INTERNATIONAL UNION OF THEORETICAL AND APPLIED MECHANICS

TOPOLOGICAL FLUID MECHANICS

Proceedings of the IUTAM Symposium

Cambridge, UK 13–18 August 1989

Edited by H. K. MOFFATT and A. TSINOBER

CAMBRIDGE UNIVERSITY PRESS
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Melbourne Sydney

Chaotic Advection by a Point Vortex in a Semidisk

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§1. Introduction. We consider the motion of a particle which is advected by a point vortex in a semi-disk. The purpose of this paper is to show how the motion of the advected particle changes from a periodic one to a chaotic one. We actually present an alternative perspective to what is observed in [**1**], where it is shown, by numerical computations, that two point vortices in a semi-disk behave chaotically if the energy of the orbits are sufficiently high, while they move quasi-periodically if the energy is low. One of the points in [**1**] is : even two vortices give rise to chaos if they are confined in a semi-disk, while three vortices are necessary to cause a chaos in the case of a full-disk and four vortices necessary in the case of the whole plane.

In this paper, we present a mathematical framework which we believe to give a clearer understanding of the dynamical system governing two vortices. In this framework, we obtain differential equations which depend on a certain parameter $\alpha \in [-1, 1]$. The differential equation studied in [**1**] is the one given here with $\alpha = -1$. It is therefore important to understand the structural change of the phase portrait as α runs in $[-1, 1]$. As a first step toward this, we consider in this paper the case where $\alpha = 0$. Our method is classical: the Poincaré map. We study the transition from periodic motions to chaotic ones.

§2. The equation and its nondimensionalization. In this section we write the governing equation and suitably nondimensionalize it. We put

$$D_R = \{z \in \mathrm{C}; |z| < R, \mathrm{Im}(z) > 0\},$$

which is an open semidisk of radius R in the complex plane. Suppose that there are two point vortices $z(t)$ and $w(t)$ $(-\infty < t < \infty, \quad z, w \in D_R)$. Let κ_1 and κ_2 denote the intensity of the vortices z and w, respectively. Then the motion of these two vortices in D_R are governed by the following (2.1,2) (see [**1**]) :

$$(2.1) \quad \dot{z} = \frac{-i}{2\pi}\left[\frac{\kappa_1}{\overline{z} - z} + \frac{\kappa_1}{\overline{z} - \frac{R^2}{z}} - \frac{\kappa_1}{\overline{z} - \frac{R^2}{\overline{z}}} - \frac{\kappa_2}{\overline{z} - \overline{w}} + \frac{\kappa_2}{\overline{z} - \frac{R^2}{w}} + \frac{\kappa_2}{\overline{z} - w} - \frac{\kappa_2}{\overline{z} - \frac{R^2}{\overline{w}}}\right],$$

$$(2.2)\quad \dot{w} = \frac{-i}{2\pi}\left[\frac{\kappa_2}{\overline{w}-w} + \frac{\kappa_2}{\overline{w}-\frac{R^2}{w}} - \frac{\kappa_2}{\overline{w}-\frac{R^2}{\overline{w}}} - \frac{\kappa_1}{\overline{w}-\overline{z}} + \frac{\kappa_1}{\overline{w}-\frac{R^2}{z}} + \frac{\kappa_1}{\overline{w}-z} - \frac{\kappa_1}{\overline{w}-\frac{R^2}{\overline{z}}}\right],$$

where the dot means differentiation with respect to time t. We change the variables to nondimensional ones by $z \to Rz, \quad w \to Rw, \quad t \to 2\pi R^2 t/\kappa_1$. Then we have

$$(2.3)\quad \dot{z} = \frac{-i}{\overline{z}-z} + \frac{-i}{\overline{z}-1/z} + \frac{i}{\overline{z}-1/\overline{z}} + \frac{\alpha i}{\overline{z}-\overline{w}} + \frac{-\alpha i}{\overline{z}-1/w} + \frac{-\alpha i}{\overline{z}-w} + \frac{\alpha i}{\overline{z}-1/\overline{w}},$$

$$(2.4)\quad \dot{w} = \frac{-\alpha i}{\overline{w}-w} + \frac{-\alpha i}{\overline{w}-1/w} + \frac{\alpha i}{\overline{w}-1/\overline{w}} + \frac{i}{\overline{w}-\overline{z}} + \frac{-i}{\overline{w}-1/z} + \frac{-i}{\overline{w}-z} + \frac{i}{\overline{w}-1/\overline{z}},$$

where $\alpha = \kappa_2/\kappa_1$. These are the equations which we wish to analyse. Note that the phase space of this dynamical system is $(D_1 \times D_1) \setminus \{(z,w); z = w\}$ and that the only α appears as a nondimensional parameter running from $-\infty$ to $+\infty$.

