mathcal{H}}_{\sqrt{t}}$. Then for $t\in(0,1)$, $p_\mu(t,\pi(\omega),\pi(\omega))\asymp\mu\bigl(K_{[\omega]_{m(t)}}\bigr)^{-1}$ by (4.2) and (6.7), and (3.4) yields $\|T_{[\omega]_{m(t)}}\| \leq \sqrt{t} \leq 5\|T_{[\omega]_{m(t)}}\|$. Moreover, $m : (0,1) \to \mathbb{N}$ is a non-decreasing surjection since $\mathbb{N} \ni m \mapsto l_{\mathcal{H}}([\omega]_m)$ is strictly decreasing by (3.4). It follows from these facts that

$$\limsup_{t\downarrow 0} \frac{2\log p_\mu(t,\pi(\omega),\pi(\omega))}{-\log t} = \limsup_{m\to\infty} \frac{-\log \mu(K_{[\omega]_m})}{-\log \|T_{[\omega]_m}\|} = \limsup_{m\to\infty}\biggl(2+\frac{m\log\frac{5}{3}}{\log\|T_{[\omega]_m}\|}\biggr)$$

and similarly for lim inf, which together with (6.2) immediately shows the assertion. □

Proposition 6.5. *Let $h \in \mathcal{S}_{\mathcal{H}_0}$ and take $\zeta_h = (\zeta_h^1, \zeta_h^2) \in \mathbb{R}^2$ so that $h - \zeta_h^1 h_1 - \zeta_h^2 h_2 \in \mathbb{R}\mathbf{1}$. Let $\omega \in \Sigma$. Then it holds that*

$$\begin{aligned} \liminf_{t\downarrow 0} \frac{2\log p_{\mu_{\langle h\rangle}}(t,\pi(\omega),\pi(\omega))}{-\log t} &= 2 + \frac{\log\frac{5}{3}}{\limsup_{m\to\infty}\frac{1}{m}\log\bigl|T^*_{[\omega]_m}\zeta_h\bigr|} \geq 1, \\ \limsup_{t\downarrow 0} \frac{2\log p_{\mu_{\langle h\rangle}}(t,\pi(\omega),\pi(\omega))}{-\log t} &= 2 + \frac{\log\frac{5}{3}}{\liminf_{m\to\infty}\frac{1}{m}\log\bigl|T^*_{[\omega]_m}\zeta_h\bigr|} \leq \kappa. \end{aligned} \tag{6.8}$$

In particular, the limit $\lim_{t\downarrow 0} 2\log p_{\mu_{\langle h\rangle}}(t,\pi(\omega),\pi(\omega))/(-\log t)$ exists if and only if so does $\lim_{m\to\infty}\frac{1}{m}\log\bigl|T^*_{[\omega]_m}\zeta_h\bigr|$, and if either of these two limits exists then

$$\lim_{t\downarrow 0} \frac{2\log p_{\mu_{\langle h\rangle}}(t,\pi(\omega),\pi(\omega))}{-\log t} = 2 + \frac{\log\frac{5}{3}}{\lim_{m\to\infty}\frac{1}{m}\log\bigl|T^*_{[\omega]_m}\zeta_h\bigr|} \in [1,\kappa]. \tag{6.9}$$

Proof. The proof goes in exactly the same way as that of Proposition 6.4 by using (3.5), (3.12) and (6.3) instead of (3.4), (3.11) and (6.2) respectively. □

Proposition 6.6. *Let ν be a Borel probability measure on Σ which satisfies $\nu\circ\sigma^{-1} = \nu$ and is ergodic with respect to the shift map $\sigma : \Sigma \to \Sigma$. Define*

$$\eta(\nu,\{T_i\}_{i\in S}) := \inf_{m\in\mathbb{N}} \frac{1}{m}\sum_{w\in W_m} \nu(\Sigma_w)\log\|T_w\|. \tag{6.10}$$

Then $\eta(\nu,\{T_i\}_{i\in S}) = \lim_{m\to\infty}\frac{1}{m}\sum_{w\in W_m}\nu(\Sigma_w)\log\|T_w\| \in \bigl[\log\frac{\sqrt{3}}{5}, \log\frac{3}{5}\bigr]$ and

$$\lim_{t\downarrow 0} \frac{2\log p_\mu(t,\pi(\omega),\pi(\omega))}{-\log t} = 2 + \frac{\log\frac{5}{3}}{\eta(\nu,\{T_i\}_{i\in S})} \qquad \nu\text{-a.e. } \omega\in\Sigma. \tag{6.11}$$

Moreover, $\eta(\nu,\{T_i\}_{i\in S}) = \log\frac{3}{5}$ if and only if $\nu(\{1^\infty,2^\infty,3^\infty\}) = 1$.

