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Notes on Microstate Free Entropy of Projections

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Abstract

We study the microstate free entropy $\chi_{\mathrm{proj}}(p_1,\dots,p_n)$ of projections, and establish its basic properties similar to the self-adjoint variable case. Our main contribution is to characterize the pair-block freeness of projections by the additivity of χ_{proj} (Theorem 4.1), in the proof of which a transportation cost inequality plays an important role. We also briefly discuss the free pressure in relation to χ_{proj}.

Introduction

The theory of free entropy, initiated and mostly developed by D. Voiculescu in his series of papers [20]–[25], has become one of the most essential disciplines of free probability theory. The microstate free entropy $\chi(X_1,\dots,X_n)$ introduced in [21] for self-adjoint non-commutative random variables $X_1,\dots,X_n$ is defined as a certain asymptotic growth rate (as the matrix size N goes to ∞) of the Euclidean volume of the set of $N\times N$ self-adjoint matrices $(A_1,\dots,A_n)$ approximating $(X_1,\dots,X_n)$ in moments. It is this microstate theory that settled some long-standing open questions in von Neumann algebras (see the survey [26]). On the other hand, the microstate-free free entropy $\chi^*(X_1,\dots,X_n)$ was also introduced in [23] based on the non-commutative Hilbert transform and

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the notion of conjugate variables, avoiding use of microstates or so-called matrix integrals which are rather hard to handle. Although it is believed that both approaches should be unified and give the same quantity, only the inequality $\chi \le \chi^*$ is known to hold true due to Biane, Capitaine and Guionnet [3] based on an idea of large deviation principle for several random matrices. In his work [25] Voiculescu developed another kind of microstate-free approach to free entropy, the so-called free liberation theory, and introduced the mutual free information i^* for subalgebras rather than random variables. He suggested there the need to apply the microstate approach to projection random variables because the usual microstate free entropy χ becomes always $-\infty$ for projections while i^* does not in general. Following the suggestion, we here study the microstate free entropy $\chi_{\mathrm{proj}}(p_1, \dots, p_n)$ of projections $p_1, \dots, p_n$ in the same lines as in [21] and [22] to provide the basis for future research.

The large deviation principle for random matrices as mentioned above started with the paper of Ben Arous and Guionnet [2] and has been almost completed in the single random matrix case (corresponding to the study of $\chi(X)$ for single random variable X), see the survey [8]. We note that such large deviation principle played quite an important role not only for the foundation of free entropy theory but also for getting free analogs of several probability theoretic inequalities (see [14] and the references therein). Recently, one more large deviation was shown in [12] for an independent pair of random projection matrices, including the explicit formula of the free entropy $\chi_{\mathrm{proj}}(p, q)$ of a projection pair (p, q). This is one of a few large deviation results (indeed the first full large deviation result) in the setting of several random matrices, though the method of the proof is based on the single variable case. Moreover, in [15] we applied it to get a kind of logarithmic Sobolev inequality between the free entropy $\chi_{\mathrm{proj}}(p, q)$ and the mutual free Fisher information $\varphi^*(W^*(p) : W^*(q))$ (see [25]) for a projection pair. The large deviation result in [12] also plays a crucial role in our study of χ_{proj} here.

The paper is organized as follows. After giving the definition and basic properties of $\chi_{\mathrm{proj}}(p_1, \dots, p_n)$ in §1, we recall in §2 the formula in the case of two variables. In §3 we introduce a certain functional calculus for a projection pair (p, q) and provide a technical tool of separate change of variable formula. This tool is essential in §4 to prove the additivity theorem characterizing the pair-block freeness of projections by the additivity of their free entropy. §5 treats a free analog of transportation cost inequalities for tracial distributions of projections. Such a free analog is of interest by itself while its simplest case is needed in the proof of the above-mentioned additivity theorem. Finally, along the same lines as in [10], we introduce in §6 the notion of free pressure and compare its Legendre transform with $\chi_{\rm proj}(p_1,\dots,p_n)$, thus giving a variational expression of free entropy for projections.

