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Abstract. We consider a free boundary problem for incompressible perfect fluid with surface tension. The problem to be considered is as follows: A perfect fluid is circulating around a circle Γ (see Fig. 1). The outward curve γ is a free boundary to be sought. We assume that the flow, which is confined between Γ and γ, is irrotational. On the free boundary, surface tension works and makes the free boundary circular. On the other hand, the centrifugal force caused by the circulation of the flow makes the fluid go outward. Hence the balance of these two kinds of forces determines the geometrical properties of the free boundary. We show that there exist progressive waves, which are periodic motions of the fluid and are the exact solutions corresponding to the solitary waves.

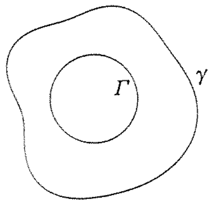

Fig. 1.

Key words: free boundary, bifurcation, progressive wave, surface tension

§ 1. Introduction

We consider a nonstationary flow of perfect fluid with a free boundary around a circle. The problem to be considered here can be regarded as a model for a flow around a celestial body. We consider a plane through an equator of a celestial body and a two-dimensional flow in this plane. We assume that for a fixed time t the flow region is enclosed by two closed Jordan curves Γ and $\gamma(t)$. Γ is an equator of a celestial body and $\gamma(t)$ is a free

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boundary which is outside Γ. These two curves enclose a doubly connected domain, which is denoted by $\Omega_{\gamma(t)}$. Then the fluid lies in $\Omega_{\gamma(t)}$ (see Fig. 1). For simplicity we assume that the inward curve Γ is the unit circle in the plane. We also assume that $\gamma(t)$ is represented as $\gamma(t)=\{(r, \theta);\ r=\gamma(t, \theta)\}$ in terms of a function $\gamma=\gamma(t, \theta)$ and the polar coordinates (r, θ). Then the problem is formulated as follows.

Problem. Find a time-dependent closed Jordan curve $\gamma(t)$, a stream function $V=V(t, r, \theta)$ and the pressure $P(t, r, \theta)$ satisfying the conditions (1.1–8) below.

(1.1) $$\Delta V=0 \quad \text{in} \quad Q_{T,\gamma}\equiv \bigcup_{0<t<T} \Omega_{\gamma(t)} ,$$

(1.2) $$V(t, 1, \theta)=0 \qquad \text{on} \quad [0, T]\times[0, 2\pi) ,$$

(1.3) $$\frac{\partial}{\partial\theta} V(t, \gamma(t, \theta), \theta)=\gamma(t, \theta)\frac{\partial\gamma}{\partial t}(t, \theta) \qquad \text{on} \quad [0, T]\times[0, 2\pi) ,$$

(1.4) $$\frac{1}{r}\frac{\partial^2 V}{\partial\theta\partial t}+\frac{\partial}{\partial r}\left(\frac{1}{2}|\nabla V|^2+P-\frac{g}{r}\right)=0 \qquad \text{in} \quad Q_{T,\gamma} ,$$

(1.5) $$-\frac{\partial^2 V}{\partial r\partial t}+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{2}|\nabla V|^2+P-\frac{g}{r}\right)=0 \qquad \text{in} \quad Q_{T,\gamma} ,$$

(1.6) $$P=\sigma K_{\gamma(t)} \qquad \text{on} \quad \gamma(t) ,$$

(1.7) $$V(0, r, \theta)=V_0(r, \theta) , \qquad \gamma(0, \theta)=\gamma_0(\theta) ,$$

(1.8) $$|\Omega_{\gamma(t)}|=\omega_0 .$$

Here σ, g and ω_0 are prescribed positive constants. Δ is the Laplace operator with respect to (r, θ). $K_{\gamma(t)}$ is the curvature of $\gamma(t)$, the sign of which is chosen to be positive if it is convex. For the physical meaning of this problem and derivation of these equations, see [7] or [8]. We only note that V is a stream function for the flow, i.e., the velocity vector is given by $(\partial V/\partial y, -\partial V/\partial x)$ and that (1.4, 5) is the Euler equation written in terms of V.

Our purpose in this paper is to show the existence of progressive wave solutions, i.e., solutions of the following form

$$\gamma=\tilde{\gamma}(\theta-ct) , \qquad V=\tilde{V}(r, \theta-ct) ,$$

where c is a constant. When $c=0$, this solution reduces to a stationary solution, which is studied in [7]. We can easily find a simple stationary solution given by

$$\gamma(t, \theta)\equiv r_0 , \qquad V=\frac{a}{\log r_0}\log r ,$$

where $r_0>1$ and $a>0$ are constants (see § 2 below). This is a radially symmetric solution and we call it a trivial solution.

