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The Stokes Expansion Method to the Bifurcation Problem of Plane Progressive Water Waves

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§0. Introduction.

In this paper, we consider a problem of progressive waves, by which we mean a fluid motion with free surface whose shape looks constant in a moving frame. Although this classical problem has a long history, the global picture of the set of the solutions (bifurcation diagrams) is not completely known. Chen and Saffman [1,2] computed several types of bifurcating solutions numerically. Following them, the first author presented in [**12,13**] numerical computations with an emphasis on the bifurcation diagrams, on which [1,2] did not pay much attention. It is our objective in this paper to give a mathematical legality to the analysis in [**12,13**].

Our study is based on the theory of normal forms of bifurcation equations. We can show that the structure of the progressive water waves in [**12,13**] is explained by the following sets of algebraic equations:

$$(A)\quad \begin{cases} (\epsilon\lambda+\eta+\alpha_1x^2+\alpha_2z^2+\alpha_3x^2z)x+(\gamma_0+\gamma_1z^2)xz=0,\\ (\delta\lambda+\beta_1x^2+\beta_2z^2)z+x^2=0,\end{cases}$$

and

$$(B)\quad \begin{cases} (\epsilon\lambda+\eta+\alpha_1x^2+\alpha_2z^2+\alpha_3x^3z+\alpha_4x^4+\alpha_5x^2z^2+\alpha_6z^4)x+x^2z=0,\\ (\delta\lambda+\beta_1x^2+\beta_2z^2+\beta_3x^3z+\beta_4x^4+\beta_5x^2z^2+\beta_6z^4)z+x^3=0,\end{cases}$$

where λ, x and z are real variables, and $\epsilon, \delta, \alpha_j$ and $\beta_j\,(j = 1, 2, \cdots, 6)$ are real constants. The equation (A) corresponds to the case of mode (1,2), which is derived by the second author ([**9,10**]) from a certain degeneration of the equations given in Fujii, Mimura and Nishiura [4]. The meaning of the degeneration is that a certain coefficient of the bifurcation equation is very small. The equation (B) corresponds to the case of the mode (1,3) which is derived in [**10**]. To study a normal form, we followed the method given in [**4,5**]. To apply their theory, we must prove that the present problem has an O(2)-symmetry. Due to the O(2)-symmetry, the bifurcation equation can be simplified and we can get to the polynomial equations above. Detail of how to derive the above polynomials is given in [**10**].

In this paper we formulate the problem by using the Stokes expansion, while Levi-Civita's formulation is used in [**9,10**]. Main result is to prove that the problem is reduced to a bifurcation problem with O(2)-equivariance. This is done in §2 after we have introduced the Stokes expansion method in §1. The O(2)-equivariance of the present problem is proved in [**9**]. However, the proof in [**9**] uses Levi-Civita's formulation, which differs from the Stokes expansion method used in [**12,13**] and the present paper. Consequently, we give a proof of O(2)-equivariance for the present formulation. We show in §3 how the equations (A,B) explain the bifurcation diagrams in [**12,13**].

§1. Formulation.

In this section we introduce our formulation. We take an x-y coordinate system moving in the same direction as the progressive wave with the same speed c. The free boundary is represented by a function H as $\{(x, y) \mid y = H(x)\}$ and the fluid region is $\{(x, y) \mid -\infty < x < \infty, 0 < y < H(x)\}$. We assume a usual hypothesis that the wave profile is periodic in x of the period, say, L. By the periodicity, we may restrict our consideration to the flow in $\Omega_H \equiv \{(x, y) \mid |x| < \frac{L}{2}, 0 < y < H(x)\}$. We, however, do not assume another well-accepted assumption that the wave profile is symmetric (this implies the function $H(x)$ is even function). This inclusion of nonsymmetric waves is necessary so as to see O(2)-equivariance in §2. This is also natural in that there is a numerical evidence of the existence of nonsymmetric waves (Zufiria [**14**]).

Problem. Find functions $H = H(x)\quad (-\frac{L}{2} < x < \frac{L}{2}),\quad U(x, y)$ and $V(x, y)$

$\left((x,y)\in\Omega_H=\left\{-\frac{L}{2}<x<\frac{L}{2}, 0<y<H(x)\right\}\right)$ satisfying the followings:

(1) U and V are harmonic in Ω_H and $w=w(z)=U+iV$ is a complex analytic function of $z=x+iy$,

(2) $H(x)$ and $\dfrac{dw}{dz}$ are periodic functions of x with a period L,

(3) $V=0$ on $y=H(x)$,

(4) $V=-a$ on $y=0$,

(5) $$\frac{1}{2}\left|\frac{dw}{dz}\right|^2+gH-T\left(\frac{H_x}{(1+H_x^2)^{1/2}}\right)_x=\text{constant} \qquad \text{on } y=H(x),$$

where a,c,g and T are positive constants. Subscripts mean differentiations. c is a propagation speed. g is the gravity acceleration and T is the surface tension coefficient. a is determined by c and the mean depth of the flow and $a=\infty$ corresponds to infinite depth. We remark that the constant of the right hand side of (5) depends on a, c and the choice of origin.

We formulate the problem by what is called the Stokes expansion method. Following the idea due to Stokes that z is regarded as a function of w, we can overcome the mathematical difficulty caused by the fact that the boundary portion $\{y=H(x)\}$ is unknown. The following (6) is a modification of the Stokes expansion so that $w=U+iV=U-ia$ corresponds to the x-axis: we seek a solution of the following form:

