ADVANCED SERIES IN NONLINEAR DYNAMICS

VOLUME 7

Proceedings of the
IUTAM/ISIMM Symposium
on

STRUCTURE AND DYNAMICS OF NONLINEAR WAVES IN FLUIDS

A. Mielke

Univ. Hannover

K. Kirchgässner

Univ. Stuttgart

World Scientific

Singapore • New Jersey • London • Hong Kong

Two Dimensional, Periodic, Capillary-Gravity Waves With Negative Surface Tension

Hisashi OKAMOTO
Research Inst. Math. Sci., Kyoto University,
Kyoto, 606-01 Japan

and

Mayumi SHŌJI
Dept. of General Education, College of Sci. & Tech.,
Nihon University, Chiyoda-ku, Tokyo, 101 Japan

ABSTRACT

The problem discussed here is to determine the shape of the steadily moving water wave on two dimensional irrotational flow of incompressible fluid. This classical problem involves two parameters: the surface tension coefficient and the gravity constant. We consider the solutions in the range where the surface tension coefficient is negative. We performed numerical computations and found that pitchfork bifurcations occur and the branches extends toward smaller surface tension. In the class of symmetric waves, we found no secondary bifurcation.

1. Introduction

We study the problem of determining the shape of the steadily moving water wave on two dimensional irrotational flow of inviscid incompressible fluid. This classical problem involves two parameters: the surface tension coefficient and the gravity constant. It is now well-known that there exists an amazingly large number of 2D capillary-gravity waves[1, 3, 6, 7, 10, 11]. Among these, Chen and Saffman[3] is a pioneering paper which opened the world of rich structure of the capillary-gravity waves. Stimulated by this, we computed[6, 7, 10, 11] numerically the bifurcation diagrams rather than particular solutions. We obtained many complicated diagrams which were new to us. An independent research by Aston[1] also shows very many solutions.

In the present paper, we continue to numerically compute permanent waves but we change the sign of the surface tension and look for what phenomena can be found. As far as we know, the case of negative surface tension has not been considered yet, presumably because the case is unphysical. However, it seems worthwhile drawing attention to the fact that the governing equation (for the permanent states) has a perfectly rigorous mathematical meaning. Our ultimate goal is to understand the global structure of the complex bifurcation of the capillary gravity waves with *positive* surface tension. However, we hope that capturing waves with negative surface tension may well contribute to the goal, as overlapping (thus totally unphysical) solutions clarifies the global bifurcation and led to a discovery of new, non-overlapping, physically meaningful solutions in our previous papers[5, 6, 10].

We are also encouraged by Prandtl and Tietjens[9] in which the authors briefly consider the free surface between water and alcohol (art. 28 of the book), state the surface tension is negative in the case, and hint the existence of unstable equilibrium figures. As a final comment, we note that Peregrine[8] derived an equation which is related to ours and found exact solutions to the equation. His solutions are interesting in that they correspond to the limiting case of our equation, see the next section. If we compare our present research with Peregrine's waves, then our hope mentioned above would not be an imaginary one.

Our conclusion is, roughly stated, as follows: (1) there are bifurcation points at which the surface tension coefficient is negative; (2) the bifurcations from them are pitchforks in all the cases; (3) secondary bifurcations were not captured as far as we computed.

2. Levi-Civita equation

Since we deal with 2D stationary waves only, it is most convenient to use the nonlinear equation of Levi-Civita. It has some variants and we employ here an integro-differential equation given by the first author[4]. The equation, which we will introduce below in a moment, is valid under the following hypotheses:

- the fluid is incompressible and inviscid;
- the flow is two dimensional and irrotational;
- the flow depth is infinite;
- acting forces are the surface tension and the gravity, only;
- the wave profile is stationary in a coordinate system moving with the same speed as the wave speed;
- the wave profile is periodic in the horizontal coordinate.