Proof. Apart from the final assertion, this is immediate from (6.4), Proposition 6.4 and Kingman's subadditive ergodic theorem [9, Theorem 10.7.1], and the same results are valid with $\eta(\nu,\{T_i\}_{i\in S})$ unchanged if the norm $\|\cdot\|$ is replaced by the operator norm $\|\cdot\|_{\mathrm{op}}$ given by $\|A\|_{\mathrm{op}} := \sup_{x\in\mathbb{R}^2,\,|x|\leq 1}|Ax|$, $A\in\mathcal{L}(\mathbb{R}^2)$; note that $\|AB\| \leq \|A\|\|B\|$ and $\|AB\|_{\mathrm{op}} \leq \|A\|_{\mathrm{op}}\|B\|_{\mathrm{op}}$ for $A,B\in\mathcal{L}(\mathbb{R}^2)$. If $\nu(\{1^\infty,2^\infty,3^\infty\}) = 1$ then clearly $\eta(\nu,\{T_i\}_{i\in S}) = \log\frac{3}{5}$. Conversely suppose $\eta(\nu,\{T_i\}_{i\in S}) = \log\frac{3}{5}$. Let $m\in\mathbb{N}$. Since $\|T_i\|_{\mathrm{op}} = 3/5$, $i\in S$, we have $\frac{1}{m}\log\|T_w\|_{\mathrm{op}} \leq \log\frac{3}{5}$ for $w\in W_m$ and hence $\frac{1}{m}\sum_{w\in W_m}\nu(\Sigma_w)\log\|T_w\|_{\mathrm{op}} \leq \log\frac{3}{5}$, where actually the equality holds by $\eta(\nu,\{T_i\}_{i\in S}) = \log\frac{3}{5}$ and (6.10) for the norm $\|\cdot\|_{\mathrm{op}}$. Therefore for each $w\in W_m$, $\nu(\Sigma_w)\bigl(\frac{1}{m}\log\|T_w\|_{\mathrm{op}} - \log\frac{3}{5}\bigr) = 0$, i.e. either $\nu(\Sigma_w) = 0$ or $\|T_w\|_{\mathrm{op}} = (3/5)^m$, but the latter holds if and only if $w = i^m$ for some $i\in S$ since $\|T_{jk}\|_{\mathrm{op}} < (3/5)^2$ for $j,k\in S$ with $j\neq k$. Thus $\nu\bigl(\bigcup_{i\in S}\Sigma_{i^m}\bigr) = 1$, and letting $m\to\infty$ yields $\nu(\{1^\infty,2^\infty,3^\infty\}) = 1$. □

Proof of Theorem 6.1. Since $\lambda \circ \sigma^{-1} = \lambda$ by Proposition 2.14, λ is ergodic with respect to σ by [29, Example 1] (see also Theorem 6.8 below), $\lambda(\{1^\infty, 2^\infty, 3^\infty\}) = 0$ and $\mu \circ \pi = \lambda$, Proposition 6.6 applies to $\lambda/2$ to imply that

$$\lim_{t\downarrow 0} \frac{2\log p_\mu(t,x,x)}{-\log t} = 2 + \frac{\log \frac{5}{3}}{\eta(\lambda/2, \{T_i\}_{i\in S})} =: d_{\mathrm{S}}^{\mathrm{loc}} \qquad \mu\text{-a.e. } x \in K \tag{6.12}$$

and that $\eta(\lambda/2, \{T_i\}_{i\in S}) \in \bigl[\log \frac{\sqrt{3}}{5}, \log \frac{3}{5}\bigr)$. Thus $d_{\mathrm{S}}^{\mathrm{loc}} \in (1, 2\log_{25/3} 5]$.

Let $h \in \mathcal{S}_{\mathcal{H}_0}$ and set $K_{Z,h} := \{x \in K_Z \mid Z_x\zeta_h \neq 0\}$, where $\zeta_h \in \mathbb{R}^2$ is as in Proposition 6.5. Then $\mu(K\setminus K_{Z,h}) = 0$ since $d\mu_{\langle h\rangle} = |Z\zeta_h|^2 d\mu$ by (2.10) and μ and $\mu_{\langle h\rangle}$ are mutually absolutely continuous. Now for $x \in K_{Z,h}$ and $\omega \in \pi^{-1}(x)$, we easily see $\lim_{m\to\infty}\bigl|T^*_{[\omega]_m}\zeta_h\bigr|/\|T_{[\omega]_m}\| = |Z_x\zeta_h|$ and hence $\lim_{t\downarrow 0} 2\log p_{\mu_{\langle h\rangle}}(t,x,x)/(-\log t) = d_{\mathrm{S}}^{\mathrm{loc}}$ if and only if $\lim_{t\downarrow 0} 2\log p_\mu(t,x,x)/(-\log t) = d_{\mathrm{S}}^{\mathrm{loc}}$ by Propositions 6.4 and 6.5, proving (6.1) by virtue of (6.12) and $\mu(K \setminus K_{Z,h}) = 0$. □

Remark 6.7. (1) We can estimate $d_{\mathrm{S}}^{\mathrm{loc}}$ numerically by using (6.10); numerical computations of the right-hand side of (6.10) with $\nu = \lambda/2$ tell us that $d_{\mathrm{S}}^{\mathrm{loc}} \geq 1.27695\ldots$ for $m = 14$, $d_{\mathrm{S}}^{\mathrm{loc}} \geq 1.27790\ldots$ for $m = 15$ and $d_{\mathrm{S}}^{\mathrm{loc}} \geq 1.27874\ldots$ for $m = 16$.
(2) Barlow and Kumagai [4, Corollary 3.6] have proved the $\hat{\nu}$-a.e. existence of the (constant) local spectral dimension $d_{\mathrm{S}}^{\mathrm{loc}}(\hat{\mu}, \hat{\nu})$ and have explicitly calculated it for the heat kernels $p_{\hat{\mu}}(t,x,y)$ on post-critically finite self-similar sets and Sierpinski carpets, when both the reference measure $\hat{\mu}$ of the Dirichlet space and another measure $\hat{\nu}$ are self-similar measures. In their case, the self-similarity of $\hat{\mu}$ has made the explicit calculation of $d_{\mathrm{S}}^{\mathrm{loc}}(\hat{\mu}, \hat{\nu})$ possible and we easily see how it varies depending on the weight of $\hat{\nu}$, whereas it seems very difficult to estimate $\eta(\nu, \{T_i\}_{i\in S})$ and see its dependence on ν in the situation of Proposition 6.6 above, even when ν is a Bernoulli measure on Σ.