(6) $$\begin{aligned} z &= x+iy \\ &= \frac{w}{c}+\frac{iL}{2\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[A_n\frac{\sinh(-\frac{2n\pi(iw-a)}{cL})}{\sinh(\frac{2n\pi a}{cL})}+iB_n\frac{\cosh(-\frac{2n\pi(iw-a)}{cL})}{\cosh(\frac{2n\pi a}{cL})}\right] \\ &\qquad +\frac{iL(A_0+iB_0)}{2\pi}, \end{aligned}$$

where $A_n, B_n\in\mathbb{R}$ are unknowns to be sought and $i=\sqrt{-1}$. The constant terms, A_0 and B_0, are determined by the positioning of the origin. Therefore they do not affect wave profiles. The function $z=z(w)$ is defined in $\{|U|\le cL/2, -a\le V\le 0\}$. Consequently the free boundary problem is transformed to a problem of fixed domain. By (3), the free surface $\{(x,y)\mid y=H(x)\}$ is obtained by putting $V=0$ in (6). This gives us

$$\begin{cases} x=\dfrac{U}{c}+\dfrac{L}{2\pi}\displaystyle\sum_{1}^{\infty}\left[\frac{A_n}{n}\coth(\frac{2n\pi a}{cL})\sin(\frac{2n\pi U}{cL})-\frac{B_n}{n}\cos(\frac{2n\pi U}{cL})\right]+\dfrac{LB_0}{2\pi}, \\ y=\dfrac{L}{2\pi}\displaystyle\sum_{1}^{\infty}\left[\frac{A_n}{n}\cos(\frac{2n\pi U}{cL})+\frac{B_n}{n}\tanh(\frac{2n\pi a}{cL})\sin(\frac{2n\pi U}{cL})\right]+\dfrac{LA_0}{2\pi}. \end{cases}$$

Our task is, therefore, to determine $(A_n, B_n, n = 1, 2, \cdots)$. We determine them by the condition (5). On the free surface, the stream function V is constant, hence we have $dw \equiv dU$ there. So $|dz/dw|^2 = (dx/dU)^2 + (dy/dU)^2$. We define non-dimensional variables $\xi = 2\pi U/cL, X(\xi) = 2\pi x(U)/L$ and $Y(\xi) = 2\pi y(U)/L$. Then we have the following equivalent expression for (5).

$$\Psi(\xi) = \text{constant}, \tag{7}$$

where

$$\left\{\begin{aligned} &\Psi(\xi) = \frac{\mu}{2}\frac{1}{X'^2 + Y'^2} + Y - \kappa\frac{X'Y'' - X''Y'}{(X'^2 + Y'^2)^{\frac{3}{2}}}, \\ &X(\xi) = \xi + \sum_1^\infty \left[\frac{A_n}{n}\coth(n\nu)\sin(n\xi) - \frac{B_n}{n}\cos(n\xi)\right] + B_0, \\ &Y(\xi) = \sum_1^\infty \left[\frac{A_n}{n}\cos(n\xi) + \frac{B_n}{n}\tanh(n\nu)\sin(n\xi)\right] + A_0, \\ &\qquad \nu = 2\pi a/(cL), \qquad \mu = 2\pi c^2/(gL), \qquad \kappa = 4\pi^2 T/(gL^2) \end{aligned}\right. \tag{8}$$

Here $'$ means differentiations about ξ. Chen and Saffman used the above equation (7-8). We, however, prefer the differential form (7') of (7) since it is convenient in order to apply bifurcation theory given by Crandall and Rabinowitz ([**3**]) and others. Then the task is to solve

$$\frac{d}{d\xi}\Psi(\xi) = 0 \tag{7'}$$

for given κ, μ and ν. Since only derivatives of X and Y appear in $d\Psi/d\xi$, we note that the constant terms, A_0 and B_0, disappear and we have a formulation which is closed in $(A_1, A_2, \cdots, B_1, B_2, \cdots)$. To state in a more mathematical fashion, we rewrite this problem as a problem to seek zeros of a mapping F which we define now. Suppose we are given two sequences $u = (A_1, A_2, \cdots)$ and $v = (B_1, B_2, \cdots)$. We define sequences $(A_1^*, A_2^*, \cdots)$ and $(B_1^*, B_2^*, \cdots)$ by

$$A_n^* = \frac{1}{\pi}\left(\frac{d}{d\xi}\Psi(\xi), \sin(n\xi)\right) = \frac{1}{\pi}\left(\Phi_{uv}(\xi), \sin(n\xi)\right), \tag{9}$$

and

$$B_n^* = -\frac{\coth(n\nu)}{\pi}\left(\frac{d}{d\xi}\Psi(\xi), \cos(n\xi)\right) = -\frac{\coth(n\nu)}{\pi}\left(\Phi_{uv}(\xi), \cos(n\xi)\right), \tag{10}$$

where

$$(8')\qquad \begin{cases} \Phi_{uv}(\xi) = \dfrac{\mu}{2}\left(\dfrac{1}{X'^2+Y'^2}\right)' + Y' - \kappa\left(\dfrac{X'Y''-X''Y'}{(X'^2+Y'^2)^{\frac{3}{2}}}\right)', \\ X'(\xi) = 1 + \displaystyle\sum_1^\infty [A_n \coth(n\nu)\cos(n\xi) + B_n \sin(n\xi)] + B_0, \\ Y'(\xi) = \displaystyle\sum_1^\infty [-A_n \sin(n\xi) + B_n \tanh(n\nu)\cos(n\xi)] + A_0, \end{cases}$$

Then $\Phi_{uv}(\xi)$ is written as

$$(11)\qquad \Phi_{uv}(\xi) = \sum_{j=1}^\infty A_j^* \sin(j\xi) - \sum_{j=1}^\infty B_j^* \tanh(j\nu)\cos(j\xi).$$

We define a mapping F by $F(\kappa,\mu,\nu;u,v) = (A_1^*, A_2^*, \cdots, B_1^*, B_2^*, \cdots)$.Consequently, our task is to solve

$$(12)\qquad F(\kappa,\mu,\nu;u,v) = 0$$

for given κ, μ and ν.
Let us define Banach spaces of series by

$$\mathrm{V}_k = \{u = (A_1, A_2, \cdots) \mid \sum_{n=1}^\infty n^k |A_n| < \infty\} \qquad (k = 1, 2, \cdots),$$

with the following norm:

$$\|u\|_k \equiv \sum_{n=1}^\infty n^k |A_n|.$$

Let X_k be Banach spaces of functions having absolutely convergent Fourier series defined by

$$\mathrm{X}_k = \{f = \sum_{n=1}^\infty A_n \cos(n\xi) + \sum_{n=1}^\infty B_n \sin(n\xi) \in C^0(S)/\mathrm{R} \mid (A_n), (B_n) \in \mathrm{V}_k\}.$$

