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PARTICLE TRAJECTORIES AROUND A RUNNING CYLINDER IN BRINKMAN'S POROUS-MEDIA FLOW

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ABSTRACT. The movement of particles around a running cylinder is considered. In 1870, J. C. Maxwell considered the problem in irrotational flow of inviscid fluid, and found that the trajectory of a particle is a curve of elastica. We consider here a similar problem in Brinkman's porous-media flow. In this case, our numerical examinations reveals some new interesting features of the particle trajectories, which are not observed in the case of irrotational flow. This is a brief review of HO's talk in NIMS conference in October, 2008.

1. INTRODUCTION

Motion of fluid particles provides us with interesting problems of dynamical systems. They are not only mathematically intriguing but also important in applications, say, to pollution problems and engineering for particle mixing. There are many references on study of motions of particles in fluid confined in a fixed domain (see, for instance, [8] or [3]). On the other hand, we know only a few studies in moving domains. One of the oldest of them is J.C. Maxwell [6], in which this eminent physicist studied the trajectories of fluid particles when a circular cylinder moves through an incompressible perfect fluid with a constant speed. Assuming that the flow is irrotational, it was shown that the complete solution is given by the elliptic functions and the trajectory forms one of the 'elastica' curves. For more details, see [6] or pages 243–246 of [7]. C. Darwin [2] considered a similar problem for a moving sphere. In this case, the solution can not be written in terms of elliptic functions but can be expressed by a simple definite integral.

One may well wonder what happens if we consider a similar problem for different fluid motion, such as rotational flow and/or viscous fluid. Here we note that we must carefully select the mathematical setting. For instance, Stokes's paradox prevents us from considering an analogous 2D problem for the Stokes equation. For large Reynolds numbers, the steady-state may not be unique, and no definite answer is expected for the Navier-Stokes equation. After surveying a general theory in the next section, we examine several examples including Maxwell's. Then we would like to turn reader's attention to a linear equation describing a viscous flow in porous media, in which an analogous problem can be solved completely. The equation which we are going to consider is the one proposed by Brinkman [1].

Key words and phrases. Brinkman's porous-media flow, particle trajectory.
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2. Problem in general

Suppose that a fluid occupies the plain (2D flow) and suppose that an obstacle (= a section of a cylinder) is running with a constant velocity. The problem is to determine the trajectories of particles in the fluid. Let (x, y) be the plain coordinates and let time t be defined so that the center of gravity of the obstacle moves on the x-axis from left to right and passes through the origin at $t = 0$. Let (X, Y) be the moving frame attached to the obstacle. If the velocity is denoted by c, we have $(x, y) = (X + ct, Y)$.

In order to determine the particle movement, Maxwell [6] considered as follows. The particle, whose coordinates in the moving frame is (X, Y), is subject to the following equation

$$\dot{X} = \frac{\partial \Psi}{\partial Y}, \qquad \dot{Y} = -\frac{\partial \Psi}{\partial X}, \tag{1}$$

where $\Psi = \Psi(X, Y)$ is the stream function. In the absolute coordinates, they are written as

$$\dot{x} = c + \frac{\partial \Psi}{\partial Y}(x - ct, y), \qquad \dot{y} = -\frac{\partial \Psi}{\partial X}(x - ct, y). \tag{2}$$

If we solve these equationsnumerically, say by the Runge-Kutta method, then we are given the trajectory. However, this strategy is not suitable to compute accurately. In fact, the accumulation of errors in lareg $|t|$ usually deteriorate the result. Instead we compute as Maxwell did, which is suumarized as follows. If an initial data (X_0, Y_0) is given, then $\Psi(X, Y) = \Psi(X_0, Y_0)$ is a constant. Suppose that we can solve this equation to have $Y = \varphi(X, X_0, Y_0)$. Substituting this into the first equation of (1), we obtain an equation of the following form:

$$\dot{X} = \Psi(X, \varphi(X, X_0, Y_0)) \equiv \Phi(X, X_0, Y_0),$$

or equivalently

$$\int \frac{\mathrm{d}X}{\Phi(X, X_0, Y_0)} = t. \tag{3}$$

This equation and $\Psi(X, Y) = \Psi(X_0, Y_0)$ determine trajectories completely in the moving coordinates.

In the absolute coordinates, we have

$$y(t) = Y(t) = \varphi(X(t), X_0, Y_0)$$
$$x(t) = \int_0^t \dot{x}(s)\mathrm{d}s = \int_0^X \left(\dot{X}(s) + c\right) \frac{\mathrm{d}s}{\dot{X}(s)}.$$

By carrying out this integral, we obtain the trajectory in the absolute coordinates if we carry out this integral.