We are now in a position to introduce the Levi-Civita equation. It is written as follows:

$$\frac{d}{d\sigma}\left(\frac{e^{2H\theta}}{2}\right) - pe^{-H\theta}\sin\theta + q\frac{d}{d\sigma}\left(e^{H\theta}\frac{d\theta}{d\sigma}\right) = 0. \tag{1}$$

Here, $\theta = \theta(\sigma)$ is an unknown function of $\sigma \in [0, 2\pi)$; p and q are parameters. H is the Hilbert transform defined as follows

$$H\left(\sum_{k=1}^{\infty}(a_n \sin n\sigma + b_n \cos n\sigma)\right) = \sum_{k=1}^{\infty}(-a_n \cos n\sigma + b_n \sin n\sigma)$$

Two non-dimensional parameters are defined as

$$p = \frac{gL}{2\pi c^2}, \qquad q = \frac{2\pi T}{mc^2 L},$$

where L is the wave length, c the wave velocity, g the gravity acceleration, m the mass density, and T is the surface tension coefficient. Details on the equation (1) can be found in the reference[4] (see also Bridges and Dias[2]) and we do not repeat them here. However, it may be useful to recall the following theorem:

Theorem 1 (O.[4]) *Define*

$$F(p, q, \theta) = \frac{d}{d\sigma}\left(\frac{e^{2H\theta}}{2}\right) - pe^{-H\theta}\sin\theta + q\frac{d}{d\sigma}\left(e^{H\theta}\frac{d\theta}{d\sigma}\right).$$

Then F is a smooth mapping from $\mathbf{R}^2 \times (H^2(S^1)/\mathbf{R})$ into $H^2(S^1)/\mathbf{R}$, where S^1 denotes the circle and H^s denotes usual Sobolev spaces of order s. The symbol $/\mathbf{R}$ implies the function spaces with zero mean on S^1. The mapping F is O(2)-equivariant[4].

Thus the problem is reduced to searching for zeros of F. Once a zero $(p, q, \theta(\sigma))$ is obtained, then we can draw the free boundary by a certain simple formula[4]. All the researches until now were concerned with the case where $q > 0$. We consider in what follows the case where $q < 0$.

As is already proved[4], the bifurcation may occur if and only if p and q satisfy $n^2 q + p = n$ for some positive integer n. We consider the case of $n = 1$. This means that we consider bifurcating solutions which have one trough and one crest in the fundamental wave length.

Although Levi-Civita's equation (1) has no bearing on physical phenomena if $q < 0$, the following limit is interesting: Let $r > 0$ be a fixed parameter and let $p = -qr$. If we divide (1) by q and let $q \to -\infty$, then the equation (1) becomes

$$re^{-H\theta}\sin\theta + \frac{d}{d\sigma}\left(e^{H\theta}\frac{d\theta}{d\sigma}\right) = 0.$$

This is essentially the same as the equation which was derived by Peregrine[8] in the analysis of surface shear waves: For, this equation is transformed to

$$rh(x) + \frac{d}{dx}\left(\frac{dh/dx}{\sqrt{1 + (dh/dx)^2}}\right) = 0, \qquad (2)$$

by introducing x and h through $dx = e^{-H\theta}\cos\theta d\sigma$, $\frac{dh}{dx} = \tan\theta$. The equation (2) is the same as Peregrine's equation except for scaling constants. He noted that the equation is the same as for Euler's elastica. The situation where Peregrine derived equation (2) is totally different from ours but this unexpected relation is interesting and may have possible applications.

3. Numerical experiment

We discretize the equation (1) as before[6, 7]. It is a spectral-collocation method with H.B. Keller's path continuation algorithm. We refer the reader to these papers as for the numerical method, since they are the same. Since the equation (1) is O(2)-equivariant[4, 2], restriction to some subclass is necessary to find a branch. In this paper, we restrict our attention to certain symmetric waves. This means that we look for solutions $\theta(\sigma)$ which are odd in σ. This restriction gives us a pitchfork bifurcation from simple eigenvalue. To sum up, we look for approximate solutions of the following form:

$$\theta(\sigma) = \sum_{n=1}^{N} a_n \sin n\sigma, \tag{3}$$

where a_n $(1 \leq n \leq N)$ are unknown. The number N determines the degree of the approximation: we took $N = 255$ for the computations of $-0.5 < q < -0.1$ and increased it until $N = 1023$ for the computations of $q = -3.0$.