REMARK 1. *If $u, v \in \mathrm{V}_k$ $(k \geq 1)$, then the functions $X(\xi)$ and $Y(\xi)$ are C^k-functions of ξ.*

PROPOSITION 1. *F is a smooth mapping from $\mathcal{U}$ into $\mathrm{V}_0 \times \mathrm{V}_0$, where $\mathcal{U}$ is some neighborhood of $\mathrm{R}^3 \times \{(0,0)\}$ in $\mathrm{R}^3 \times \mathrm{V}_2 \times \mathrm{V}_2$.*

PROOF: First we remark that $\mathrm{X}_0 \oplus \mathrm{R}$ is a Banach algebra, namely if $f, g \in \mathrm{X}_0 \oplus \mathrm{R}$, then $f \cdot g \in \mathrm{X}_0 \oplus \mathrm{R}$. $\mathrm{X}_k \oplus \mathrm{R}$ $(k \geq 1)$ are also Banach algebras. Note that if $u = (A_1, A_2, \cdots), v = (B_1, B_2, \cdots) \in \mathbf{V}_2$, then $X', Y' \in \mathbf{X}_2, X'', Y'' \in \mathbf{X}_1$ and $X''', Y''' \in \mathbf{X}_0$. If $\|u\|_2 + \|v\|_2 < \delta < \tanh(\nu_0)$ for $0 < \nu_0 \le \nu$, then $(X'^2 + Y'^2)^{-1} < \infty$, since $|X'(\xi)| \ge 1 - \coth(\nu_0)(\|u\|_0 + \|v\|_0)$ and $|Y'(\xi)| \le \|u\|_0 + \|v\|_0$. By this we see that $(X'^2 + Y'^2)^{-1} \in \mathbf{X}_0 \oplus \mathbf{R}$. These facts and

$$\frac{d}{d\xi}\Psi_{uv}(\xi) = -\mu\frac{X'X'' + Y'Y''}{(X'^2 + Y'^2)^2} + Y' - \kappa\frac{X'Y''' - X'''Y'}{(X'^2 + Y'^2)^{\frac{3}{2}}} + \kappa\frac{(X'Y'' - X''Y')(X'X'' + Y'Y'')}{(X'^2 + Y'^2)^{\frac{5}{2}}}.$$

prove that $\Phi \in \mathbf{X}_0$. On the other hand, we have $\tanh(\nu) \le \tanh(n\nu) < 1\,(1 \le n)$. These fact and (9,10) show that (A_n^*) and $(B_n^*) \in \mathbf{V}_0$. ∎

Our problem is now to solve the nonlinear equation $F(\kappa, \mu, \nu; u, v) = 0$ in $\mathcal{U}$. To put our problem into a bifurcation theoretic formulation, we consider a Fréchet derivative of F at $(u, v) = (0, 0)$, which is denoted by $DF(\kappa, \mu, \nu; 0, 0)$. We prove the following

PROPOSITION 2. *$(u, v) = (0, 0)$ is a solution for all κ, μ and ν. $DF(\kappa, \mu, \nu; 0, 0) : \mathbf{V}_2 \times \mathbf{V}_2 \to \mathbf{V}_0 \times \mathbf{V}_0$ fails to be an isomorphism if and only if κ, μ and ν satisfy the following (13) for some positive integer m:*

$$\mu = \tanh(m\nu)(\frac{1}{m} + m\kappa). \tag{13}$$

PROOF: $F(\kappa, \mu, \nu; 0, 0) \equiv 0$ is obvious. By linearizing (7') at $(u, v) = (0, 0)$ we have

$$-\mu X'' + Y' - \kappa Y''' = 0.$$

By this equation and (8), we obtain

$$\begin{aligned} &DF(\kappa, \mu, \nu; 0, 0)(u, v) \\ &\quad = \sum \{\mu n \coth(n\nu) - 1 - \kappa n^2\}[A_n \sin(n\xi) - B_n \tanh(n\nu)\cos(n\xi)], \\ &\qquad\qquad ((u, v) \in \mathbf{V}_2 \times \mathbf{V}_2), \end{aligned} \tag{14}$$

and (13) follows immediately. ∎

Now we introduce the following symbol:

$$S_m = \{(\kappa, \mu, \nu) \mid \mu = \tanh(m\nu)(\frac{1}{m} + m\kappa)\}.$$

DEFINITION.

(i) If $(\kappa, \mu, \nu) \in S_n$ and $(\kappa, \mu, \nu) \notin S_m \quad (\forall m \ne m)$, then we call (κ, μ, ν) a simple

bifurcation point of mode n.
(ii) Let m, n be integers such that $0 < m < n$. If $(\kappa, \mu, \nu) \in S_m \cap S_n \equiv S_{m,n}$ and $(\kappa, \mu, \nu) \notin S_l \quad (\forall l \neq m, n)$, then we call (κ, μ, ν) a double bifurcation point of mode (m,n).

For fixed $\nu = \nu_0$ and $m \in \mathbb{N}$, $\{(\kappa, \mu) \mid (\kappa, \mu, \nu_0) \in S_m\}$ forms straight lines. Let m and n be fixed integers such that $0 < m < n$ and let (κ_0, μ_0, ν_0) be the double bifurcation point of mode (m, n). Then we have

$$\kappa_0 = \frac{n \tanh(m\nu_0) - m \tanh(n\nu_0)}{mn(n \tanh(n\nu_0) - m \tanh(m\nu_0))}, \tag{15}$$

$$\mu_0 = \frac{(n^2 - m^2) \tanh(m\nu_0) \tanh(n\nu_0)}{mn(n \tanh(n\nu_0) - m \tanh(m\nu_0))}. \tag{16}$$

§2. O(2)-equivariance.