§1. Definition

Let $\mathrm{U}(N)$ be the unitary group of order N. Let $G(N,k)$ denote the set of all $N\times N$ orthogonal projection matrices of rank k, that is, $G(N,k)$ is identified with the Grassmannian manifold consisting of k-dimensional subspaces in $\mathbb{C}^N$. With the diagonal matrix $P_N(k)$ of the first k diagonals 1 and the others 0, each $P\in G(N,k)$ is diagonalized as

$$P=UP_N(k)U^*, \tag{1.1}$$

where $U\in\mathrm{U}(N)$ is determined up to the right multiplication of elements in $\mathrm{U}(k)\oplus\mathrm{U}(N-k)$. Hence $G(N,k)$ is identified with the homogeneous space $\mathrm{U}(N)/(\mathrm{U}(k)\oplus\mathrm{U}(N-k))$, and we have a unique probability measure $\gamma_{G(N,k)}$ on $G(N,k)$ invariant under the unitary conjugation $P\mapsto UPU^*$ for $U\in\mathrm{U}(N)$. In the homogeneous space description, this is the unique probability measure on $\mathrm{U}(N)/(\mathrm{U}(k)\oplus\mathrm{U}(N-k))$ invariant under the left multiplication of elements in $\mathrm{U}(N)$ or in other words the induced measure from the Haar probability measure $\gamma_{\mathrm{U}(N)}$ on $\mathrm{U}(N)$. Let $\xi_{N,k}:\mathrm{U}(N)\to G(N,k)$ be the (surjective continuous) map defined by (1.1), i.e., $\xi_{N,k}(U):=UP_N(k)U^*$. Then the measure $\gamma_{G(N,k)}$ is more explicitly written as

$$\gamma_{G(N,k)}=\gamma_{\mathrm{U}(N)}\circ\xi_{N,k}^{-1}. \tag{1.2}$$

Throughout the paper $(\mathcal{M},\tau)$ is a tracial W^*-probability space. Let $(p_1,\dots,p_n)$ be an n-tuple of projections in $(\mathcal{M},\tau)$. Following Voiculescu's proposal in [25, 14.2] we define the *free entropy* $\chi_{\rm proj}(p_1,\dots,p_n)$ of $(p_1,\dots,p_n)$ as follows. Choose $k_i(N)\in\{0,1,\dots,N\}$ for each $N\in\mathbb{N}$ and $1\le i\le n$ in such a way that $k_i(N)/N\to\tau(p_i)$ as $N\to\infty$ for $1\le i\le n$. For each $m\in\mathbb{N}$ and $\varepsilon>0$ we set

(1.3)

$$\begin{aligned}&\Gamma_{\rm proj}(p_1,\dots,p_n;k_1(N),\dots,k_n(N);N,m,\varepsilon)\\ &\quad:=\Bigg\{(P_1,\dots,P_n)\in\prod_{i=1}^n G(N,k_i(N)):\left|\frac{1}{N}\mathrm{Tr}_N(P_{i_1}\cdots P_{i_r})-\tau(p_{i_1}\cdots p_{i_r})\right|<\varepsilon\\ &\qquad\qquad\text{for all } 1\le i_1,\dots,i_r\le n,\ 1\le r\le m\Bigg\},\end{aligned}$$

where Tr_N stands for the usual (non-normalized) trace on the $N \times N$ matrices. We then define

(1.4)

$$\chi_{\mathrm{proj}}(p_1, \dots, p_n) := \lim_{\substack{m\to\infty \\ \varepsilon\searrow 0}} \limsup_{N\to\infty} \frac{1}{N^2} \log\left(\bigotimes_{i=1}^{n} \gamma_{G(N,k_i(N))}\right)\bigl(\Gamma_{\mathrm{proj}}(p_1, \dots, p_n; k_1(N), \dots, k_n(N); N, m, \varepsilon)\bigr).$$

To justify the definition of χ_{proj}, here arises a natural question whether or not the quantity $\chi_{\mathrm{proj}}(p_1, \dots, p_n)$ depends on the particular choice of $k_i(N)$. The answer is the following:

Proposition 1.1. *The above definition of $\chi_{\mathrm{proj}}(p_1, \dots, p_n)$ is independent of the choices of $k_i(N)$ with $k_i(N)/N \to \alpha_i$ for $1 \le i \le n$.*