At the end of this section, we give a new simple proof of the ergodicity of the measure $\lambda = \mu \circ \pi$.

Theorem 6.8 **([29]).** *The measure λ is ergodic with respect to the shift map $\sigma : \Sigma \to \Sigma$.*

Proof. Let $A \in \mathcal{B}(\Sigma)$ satisfy $\sigma^{-1}(A) = A$. Set $\mathcal{E}_A(u,v) := \lambda_{\langle u,v\rangle}(A)/2$ for $u, v \in \mathcal{F}$, so that $\mathcal{E}_A : \mathcal{F}\times\mathcal{F} \to \mathbb{R}$ is a non-negative definite symmetric bilinear form satisfying $\mathcal{E}_A(u,u) \leq \mathcal{E}(u,u)$, $u \in \mathcal{F}$. We claim that there exists $c_A \in [0,1]$ such that

$$\mathcal{E}_A(u,v) = c_A\mathcal{E}(u,v), \qquad u, v \in \mathcal{F}. \tag{6.13}$$

Note that $\lambda_{\langle u,v\rangle} \circ \sigma_i = (5/3)\lambda_{\langle u\circ F_i, v\circ F_i\rangle}$ for $u, v \in \mathcal{F}$ and $i \in S$. Since $A = \sigma^{-1}(A) = \bigcup_{i\in S}\sigma_i(A)$ we see that for any $u, v \in \mathcal{F}$,

$$\mathcal{E}_A(u,v) = \frac{1}{2}\sum_{i\in S}\lambda_{\langle u,v\rangle}(\sigma_i(A)) = \frac{5}{3}\sum_{i\in S}\frac{1}{2}\lambda_{\langle u\circ F_i, v\circ F_i\rangle}(A) = \frac{5}{3}\sum_{i\in S}\mathcal{E}_A(u\circ F_i, v\circ F_i). \tag{6.14}$$

By $\mathcal{E}_A(\mathbf{1},\mathbf{1}) = 0$ we can regard $\mathcal{E}_A$ as a non-negative definite symmetric bilinear form on $\mathcal{H}_0/\mathbb{R}\mathbf{1}$, and let Q_A be its matrix representation through the basis $\{h_1, h_2\}$ of $\mathcal{H}_0/\mathbb{R}\mathbf{1}$. Then (6.14) together with Proposition 2.12-(ii) yields $Q_A = (5/3)\sum_{i\in S} T_i Q_A T_i^*$, based on which a direct calculation using Proposition 2.12-(i) easily shows that $Q_A = c_A\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ for some $c_A \in [0,1]$. Thus (6.13) holds for any $u, v \in \mathcal{H}_0$, hence for any $u, v \in \bigcup_{m\in\mathbb{N}}\mathcal{H}_m$ by (6.14) and (2.2), and then also for any $u, v \in \mathcal{F}$ since $\bigcup_{m\in\mathbb{N}}\mathcal{H}_m/\mathbb{R}\mathbf{1}$ is dense in $(\mathcal{F}/\mathbb{R}\mathbf{1}, \mathcal{E})$ and $\mathcal{E}_A(u,u) \leq \mathcal{E}(u,u)$, $u \in \mathcal{F}$.

Let $u, f \in \mathcal{F}$. By [13, Lemma 3.2.5] and the strong locality of $(\mathcal{E},\mathcal{F})$,

$$d\Big(\mu_{\langle uf,u\rangle} - \frac{1}{2}\mu_{\langle u^2,f\rangle}\Big) = ud\mu_{\langle f,u\rangle} + fd\mu_{\langle u\rangle} - ud\mu_{\langle u,f\rangle} = fd\mu_{\langle u\rangle}. \tag{6.15}$$

By Propositon 2.8, (6.13) and (2.4), the value of (6.15) on $\pi(A)$ results in

$$\int_{\pi(A)} fd\mu_{\langle u\rangle} = \lambda_{\langle uf,u\rangle}(A) - \frac{1}{2}\lambda_{\langle u^2,f\rangle}(A) = 2\mathcal{E}_A(uf,u) - \mathcal{E}_A(u^2,f) = c_A\int_K fd\mu_{\langle u\rangle},$$

which implies that $\mathbf{1}_{\pi(A)}\cdot\mu_{\langle u\rangle} = c_A\mu_{\langle u\rangle}$ since $\mathcal{F}$ is dense in $(C(K),\|\cdot\|_\infty)$. In particular, we have $0 = c_A\mu_{\langle u\rangle}(K\setminus\pi(A)) = c_A\lambda_{\langle u\rangle}(\Sigma\setminus A)$. Now suppose $\lambda(A) > 0$. Then $c_A > 0$ by (6.13) and hence $\lambda_{\langle u\rangle}(\Sigma\setminus A) = 0$ for any $u\in\mathcal{F}$. Thus $\lambda(\Sigma\setminus A) = 0$. □

7 Eigenvalues of the Laplacian

In this last section, we show that the Hausdorff and box-counting dimensions of $(K,\rho_{\mathcal{H}})$ naturally arise as the asymptotic order of the eigenvalues of the Laplacian associated with $(K,\mu,\mathcal{E},\mathcal{F})$ and that those dimensions are *not integers*, as in Theorem 1.3-(3).

Let us first recall the following standard notations and definitions. See e.g. [10, Section 2.1] and references therein for details of Hausdorff measure, Hausdorff dimension and box-counting dimension; note that the definitions there apply to any metric space although they are stated only for subsets of the Euclidean spaces.