We will prove that there exist progressive wave solutions in a neighborhood of the trivial solution. The propagation speed c is determined by the wave number and the magnitude of circulation. The important consequence of this result is that the trivial solution is not asymptotically stable, since the progressive wave solution is a time periodic solution. This fact constitutes a striking contrast to the case of viscous fluid (see Beale [1]). Indeed, in [1], Beale considers viscous incompressible fluid flow above a plane-like bottom with finite depth (a model of flow in the ocean). He proves that the rest state (motionless fluid with a horizontal plane as a free surface) is asymptotically stable by virtue of the viscous and capillary forces. Furthermore the rate of convergence to the motionless state is proved to be $O(t^{-1/2})$ (see, Beale and Nishida [13]).

We also show that if the circulation of the flow is zero, there is no stationary solution other than one in which the free boundary is a circle. This fact is worthy of notice because even if the circulation is zero there are progressive wave solutions which are not circles.

Finally we remark that we show the existence of the progressive waves by the bifurcation theory. In utilizing the theory, we take the propagation speed as a bifurcation parameter. This fact provides for distinction between our problem and the problem for flows in an infinite domain over the straight line or plane (i.e., the ocean problem). Indeed, the wave number ranges over all the real numbers in the case of the ocean problem. On the contrary, in our problem, the wave number can take only the integer multiples of 2π. This makes the spectrum discrete, whence we can use the bifurcation theory.

Our problem differs from the dissipative system of evolution equations like reaction-diffusion equations defined on **R**. In some dissipative system, the existence of the progressive wave solution is known but its propagation speed is uniquely determined by the system. Therefore such progressive waves do not fall into the framework of the bifurcation theory (see [4, 6]).

This paper is composed of four sections. In section 2 we reformulate the problem and give a precise version of the theorem. Section 3 is devoted to the proof of the theorem concerning the existence of progressive waves. In section 4 we show the uniqueness theorem for the stationary flow when the fluid does not move but only the surface tension works on the free surface. Finally we give a derivation of the formula used in § 3 concerning variation of domain.

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§ 2. Existence of Progressive Wave

We begin with the fact that there is a trivial stationary solution in which the free boundary is a circle. The stationary problem is to find a closed Jordan curve γ and a function V such that the conditions (2.1–5) below are satisfied:

$$\Delta V=0 \quad \text{in } \Omega_\gamma , \tag{2.1}$$

$$V=0 \quad \text{on } \Gamma , \tag{2.2}$$

$$V=a \quad \text{on } \gamma , \tag{2.3}$$

$$\frac{1}{2}|\nabla V|^2-\frac{g}{r}+\sigma K_\gamma=\text{constant} \qquad \text{on } \gamma , \tag{2.4}$$

$$|\Omega_\gamma|=\omega_0 . \tag{2.5}$$

Here Ω_γ is a doubly connected domain bounded by Γ and γ. Note that in the stationary problem the condition (1.3) reads that V is constant on γ. So we denote the constant by a. The constant a represents the magnitude of the circulation. The case of $a=0$ corresponds to the case where the circulation vanishes. If $a=0$, then $V\equiv 0$ by virtue of (2.1–3). Hence, in this case, the fluid does not move anywhere. The conditions (1.4–6) reduce to (2.4), which is known as the Bernoulli equality.

We now define $r_0>1$ by $\pi(r_0{}^2-1)=\omega_0$. Let γ_0 be a circle of radius r_0 with the origin as its center. We put $V_0(=V_0(r))=(a/\log r_0)\log r$ for $1<r<r_0$. Then $\{\gamma_0, V_0\}$ is a stationary solution, i.e., satisfies (2.1–5).