Remark 1. It is enough to consider only $-1 \le \alpha \le 1$. For, if $G(\alpha, z, w)$ denotes the right hand side of (2.3), then the right hand side of (2.4) is $\alpha G(1/\alpha, w, z)$. This implies that the dynamics of (α, z, w) is the same as $(1/\alpha, w, z)$, if we change the time scale.

In [**1**] orbits of (2.3,4) are numerically computed in the case of $\alpha = -1$. Some of them with a high energy are chaotic, i.e., they have continuous power spectra. On the other hand, as far as the authors know, no chaotic motion has been found if α is positive. Accordingly it is important to consider the structural change of the phase portrait as α runs from -1 to $+1$. For instance, we should determine where in the parameter space chaotic motions appear and where they do not. In this paper we consider the case of $\alpha = 0$, which enables us to use a mathematical theory. When $\alpha = 0$, we have

$$(2.5)\qquad \dot{z} = \frac{-i}{\overline{z}-z} + \frac{-i}{\overline{z}-1/z} + \frac{i}{\overline{z}-1/\overline{z}}$$

and

$$(2.6)\qquad \dot{w} = \frac{i}{\overline{w}-\overline{z}} + \frac{-i}{\overline{w}-1/z} + \frac{-i}{\overline{w}-z} + \frac{i}{\overline{w}-1/\overline{z}}.$$

The meaning of this system is that the intensity of w is infinitely small compared with that of z. Therefore z moves irrelevantly to w, while the motion of w is influenced by z. Note that (2.5) is independent of w. We may alternatively say that w moves as a passive particle in a vector field created by z. We now prove some elementary properties of (2.5,6). We introduce Hamiltonians

$$H(z) = \frac{1}{2}\log\frac{|1-z^2|}{|1-z\overline{z}||z-\overline{z}|} \quad \text{and} \quad \widetilde{H}(w,t) = \frac{1}{2}\log\frac{|w-z(t)||w-1/z(t)|}{|w-\overline{z(t)}||w-1/\overline{z(t)}|}.$$

Then (2.5,6) are written as the following Hamiltonian systems, respectively:

$$\dot{z} = 2i\frac{\partial H}{\partial \overline{z}}, \tag{2.7}$$

$$\dot{w} = 2i\frac{\partial \widetilde{H}}{\partial \overline{w}}. \tag{2.8}$$

PROPOSITION 1. *The system (2.7) is completely integrable and has a unique equilibrium:*

$$z = i\sqrt{\sqrt{5}-2} \tag{2.9}$$

Other orbits of (2.7) are periodic ones which surround this equilibrium, see Figure 1.

PROOF: The essential part of the proof is given in [**2**]. We, however, give a complete proof in our framework. Let us use the polar coordinates (I, σ) defined by $\sqrt{2I}e^{i\sigma} = z$. Then, by the definition of the Hamiltonian, we have

$$e^{-4H} = \frac{(1-2I)^2 8I\sin^2\sigma}{4I^2+1-4I\cos 2\sigma}. \tag{2.10}$$

We now introduce some symbols. We put

$$A = e^{-4H}, \qquad \xi = 1-2I, \qquad f(\xi) = -4\xi^3 + (4-A)\xi^2 + 4A\xi - 4A.$$

Then (2.10) is rewritten as :

$$\cot\sigma = \frac{\sqrt{f(\xi)}}{\sqrt{A}\xi}, \tag{2.11}$$

This equation defines a family of closed curves in D_1. If we regard the right hand side of (2.10) as a function of (I,σ), then we see that it has one and only one maximum at $\sigma=\pi/2$, $I=(\sqrt{5}-2)/2$. At this point A takes it maximum value $10\sqrt{5}-22$. If $0<A<10\sqrt{5}-22$, then (2.11) defines a closed curve enclosing the point (2.9) inside it. On these curves, the motion of z is described as follows. Taking the real part of (2.5) multiplied by $\overline{z}$, we have

$$\dot{I}=\frac{1}{2}\cot\sigma-\frac{4I\sin\sigma\cos\sigma}{4I^2+1-4I\cos2\sigma} \tag{2.12}$$

By (2.11,12) we have $\dot{\xi}=(A-\xi^2)\sqrt{f(\xi)}/(\sqrt{A}\xi^3)$. This equation defines the time evolution of the vortex $z(t)$ on the closed curves given by (2.11). We can solve this equstion by means of elliptic functions and see that the solutions are periodic.