Notation. Let (E,ρ) be a metric space and let $A\subset E$ be non-empty.
(1) For $\alpha\in(0,\infty)$, the α-dimensional Hausdorff measure and the Hausdorff dimension of A with respect to ρ are denoted by $\mathcal{H}^\alpha(A,\rho)$ and $\dim_{\mathrm{H}}(A,\rho)$, respectively.
(2) The lower and upper box-counting dimensions of A with respect to ρ are denoted by $\underline{\dim}_{\mathrm{B}}(A,\rho)$ and $\overline{\dim}_{\mathrm{B}}(A,\rho)$, respectively. If they are equal, their common value, called the box-counting dimension of A with respect to ρ, is denoted by $\dim_{\mathrm{B}}(A,\rho)$.

Note that $0\le\dim_{\mathrm{H}}(A,\rho)\le\underline{\dim}_{\mathrm{B}}(A,\rho)\le\overline{\dim}_{\mathrm{B}}(A,\rho)\le\infty$ by [10, (2.14)].

Definition 7.1. Let ν be a finite Borel measure on K with full support. Noting that the non-positive self-adjoint operator Δ_ν of $(K,\nu,\mathcal{E},\mathcal{F})$ (the generator of $\{T^\nu_t\}_{t\in(0,\infty)}$) has discrete spectrum and that $\operatorname{tr}T^\nu_t < \infty$ for $t\in(0,\infty)$ by [8, Theorem 2.1.4], let $\{\lambda^\nu_n\}_{n\in\mathbb{N}}$ be the non-decreasing enumeration of all the eigenvalues of $-\Delta_\nu$, where each eigenvalue is repeated according to its multiplicity. The *eigenvalue counting function* $\mathcal{N}_\nu$ and the *partition function* $\mathcal{Z}_\nu$ *of the Dirichlet space* $(K,\nu,\mathcal{E},\mathcal{F})$ are defined respectively by

$$\mathcal{N}_\nu(s) := \#\{n\in\mathbb{N}\mid\lambda^\nu_n\le s\},\quad s\in\mathbb{R}, \tag{7.1}$$

$$\mathcal{Z}_\nu(t) := \sum_{n\in\mathbb{N}} e^{-t\lambda^\nu_n} = \int_{[0,\infty)} e^{-ts}d\mathcal{N}_\nu(s) = \int_K p_\nu(t,x,x)d\nu(x),\quad t\in(0,\infty). \tag{7.2}$$

In the situation of Definition 7.1, $\mathcal{N}_\nu(0) = 1$ by $\lambda^\nu_1 = 0 < \lambda^\nu_2$, and $\mathcal{N}_\nu(s) < \infty$ for any $s\in[0,\infty)$ since $\lim_{n\to\infty}\lambda^\nu_n = \infty$. Moreover, $\mathcal{Z}_\nu$ is $(0,\infty)$-valued and continuous.

We now state the main theorem of this section. Recall the constant $d^{\mathrm{loc}}_{\mathrm{S}}\in(1,2\log_{25/3}5]$ given in Theorem 6.1.

Theorem 7.2. *Set $d_{\mathrm{S}} := \dim_{\mathrm{H}}(K, \rho_{\mathcal{H}})$. Then $\mathcal{H}^{d_{\mathrm{S}}}(K, \rho_{\mathcal{H}}) \in (0, \infty)$, and for any $h \in \mathcal{S}_{\mathcal{H}_0}$,*

$$d_{\mathrm{S}} = \dim_{\mathrm{B}}(K, \rho_{\mathcal{H}}) = \dim_{\mathrm{B}}(K, \rho_h) \in [d_{\mathrm{S}}^{\mathrm{loc}}, 2\log_{25/3} 5]. \tag{7.3}$$

Moreover, there exist $c_{7.1}, c_{7.2} \in (0, \infty)$ such that for any $h \in \mathcal{S}_{\mathcal{H}_0}$, any $s \in [1, \infty)$ and any $t \in (0, 1]$,

$$c_{7.1} s^{d_{\mathrm{S}}/2} \leq \mathcal{N}_{\mu}(s) \leq c_{7.2} s^{d_{\mathrm{S}}/2}, \qquad c_{7.1} s^{d_{\mathrm{S}}/2} \leq \mathcal{N}_{\mu_{\langle h \rangle}}(s) \leq c_{7.2} s^{d_{\mathrm{S}}/2}, \tag{7.4}$$
$$c_{7.1} t^{-d_{\mathrm{S}}/2} \leq \mathcal{Z}_{\mu}(t) \leq c_{7.2} t^{-d_{\mathrm{S}}/2}, \qquad c_{7.1} t^{-d_{\mathrm{S}}/2} \leq \mathcal{Z}_{\mu_{\langle h \rangle}}(t) \leq c_{7.2} t^{-d_{\mathrm{S}}/2}. \tag{7.5}$$

Remark 7.3. *The author has no idea whether $d_{\mathrm{S}} = \dim_{\mathrm{H}}(K, \rho_h)$ for $h \in \mathcal{S}_{\mathcal{H}_0}$.* Also the estimate $d_{\mathrm{S}} \leq 2\log_{25/3} 5$ is *by no means* best possible.

The limits $\lim_{s\to\infty} 2\log \mathcal{N}_{\nu}(s)/\log s$ and $\lim_{t\downarrow 0} 2\log \mathcal{Z}_{\nu}(t)/(-\log t)$, if exist, are usually called the *(global) spectral dimension of the Dirichlet space* $(K, \nu, \mathcal{E}, \mathcal{F})$. Theorem 7.2 in particular implies that the spectral dimensions of $(K, \mu, \mathcal{E}, \mathcal{F})$ and $(K, \mu_{\langle h \rangle}, \mathcal{E}, \mathcal{F})$, where $h \in \mathcal{S}_{\mathcal{H}_0}$, exist and are equal to $\dim_{\mathrm{H}}(K, \rho_{\mathcal{H}})$, $\dim_{\mathrm{B}}(K, \rho_{\mathcal{H}})$ and $\dim_{\mathrm{B}}(K, \rho_h)$.

