A solution to the Basel problem

Takuya Ooura

We show a simple elementary proof of the formula (solution to the Basel problem):

$$\frac{\pi^2}{6} = \sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots \tag{1}$$

Proof. Repeatedly using the identities

$$\frac{2}{3} = \frac{1}{4}\left(\frac{2}{3} + \frac{1}{\sin^2 \frac{\pi}{4}}\right), \quad \frac{1}{\sin^2\theta} = \frac{\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}} = \frac{1}{4}\left(\frac{1}{\sin^2\frac{\theta}{2}} + \frac{1}{\sin^2(\frac{\pi}{2} - \frac{\theta}{2})}\right)$$

gives

$$\begin{aligned}
\frac{2}{3} &= \frac{1}{4}\left(\frac{2}{3} + \frac{1}{\sin^2\frac{\pi}{4}}\right) = \frac{1}{4}\left(\frac{1}{4}\left(\frac{2}{3} + \frac{1}{\sin^2\frac{\pi}{4}}\right) + \frac{1}{4}\left(\frac{1}{\sin^2\frac{\pi}{8}} + \frac{1}{\sin^2\frac{3\pi}{8}}\right)\right) \\
&= \frac{1}{16}\left(\frac{2}{3} + \frac{1}{\sin^2\frac{\pi}{8}} + \frac{1}{\sin^2\frac{2\pi}{8}} + \frac{1}{\sin^2\frac{3\pi}{8}}\right) \\
&= \cdots \\
&= \frac{1}{4^n}\left(\frac{2}{3} + \sum_{k=1}^{2^n-1} \frac{1}{\sin^2\frac{\pi k}{2^{n+1}}}\right).
\end{aligned}$$

Since $\sin\theta_k < \theta_k < \tan\theta_k$ ($\theta_k = \pi k/2^{n+1}$, $k = 1, 2, \cdots, 2^n - 1$), we have

$$\frac{1}{\sin^2\theta_k} > \frac{1}{\theta_k^2} > \frac{1}{\tan^2\theta_k} = \frac{1}{\sin^2\theta_k} - 1.$$

Summing this inequality from $k = 1$ to $2^n - 1$ and multiplying by $1/4^n$ yield

$$\frac{2}{3} - \frac{2}{3\cdot 4^n} > \frac{4}{\pi^2}\sum_{k=1}^{2^n-1}\frac{1}{k^2} > \frac{2}{3} - \frac{2}{3\cdot 4^n} - \frac{2^n - 1}{4^n}.$$

Since $\frac{2}{3\cdot 4^n} \to 0$, $\frac{2^n-1}{4^n} \to 0$ as $n \to \infty$, we have (1). □

This proof is an improved version of [1] and [2].

References

[1] A. M. Yaglom, I. M. Yaglom, An elementary derivation of the formulas of Wallis, Leibnitz and Euler for the number π, Uspekhi Mat. Nauk **57** (1953) 181-187.

[2] J. Hofbauer, A simple proof of $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$ and related identities, Amer. Math. Monthly **109** (2002) 196-200.