By the periodicity of z, the equation (2.8) is a system whose Hamiltonian depends periodically on t.

PROPOSITION 2. *The differential equation (2.6) is definable on the boundary of D_1. The boundary of D_1 is invariant with respect to the flow given by (2.6). $w=1,-1$ are unstable equilibria.*

PROOF: The right hand side of (2.6) is equal to the following

$$\frac{i(\overline{z}-z)(1-|z|^2)(1-\overline{w}^2)}{\{\overline{w}^2-(z+\overline{z})\overline{w}+|z|^2\}\{\overline{w}^2|z|^2-(z+\overline{z})\overline{w}+1\}}. \tag{2.13}$$

It is clear from (2.13) that $w=-1,+1$ are equilibria. On the boundary circumference, we have $w=e^{i\gamma}$ $(0\leq\gamma\leq\pi)$. In this case, (2.13) is equal to $c(e^{2i\gamma}-1)=2c\sin\gamma ie^{i\gamma}$, where $c\in\mathbf{R}$. This means the vector field is tangent to the boundary. Similarly it is tangent in the case of $w\in[-1,1]$, since (2.13) $\in\mathbf{R}$ when $w\in\mathbf{R}$. Therefore the boundary of D_1 is an invariant set.

Thus the equation (2.5,6) has nice properties which (2.3,4) with $\alpha\neq0$ does not share. Notice that (2.4) can not be defined on the boundary for $\alpha\neq0$. Although (2.5,6) are simple, it is connected through α to the equation considered in [**1**].

Since a similar problem is considered in Aref and Pomphrey [**5**,6], we would like to mention our motivation here. In [**5**,6], they consider the motion of a passive vortex stirred by three identical vortices. Since this problem is a special case of three vortices with different intensities, it seems to us that our problem is simpler than theirs. Note that the vortex z by which the motion of w is take place, can move periodically or stationary and there is no motion of other kind. On the other hand, three vortices can move with more varieties, e.g., they can collide ([7]).

Suppose that z is the equilibrium (2.9). Then (2.8) is independent of time, which implies that the Hamiltonian $\widetilde{H}$ is constant along individual orbits. Consequently (2.8) is completely integrable and the orbits of (2.8) consist only of closed Jordan curves defined by $\widetilde{H}(w, i\sqrt{\sqrt{5}-2}) = constant$. Furthermore, they occupy the whole phase space of (2.8) except for the boundary (see Figure 2). If the initial position of z is placed slightly apart from (2.9), then z moves on a small closed curve surrounding the equilibrium. In this case $\widetilde{H}$ is no longer independent of time and complicated orbits may appear. Let T be the period of z. Then we can obtain a Poincaré map in a usual way:

$$f : w(0) \to w(T). \tag{2.14}$$

We give in APPENDIX a theorem by which the map (2.14) becomes well-defined in $\Omega \equiv \overline{D_1} \setminus \{z(0)\}$. This is equivalent to saying that

$$\text{if} \quad w(0) \neq z(0), \text{ then } \quad w(t) \neq z(t) \text{ for all } \quad t.$$

If this is proved, it is clear that the map (2.14) is one-to-one, onto and continuous. Furthermore it preserves the area. Although our "proof " is not complete, we think the account in APPENDIX is a strong evidence of the correctness of the theorem.

We now examine the properties of the Poincaré map. It is enough to consider the case where $z(0) = iq + i\sqrt{\sqrt{5}-2}$ $(0 < q < 1 - \sqrt{\sqrt{5}-2})$. Let the mapping be denoted by f_q when $z(0) = iq + i\sqrt{\sqrt{5}-2}$. Several orbits are drawn on each figures 3-8. Figure 3, $\cdots$, 8 correspond to $q = 0.01, 0.05, 0.1, 0.25, 0.3, 0.4$, respectively.