In this section we define an action of the orthogonal group O(2) on $V_0 \times V_0$ and prove that the mapping F commutes with it. Let us recall that O(2) is generated by rotation with angle $\beta \, (0 \leq \beta < 2\pi)$ and the reflection with respect to the x-axis. Accordingly,

$$w \longrightarrow w + \frac{cL\beta}{2\pi} \qquad \text{and} \qquad w \longrightarrow \overline{w}$$

define actions of O(2) on w. These actions define the following actions on $(A_1, A_2, \cdots, B_1, B_2, \cdots)$:

$$(A_1, A_2, \cdots, B_1, B_2, \cdots) \longrightarrow (A_1', A_2', \cdots, B_1', B_2', \cdots), \tag{17}$$

where the entries of the right hand side are:

$$\begin{cases} A_n' = A_n \cos(n\beta) + B_n \tanh(n\nu) \sin(n\beta) \\ B_n' = -A_n \coth(n\nu) \sin(n\beta) + B_n \cos(n\beta) \end{cases} \quad \text{for the } \beta\text{-rotation} \tag{18}$$

and

$$A_n' = A_n, \quad B_n' = -B_n \qquad \text{for the reflection.} \tag{19}$$

Then we have the following theorem which is a basis of the subsequent analysis.

THEOREM 1. *F commutes with the above actions of O(2). Namely the following relation holds*

$$F(\kappa, \mu, \nu; \gamma(u, v)) = \gamma F(\kappa, \mu, \nu; u, v) \qquad (\gamma \in O(2)).$$

PROOF: By the β-rotation, $X'(\xi)$ is sent to $X'(\xi+\beta)$ and it holds that

$$X'(\xi+\beta) = 1 + \sum_1^\infty [A_n \coth(n\nu)\cos(n(\xi+\beta)) + B_n \sin(n(\xi+\beta))],$$
$$= 1 + \sum_1^\infty [A'_n \coth(n\nu)\cos(n\xi) + B'_n \sin(n\xi)].$$

We have similar expressions for the other derivatives of X and Y. Therefore sending ξ to $\xi+\beta$ is equivalent to sending $((A_j),(B_j))$ to $((A'_j),(B'_j))$ by the rule (18). On the other hand $\Phi(\xi)$ is, by the action of β-rotation, sent to

$$\Phi(\xi+\beta) = \sum A^*_n \sin(n(\xi+\beta)) - \sum B^*_n \tanh(n\nu)\cos(n(\xi+\beta)),$$
$$= \sum\{A^*_n \cos(n\beta) + B^*_n \tanh(n\nu)\sin(n\beta)\}\sin(n\xi)$$
$$- \sum\{B^*_n \cos(n\beta) - A^*_n \coth(n\nu)\sin(n\beta)\}\tanh(n\nu)\cos(n\xi).$$

This shows that $((A^*_n),(B^*_n))$ is also transformed by the rule (18). Hence (18) is satisfied for the β-rotation. Similarly, it is easy to see the commutativity with the reflection. ▮

Although this theorem is simple, it is valuable in that the presence of a symmetry group considerably simplifies the bifurcation equation. Namely the theorem enables us to use a classification of O(2)-equivariant mappings in [**5,7**]. Let integers m and n be fixed as $0 < m < n$ and let (κ_0, μ_0, ν_0) be the double bifurcation point of mode (m,n). We now have

THEOREM 2. *The kernel of $DF(\kappa_0,\mu_0,\nu_0;0,0)$ is of 4-dimensions, and it is spanned by $(e_m,0),(0,e_m),(e_n,0)$ and $(0,e_n)$, where $e_m = (0,\cdots,0,\overset{m}{\check{1}},0,\cdots)$ and $0=(0,0,\cdots)\in V_0$.*

PROOF: (14) easily proves this. ▮

Let N denote the kernel and represent it as

$$N = \{x(e_m,0)+y\coth(m\nu)(0,e_m)+z(e_n,0)+w\coth(n\nu)(0,e_n) \mid x,y,z,w \in \mathbb{R}\}.$$

Introducing complex variables $\sigma = x+iy$ and $\zeta = z+iw$, we identify N with $\mathbb{C}^2 = \{(\sigma,\zeta) \mid \sigma,\zeta \in \mathbb{C}\}$. Then the actions of O(2) on N are expressed as follows:

(20) $\quad (\sigma,\zeta) \longrightarrow (\overline{\sigma},\overline{\zeta})$ for the reflection,

(21) $\quad (\sigma,\zeta) \longrightarrow (e^{-im\beta}\sigma, e^{-in\beta}\zeta)$ for the rotation with angleβ.

Since $(x, y\coth(m\nu), z, w\coth(n\nu))$ is sent to $(x, -y\coth(m\nu), z, -w\coth(n\nu))$ by reflection (19), (20) is obvious. $(x, y\coth(m\nu), z, w\coth(n\nu))$ is sent by (18) to

$$\begin{aligned}&\big(x\cos(m\beta) + y\sin(m\beta), \{y\cos(m\beta) - x\sin(m\beta)\}\coth(m\nu),\\ &\qquad z\cos(n\beta) + w\sin(n\beta), \{w\cos(n\beta) - z\sin(n\beta)\}\coth(n\nu)\big)\\ &= \big(Re(e^{-im\beta}\sigma), Im(e^{-im\beta}\sigma)\coth(m\nu), Re(e^{-in\beta}\zeta), Im(e^{-in\beta}\zeta)\coth(n\nu)\big),\end{aligned}$$

which proves (21).