This method can be used in any case of incompressible fluid flow. Suppose that a cylinder with a radius $a > 0$ is moving in a porous-media. It is supposed to run on the x-axis with a constant speed $U > 0$. In a moving flame (X, Y), which is attached

to the cylinder, the Brinkman's flow is governed by the following equations:

$$\begin{aligned}
&(4) & \mu \triangle \mathbf{V} - \frac{\mu^*}{K}\mathbf{V} - \nabla P &= 0 & &(a < R < \infty),\\
&(5) & \operatorname{div} \mathbf{V} &= 0 & &(a < R < \infty),\\
&(6) & \mathbf{V}\big|_{R=a} &= 0, & &\\
&(7) & \lim_{r \to \infty} \mathbf{V} &= (-U_\infty, 0), & &
\end{aligned}$$

where $\mathbf{V} = (V_1, V_2)$ is a velocity vector, $\mu > 0$ is the viscosity, $\mu^* > 0$ is the second viscosity, $K > 0$ is a constant called the permeability, and $R = \sqrt{X^2 + Y^2}$. These equations were derived by Brinkman [1] as a model for fluid motion in porous media. Their interesting applications can be found in [4, 9, 10, 11]. For derivation of (4)–(7), see [1] or [11]. Note that (4) is a model rather than a rigorously derived equation. It is nothing but d'Arcy's law if $\mu = 0$. On the other hand, it is the Stokes equation if $\mu^* = 0$ or if $K = \infty$. The solution of Brinkman's equation has some interesting properties which are not shared by the solution of Stokes equation (see [9, 11]). Disappearance of Stokes' paradox, which we explain immediately, is one of them.

We employ the following non-dimensionalization:

$$R \mapsto aR, \qquad \mathbf{V} \mapsto U\mathbf{V}.$$

We then introduce the stream function Ψ by

$$\mathbf{V} = \left(\frac{\partial \Psi}{\partial Y}, -\frac{\partial \Psi}{\partial X}\right).$$

It satisfies

$$\begin{aligned}
&(8) & &\triangle^2 \Psi - \lambda^2 \triangle \Psi = 0,\\
&(9) & &\Psi\big|_{R=1} = \frac{\partial \Psi}{\partial r}\bigg|_{R=1} = 0,\\
&(10) & &\Psi \sim -Y \qquad (R \to \infty),
\end{aligned}$$

where

$$\lambda = \sqrt{\frac{\mu^*}{\mu K}}. \tag{11}$$

We call λ Brinkman's constant. It is known as Stokes' paradox that the equations (8)–(10) have no solution for $\lambda = 0$. However, if $\lambda > 0$, the equations of (8)–(10) is uniquely solvable. If we follow the computation of [9, 11, 13], we easily find that

$$\Psi = \Psi(R, \Theta) = -\left(R - \frac{K_2(\lambda)}{K_0(\lambda)R} + \frac{2K_1(\lambda R)}{\lambda K_0(\lambda)}\right)\sin\Theta, \tag{12}$$

where Θ is defined by $X = R\cos\Theta$ and $Y = R\sin\Theta$, and K_n $(n = 0, 1, 2)$ are the modified Bessel functions of order n.

We now consider the motion of particles in the fluid which moves passively by this flow. For this purpose, it is more convenient to use the polar coordinates (R, Θ) rather than the Cartesian coordinates (X, Y). Let (R, Θ) be the polar coordinates of the particle in the moving frame. Then they are governed by:

$$\dot{R} = \frac{1}{R}\frac{\partial \Psi}{\partial \Theta} = -\left(1 - \frac{K_2(\lambda)}{K_0(\lambda)R^2} + \frac{2K_1(\lambda R)}{\lambda K_0(\lambda)R}\right)\cos\Theta, \tag{13}$$

$$R\dot{\Theta} = -\frac{\partial \Psi}{\partial R} = \left(1 + \frac{K_2(\lambda)}{K_0(\lambda)R^2} + \frac{2K_1'(\lambda R)}{K_0(\lambda)}\right)\sin\Theta. \tag{14}$$

Let a quantity η_∞ be defined by

$$\eta_\infty = \left(R - \frac{K_2(\lambda)}{K_0(\lambda)R} + \frac{2K_1(\lambda R)}{\lambda K_0(\lambda)}\right)\sin\Theta. \tag{15}$$

3. Discussions

Finally we will discuss our numerical results of Brinkman's porous-media flow. Before that we would like to view the result of the problem for the Stokes equations which corresponds to $\lambda = 0$. If the moving body is a sphere, it is known that fluid particles are drifted infinitely in the case of the Stokes flow. All particles are drifted infinitely from the left to the right while the sphere moves from $-\infty$ to ∞.

When $\lambda > 0$, fluid particles around a moving cylinder are drifted within finite range while the cylinder moves from $-\infty$ to ∞. It is similar to the results in the case of irrotational flow except that drifted range of particles and profile of trajectories change depending on λ. Brinkman's porous-media flow has two distinguishing characters. Though we cannot compute the case of $\lambda = 0$ in the Brinkman's equation, our results suggest that drifted range of particles increase to the infinite as $\lambda \to 0$. Considering the case of the Stokes flow, we think it is a convincing result. The drifted range of particles becomes smaller as the parameter λ increases.

On the other hand we have another phenomenon in the case of Brinkman's porous-media flow. For very small λ, trajectories have profiles similar to the Stokes case in a neighborhood of the cylinder.

If λ is not so small, the shape of trajectory changes in the following way as η_∞ increases: gently sloping curve $\Longrightarrow$ a curve with a cusp $\Longrightarrow$ self-intersecting curve $\Longrightarrow$ non-self-intersecting curve $\Longrightarrow$ diminishes to a point. For larger λ the change to a self-intersection curve begins with a smaller η_∞.

If both $\lambda > 0$ and η_∞ are small, the trajectory has no self-intersection. It is difficult for us to see which (λ, η_∞) this actually happens.

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