It should be noticed that there is a fixed point in a lower part of the imaginary axis and that it is enclosed by a layer of closed curves. This shows that there is a periodic orbit which has exactly the same period as that of $z(t)$ and that it is stable. Some topological argument shows that there must be an unstable fixed point. Figure 1 shows that the unstable fixed point is on the upper side of the imaginary axis and that the stable fixed point are connected to the unstable one by a homoclinic orbit. We can observe that the region occupied by the invariant circles reduces and the islands grows up in accordance with the increase of q. We also notice that, even in the case of a large q, there are KAM tori around the point $z(0)$. The reason is that, when $w(0)$ is close to $z(0)$, the interaction of w with the boundary is negligibly small compared with the interaction between w and z (see the definition of F_0 in the APPENDIX).

Conclusion. Our equation (2.6), despite its simple appearance, exhibits chaotic orbits. It seems to the authors that ours is one of the simplest equation among the chaos-displaying vortex systems. As is shown in [4], streamlines of a stationary 3-D Euler flow can be chaotic. Our example shows that 2-D time-periodic Euler flow may have chaotic trajectories of particles.

APPENDIX 1. Here we prove :

THEOREM A. *For any tubular neighborhood N of $O = \{(z(t), t); 0 \le t < T\}$, there is an invariant torus such that it lies in N and that O lies inside it*

The precise meaning of this theorem is as follows: The phase space of (2.8) is $\bigcup_{0 \le t \le T}(\overline{D_1} \setminus \{z(t)\})$, where the sections $t = 0$ and $t = T$ are identified. Therefore it is homeomorphic to $(\overline{D_1} \setminus \{z(0)\}) \times S^1$, where S^1 is a circle. Note that $\{z(0)\} \times S^1$ corresponds to the orbit of z. The above theorem asserts that all the neighborhood of $\{z(0)\} \times S^1$ has an invariant torus which contains $\{z(0)\} \times S^1$ inside.

FORMAL PROOF OF THEOREM A: Let us introduce $U + iV = u + iv - z(t)$ where $u + iv = w$. Then (2.8) is rewritten as

$$\dot{U} = -\frac{\partial K}{\partial V}, \qquad \dot{V} = \frac{\partial K}{\partial U}, \tag{A.3}$$

where we have put

$$K(U, V, t) = \frac{1}{2} \log \frac{|U + iV||U + iV + z - 1/z|}{|U + iV + z - \overline{z}||U + iV + z - 1/\overline{z}|} + V\mathrm{Re}(\dot{z}) - U\mathrm{Im}(\dot{z}).$$

Note that the right hand side depends on t through $z = z(t)$. If we define $K_0(U, V)$ and $K_1(U, V, t)$ by $K_0(U, V) = \frac{1}{4}\log(U^2 + V^2)$, $K_1(U, V, t) = K(U, V, t) - K_0(U, V)$ then, K_1 is continuous on $\overline{D_1}$, the closure of D_1. Note that the orbits of $\dot{U} = -\frac{\partial K_0}{\partial V}$, $\dot{V} = \frac{\partial K_0}{\partial U}$, are simply the circles about the origin. We attempt to apply the KAM theory to the Hamiltonian system (A.3). Let $\epsilon > 0$ be a small parameter. We introduce canonical variables (p, q) by $p = \frac{U^2 + V^2}{2\epsilon^2}$, $q = \arg(U + iV)$. We further change t to $\epsilon^2 t$. Then (A.3) becomes :

$$\dot{p} = -\frac{\partial F}{\partial q}, \qquad \dot{q} = \frac{\partial F}{\partial p}, \tag{A.4}$$

where we have put

$$(A.5) \qquad F = F_0(p) + F_1(p,q,t,\epsilon), \quad \text{with} \quad F_0(p,q,t,\epsilon) = \frac{1}{4}\log p,$$

$$F_1(p,q,t,\epsilon) = \frac{1}{2}\log \frac{|\epsilon\sqrt{2p}e^{iq} + z(\tau) - 1/z(\tau)|}{|\epsilon\sqrt{2p}e^{iq} + z(\tau) - \overline{z(\tau)}||\epsilon\sqrt{2p}e^{iq} + z(\tau) - 1/\overline{z(\tau)}|}$$

$$+ \epsilon\sqrt{2p}\sin q \mathrm{Re}(\dot{z}(\tau)) - \epsilon\sqrt{2p}\cos q \mathrm{Im}(\dot{z}(\tau)),$$

where $\tau = \epsilon^2 t$. These are defined on $q \in \mathbf{R}/2\pi\mathbf{Z}$ and $p \sim 1$. In this setting we wish to use Theorem 2 in Arnold [**3**]. This theorem garantees the existence of invariant tori for $\epsilon > 0$ which is close to unperturbed torus $p = p_0(\in [1/2, 2])$ where p_0 is sufficiently incommensurable. There is, however, one difficulty that the slowly changing parameter is ϵt in [**3**], while it is $\epsilon^2 t$ in (A.5). We hope that this diffuiculty is overcome if we follow the method of [**3**] in detail. Accordingly we are satisfied by the form (A.4) and stop here rather than pursuing rigorous proof, which seems to require a formidable calculation.