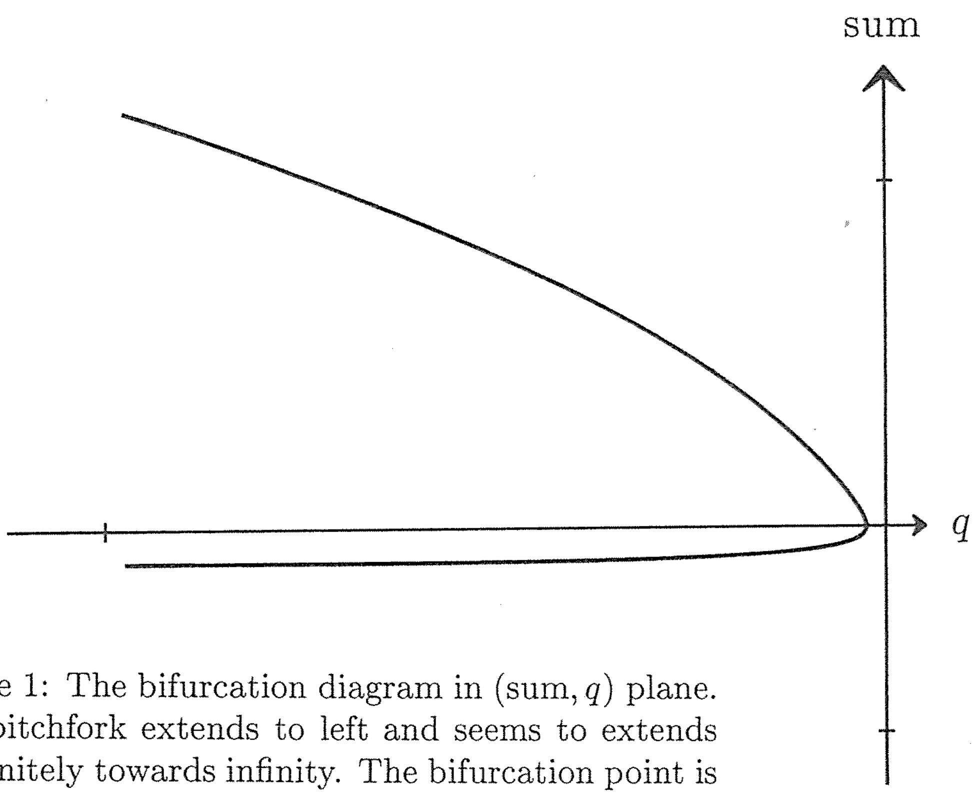

Figure 1: The bifurcation diagram in (sum, q) plane. The pitchfork extends to left and seems to extends indefinitely towards infinity. The bifurcation point is $q = -0.1$. No secondary bifurcation was found.

