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Degenerate Bifurcations in the Taylor-Couette Problem

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§1. **Introduction.** The purpose of this paper is to explain certain bifurcation equations which describe newly discovered phenomena in the Taylor-Couette problem of the fluid motion between two concentric cylinders. The equations to be considered here are:

$$(1.1)\qquad \begin{cases} x(\epsilon\lambda + \alpha + ax^2 + bz^2 + cx^2z) + (\beta + ez^2)xz = 0, \\ z(\delta\lambda + \hat{a}x^2 + \hat{b}z^2) \pm x^2 = 0, \end{cases}$$

and

$$(1.2)\qquad \begin{cases} x(\epsilon\lambda + \alpha + ax^2 + bz^2) \pm xz = 0, \\ z(\delta\lambda + \hat{a}x^2 + \hat{b}z^2 + \hat{c}x^2z) + (\beta + dx^2)x^2 = 0, \end{cases}$$

where λ, x and z are real variables, $\epsilon, \delta, a, b, c, d, e, \hat{a}, \hat{b}$ and $\hat{c}$ are real constants. We explain in the subsequent sections how the solutions to (1.1,2) fit the bifurcation diagram given in Tavener and Cliffe [**7**] in which Taylor vortices of new type bifurcating from the Couette flow are computed numerically. The equations (1.1,2) are derived from a certain degeneration of the equations given in Fujii, Mimura and Nishiura [**1**]. The equation (1.1) are considered in Fujii, Nishiura and Hosono [**2**], but (1.2) seems to be new. Although they considers in [**1**,2] a reaction diffusion system which has nothing to do with the Taylor-Couette problem, the local structure of the bifurcation is of the same category. This is because the orthogonal group O(2) acts on both problems.

In this paper we announce a results in [**5**,6] in which we employed the singularity theoretic approach by Golubitsky and Schaeffer ([**3**,4]) to see systematically the structure of the equations. In §2 we state a precise statement of the Taylor-Couette problem. In §3 the relation between (1.1,2) and the Taylor-Couette problem is given. §4 is a final section for discussions.

§2. **The Taylor-Couette problem.** In this section we recall some newly developped analysis for the mechanism of vortex number exchange in [**7**]. The problem considered by them is to determine a fluid velocity field (u, v, w) and the pressure p which satisfy the following stationary Navier-Stokes equation (2.1-4):

$$\Delta u - \frac{u}{r^2} - R\left[u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} + \frac{\partial p}{\partial r}\right] = 0, \tag{2.1}$$

$$\Delta v - \frac{v}{r^2} - R\left[u\frac{\partial v}{\partial r} + w\frac{\partial v}{\partial z} + \frac{uv}{r}\right] = 0, \tag{2.2}$$

$$\Delta w - R\left[u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z}\right] = 0, \tag{2.3}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0, \tag{2.4}$$

where the cylindrical coordinates are adopted, u, v, w are r, θ, z - component, respectively and R is the Reynolds number. In this paper, as in [**7**], we consider only velocity fields which are independent of the azimutal coodinate θ and the time t. We have suitably nondimensionalized the quantities so that (2.1-4) are satisfied in

$$\{(r, z); \quad 1 < r < \eta, \quad 0 < z < \Gamma\}.$$

The following boundary conditions are imposed :

$$(u, v, w) = (0, 1, 0) \qquad (r = 1) \tag{2.5}$$

$$(u, v, w) = (0, 0, 0) \qquad (r = \eta) \tag{2.6}$$

$$\left(\frac{\partial u}{\partial z}, \frac{\partial v}{\partial z}, w\right) = (0, 0, 0) \qquad (z = 0, \Gamma) \tag{2.7}$$