References

1. Y. Kimura and H. Hasimoto, J. Phys. Soc. Japan **55** (1986), 5–8.
2. Y. Kimura, Y. Kusumoto and H. Hasimoto, J. Phys. Soc. Japan **53** (1984), 2988–2995.
3. V.I. Arnold, Soviet Math. Dokl. **3** (1962), 136-140.
4. V.I. Arnold, "Mathematical Methods of Classical Mechanics," Springer Verlag, New York, Heidelberg, Berlin, 1978.
5. H. Aref and N. Pomphrey, Proc. R. Soc. London A **380** (1982), 359–387.
6. H. Aref and N. Pomphrey, Phys. Lett. A **78** (1980), 297–300.
7. H. Aref, Ann. Rev. Fluid Mech. **15** (1983), 345–389.

Keywords. point vortex, homoclinic orbit, Chaos

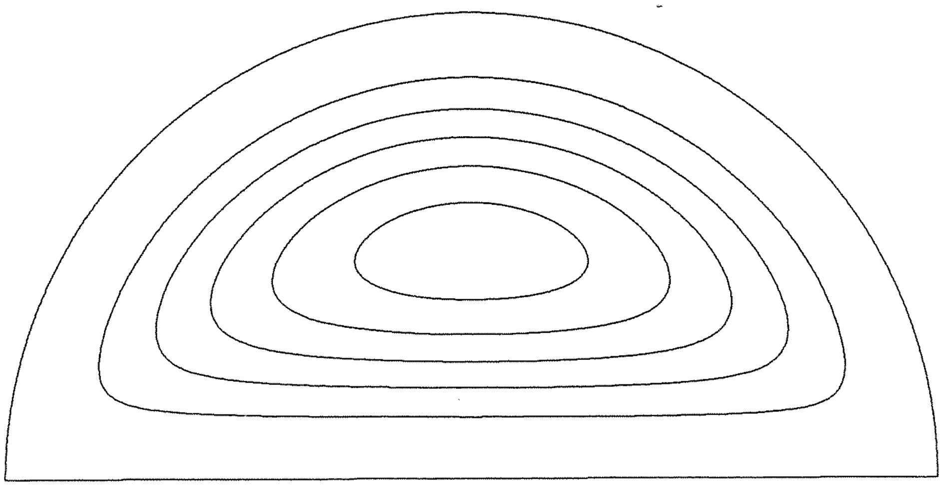

Figure 1. Orbits of $z(t)$

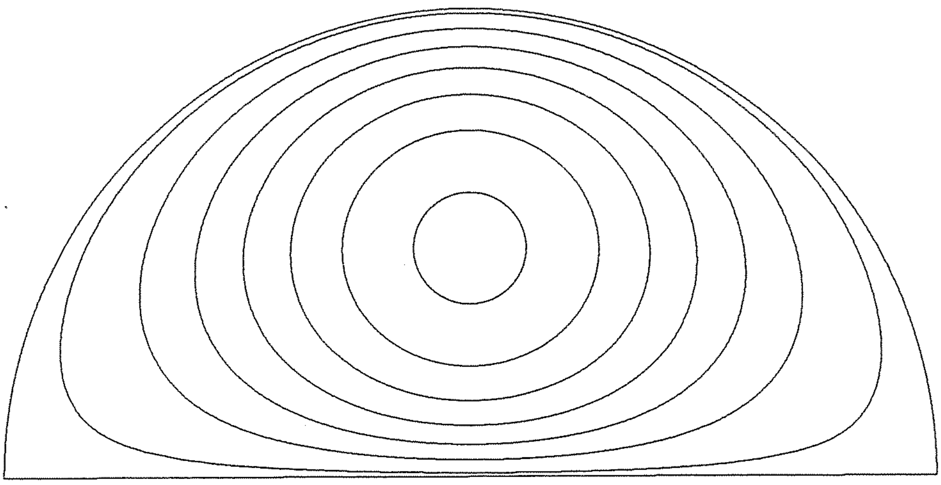

Figure 2. Orbits of $w(t)$ when $z \equiv \sqrt{\sqrt{5} - 2}\, i$

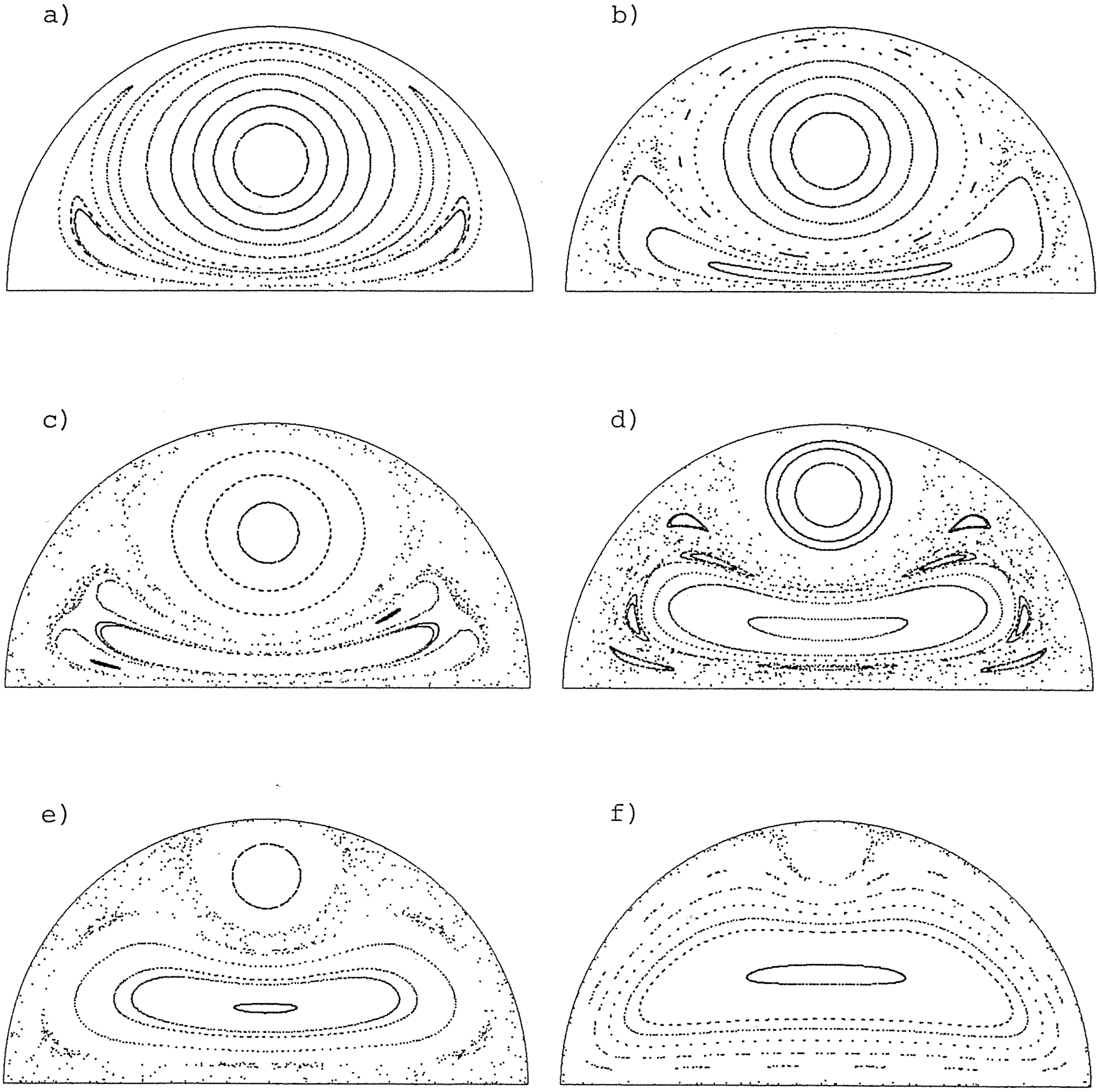

Figure 3. Poincaré map:

a) $q = 0.01$, b) $q = 0.05$, c) $q = 0.1$, d) $q = 0.25$,
e) $q = 0.3$, f) $q = 0.4$