The rest of this section is devoted to the proof of Theorem 7.2, for which the following proposition is fundamental.

Proposition 7.4. *There exist $c_{7.3}, c_{7.4} \in (0, \infty)$ such that for any $h \in \mathcal{S}_{\mathcal{H}_0}$, any $s \in [1, \infty)$ and any $t \in (0, 1]$,*

$$c_{7.3}\#\Lambda^{\mathcal{H}}_{s^{-1/2}} \leq \mathcal{N}_{\mu}(s) \leq c_{7.4}\#\Lambda^{\mathcal{H}}_{s^{-1/2}}, \quad c_{7.3}\#\Lambda^{h}_{s^{-1/2}} \leq \mathcal{N}_{\mu_{\langle h \rangle}}(s) \leq c_{7.4}\#\Lambda^{h}_{s^{-1/2}}, \tag{7.6}$$
$$c_{7.3}\#\Lambda^{\mathcal{H}}_{\sqrt{t}} \leq \mathcal{Z}_{\mu}(t) \leq c_{7.4}\#\Lambda^{\mathcal{H}}_{\sqrt{t}}, \qquad c_{7.3}\#\Lambda^{h}_{\sqrt{t}} \leq \mathcal{Z}_{\mu_{\langle h \rangle}}(t) \leq c_{7.4}\#\Lambda^{h}_{\sqrt{t}}. \tag{7.7}$$

Proof. (7.6) follows from [19, Theorem 4.3 and Proposition 4.4], (3.4), (3.5) and $\mathcal{N}_{\mu}(0) = \mathcal{N}_{\mu_{\langle h \rangle}}(0) = 1$. Then noting that $\#\Lambda^{\mathcal{H}}_{3s/5} \leq 3\#\Lambda^{\mathcal{H}}_{s}$ and $\#\Lambda^{h}_{3s/5} \leq 3\#\Lambda^{h}_{s}$ for $s \in (0, 1)$ by (3.4), (3.5) and [19, Proposition 2.7], we can easily verify (7.7) from (7.6); note also that $\mathcal{Z}_{\nu}(t) = \int_0^\infty e^{-s}\mathcal{N}_{\nu}(s/t)ds$ for $\nu \in \{\mu, \mu_{\langle h \rangle}\}$ and $t \in (0, \infty)$. □

Lemma 7.5. *Let $h \in \mathcal{S}_{\mathcal{H}_0}$. Then $\#\Lambda^{h}_{st} \leq \#\Lambda^{h}_{t}\#\Lambda^{\mathcal{H}}_{s} \leq 3^9\#\Lambda^{h}_{st}$ for any $s, t \in (0, 1]$. In particular, $\#\Lambda^{h}_{s} \leq \#\Lambda^{\mathcal{H}}_{s} \leq 3^9\#\Lambda^{h}_{s}$ for any $s \in (0, 1]$.*

Proof. Since $\Lambda^{\mathcal{H}}_1 = \Lambda^h_1 = \{\emptyset\}$, the latter assertion follows by setting $t = 1$ in the former, which in turn is trivial for $s = 1$. Let $s, t \in (0, 1]$, $s < 1$ and take $\zeta_h = (\zeta_h^1, \zeta_h^2) \in \mathbb{R}^2$ such that $h - \zeta_h^1 h_1 - \zeta_h^2 h_2 \in \mathbb{R}\mathbf{1}$. Then for each $(v, w) \in \Lambda^h_t \times \Lambda^{\mathcal{H}}_s$, $l_h(vw) = |T_w^* T_v^* \zeta_h| \leq \|T_w\||T_v^*\zeta_h| = l_{\mathcal{H}}(w) l_h(v) \leq st$ and hence $vw \leq \tau(v, w)$ for a unique $\tau(v, w) \in \Lambda^h_{st}$. Thus we have a mapping $\tau : \Lambda^h_t \times \Lambda^{\mathcal{H}}_s \to \Lambda^h_{st}$, which is surjective; indeed, if $u \in \Lambda^h_{st}$, then $u1^\infty \in \Sigma_v$ for some $v \in \Lambda^h_t$ and $\sigma^{|v|}(u1^\infty) \in \Sigma_w$ for some $w \in \Lambda^{\mathcal{H}}_s$, so that $u1^\infty \in \Sigma_u \cap \Sigma_{\tau(v,w)}$ and $u = \tau(v, w)$. Therefore $\#\Lambda^h_{st} \leq \#\Lambda^h_t\#\Lambda^{\mathcal{H}}_s$.

Let $\iota : S \to S$ denote the bijection $i \mapsto i + 1 \bmod 3$, so that ι naturally defines a bijection $W_* \to W_*$ given by $w_1 \ldots w_m \mapsto \iota(w_1) \ldots \iota(w_m)$, which we also write as ι. Let $R := \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$. Then for $w \in W_*$, clearly $T_{\iota(w)} = R T_w R^{-1}$ and $\|T_{\iota(w)}\| = \|R T_w R^{-1}\| = \|T_w\|$, and therefore $w \in \Lambda^{\mathcal{H}}_s$ if and only if $\iota(w) \in \Lambda^{\mathcal{H}}_s$. Thus, with $w_1 \in S$ denoting the first component of $w \in W_* \setminus \{\emptyset\}$,

$$\Lambda^{\mathcal{H}}_s = \bigcup_{w \in \Lambda^{\mathcal{H}}_s,\, w_1 = 1} \{w, \iota(w), \iota^2(w)\} \quad \text{(disjoint union)}. \tag{7.8}$$