In this circumstance, the Couette flow

$$(u, v, w, p) = (0, v_0(r), 0, p_0(r)) \qquad \text{with}$$

$$v_0(r) = Ar + \frac{B}{r}, \qquad p_0(r) = \int_1^r \frac{v_0(r)^2}{r} dr$$

satisfies all the requirements if $A = 1/(1-\eta^2)$, $B = \eta^2/(\eta^2-1)$. The problem is to study the solutions bifurcating from this Couette flow. Notice that the condition (2.7) makes it difficult to examine the results by a laboratory experiment but there is no difficulty in performing computer simulation. In fact, [7] presents a penetrating description of the stability exchange of the solutions. Let us state briefly the results in [7]. They fix the parameter $\eta = 1.0/0.615$ and let the nondimensional height of the cylinder Γ vary. It is known that the primary bifurcation branch consists of the two-cell Taylor vortices for small value of Γ and that it consists of the four-cell Taylor vortices for larger value of Γ. The number of the vortices, however, is integer and Γ can vary continuously. Therefore, there exist flows of "mixed type" when Γ is in a intermediate range. The bifurcation diagrams in [7] describe qualitatively how this exchange from two-cell to four cell occurs.

Although they consider the exchange mechanism between two-cell and six-cell or four-cell and six-cell, we consider only the exchange of two and four, which requires the least mathematical technique to theorize.

§3. Degenerate bifurcation equations. In this section we show how (1.1,2) are derived. Our starting point is to observe a hidden symmetry in (2.1-7). Let us consider another problem to seek (u^*, v^*, w^*, p^*) which satisfies (2.1-4) in $1 < r < \eta, -\Gamma < z < \Gamma$, the boundary condition (2.5,6) and the periodic boundary condition on $z = -\Gamma, \Gamma$. (Note that the height of the cylinder is double.) We call this problem $[2\Gamma]$ and the original problem $[\Gamma]$. This new problem $[2\Gamma]$ has an advantage that it is O(2)-equivariant in the following sense: Let us define an action of the orthogonal group O(2) by

$$(3.2) \qquad \gamma(u(r,z), v(r,z), w(r,z), p(r,z)) \to (u(r,z+\alpha), v(r,z+\alpha), w(r,z+\alpha), p(r,z+\alpha))$$

if $\gamma \in O(2)$ is a rotation with angle α,

$$(3.3) \qquad \gamma(u(r,z), v(r,z), w(r,z), p(r,z)) \to (u(r,-z), v(r,-z), -w(r,-z), p(r,-z))$$

if $\gamma \in O(2)$ is a reflection. Then it is easily checked that the problem $[2\Gamma]$ is O(2)-equivariant: in other words, the governing equation is covariant with the O(2)-action (3.2,3). We recall that the larger the symmetry group is, the simpler the equation becomes. Therefore the introduction of $[2\Gamma]$ is an advantage, since $[\Gamma]$ is covariant with only a discrete group. The relation between $[2\Gamma]$ and $[\Gamma]$ is given by

PROPOSITION 3.1. *If (u^*, v^*, w^*, p^*) is a solution to* $[2\Gamma]$ *and if (u^*, v^*, w^*, p^*) is invariant with respect to (3.3), then it satisfies* $[\Gamma]$ *in $0 < z < \Gamma$. Conversely, if (u, v, w, p) satisfies* $[\Gamma]$ *and if we extend it to (u^*, v^*, w^*, p^*) in such a way that u^*, v^*, p^* are even extention of u, v, p, respectively, and w^* is an odd extension of w, then (u^*, v^*, w^*, p^*) is a solution to* $[2\Gamma]$.

In short, finding solutions to $[\Gamma]$ is equivalent to finding "symmetric" solutions to $[2\Gamma]$. From now we consider $[2\Gamma]$. In the remaining part of this section, we explain how the analysis in [**5**] is applied to $[2\Gamma]$.