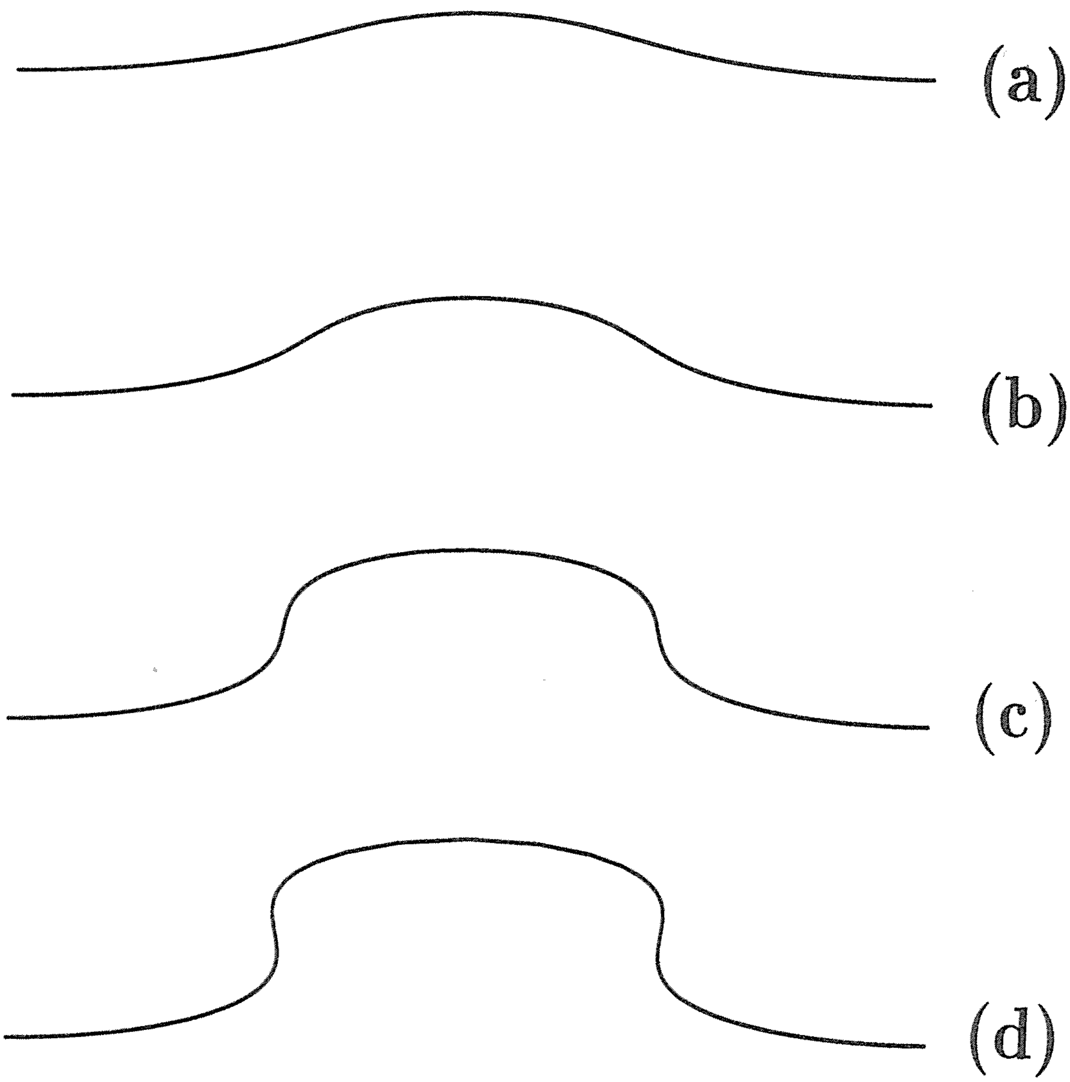

Figure 2: Wave profiles for $q = -0.5(a)$, $-1.0(b)$, $-2.25(c)$, $-3.0(d)$. N is equal to $255, 511, 1023, 1023$, respectively.

Because of the size limitation of the paper we focus on the result. We have bifurcation points $\{(p,q)\ ; q = 1-p\}$. We fixed $p = 1.1$ and considered the set of solutions (θ, q), with q as a bifurcation parameter. The bifurcation is a pitchfork emanating from $q = -0.1$, see Figure 1. We plotted the solutions in the (sum, q) plane, where $\mathrm{sum} = a_1 + a_2 + \cdots + a_N$ with a_n being given in (3). This quantity measures the largeness of the solutions, since

$$\mathrm{sum} = -(H\theta)(0),$$

and since $\exp((H\theta)(\sigma))$ is equal to the absolute value of the flow velocity divided by the wave velocity[4]. The pitchfork is subcritical; namely the branch exists in the range $q < -0.1$. As we trace the path of solutions, q becomes smaller and smaller. The wave profiles depart from nearly sinusoidal forms to some particular forms presented in Figure 2.