We now use the Lyapounov-Schmidt method to obtain a bifurcation equation. Denoting by P the L^2-projection onto N, we consider an equation of $\phi = \phi(\kappa, \mu, \nu; x, y, z, w)$:

$$\begin{aligned}(I - P)F(\kappa, \mu, \nu; x(e_m, 0) + y\coth(m\nu)(0, e_m) + z(e_n, 0)\\ + w\coth(n\nu)(0, e_n) + \phi(\kappa, \mu, \nu, x, y, z, w)) = 0.\end{aligned}$$

In a standard fashion, this is solved uniquely in some neighborhood of $(\kappa_0, \mu_0, \nu_0; 0, 0, 0, 0)$ and we have a function

$$\phi : D \longrightarrow (I - P)(V_2 \times V_2),$$

where D is a neighborhood of $(\kappa_0, \mu_0, \nu_0; 0, 0, 0, 0)$. We now define G by

$$\begin{aligned}G(\kappa, \mu, \nu; x, y, z, w) = PF(\kappa, \mu, \nu; x(e_m, 0) + y\coth(m\nu)(0, e_m) + z(e_n, 0)\\ + w\coth(n\nu)(0, e_n) + \phi(\kappa, \mu, \nu; x, y, z, w)).\end{aligned}$$

As above, we use the notation by complex variables σ and ζ and we identify the range of G with $\mathbb{C}^2$ in the same way. As for G, we have the following important property:

THEOREM 3. *The mapping G is defined in a neighborhood of $(\kappa_0, \mu_0, \nu_0; 0, 0)$ in $\mathbb{R}^3 \times \mathbb{C}^2$ and takes its value in $\mathbb{C}^2$. If we denote its components (G_1, G_2), it holds that*

$$G(\kappa, \mu, \nu; e^{-im\beta}\sigma, e^{-in\beta}\zeta) = \big(e^{-im\beta}G_1(\kappa, \mu, \nu; \sigma, \zeta), e^{-in\beta}G_2(\kappa, \mu, \nu; \sigma, \zeta)\big),$$

for all $\beta \in [0, 2\pi)$, *and*

$$G(\kappa, \mu, \nu; \overline{\sigma}, \overline{\zeta}) = \big(\overline{G_1(\kappa, \mu, \nu; \sigma, \zeta)}, \overline{G_2(\kappa, \mu, \nu; \sigma, \zeta)}\big).$$

PROOF: As is proved in Sattinger [11], the bifurcation equation inherits the O(2)-equivariance of the original equation. The above property is O(2)-equivariance in the present notation. ∎

In this way, we have the same property as in Theorem 3.4 in Okamoto ([**9**]), where Levi-Civita's formulation is used instead of the Stokes expansion here.

Next we study the bifurcation equation G by making use of theorems in [**4**,**5**]. We introduce a parameters λ_1 and λ_2 by $\lambda_1 = \kappa - \kappa_0$ and $\lambda_2 = \mu - \mu_0$ and we fix ν. Putting $\lambda = (\lambda_1, \lambda_2)$, we can write bifurcation equation as $G = G(\lambda; \sigma, \zeta)$. The following Theorem is proved in [**4**].

THEOREM 4. *If $G = G(\lambda; \sigma, \zeta)$ is a smooth mapping from a neighborhood of the origin in $\mathbb{R}^2 \times \mathbb{C}^2$ into $\mathbb{C}^2$, then it must be of the following form:*

$$G_1(\lambda; \sigma, \zeta) = f_1(\lambda, U, V, R)\sigma + f_2(\lambda, U, V, R)\overline{\sigma}^{n'-1}\zeta^{m'},$$
$$G_2(\lambda; \sigma, \zeta) = f_3(\lambda, U, V, R)\zeta + f_4(\lambda, U, V, R)\sigma^{n'}\overline{\zeta}^{m'-1},$$

where $f_j\,(1 \le j \le 4)$ is a smooth function from a neighborhood of the origin in $\mathbb{R}^5$ into $\mathbb{R}$. U, V and R are defined by

$$U = |\sigma|^2, V = |\zeta|^2, R = \mathbf{Re}(\overline{\sigma}^{n'}\zeta^{m'}).$$

m' and n' are positive integers with no common divisor such that $n'/m' = n/m$.

The equation (A) in §0 is derived in [**9**] from degeneration of f_4 for the case of $n/m = 2$. The equation (B) in §0 is the case of $n/m = 3$. For the derivation, see [**10**,13].

In the next section, we compare the bifurcation diagrams obtained by our simulation with those of zero sets of (A) and (B).

§3. Bifurcation diagrams.

In this section, we show our results and compare them with the diagrams of zeros of the equations (A) and (B). Here we consider only symmetric waves, so we put $B_j = 0$ for all $j \ge 1$ in (6). The figures in this paper are those of infinite depth, i.e $\nu = \infty$. The waves of finite depth are presented in [**13**].

The outline of our numerical algorithm is as follows. By replacing $(A_1, A_2, \cdots)$ with $(A_1, A_2, \cdots, A_N, 0, \cdots)$ and truncating, $(A_1^*, A_2^*, \cdots)$ at $n = N$, we have the discrete version of the equation (12). The resulting nonlinear equations, combined with one more equation of controlling bifurcation parameter, can be solved by the Euler-Newton method. In order to follow the bifurcation branch, we employed H.B. Keller's method ([**6**]). Details of numerical procedure are written in [**13**]. We computed bifurcating solutions from the double bifurcation points of mode (1,2) and (1,3).

First we explain the case of mode (1,2). By (15,16), $(\kappa_0, \mu_0, \nu_0) = (0.5, 1.5, \infty)$ is the double bifurcation point. Figure 1 shows wave profiles of bifurcating solutions for $\kappa = 0.7$ and each wave is drawn in one wavelength $(0 \le x \le L)$.

Wave configurations of solutions, which bifurcate from the simple bifurcation point of mode 2 or mode 1, are transformed as figures a_1-a_5 or c_1-c_3 respectively. From the branch of mode 2 there emanates a secondary bifurcation, the wave configuration of which is transformed as figures b_1-b_3 and a_5. The secondary branch rejoins the branch of mode 2 again. Waves of figures a_4, a_5, b_3 and c_3 have overlapped fluid regions like Figure 2. Such solutions are physically meaningless, but they have equally rigorous meaning as solutions of (12). Although Chen and Saffman stopped computing when waves touch themselves in such a way as Figures a_3, b_2 and c_3, we continued the computation further. We would like to emphasize the importance of such unphysical solutions for the understanding of the global bifurcation diagrams. It is important to notice the difference of the diagram in Figure 1 and that of Figure 4 (ii): a turning point appears or disappears as the bifurcation parameter κ changes. Such a difference is clarified only when we consider all the solutions to (12). In Figure 1, all the lowest parts of self-intersecting waves appear to have sharp corners, but they are actually smooth as shown within the dotted circle. Figure 3 shows the bifurcation diagram of these drawn in A_1-A_2-μ-space. In Figure 4, we show bifurcation diagrams for various κ near κ_0. Here dotted parts of branches indicate solutions which have overlapped regions as above.