Let $(v,w)\in\Lambda^h_t\times\Lambda^{\mathcal{H}}_s$. Since $\bigl|\sum_{k=0}^{2}R^kAR^{-k}x\bigr|=(3/2)\sqrt{\|A\|^2+2\det A}|x|$ for $A\in\mathcal{L}(\mathbb{R}^2)$ and $x\in\mathbb{R}^2$ by a direct calculation, we obtain

$$\sum_{k=0}^{2}l_h(v\iota^k(w))=\sum_{k=0}^{2}\bigl|T^*_{\iota^k(w)}T^*_v\zeta_h\bigr|\geq\left|\sum_{k=0}^{2}R^kT^*_wR^{-k}T^*_v\zeta_h\right|>\frac{3}{2}\|T_w\||T^*_v\zeta_h|>\frac{3st}{50}$$

by $\det T_w=(3/25)^{|w|}>0$, (3.4) and (3.5). Thus $l_h(v\iota^k(w))>st/50$ for at least one $k\in\{0,1,2\}$, and it follows from (7.8) that $3\#\Lambda_{t,s}\geq\#\Lambda^h_t\#\Lambda^{\mathcal{H}}_s$, where we set $\Lambda_{t,s}:=\{(v,w)\in\Lambda^h_t\times\Lambda^{\mathcal{H}}_s\mid l_h(vw)>st/50\}$. Each $(v,w)\in\Lambda_{t,s}$ admits $u(v,w)\in\Lambda^h_{st/50}$ such that $u(v,w)\leq vw$, and then the mapping $\Lambda_{t,s}\ni(v,w)\mapsto u(v,w)\in\Lambda^h_{st/50}$ is clearly injective. Therefore, noting also that $\#\Lambda^h_{3r/5}\leq 3\#\Lambda^h_r$ for $r\in(0,1]$ by (3.5) and [19, Proposition 2.7], we get $\#\Lambda^h_t\#\Lambda^{\mathcal{H}}_s\leq 3\#\Lambda_{t,s}\leq 3\#\Lambda^h_{st/50}\leq 3^9\#\Lambda^h_{st}$. □

Proposition 7.6. *There exists $d_{\mathrm{B}}\in[1,2\log_{25/3}5]$ such that for any $h\in\mathcal{S}_{\mathcal{H}_0}$,*

$$3^{-10}s^{-d_{\mathrm{B}}}\leq\#\Lambda^h_s\leq\#\Lambda^{\mathcal{H}}_s\leq 3^{19}s^{-d_{\mathrm{B}}},\qquad s\in(0,1]. \tag{7.9}$$

Proof. Let $s\in(0,1)$. Noting (3.4) and that $\|A\|^2\geq 2|\det A|$ for $A\in\mathcal{L}(\mathbb{R}^2)$, we have $s\wedge(3/5)^{|w|-1}\geq\|T_w\|\geq(s/5)\vee(3/25)^{|w|/2}$ and $(5/3)^{|w|}\geq s^{-2\log_{25/3}\frac{5}{3}}$ for $w\in\Lambda^{\mathcal{H}}_s$. Therefore

$$2s\#\Lambda^{\mathcal{H}}_s\geq 2=\sum_{w\in\Lambda^{\mathcal{H}}_s}\left(\frac{5}{3}\right)^{|w|}\|T_w\|^2\geq s^{-2\log_{25/3}\frac{5}{3}}\frac{s^2}{25}\#\Lambda^{\mathcal{H}}_s=\frac{s^{2\log_{25/3}5}}{25}\#\Lambda^{\mathcal{H}}_s. \tag{7.10}$$

Let $h\in\mathcal{S}_{\mathcal{H}_0}$. Then since $3^{-9}\#\Lambda^h_s\#\Lambda^h_t\leq\#\Lambda^h_{st}\leq 3^9\#\Lambda^h_s\#\Lambda^h_t$ for any $s,t\in(0,1]$ by Lemma 7.5, a standard argument for subadditive and superadditive sequences together with (7.10) immediately shows the assertion; recall that $\#\Lambda^h_{3s/5}\leq 3\#\Lambda^h_s$ for $s\in(0,1]$ by (3.5) and [19, Proposition 2.7] and that $\#\Lambda^h_s\leq\#\Lambda^{\mathcal{H}}_s\leq 3^9\#\Lambda^h_s$ for $s\in(0,1]$ by Lemma 7.5. □

Lemma 7.7. *Let Λ be a finite subset of W_* satisfying $K=\bigcup_{w\in\Lambda}K_w$. Then there exists a subset Λ_0 of Λ which is a partition of Σ.*

Proof. $K=\bigcup_{w\in\Lambda}K_w$ and $K\neq V_0$ imply $\Sigma=\bigcup_{w\in\Lambda}\Sigma_w$, and then an induction on $\#\Lambda$ easily shows the lemma. □

Lemma 7.8. *Let $\alpha,\delta,M\in(0,\infty)$, and let $\mathcal{H}^\alpha_\delta(\cdot,\rho_{\mathcal{H}})$ be the α-dimensional pre-Hausdorff measure on $(K,\rho_{\mathcal{H}})$ as defined in* [10, (2.7)] *and* [23, *Definition* 1.5.1]. *If $\delta\in(0,\sqrt{2}/50)$ and $\mathcal{H}^\alpha_\delta(K,\rho_{\mathcal{H}})<M$, then there exists a partition Λ of Σ such that $\sum_{w\in\Lambda}\|T_w\|^\alpha<4(25\sqrt{2})^\alpha M$ and $\max_{w\in\Lambda}\|T_w\|\leq 25\sqrt{2}\delta$.*