There is a numerical evidence that, at some value of Γ, say Γ_0, the linearlized operator of (2.1-4) with (2.5,6) and the periodic boundary condition has a four dimensional null space spanned by $g_1 \sim g_4$ which are of the following form:

$$g_1(r, z) = (U_2(r)\cos(2\psi), V_2(r)\cos(2\psi), W_2(r)\sin(2\psi), P_2(r)\cos(2\psi))$$

where $\psi = 2\pi z/\Gamma$;

$$g_2(r, z) = (U_2(r)\sin(2\psi), V_2(r)\sin(2\psi), W_2(r)\cos(2\psi), P_2(r)\sin(2\psi)),$$

$$g_3(r, z) = (U_4(r)\cos(4\psi), V_4(r)\cos(4\psi), W_4(r)\sin(4\psi), P_4(r)\cos(4\psi)),$$

$$g_4(r, z) = (U_4(r)\sin(4\psi), V_4(r)\sin(4\psi), W_4(r)\cos(4\psi), P_4(r)\sin(4\psi))$$

In these expressions $U_j(r), V_j(r), W_j(r), P_j(r)$ are functions of r only and are determined through a certain ordinary differential equation (see [**6**]).

Let $F = F(R, \Gamma; u, v, w, p)$ be a nonlinear functional for $[2\Gamma]$ realized in some Banach space, and let P be a projection onto the 4-dimensional space spanned by $g_1, \cdots, g_4$. Then the bifurcation of Taylor vortices is governed in a neighborhood of $(R, \Gamma_0; 0, v_0, 0, p_0)$ by

$$G(R, \Gamma; x, y, z, w) = PF(R, \Gamma; (0, v_0, 0, p_0) + xg_1 + yg_2 + zg_3 + wg_4 + \phi),$$

where ϕ is in a complement of the range of P. Since O(2)-equivariance is reflected in this bifurcation equation, G must be of a special form given in [**1**,5].

In order to explain the form in [**1**,5], we identify real (x, y, z, w) space with $\mathbf{C}^2$ by $\xi = x + iy, \zeta = z + iw$. Then G must be of the following form: $G = (G_1, G_2)$,

$$G_1 = f_1\xi + f_2\overline{\xi}\zeta, \tag{3.4}$$

$$G_2 = f_3\zeta + f_4\xi^2, \tag{3.5}$$

where $f_j (j = 1,2,3,4)$ are functions of $R, \Gamma, |\xi|^2, |\zeta|^2$ and $\mathrm{Re}(\xi^2\overline{\zeta})$ only. For the proof, see [1]. Since G is a bifurcation equation, it holds that $f_1(0,0;0,0,0,0) = f_3(0,0;0,0,0,0) = 0$. Under this conditions, the most general case is

(A) $\quad f_2(0,0;0,0,0,0) \neq 0, \qquad f_4(0,0;0,0,0,0) \neq 0$

which is considered in [1]. The degeneration which we mentioned at the beginning of the present paper is

(B) $\quad f_2(0,0;0,0,0,0) = 0, \qquad f_4(0,0;0,0,0,0) \neq 0,$

and

(C) $\quad f_2(0,0;0,0,0,0) \neq 0, \qquad f_4(0,0;0,0,0,0) = 0.$

The case (B) is considered in [2] but (C) seems to be new. Since (3.4,5) is applicable to a number of problems, we think it is useful to study (3.4,5) systematically. To this end, the machinaries by Golubitsky and Schaeffer [3,4] are easy to handle for application-oriented mathematicians. Below we summarize the results in [5], where a computations of normal forms for (3.4,5) via the method in [3,4] are given. In the case of (A), the bifurcation equation is, when slightly perturbed, generically O(2)-equivalent to

$$\begin{cases} \xi(\epsilon\lambda + \alpha + b|\zeta|^2) \pm \overline{\xi}\zeta = 0, \\ \zeta(\delta\lambda + \hat{b}|\zeta|^2) \pm \xi^2 = 0, \end{cases} \tag{3.6}$$

where α is a perturbation (unfolding) parameter. Note that we only consider an O(2)-equivariant perturvation. By the O(2)-equivalence we mean that the bifurcation equation is transformed to one of this form by a suitable coordinates change which preserves the O(2)- equivariance. Roughly, we can say that (3.6) is a normal form in the case of (A). In the case of (B),