Our aim is to prove the existence of progressive wave solutions. We exclusively consider the solutions near the trivial solution. Hence what we really do is to show the existence of functions $u\in C^{3+\alpha}(S^1)$ and $V\in C^{3+\alpha}(\bar{\Omega}_\gamma)$ such that γ is given by $\gamma(t,\theta)=r_0+u(\theta-ct)$. Here the symbol $C^{3+\alpha}$ implies the Hölder space. These functions are governed by (1.1–8). If we introduce a new variable $\phi=\theta-ct$ and if we note that $\partial/\partial t$ is replaced by $-c(\partial/\partial\phi)$, then these governing equations are, in the present case, expressed as follows:

$$r\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}V\right)+\frac{\partial^2 V}{\partial\phi^2}=0 \qquad \text{in } \Omega_\gamma , \tag{2.6}$$

$$V(1,\phi)=0 \qquad \text{on } [0, 2\pi) , \tag{2.7}$$

$$\frac{\partial}{\partial\phi}V(\gamma(\phi),\phi)=-c\gamma(\phi)\frac{\partial\gamma}{\partial\phi}(\phi) \qquad \text{on } [0, 2\pi) , \tag{2.8}$$

$$-\frac{c}{r}\frac{\partial^2 V}{\partial\phi^2}+\frac{\partial}{\partial r}\left(\frac{1}{2}|\nabla V|^2+P-\frac{g}{r}\right)=0 \qquad \text{in } \Omega_\gamma , \tag{2.9}$$

(2.10) $$c\frac{\partial^2 V}{\partial r\partial\phi}+\frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{1}{2}|\nabla V|^2+P-\frac{g}{r}\right)=0 \qquad \text{in} \quad \Omega_\gamma\,,$$

(2.11) $$P=\sigma K_\gamma \qquad \text{on} \quad \gamma\,,$$

(2.12) $$\frac{1}{2}\int_0^{2\pi}(r_0+u(\phi))^2 d\phi-\pi=\omega_0\,.$$

The equation (2.10) is equivalent to saying that $cr(\partial V/\partial r)+(1/2)|\nabla V|^2+P-g/r$ does not depend on ϕ. On the other hand, by (2.6), the equation (2.9) is rewritten as follows:

$$c\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{\partial}{\partial r}\left(\frac{1}{2}|\nabla V|^2+P-\frac{g}{r}\right)=0 \qquad \text{in } \Omega_\gamma\,.$$

Hence we see that $cr(\partial V/\partial r)+(1/2)|\nabla V|^2+P-g/r$ does not depend on r. Therefore the condition (2.9–11) is reduced to the equation below:

(2.13) $$cr\frac{\partial V}{\partial r}+\frac{1}{2}|\nabla V|^2+\sigma K_\gamma-\frac{g}{r}=\text{constant} \qquad \text{on} \quad \gamma\,.$$

Next we consider the equation (2.8), which is rewritten as

$$\frac{\partial}{\partial\phi}V(\gamma(\phi),\phi)=-\frac{c}{2}\frac{\partial}{\partial\phi}\gamma^2\,.$$

Hence we have $V(\gamma(\phi),\phi)=-(c/2)\gamma(\phi)^2+\text{constant}$.

Now we can reformulate the problem. We first define symbols:

γ_u: a closed Jordan curve represented by $(r_0+u(\phi),\phi)$.

$\Omega_u=\{(r,\phi)\in\mathbf{R}^2;\ 1<r<r_0+u(\phi),\ 0\leqq\phi<2\pi\}$,

K_u: the curvature of γ_u.

Problem. Find a function $u\in C^{3+\alpha}(S^1)$ and a function $V\in C^{3+\alpha}(\bar{\Omega}_u)$ satisfying the following conditions:

(2.14) $$r\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{\partial^2 V}{\partial\phi^2}=0 \qquad \text{in} \quad \Omega_u\,,$$

(2.15) $$V(1,\phi)=0 \qquad \text{for} \quad 0\leqq\phi<2\pi\,,$$

(2.16) $$V(\gamma_u(\phi),\phi)=-\frac{c}{2}\gamma_u(\phi)^2+a+\frac{c}{2}r_0^2 \qquad \text{for} \quad 0\leqq\phi<2\pi\,,$$

(2.17) $$cr\frac{\partial V}{\partial r}+\frac{1}{2}|\nabla V|^2-\frac{g}{r}+\sigma K_\gamma=\text{constant} \qquad \text{on} \quad \gamma_u\,,$$