Proof. For $1 \le i \le n$ let $l_i(N)$, $N \in \mathbb{N}$, be another sequence such that $l_i(N)/N \to \alpha_i$ as $N \to \infty$. In what follows we will denote, for brevity, the microstate set in (1.3) by $\Gamma(\vec{k}(N), m, \varepsilon)$ with $\vec{k}(N) := (k_1(N), \dots, k_n(N))$. Moreover, let $\xi_{\vec{k}(N)}(\vec{U}) := \bigl(\xi_{N,k_1(N)}(U_1), \dots, \xi_{N,k_n(N)}(U_n)\bigr)$ for $\vec{U} = (U_1, \dots, U_n) \in \mathrm{U}(N)^n$, and consider the subset $\widetilde{\Gamma}(\vec{l}(N), m, \varepsilon) := \xi_{\vec{l}(N)} \circ \xi_{\vec{k}(N)}^{-1}\bigl(\Gamma(\vec{k}(N), m, \varepsilon)\bigr)$ of $\prod_{i=1}^n G(N, l_i(N))$. Since

$$\xi_{N,l_i(N)}(U) - \xi_{N,k_i(N)}(U) = U\bigl(P_N(k_i(N)) - P_N(l_i(N))\bigr)U^*,$$

we get

$$\bigl\|\xi_{N,l_i(N)}(U) - \xi_{N,k_i(N)}(U)\bigr\|_1 = \frac{|l_i(N) - k_i(N)|}{N},$$

where $\|\cdot\|_1$ denotes the trace-norm with respect to $N^{-1}\mathrm{Tr}_N$. For every $m \in \mathbb{N}$ and $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that $N^{-1}|l_i(N) - k_i(N)| < \varepsilon/m$ for all $N \ge N_0$ and $1 \le i \le n$. Let us prove that $\widetilde{\Gamma}(\vec{l}(N), m, \varepsilon) \subset \Gamma(\vec{l}(N), m, 2\varepsilon)$ whenever $N \ge N_0$. Assume that $N \ge N_0$ and $\vec{Q} = (Q_1, \dots, Q_n) \in \widetilde{\Gamma}(\vec{l}(N), m, \varepsilon)$; then there is $\vec{U} = (U_1, \dots, U_n) \in \mathrm{U}(N)^n$ so that $\vec{Q} = \xi_{\vec{l}(N)}(\vec{U})$ and $\vec{P} = (P_1, \dots, P_n) := \xi_{\vec{k}(N)}(\vec{U}) \in \Gamma(\vec{k}(N), m, \varepsilon)$. Since

$$\|Q_i - P_i\|_1 = \bigl\|\xi_{N,l_i(N)}(U_i) - \xi_{N,k_i(N)}(U_i)\bigr\|_1 < \frac{\varepsilon}{m}, \qquad 1 \le i \le n,$$

we get for $1 \le i_1, \dots, i_r \le n$ and $1 \le r \le m$

$$\left|\frac{1}{N}\mathrm{Tr}_N(Q_{i_1} \cdots Q_{i_r}) - \frac{1}{N}\mathrm{Tr}_N(P_{i_1} \cdots P_{i_r})\right| \le \sum_{j=1}^{r} \|Q_{i_j} - P_{i_j}\|_1 < \varepsilon,$$

and thus

$$\left|\frac{1}{N}\mathrm{Tr}_N(Q_{i_1}\cdots Q_{i_r}) - \tau(p_{i_1}\cdots p_{i_r})\right| < 2\varepsilon,$$

implying $\vec{Q} \in \Gamma(\vec{l}(N), m, 2\varepsilon)$. Setting $\gamma_{\vec{k}(N)} := \bigotimes_{i=1}^n \gamma_{G(N,k_i(N))}$, we now have, thanks to (1.2),

$$\begin{aligned}\gamma_{\vec{l}(N)}\big(\Gamma(\vec{l}(N), m, 2\varepsilon)\big) &\ge \gamma_{\vec{k}(N)}\big(\widetilde{\Gamma}(\vec{l}(N), m, \varepsilon)\big)\\ &= \big(\gamma_{\mathrm{U}(N)}\big)^{\otimes n} \circ \xi_{\vec{l}(N)}^{-1} \circ \xi_{\vec{l}(N)} \circ \xi_{\vec{k}(N)}^{-1}\big(\Gamma(\vec{k}(N), m, \varepsilon)\big)\\ &\ge \big(\gamma_{\mathrm{U}(N)}\big)^{\otimes n} \circ \xi_{\vec{k}(N)}^{-1}\big(\Gamma(\vec{k}(N), m, \varepsilon)\big)\\ &= \gamma_{\vec{k}(N)}\big(\Gamma(\vec{k}(N), m, \varepsilon)\big)\end{aligned}$$

whenever $N \ge N_0$. This implies that

$$\limsup_{N\to\infty} \frac{1}{N^2}\log \gamma_{\vec{l}(N)}\big(\Gamma(\vec{l}(N), m, 2\varepsilon)\big) \ge \limsup_{N\to\infty} \frac{1}{N^2}\log \gamma_{\vec{k}(N)}\big(\Gamma(\vec{k}(N), m, \varepsilon)\big),$$

which says that the free entropy (1.4) given for $\vec{k}(N)$ is not greater than that for $\vec{l}(N)$. By symmetry we observe that both free entropies must coincide. ☐

The following are basic properties of χ_{proj}. We omit their proofs, all of which are essentially same as in the case of self-adjoint variables in [21] or else obvious.