$$\begin{cases} \xi(\epsilon\lambda + \alpha + a|\xi|^2 + b|\zeta|^2 + c\mathrm{Re}(\xi^2\overline{\zeta})) + (\beta + e|\zeta|^2)\overline{\xi}\zeta = 0, \\ \zeta(\delta\lambda + \hat{a}|\xi|^2 + \hat{b}|\zeta|^2) \pm \xi^2 = 0, \end{cases} \tag{3.7}$$

is a normal form. In the case of (C),

$$\begin{cases} \xi(\epsilon\lambda + \alpha + a|\xi|^2 + b|\zeta|^2) \pm \overline{\xi}\zeta = 0, \\ \zeta(\delta\lambda + \hat{a}|\xi|^2 + \hat{b}|\zeta|^2 + \hat{c}\mathrm{Re}(\xi^2\overline{\zeta})) + (\beta + d|\xi|^2)\xi^2 = 0, \end{cases} \tag{3.8}$$

is a normal form. When we consider the problem [Γ] , the solutions are invariant with respect to the reflection (3.3). Therefore we obtain the bifurcation equations for [Γ] by restricting complex variables ξ, ζ to real ones x, z. This restriction produces (1.1,2) from (3.7,8), respectively.

§4. Discussions. We first remark that, when R is taken as a bifurcation parameter, there are additional splitting parameters Γ and η. Therrefore there is a good possibility that the following scenario holds true: In the (Γ, η) plane there is a 1-dimensional variety where the critical Reynolds number of 2-cell flow and that of 4-cell flow coincide. And on this variety there are points at which (B) or (C) holds. If we assume that these points exist and are not far from the values taken in [7], then it is natural that the figures in [7] are captured by (1.1,2).

We now show how the phenomana in [7] is explained by (1.1,2). We choose α and β suitably to have zero sets of (1.1). In Fig. 2-6, we give drawings by a computer and a X-Y plotter. These explain the bifurcation diagrams in [7] for large Γ (Fig. 4.3 (D-I)). From (1.2) we obtain Fig. 1, which explains diagrams for smaller Γ (Fig. 4.3 (C) of [7]). Although there is a pitchpork bifurcation in Fig. 4.3 (A,B,C) of [7], this cannot be explained by (1.2). This ,however, may be a consequence of more global bifurcation. Except for this, our pictures fit the diagrams in [7] quite well. For the complete set of the bifurcation diagrams of (1.1,2), see [6].

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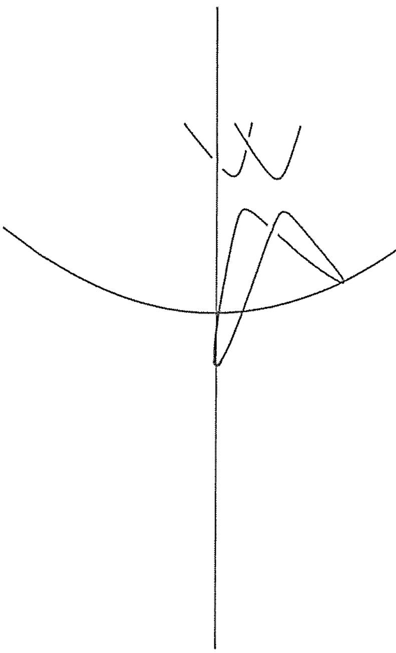