(2.18) $$\frac{1}{2}\int_0^{2\pi}(r_0+u(\phi))^2 d\phi-\pi=\omega_0\,.$$

Note that the problem above reduces to the sataionary problem (2.1–5) when the constant c vanishes. We solve this problem by a method very similar to that in [7]. Namely we regard this as a bifurcation problem. In [7, 9] the first author proved that nontrivial solutions to (2.1–5) bifurcate from the trivial solution. In this case we viewed the parameter a as the bifurcation parameter. As a consequence we see that there are an infinite number of nontrivial stationary solutions. To show the nontrivial solutions to (2.14–18), we fix a and we take the propagation speed c as a bifurcation parameter. We now give a framework by functional analysis. We define a mapping F which is defined near the origin of $\mathbf{R}\times C^{3+\alpha}(S^1)\times\mathbf{R}$ and takes its value in $C^{1+\alpha}(S^1)\times\mathbf{R}$. To define F, we first solve (2.14–16) which is a Dirichlet problem with respect to V for a given $u\in C^{3+\alpha}(S^1)$, and we denote the solution by V_u. Namely we define V_u by

$$\Delta V_u=0 \quad \text{in } \Omega_u, \qquad V_u=0 \quad \text{on } \Gamma , \qquad V_u=a+\frac{c}{2}(r_0{}^2-\gamma_u{}^2) \quad \text{on } \gamma_u .$$

We now define F as follows:

$$F_1(c;u,\xi)=\left(cr\frac{\partial V_u}{\partial r}+\frac{1}{2}|\nabla V_u|^2-\frac{g}{r}\right)\Big|_{\gamma_u}+\sigma K_u-\xi_0-\xi ,$$

$$F_2(c;u,\xi)=\frac{1}{2}\int_0^{2\pi}(r_0+u(\phi))^2 d\phi-\pi-\omega_0 ,$$

$$F(c;u,\xi)=(F_1(c;u,\xi),\,F_2(c;u,\xi)) .$$

Here the constant ξ_0 is defined so that $F(c;0,0)=(0,0)$. Namely we put

$$\xi_0=\frac{ca}{\log r_0}+\frac{1}{2}\left(\frac{a}{r_0\log r_0}\right)^2+\frac{\sigma-g}{r_0} .$$

Using a pull-back $(r_0,\theta)\to(r_0+u(\theta),\theta)$, we regard $F_1(c;u,\xi)$ as an element of $C^{1+\alpha}(S^1)$. Then it is clear that

$$\{\gamma_u, V_u\} \text{ is a solution to (2.14–18)} \Leftrightarrow F(c;u,\xi)=(0,0) .$$

Of course $(0,0)$ corresponds to the trivial solution. We show the existence of nontrivial zero-points of the mapping F, i.e., we prove the following

Theorem. *For $n\in\mathbf{N}$, define c_n by*

$$c_n=-\frac{a}{r_0{}^2\log r_0}\pm\left(\frac{(n^2-1)\sigma+g}{nr_0{}^3R_n}-\frac{a^2}{nR_nr_0{}^4(\log r_0)^2}\right)^{1/2}, \tag{2.19}$$

where $R_n=(r_0{}^n+r_0{}^{-n})/(r_0{}^n-r_0{}^{-n})$. Let $n\in\mathbf{N}$ be fixed. Suppose $c_n\in\mathbf{R}$ and that $c_n\neq c_m$ for all m which is not equal to n. Then c_n is a bifurcation point of F.

Remark 1. Note that the parameter $a \geqq 0$ is fixed. It is important to observe that in the case of $a=0$ there is no stationary solution other than the trivial solution but a progressive wave solution does exist even in this case. For the first statement, see § 4.

Remark 2. The assumption that $c_n \neq c_m$ for $m \neq n$ implies simpleness of the multiplicity in some sense. But for appropriate values of parameters σ, g, ω_0, it happens that $c_n = c_m$ for $n \neq m$. Even if this is the case, c_n is still a bifurcation point. But the behavior of the bifurcating branches becomes much more complicated. The situation is the same as that in [10], see also Fujii, Mimura and Nishiura [3]. So we only consider the bifurcation under the simpleness assumption.

§ 3. Proof of Theorem

This section is devoted to the proof of Theorem. We first show that F is a C^1-mapping and derive the concrete expression of the Fréchet derivative of F at $(c; 0, 0)$. The proof is similar to that in [7]. In fact the differentiability of F is proved in the same way as in [7]. Hence we omit it. We put $b = cr_0 + a/(r_0 \log r_0)$. Then the derivative is represented as follows.

