A proof of the Wallis product formula

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Wallis product formula (John Wallis 1655)

$$\frac{\pi}{2} = \prod_{k=1}^{\infty} \left(\frac{2k}{2k-1} \cdot \frac{2k}{2k+1} \right) = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots \tag{1}$$

was derived in a method without using calculus. We show another elementary proof.

Proof. Repeatedly using the identities

$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\sin\frac{\theta}{2}\left(1 - 2\sin^2\frac{\theta}{4}\right) = 2\sin\frac{\theta}{2}\left(1 - \frac{\sin^2\frac{\theta}{4}}{\sin^2\frac{\pi}{4}}\right)$$

and

$$\begin{aligned} 1 - \frac{\sin^2\beta}{\sin^2\alpha} &= 1 - \frac{\sin^2\frac{\beta}{2}\left(1-\sin^2\frac{\beta}{2}\right)}{\sin^2\frac{\alpha}{2}\cos^2\frac{\alpha}{2}} = \left(1 - \frac{\sin^2\frac{\beta}{2}}{\sin^2\frac{\alpha}{2}}\right)\left(1 - \frac{\sin^2\frac{\beta}{2}}{\cos^2\frac{\alpha}{2}}\right) \\ &= \left(1 - \frac{\sin^2\frac{\beta}{2}}{\sin^2\frac{\alpha}{2}}\right)\left(1 - \frac{\sin^2\frac{\beta}{2}}{\sin^2\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)}\right) \end{aligned}$$

gives

$$\begin{aligned} 1 &= \sin\frac{\pi}{2} = 2\sin\frac{\pi}{4}\left(1 - \frac{\sin^2\frac{\pi}{8}}{\sin^2\frac{\pi}{4}}\right) \\ &= 4\sin\frac{\pi}{8}\left(1 - \frac{\sin^2\frac{\pi}{16}}{\sin^2\frac{\pi}{8}}\right)\left(1 - \frac{\sin^2\frac{\pi}{16}}{\sin^2\frac{2\pi}{8}}\right)\left(1 - \frac{\sin^2\frac{\pi}{16}}{\sin^2\frac{3\pi}{8}}\right) \\ &= \cdots \\ &= 2^n \sin\frac{\pi}{2^{n+1}} \prod_{k=1}^{2^n-1}\left(1 - \frac{\sin^2\frac{\pi}{2^{n+2}}}{\sin^2\frac{\pi k}{2^{n+1}}}\right). \end{aligned} \tag{2}$$

Since $\sin 2\theta < 2\theta$ and $\sin(2k\theta) < 2k\sin\theta$ ($\theta = \pi/2^{n+2}$, $k = 1, 2, \cdots, 2^n - 1$), we have

$$1 < \frac{\pi}{2}\prod_{k=1}^{2^n-1}\left(1 - \frac{1}{4k^2}\right). \tag{3}$$

By using

$$1 - \frac{\tan^2\beta}{\tan^2\alpha} = 1 - \tan^2\beta\left(\frac{1}{\sin^2\alpha} - 1\right) = \frac{1}{\cos^2\beta}\left(1 - \frac{\sin^2\beta}{\sin^2\alpha}\right)$$

and $\sin 2\beta = 2\tan\beta\cos^2\beta$, (2) is rewritten as

$$1 = 2^{n+1}\tan\frac{\pi}{2^{n+2}}\cos^{2^{n+1}}\frac{\pi}{2^{n+2}}\prod_{k=1}^{2^n-1}\left(1 - \frac{\tan^2\frac{\pi}{2^{n+2}}}{\tan^2\frac{\pi k}{2^{n+1}}}\right).$$

Since $\cos^2\theta = 1 - \sin^2\theta > 1 - \theta^2$, $\cos^4\theta > (1-\theta^2)^2 > 1 - 2\theta^2$, $\cos^8\theta > (1-2\theta^2)^2 > 1 - 4\theta^2$, $\cdots$, $\cos^{2^{n+1}}\theta > 1 - 2^n\theta^2$, $\tan\theta > \theta$ and $\tan(2k\theta) > 2k\tan\theta$ ($\theta = \pi/2^{n+2}$, $k = 1, 2, \cdots, 2^n - 1$), we have

$$1 > \frac{\pi}{2}\left(1 - \frac{\pi^2}{2^{n+4}}\right)\prod_{k=1}^{2^n-1}\left(1 - \frac{1}{4k^2}\right). \tag{4}$$

From (3) and (4), we have

$$\frac{\pi}{2}\left(1 - \frac{\pi^2}{2^{n+4}}\right) < \prod_{k=1}^{2^n-1}\frac{4k^2}{4k^2-1} < \frac{\pi}{2}.$$

Since $\frac{\pi^2}{2^{n+4}} \to 0$ as $n \to \infty$, (1) follows immediately. $\square$