Now we show that these diagrams can be produced by (A). Figure 5 shows diagrams of zero sets of (A) for appropriately chosen coefficients. We may regard (i) - (iv) of Figure 4 are qualitatively the same as (i) - (iv) of Figure 3 respectively.

In Figure 6-7, we show the case of mode (1,3). $(\kappa_0, \mu_0, \nu_0) = (1/3, 4/3, \infty)$ is the double bifurcation point. The numerical results are shown in Figure 6. Meaning of dotted parts of bifurcation branches is the same as above. In Figure 7, we show diagrams of zero sets of the equation (B). These figures show qualitatively the same phenomena. Each secondary bifurcation branch is transcritical and forms a closed loop. As κ increases, the closed loop becomes smaller, it comes to have no intersection with the branch of mode 3 and then disappears. These are new bifurcation structures we have discovered.

§4. Conclusions.

Chen and Saffman used the Stokes expansion method in [**1**,**2**]. To write it mathematically, we modified their algorithm and used the equation (7') instead of (7). By this formulation we proved O(2)-equivariance which is a basis for the subsequent theoretic analysis.

For our results we emphasize the following facts. First, we can not see qualitative agreement of the numerical results with the mathematical analysis unless we consider not only ordinary solutions but also those which have self-intersections. Secondly, the structures for mode (1,2) are new examples for O(2)-equivariant systems which have analyzed only theoretically in [**9**]. Thirdly, new bifurcation diagrams were found in the case of mode (1,3). To explain these we introduced (B). For a mathematical proof, see [**10**].

We could hardly compute for small κ. In particular, we could not compute asymptotic behavior to the pure gravity waves. These are left to the future.

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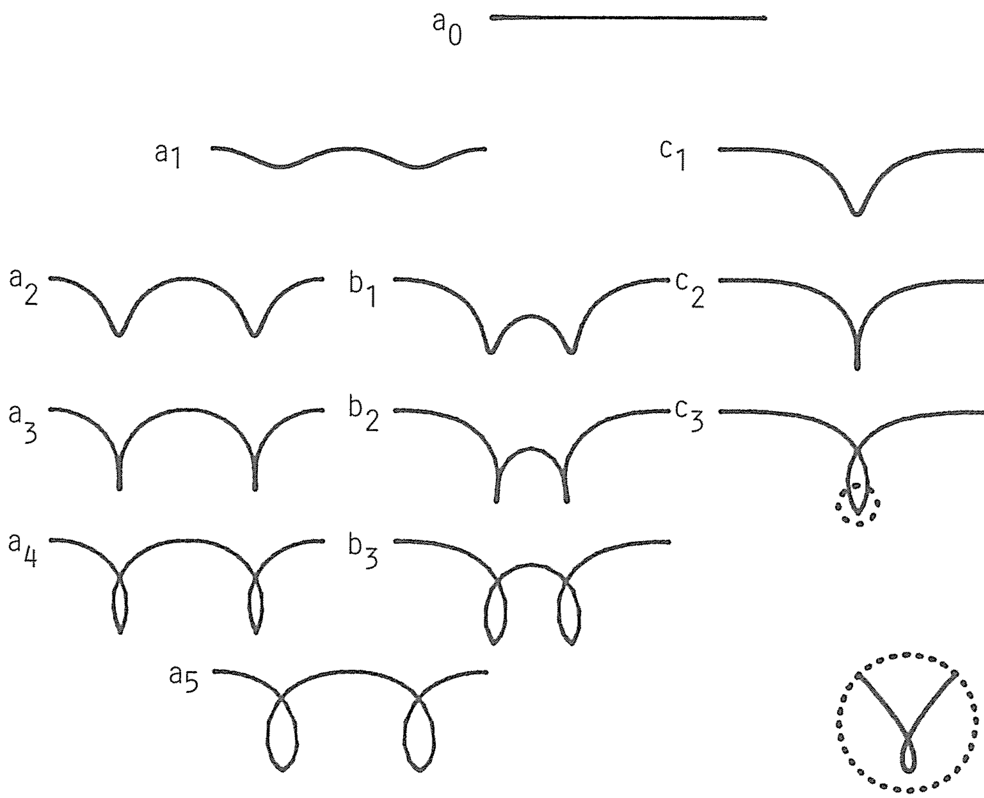