Proof. By $\mathcal{H}^\alpha_\delta(K,\rho_{\mathcal{H}})<M$ we can choose a sequence $\{A_n\}_{n\in\mathbb{N}}$ of non-empty subsets of K with $L_n:=\operatorname{diam}(A_n,\rho_{\mathcal{H}})\leq\delta$ so that $K=\bigcup_{n\in\mathbb{N}}A_n$ and $\sum_{n\in\mathbb{N}}L_n^\alpha<M$. Take $\varepsilon\in(0,(\delta/3)^\alpha]$ such that $3^\alpha\varepsilon+\sum_{n\in\mathbb{N}}L_n^\alpha<M$. For $n\in\mathbb{N}$, we set $D_n:=L_n$ if $L_n>0$ and $D_n:=3(2^{-n}\varepsilon)^{1/\alpha}$ if $L_n=0$, so that $D_n\leq\delta$ and $\sum_{n\in\mathbb{N}}D_n^\alpha<M$. We also set $B_n:=\bigcup_{x\in A_n}B_{\varepsilon_n}(x,\rho_{\mathcal{H}})$, where $\varepsilon_n:=(2^{-n}\varepsilon)^{1/\alpha}$ if $L_n=0$ and otherwise $\varepsilon_n\in(0,1]$ is chosen so that $L_n+3\varepsilon_n<\sqrt{2}/50$ and $\Lambda^{\mathcal{H}}_{25\sqrt{2}L_n}=\Lambda^{\mathcal{H}}_{25\sqrt{2}(L_n+3\varepsilon_n)}$; recall (Sr) in Definition 3.2-(1). Then $\operatorname{diam}(B_n,\rho_{\mathcal{H}})\leq L_n+2\varepsilon_n$ and

$$A_n\subset B_n\subset B_{L_n+3\varepsilon_n}(x_n,\rho_{\mathcal{H}})\subset U_{25\sqrt{2}(L_n+3\varepsilon_n)}(x_n,\mathcal{S}^{\mathcal{H}})=U_{25\sqrt{2}D_n}(x_n,\mathcal{S}^{\mathcal{H}}) \tag{7.11}$$

by (3.11), where we have chosen $x_n \in B_n \setminus V_*$. By Lemma 3.5, $U_{25\sqrt{2}D_n}(x_n, \mathcal{S}^{\mathcal{H}}) = \bigcup_{i=1}^{m_n} K_{w^{n,i}}$ for some $m_n \in \{1,2,3,4\}$ and $\{w^{n,i}\}_{i=1}^{m_n} \subset \Lambda^{\mathcal{H}}_{25\sqrt{2}D_n}$, so that $B_n \subset \bigcup_{i=1}^{m_n} K_{w^{n,i}}$ by (7.11), $\|T_{w^{n,i}}\| \le 25\sqrt{2}D_n \le 25\sqrt{2}\delta$ and $\sum_{n\in\mathbb{N}}\sum_{i=1}^{m_n}\|T_{w^{n,i}}\|^\alpha \le \sum_{n\in\mathbb{N}} 4(25\sqrt{2}D_n)^\alpha < 4(25\sqrt{2})^\alpha M$. Since K is compact, $K = \bigcup_{k=1}^N B_{n_k}$ for some $N \in \mathbb{N}$ and $\{n_k\}_{k=1}^N \subset \mathbb{N}$. Now we apply Lemma 7.7 to $\{w^{n_k,i} \mid k \in \{1,\dots,N\},\, i \in \{1,\dots,m_{n_k}\}\}$ to have a partition Λ of Σ with the desired properties. □

Proof of Theorem 7.2. Let $h \in \mathcal{S}_{\mathcal{H}_0}$. Lemma 3.5, (3.11), (3.12), Proposition 7.6 and [19, Proposition 2.24] together imply that

$$d_{\mathrm{S}} = \dim_{\mathrm{H}}(K,\rho_{\mathcal{H}}) \le \dim_{\mathrm{B}}(K,\rho_{\mathcal{H}}) = \dim_{\mathrm{B}}(K,\rho_h) = d_{\mathrm{B}} \in [1, 2\log_{25/3} 5]. \tag{7.12}$$

We follow [10, Proof of Theorem 3.1] in this paragraph. Let $\alpha \in (0,\infty)$. We suppose $\mathcal{H}^\alpha_{1/36}(K,\rho_{\mathcal{H}}) \le \frac{1}{4}6^{-2\alpha}$ and deduce $d_{\mathrm{B}} < \alpha$, from which we conclude that $d_{\mathrm{S}} = d_{\mathrm{B}}$ by letting $\alpha \downarrow d_{\mathrm{S}}$, that $\mathcal{H}^{d_{\mathrm{S}}}_{1/36}(K,\rho_{\mathcal{H}}) > \frac{1}{4}6^{-2d_{\mathrm{S}}}$ and that (7.4) and (7.5) hold by virtue of Propositions 7.4 and 7.6. By Lemma 7.8, there exists a partition Λ of Σ such that $\sum_{w\in\Lambda}\|T_w\|^\alpha < 1$. Then $\emptyset \notin \Lambda$. Choose $\beta \in (0,\alpha)$ so that $r_\Lambda(\beta) := \sum_{w\in\Lambda}\|T_w\|^\beta < 1$. Let $s \in (0,1]$. We define $W_*(\Lambda) := \{\emptyset\} \cup \bigcup_{m\in\mathbb{N}} \Lambda^m$ and

$$\Gamma^\Lambda_s := \{w \mid w = w^1 \dots w^m \in W_*(\Lambda),\ l_{\mathcal{H}}(w^1 \dots w^{m-1}) > s \ge l_{\mathcal{H}}(w)\} \tag{7.13}$$