$$D_{u,\xi}F = \begin{pmatrix} D_u F_1(c; 0, 0) & D_\xi F_1(c; 0, 0) \\ D_u F_2(c; 0, 0) & D_\xi F_2(c; 0, 0) \end{pmatrix}$$

$$D_u F_1(c; 0, 0)w = \frac{caw}{r_0 \log r_0} + b\left(\frac{\partial U}{\partial r} - \frac{aw}{r_0^2 \log r_0}\right) + \frac{gw}{r_0^2} - \frac{\sigma}{r_0^2}(w'' + w) \qquad (w \in C^{3+\alpha}(S^1)) , \tag{3.1}$$

$$D_u F_1(c; 0, 0)\lambda = -\lambda \qquad (\lambda \in \mathbf{R}) , \tag{3.2}$$

$$D_\xi F_2(c; 0, 0)w = r_0 \int_0^{2\pi} w(\theta) d\theta \qquad (w \in C^{3+\alpha}(S^1)) , \tag{3.3}$$

$$D_\xi F_2(c; 0, 0)\lambda = 0 \qquad (\lambda \in \mathbf{R}) , \tag{3.4}$$

where U is a solution of the Dirichlet problem below:

$$\begin{cases} \Delta U = 0 & \text{in} \quad 1 < r < r_0 , \\ U = 0 & \text{on} \quad \Gamma , \\ U = -bw & \text{on} \quad r = r_0 . \end{cases} \tag{3.5}$$

These formulas are derived in the Appendix. Admitting these formulas, we calculate the critical points of $D_{u,\xi}F$, i.e., we determine the conditions under which $D_{u,\xi}F(c; 0, 0)$ fails to be an isomorphism from $C^{3+\alpha}(S^1) \times \mathbf{R}$ onto $C^{1+\alpha}(S^1) \times \mathbf{R}$. For this purpose we look for $(w, \lambda) \in C^{3+\alpha}(S^1) \times \mathbf{R}$ such that $D_{u,\xi}F(c; 0, 0)(w, \lambda) =$

(0, 0). To express this equation concretely we expand w in the Fourier series:

$$w=\sum_{n=1}^{\infty} s_n \sin(n\theta)+\sum_{n=0}^{\infty} c_n \cos(n\theta) .$$

Then the solution of (3.5) is expressed as follows.

$$U=-\sum_{n=1}^{\infty} b\frac{r^n-r^{-n}}{r_0{}^n-r_0{}^{-n}}(s_n \sin n\theta+c_n \cos n\theta)-bc_0\frac{\log r}{\log r_0} .$$

Hence we have

$$\left.\frac{\partial U}{\partial r}\right|_{r=r_0}=-\sum_{n=1}^{\infty} b\frac{nR_n}{r_0}(s_n \sin n\theta+c_n \cos n\theta)-\frac{bc_0}{r_0 \log r_0} ;$$

Therefore we have

$$D_uF_1(c; 0, 0)w=c_0\times\text{“something”}$$
$$+\sum_{n=1}^{\infty}\left(\frac{ca}{r_0 \log r_0}+\frac{g-\sigma}{r_0{}^2}+\frac{n^2\sigma}{r_0{}^2}-b\left(b\frac{nR_n}{r_0}+\frac{a}{r_0{}^2 \log r_0}\right)\right)(s_n \sin n\theta+c_n \cos n\theta) ;$$

On the other hand, we easily obtain

$$D_uF_2(c; 0, 0)=2\pi r_0c_0 .$$

By these equalities we see that there is a $(w, \lambda)\neq(0, 0)$ satisfying $D_{u,\xi}F(c; 0, 0)\times(w, \lambda)=(0, 0)$ if and only if c satisfies (2.14) for some $n=1, 2, \cdots$. In fact $(w, \lambda)=(\cos n\theta, 0)$ and $(w, \lambda)=(\sin n\theta, 0)$ are eigenvectors for $c=c_n$. By the method used in [7] we can show that $D_{u,\xi}F(c; 0, 0)$ is an isomorphism if and only if $c\notin\{c_n\}$.

To show that c_n is a bifurcation point, we use Theorem 1.7 of Crandall and Rabinowitz [2]. This theorem ensures bifurcation from simple eigenvalue. So we introduce the following function space:

$$X^{m+\alpha}=\{u\in C^{m+\alpha}(S^1); u(\theta)\equiv u(2\pi-\theta)\} \qquad (m\in \mathbf{N}, 0<\alpha<1) .$$