When we trace the solutions, we computed the eigenvalues of the Jacobians at the solutions, with a hope that we might have secondary bifurcations. However, we did not find any eigenvalue which cross the origin, as far as we have computed. Similar computations are carried out in the case of $p = 1.2$ and 2.1 but we find no critical eigenvalues, either. So we conclude that secondary bifurcation is absent *in* $q < 0$ *and in the symmetric waves.* There is the possibility that there is a secondary bifurcation of non-symmetric waves. However, more careful numerical tests would be necessary to derive any further conclusion.

4. Tanaka's method

The above-mentioned numerical method has a limitation that the Fourier series of $\theta(\sigma)$ tends to have a longer tail as q decrease. At $q = -3.0$, the 1023-th Fourier coefficient (see the equation (3)) satisfies $a(1023)/a(1) \approx 3.8846 \times 10^{-9}$, while it equals approximately 2.8577×10^{-7} at $q = -4.0$. The wave profile at $q = -4.0$ with $N = 1023$ is shown in Figure 3; it clearly shows inadequacy of truncation size.

In order to go further we modified our scheme following Tanaka[12]'s method for very steep gravity waves. He uses a certain variable transformation. Using his technique we are able to trace the solution path further towards left, until $q \approx -7.0$. The experiments are now in progress and will be reported elsewhere.

Acknowledgment: The authors are very grateful to the referee who brought our attention to the Peregrine's paper. This work was partially supported by the Sumitomo Foundation.

References

1. P.J. Aston, Wave Motion, vol. 17 (1993), pp. 113 - 141.

2. T.J. Bridges and F. Dias, Nonlinear Anal., Theory, Meth., and Appl., vol. 14 (1990), pp. 733–764.
3. B. Chen and P.G. Saffman, Stud. Appl. Math., vol. 62 (1980), pp. 95–111.
4. H. Okamoto, Nonlinear Anal. Theory Meth. & Appl., vol. 14 (1990) pp. 469–481.
5. H. Okamoto and M. Shōji, Res. Inst. Math. Sci. Kokyuroku, Kyoto Univ. # 740, (1991), pp 208–242.
6. H. Okamoto and M. Shōji, in Proceedings of the First China-Japan Seminar on Numerical Mathematics, Z.-c. Shi and T. Ushijima, World Scientific, (1993) pp. 119–132.
7. H. Okamoto and M. Shōji, The resonance of modes in the problem of capillary gravity waves, submitted.
8. D.H. Peregrine, J. Hydraulic Div. ASCE, vol.100 (1974), pp. 1215–1227.
9. L. Prandtl and O.G. Tietjens, *Fundamentals of Hydro- and Aerodynamics*, Dover Publ. (1957).
10. M. Shōji, J. Fac. Sci. Univ. Tokyo, Sec. IA, vol. 36 (1989), pp. 571–613.
11. M. Shōji and H. Okamoto, to appear in Adv. Math. Sci. Appl.
12. M. Tanaka, J. Phys. Soc. Japan, vol. 52 (1983), pp. 3047–3055.

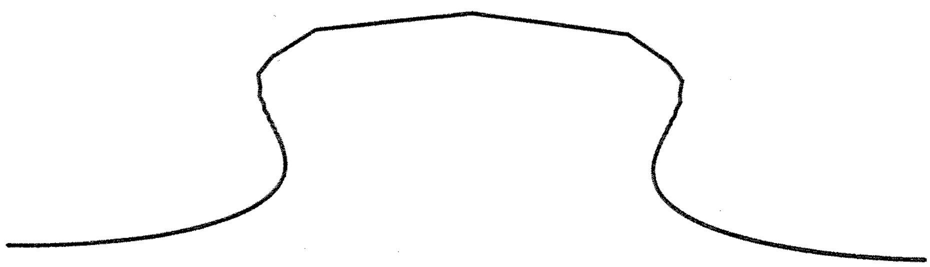

Figure 3: Wave profile for $q = -4.0$. The truncation is $N = 1023$. Insufficiency of the modes are clearly seen at the crest.