Figure 1

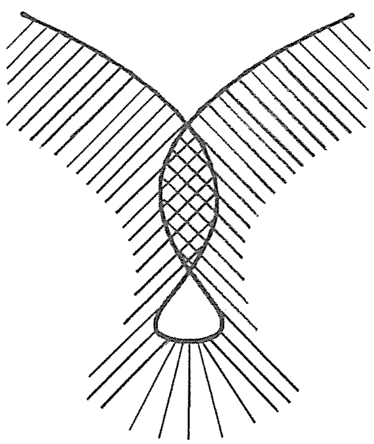

Figure 2

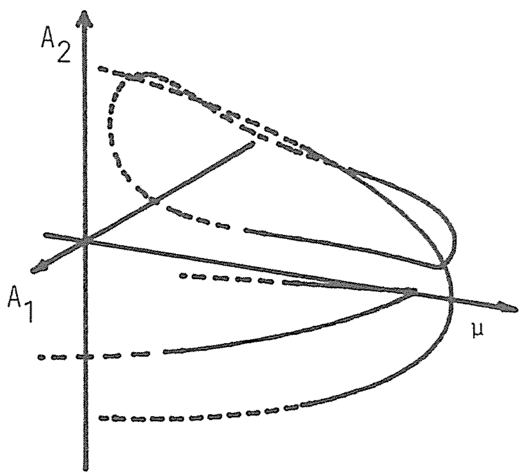

Figure 3

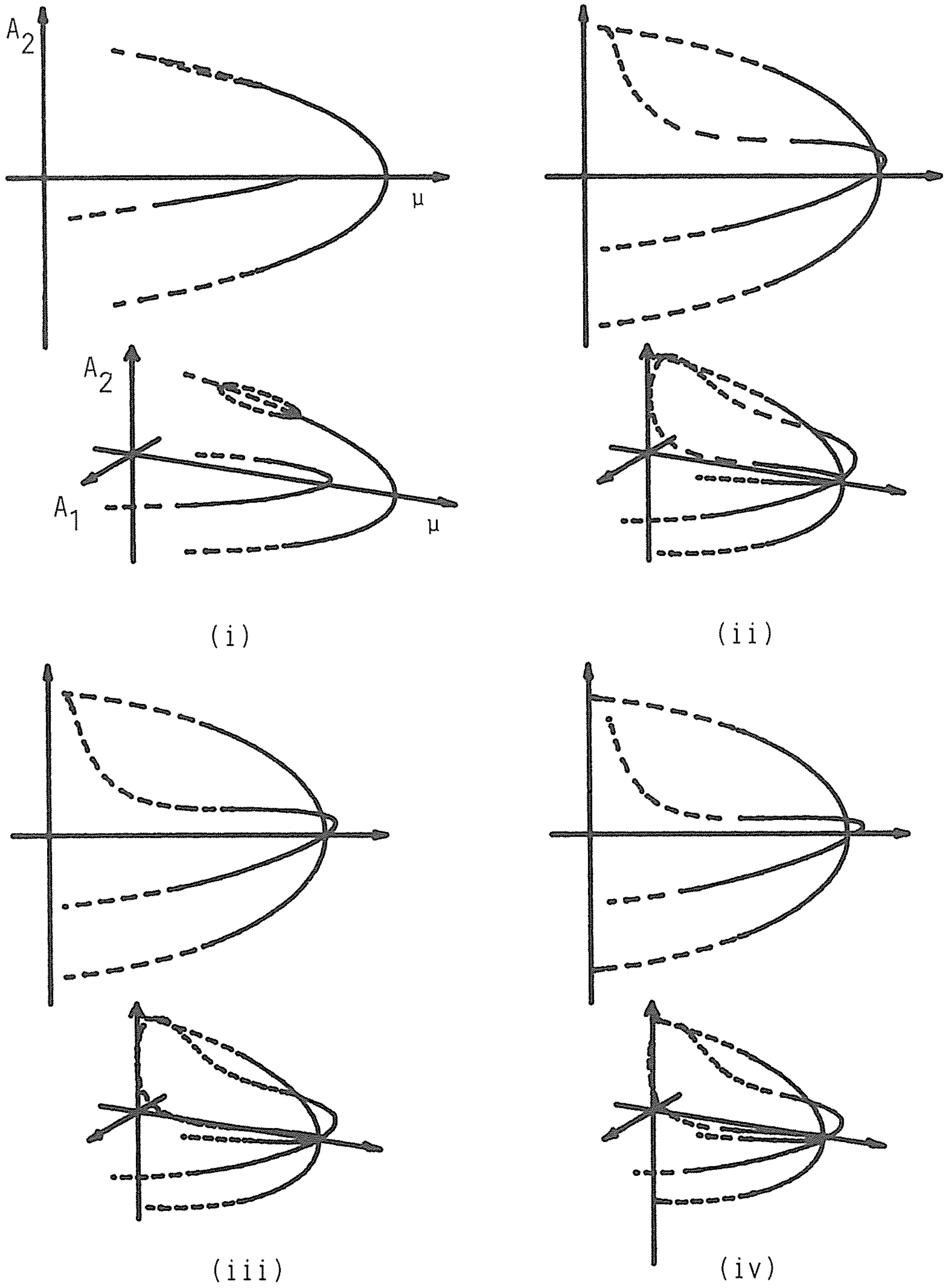

Figure 4