Fig. 1

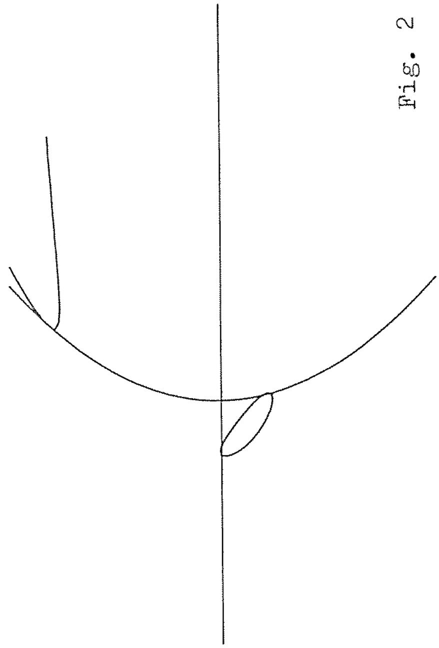

Fig. 2

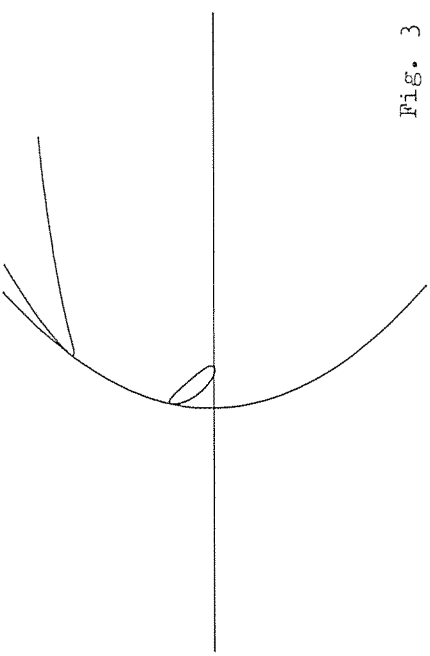

Fig. 3

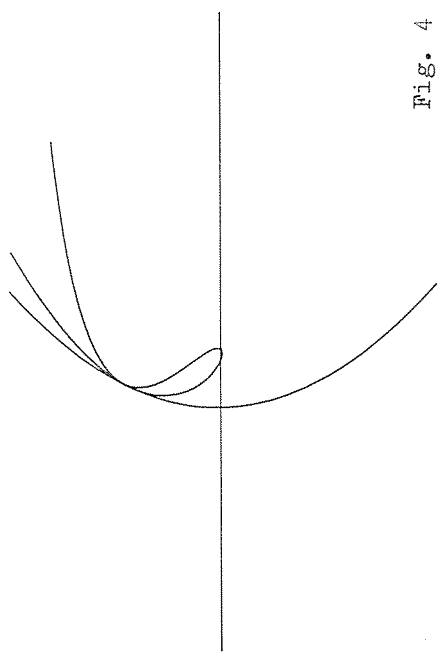

Fig. 4

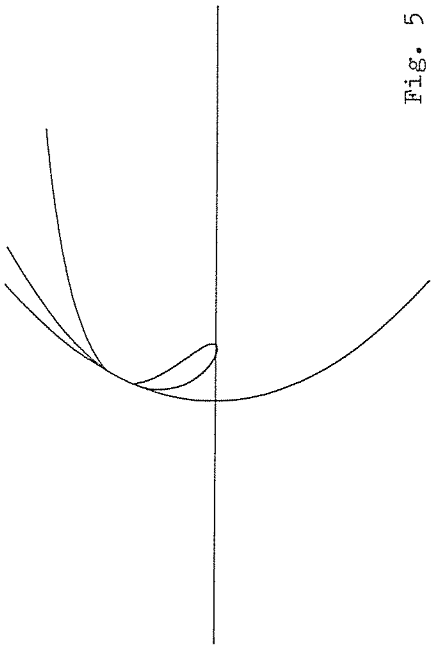

Fig. 5

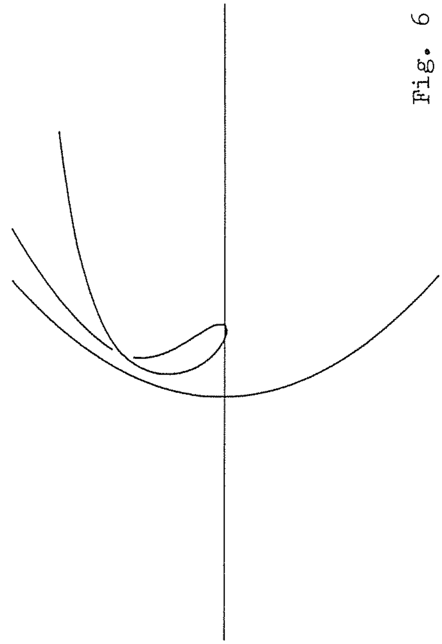

Fig. 6