Then we can easily show that F is a mapping from $\mathbf{R}\times X^{3+\alpha}\times\mathbf{R}$ into $X^{1+\alpha}\times\mathbf{R}$. Indeed $u\in X^{3+\alpha}$ if and only if the curve γ_u is symmetric with respect to the x-axis. If this symmetry holds, then the solution of (3.5) is symmetric with respect to the x-axis. Hence we see that $F: \mathbf{R}\times X^{3+\alpha}\times\mathbf{R}\to X^{1+\alpha}\times\mathbf{R}$. Henceforth we consider the mapping F restricted to $\mathbf{R}\times X^{3+\alpha}\times\mathbf{R}$. We now verify the conditions in [2]. By the assumption for c_n the kernel of $D_{u,\xi}F(c_n; 0, 0)$ is spanned only by $(\cos n\theta, 0)$. Then verification of the conditions in [2] is easy except that $D_cD_{u,\xi}F(c_n; 0, 0)(\cos n\theta, 0)\notin$ Range $D_{u,\xi}F(c_n; 0, 0)$ (nondegeneracy condition). We can show this by the formula (3.1–4). Differentiating these formulas by c, we see that $D_cD_{u,\xi}F(c_n; 0, 0)(\cos n\theta, 0)=\text{constant}\times$

$(\cos n\theta, 0)$. On the other hand, the range of $D_{u,\xi}F(c_n; 0, 0)$ is orthogonal (in the L^2-sense) to $(\cos n\theta, 0)$. Hence the condition above (the nondegeneracy condition) is verified. The proof is now completed. Q.E.D.

Remark 3.1. Numerical computation shows that the bifurcation occurs subcritically (see Fig. 2). This, however, does not imply that the bifurcating solutions are unstable. In fact, the nonstationary problem (1.1–8) is not of the form $u_t = F(u)$, where subscript means the differentiation, but we are dealing with the evolution equation of the following form:

$$u_{tt} = \Phi(u, u_t, u_{t\theta}, u_\theta, u_{\theta\theta}, u_{\theta\theta\theta}) . \tag{3.6}$$

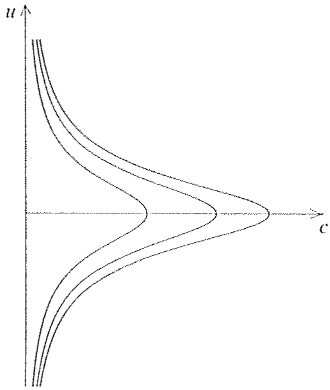

Fig. 2. A schematic bifurcation diagram. Every branch occurs subcritically.

(The reduction of (1.1–8) to (3.6) is found in [14].) Analyzing (3.6) in a way which is standard in the bifurcation theory, we can see that both the trivial solution and the bifurcating solution are marginally stable in the sense of the linearized stability. The stability analysis of the nonlinear equation (3.6) is very hard. It requires more analysis to determine the stability in the nonlinear sense.

Remark 3.2. As for the stability of the progressive wave solutions or the stationary solutions, nothing has been rigorously obtained so far. But the geometrical properties of stationary solutions are studied in [11, 12]. They show that the figures of the stationary solutions are not so simple. On the other hand, it is very difficult to simulate the nonstationary problem for a long range of time. Our computation (see Fig. 3) shows that the progressive waves exist for appropriate time. But a desicive answer for the stability is not yet known.

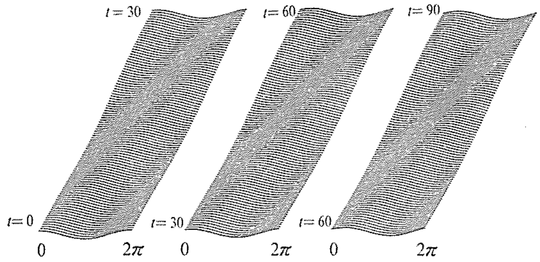

Fig. 3-a. A solitary wave which travels to the right. Difference $\gamma(t, \theta)-r_0$ is plotted from $t=0$ to $t=90$.

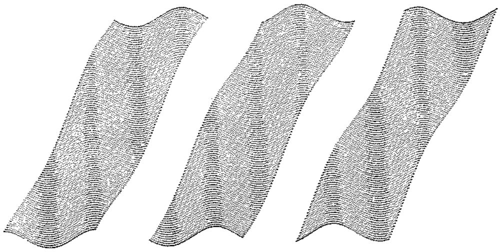

Fig. 3-b. A solitary wave which travels to the left.