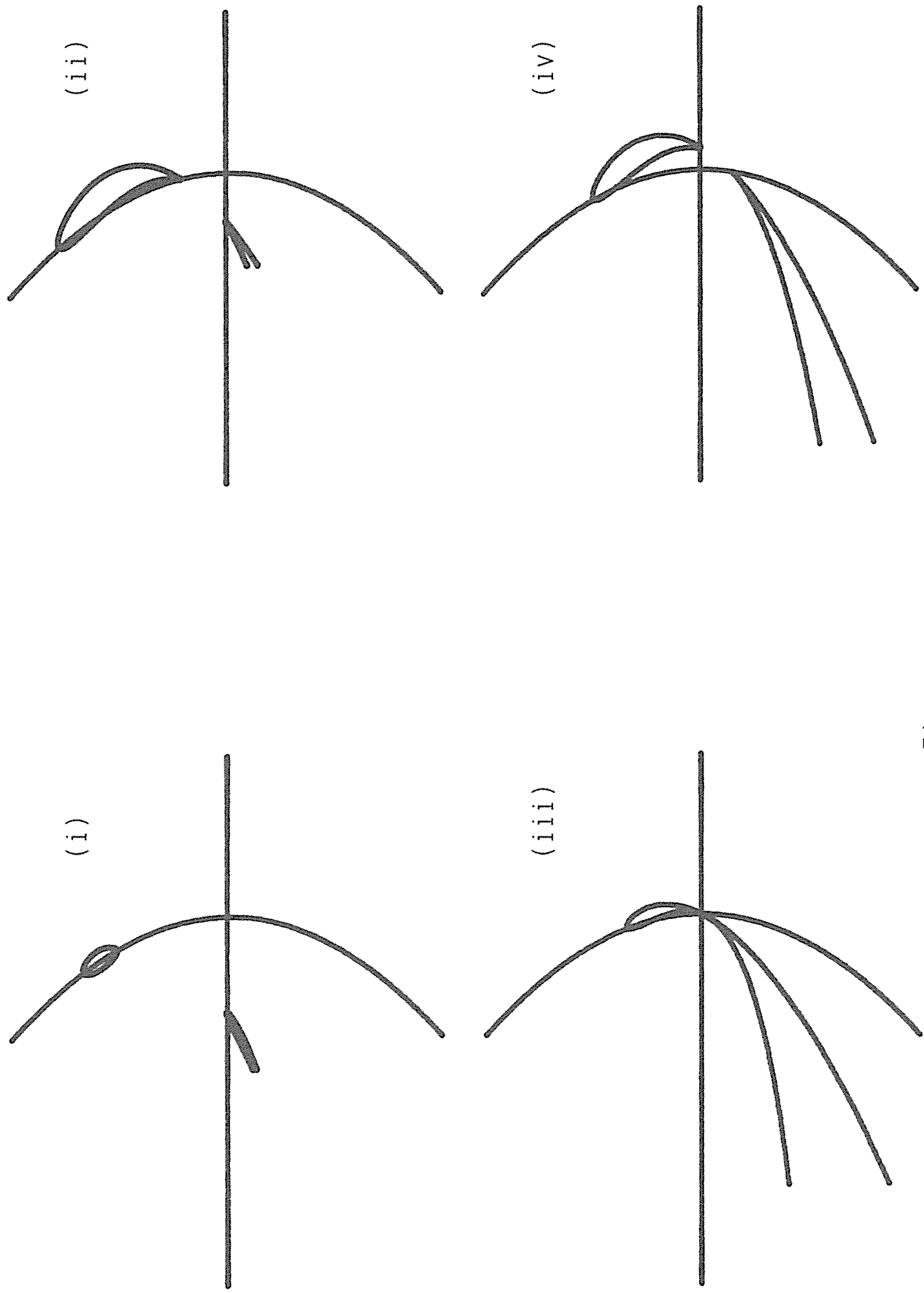

Figure 5

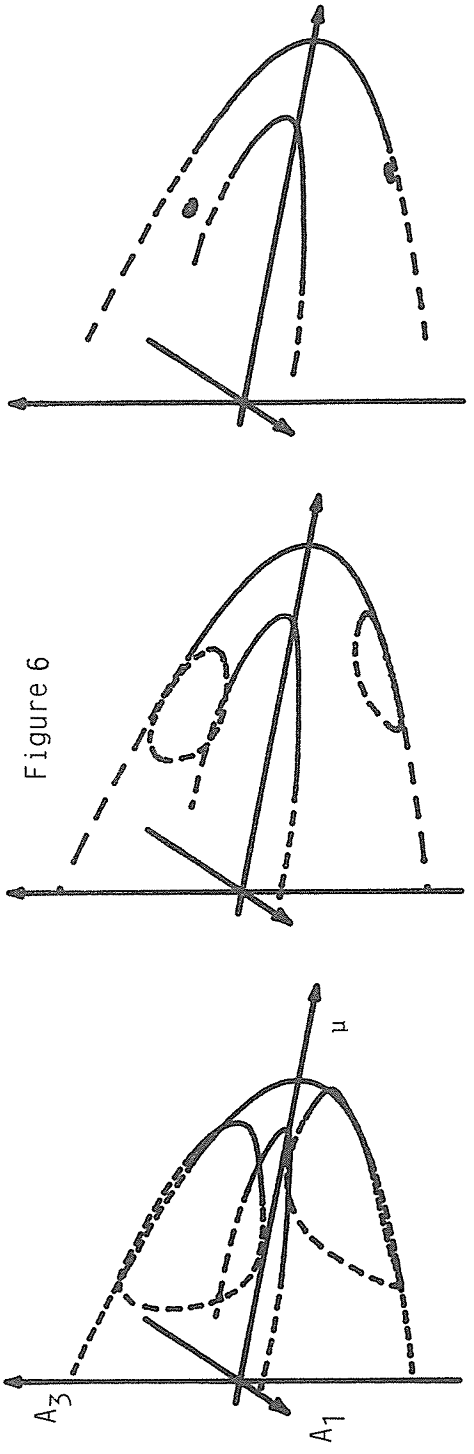

Figure 6

(i)

(ii)

(iii)

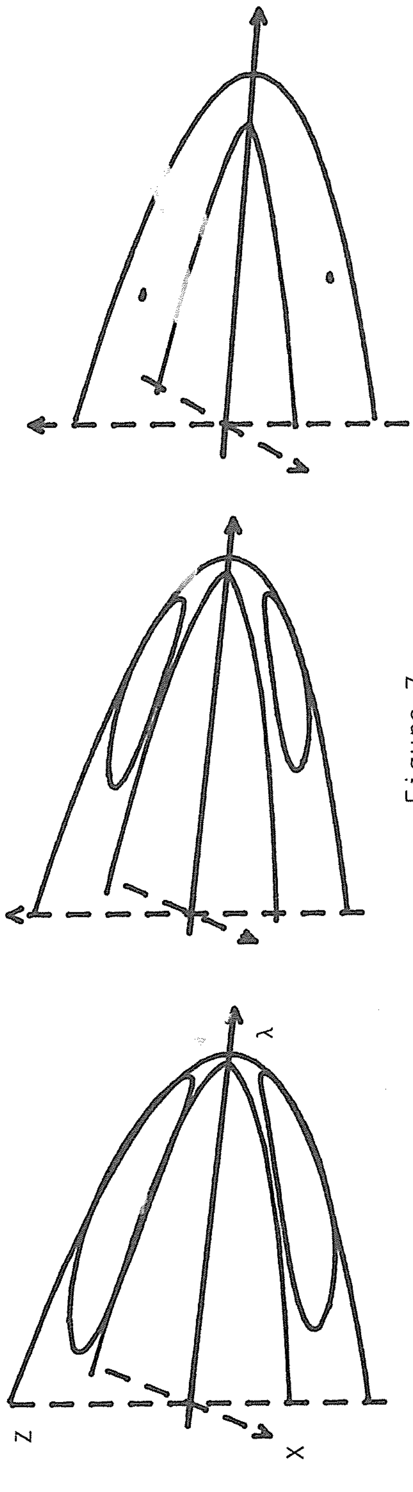

Figure 7