with the convention $l_{\mathcal{H}}(w^1\dots w^{m-1}) := 2$ for $w = \emptyset$, where we naturally regard $w = w^1 \dots w^m \in W_*(\Lambda)$ as an element of W_* in the way of Definition 3.1-(1); note that this natural identification $W_*(\Lambda) \to W_*$ is injective since Λ is a partition of Σ. Then as a subset of W_*, Γ^Λ_s is easily seen to be a partition of Σ with $\Gamma^\Lambda_s \le \Lambda^{\mathcal{H}}_s$. Since $\Gamma^\Lambda_s \subset \{w \in W_*(\Lambda) \mid b_\Lambda s < \|T_w\|\}$ by (3.4), where $b_\Lambda := 5^{-\max_{w\in\Lambda}|w|}$, we have

$$\begin{aligned}
\#\Lambda^{\mathcal{H}}_s \le \#\Gamma^\Lambda_s \le \#\{w \in W_*(\Lambda) \mid b_\Lambda s < \|T_w\|\} &\le \sum_{m\in\mathbb{N}\cup\{0\}}\sum_{w\in\Lambda^m}\|T_w\|^\beta b_\Lambda^{-\beta}s^{-\beta}\\
&\le \sum_{m\in\mathbb{N}\cup\{0\}}\sum_{w^1,\dots,w^m\in\Lambda}\|T_{w^1}\|^\beta\cdots\|T_{w^m}\|^\beta b_\Lambda^{-\beta}s^{-\beta}\\
&= \sum_{m\in\mathbb{N}\cup\{0\}} r_\Lambda(\beta)^m b_\Lambda^{-\beta}s^{-\beta} = \bigl(1-r_\Lambda(\beta)\bigr)^{-1}b_\Lambda^{-\beta}s^{-\beta},
\end{aligned}$$

which and (7.9) yield $d_{\mathrm{B}} \le \beta < \alpha$.

Next we show $\mathcal{H}^{d_{\mathrm{S}}}(K,\rho_{\mathcal{H}}) < \infty$. Let $s \in \bigl(0,1]$. Then $\mathrm{diam}(K_w,\rho_{\mathcal{H}}) \le 10 l_{\mathcal{H}}(w) \le 10s$ for $w \in \Lambda^{\mathcal{H}}_s$ by Lemma 3.7 and (3.11) and hence

$$\mathcal{H}^{d_{\mathrm{S}}}_{10s}(K,\rho_{\mathcal{H}}) \le \sum_{w\in\Lambda^{\mathcal{H}}_s}\mathrm{diam}(K_w,\rho_{\mathcal{H}})^{d_{\mathrm{S}}} \le (10s)^{d_{\mathrm{S}}}\#\Lambda^{\mathcal{H}}_s \le (10s)^{d_{\mathrm{S}}}3^{19}s^{-d_{\mathrm{B}}} = 3^{19}10^{d_{\mathrm{S}}}$$

by (7.9) and $d_{\mathrm{S}} = d_{\mathrm{B}}$. Letting $s \downarrow 0$, we obtain $\mathcal{H}^{d_{\mathrm{S}}}(K,\rho_{\mathcal{H}}) \le 3^{19}10^{d_{\mathrm{S}}} < \infty$.

Finally, we prove $d^{\mathrm{loc}}_{\mathrm{S}} \le d_{\mathrm{S}}$. By Jensen's inequality and (7.2),

$$\frac{1}{2}\int_K \frac{2\log p_\mu(t,x,x)}{-\log t}d\mu(x) \le \frac{2}{-\log t}\log\biggl(\frac{1}{2}\int_K p_\mu(t,x,x)d\mu(x)\biggr) = \frac{2\log(\mathcal{Z}_\mu(t)/2)}{-\log t}$$

for $t \in (0,1)$, and letting $t \downarrow 0$ results in $d^{\mathrm{loc}}_{\mathrm{S}} \le d_{\mathrm{S}}$ by (6.1), Fatou's lemma and (7.5); note that $t \mapsto p_\mu(t,x,x)$ is $(0,\infty)$-valued and non-increasing for each $x \in K$ by [8, Theorem 2.1.4]. □

Remark 7.9. A simple direct argument shows the following lower bound

$$d_{\mathrm{S}} = \dim_{\mathrm{H}}(K, \rho_{\mathcal{H}}) = \dim_{\mathrm{B}}(K, \rho_{\mathcal{H}}) \geq 1 + 2\log_{25/3}\tfrac{6}{5} = 1.17198\ldots \tag{7.14}$$

although it is weaker than the numerical estimate $d_{\mathrm{S}} \geq d_{\mathrm{S}}^{\mathrm{loc}} \geq 1.27874\ldots$ implied by Theorem 7.2 and Remark 6.7. Indeed, since $\sum_{i\in S} T_i = \frac{6}{5}\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ by Proposition 2.12-(i), $\sum_{w\in\Lambda}(\frac{5}{6})^{|w|}T_w = \left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ for any partition Λ of Σ. Therefore for $s \in (0,1)$,

$$\sqrt{2} \leq \sum_{w\in\Lambda_s^{\mathcal{H}}} \left(\frac{5}{6}\right)^{|w|} \|T_w\| \leq \sum_{w\in\Lambda_s^{\mathcal{H}}} \|T_w\|^{1+2\log_{25/3}\frac{6}{5}} \leq s^{1+2\log_{25/3}\frac{6}{5}} \#\Lambda_s^{\mathcal{H}} \tag{7.15}$$

by virtue of the lower bound in (6.4). Now (7.14) follows from (7.15), (7.9) and $d_{\mathrm{S}} = d_{\mathrm{B}}$.

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