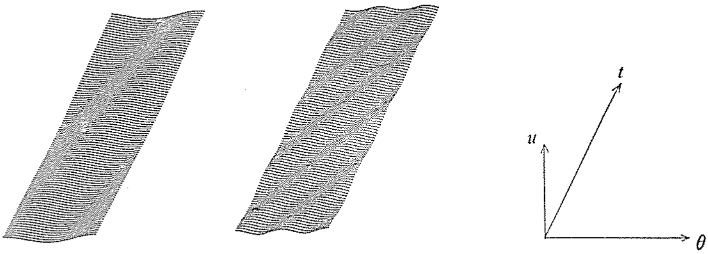

Fig. 3-c. A one-peak solitary wave and a two-peak solitary wave. In Fig. 3 the parameter a is fixed to be 0.1.

§ 4. Uniqueness of Stationary Solution

Here we prove a uniqueness theorem for the stationary problem. Our goal is to show the following

Theorem 2. *Suppose that $a=0$ and that $g>0$. Then there is no solution to* (2.1–5) *other than $\gamma=\gamma_0$, $V\equiv 0$.*

Remark. Here we do not assume that γ lies near γ_0. We only assume that γ is a C^2-curve. The assumption that $g>0$ is indispensable. In fact, if

$g<0$, existence of nontrivial solutions can be proved by the bifurcation theory.

Proof of Theorem 2. Let $\{\gamma, V\}$ be a solution. Then, by (2.1–3), we see that V vanishes identically in Ω_γ. Therefore

$$\sigma K_\gamma = \frac{g}{r} + \xi \quad \text{on} \quad \gamma\,, \tag{4.1}$$

where ξ is a constant. We have only to show that γ is a circle. We first prove the case where $\sigma<g$. Let A (resp. B) be a point on γ which has the largest (resp. smallest) distance from the origin. Then we have

$$\sigma K_\gamma(A) - \frac{g}{OA} = \sigma K_\gamma(B) - \frac{g}{OB} \quad \text{and} \quad OA \geqq OB\,.$$

By the definition of A and B, it holds that $K_\gamma(A) \geqq 1/OA$ and that $K_\gamma(B) \leqq 1/OB$. Therefore we obtain $\sigma/OA \leqq \sigma K_\gamma(A) = g/OA - g/OB + \sigma K_\gamma(B) \leqq g/OA - g/OB + \sigma/OB$. By this inequality and the assumption $\sigma<g$, we obtain $OA \leqq OB$, which implies that γ is a circle.

We now prove the case where $\sigma \geqq g$. We assume for simplicity that the curve γ is represented by the equation $r=\gamma(\theta)$. Then by (2.5) we have

$$\frac{1}{2}\int_0^{2\pi} \gamma(\theta)^2 d\theta = \pi + \omega_0\,. \tag{4.2}$$

It is known that total (oriented) curvature of a closed Jordan curve is equal to 2π, i.e.,

$$\int_0^{2\pi} K_\gamma (\gamma(\theta)^2 + \gamma'(\theta)^2)^{1/2} d\theta = 2\pi\,.$$

As for the proof, see, e.g., Hopf [5]. By (4.1) and this fact we have

$$2\pi\sigma = g\int_0^{2\pi} (\gamma(\theta)^2 + \gamma'(\theta)^2)^{1/2}/\gamma(\theta)\, d\theta + \xi L_\gamma\,, \tag{4.3}$$

where L_γ is the length of the curve γ. We multiply (4.1) by γ^2 and integrate it on $[0, 2\pi)$. Then we obtain

$$\begin{aligned} \xi\int_0^{2\pi} \gamma(\theta)^2 d\theta &= -g\int_0^{2\pi} \gamma(\theta) d\theta + \sigma\int_0^{2\pi} \gamma^2 K_\gamma \\ &= -g\int_0^{2\pi} \gamma(\theta) d\theta + \sigma L_\gamma\,. \end{aligned} \tag{4.4}$$

The second equality is proved by the formula

$$K_\gamma = \frac{1}{(\gamma(\theta)^2 + \gamma'(\theta)^2)^{1/2}} - \frac{1}{\gamma}\left(\frac{\gamma'(\theta)}{(\gamma(\theta)^2 + \gamma'(\theta)^2)^{1/2}}\right)'.$$

Eliminating ξ from (4.2–4) we obtain

$$\sigma\left(2\pi-\frac{L_\gamma^2}{2(\pi+\omega_0)}\right)=g\left(\int_0^{2\pi}(\gamma(\theta)^2+\gamma'(\theta)^2)^{1/2}/\gamma(\theta)d\theta-\frac{L_\gamma}{2(\pi+\omega_0)}\int_0^{2\pi}\gamma(\theta)d\theta\right). \tag{4.5}$$

By the isoperimetric inequality the left-hand side is nonpositive and it vanishes if and only if γ is a circle. Then, by $\sigma \geqq g>0$, we have

$$\frac{L_\gamma}{2(\pi+\omega_0)}\int_0^{2\pi}\gamma(\theta)d\theta-\int_0^{2\pi}(\gamma(\theta)^2+\gamma'(\theta)^2)^{1/2}/\gamma(\theta)d\theta \geqq \frac{L_\gamma^2}{2(\pi+\omega_0)}-2\pi .$$

On the other hand, it obviously holds that $\int_0^{2\pi}\gamma(\theta)d\theta \leqq L_\gamma$ and that $-\int_0^{2\pi}(\gamma(\theta)^2+\gamma'(\theta)^2)^{1/2}/\gamma(\theta)d\theta \leqq -2\pi$. Hence the equality must hold. This is possible if and only if $\gamma'\equiv 0$, i.e., γ is a circle, which completes the proof.

Q.E.D.

Appendix.

Here we show how the formulas (3.1–4) are derived. The procedure here is the same as that in [7]. In fact we borrow a lemma which is proved in [7]. We first prepare notation. Let f be a C^1-mapping from $C^{m+1+j+\alpha}(S^1)$ into $C^{m+1+\alpha}(S^1)$. Here $m>0$ and $j\geqq 0$ are integers and $0<\alpha<1$. For a given u, we denote by W_u the solution of the following Dirichlet problem:

$$\Delta W_u=0 \quad \text{in} \quad \Omega_u , \tag{A.1}$$

$$W_u=0 \quad \text{on} \quad \Gamma , \qquad W_u=f(u) \quad \text{on} \quad \gamma_u . \tag{A.2}$$

Of course the second equation of (A.2) implies that $W_u(r_0+u(\theta),\theta)=f(u)(\theta)$. In terms of W_u we define mappings T and T_1 by

$$T(u)=\frac{\partial W_u}{\partial \nu_u}\bigg|_{\gamma_u} , \qquad T_1(u)=\frac{\partial W_u}{\partial r}\bigg|_{\gamma_u} ,$$

where $\partial/\partial\nu_u$ is a differentiation along the outward normal to γ_u. Then the following lemma holds:

Lemma. *T and T_1 are C^1-mappings from some neighborhood of the origin in $C^{m+1+j+\alpha}(S^1)$ into $C^{m+\alpha}(S^1)$. If u belongs to C^λ for some $\lambda>m+2+j+\alpha$, then the derivative of T is represented by*

$$D_uT(u)v=\frac{\partial \Psi}{\partial \nu_u}\bigg|_{\gamma_u}+\Sigma(W_u)v-\frac{(r_0+u)v'-u'v}{\{(r_0+u)^2+(u')^2\}^{3/2}}\frac{\partial f(u)}{\partial\theta} . \tag{A.4}$$

Here Ψ is the solution of

$$\Delta\Psi=0 \quad \textit{in} \quad \Omega_u , \tag{A.5}$$

(A.6) $$\Psi=0 \quad on \quad \Gamma\,, \qquad \Psi=D_u f(u)v-\frac{\partial W_u}{\partial r}v \quad on \quad \gamma_u\,.$$

Here $D_u f$ *is the Fréchet derivative. The function* $\Sigma(W_u)$ *is given by*

$$\Sigma(W_u)=\frac{1}{((r_0+u)^2+(u')^2)^{1/2}}\left(r\frac{\partial^2 W_u}{\partial r^2}-u'\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial W_u}{\partial\theta}\right)\right).$$

The derivative of T_1 *is represented by*

$$D_u T_1(u)v=\frac{\partial\Psi}{\partial r}\bigg|_{\gamma_u}+\frac{\partial^2 W_u}{\partial r^2}v\,.$$

If we admit this lemma, the proof of (3.1–4) is easy. In fact we put $f(u)=-(c/2)(r_0+u)^2+a+(c/2)r_0^2$. Then the lemma shows that

(A.7) $$D_u T(0)w=D_u T_1(0)w=\frac{\partial U}{\partial r}+\frac{\partial^2 V_0}{\partial r^2}w\,,$$

where U is given by (3.5). Observing that $|\nabla V|=\partial V_u/\partial\nu$, we obtain (3.1) by (A.7). Other formulas are easy to see. Q.E.D.

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