Proposition 1.2. *Let $p_1, \dots, p_n$ be projections in $(\mathcal{M}, \tau)$.*

(i) *Negativity*: $\chi_{\mathrm{proj}}(p_1, \dots, p_n) \le 0$.

(ii) *Subadditivity*: *for every* $1 \le j < n$,

$$\chi_{\mathrm{proj}}(p_1, \dots, p_n) \le \chi_{\mathrm{proj}}(p_1, \dots, p_j) + \chi_{\mathrm{proj}}(p_{j+1}, \dots, p_n).$$

(iii) *Upper semi-continuity*: *if a sequence $(p_1^{(m)}, \dots, p_n^{(m)})$ of n-tuples of projections converges to $(p_1, \dots, p_n)$ in distribution, then*

$$\chi_{\mathrm{proj}}(p_1, \dots, p_n) \ge \limsup_{m\to\infty} \chi_{\mathrm{proj}}(p_1^{(m)}, \dots, p_n^{(m)}).$$

(iv) $\chi_{\mathrm{proj}}(p_1, \dots, p_n)$ *does not change when p_i is replaced by $p_i^{\perp} := \mathbf{1} - p_i$ for some i.*

Remark 1.3. We may adopt different ways to introduce the free entropy of an n-tuple $(p_1,\dots,p_n)$ of projections in $(\mathcal{M},\tau)$. For instance, consider two unitarily invariant probability measures $\gamma^{(1)}_{G(N)}$ and $\gamma^{(2)}_{G(N)}$ on $G(N) := \bigsqcup_{k=0}^N G(N,k)$ determined by the weights on $G(N,k)$, $0\le k\le N$, given as

$$\gamma^{(1)}_{G(N)}(G(N,k)) = \frac{1}{N+1}, \quad \gamma^{(2)}_{G(N)}(G(N,k)) = \frac{1}{2^N}\binom{N}{k}.$$

We set

$$\begin{aligned}&\Gamma_{\mathrm{proj}}(p_1,\dots,p_n;N,m,\varepsilon)\\ &\quad := \biggl\{(P_1,\dots,P_n)\in G(N)^n : \left|\frac{1}{N}\mathrm{Tr}_N(P_{i_1}\cdots P_{i_r}) - \tau(p_{i_1}\cdots p_{i_r})\right| < \varepsilon \\ &\qquad\qquad\qquad \text{for all } 1\le i_1,\dots,i_r\le n,\ 1\le r\le m\biggr\},\end{aligned}$$

and define for $j=1,2$

$$\chi^{(j)}_{\mathrm{proj}}(p_1,\dots,p_n) := \lim_{\substack{m\to\infty\\ \varepsilon\searrow 0}} \limsup_{N\to\infty} \frac{1}{N^2}\log\Bigl(\gamma^{(j)}_{G(N)}\Bigr)^{\otimes n}\bigl(\Gamma_{\mathrm{proj}}(p_1,\dots,p_n;N,m,\varepsilon)\bigr).$$

It is fairly easy to see (similarly to the proof of Proposition 1.1) that both $\chi^{(j)}_{\mathrm{proj}}(p_1,\dots,p_n)$, $j=1,2$, coincide with $\chi_{\mathrm{proj}}(p_1,\dots,p_n)$ given in (1.4).

§2. Case of Two Projections

Let (p,q) be a pair of projections in a tracial W^*-probability space $(\mathcal{M},\tau)$ with $\alpha := \tau(p)$ and $\beta := \tau(q)$. Set

$$E_{11} := p\wedge q, \quad E_{10} := p\wedge q^\perp, \quad E_{01} := p^\perp\wedge q, \quad E_{00} := p^\perp\wedge q^\perp,$$
$$E := \mathbf{1} - (E_{00}+E_{01}+E_{10}+E_{11}).$$

Then E and E_{ij} are in the center of $\{p,q\}''$ and $(E\{p,q\}''E, \tau|_{E\{p,q\}''E})$ is isomorphic to $L^\infty((0,1),\nu;M_2(\mathbb{C}))$, where ν is the measure on $(0,1)$ determined by

$$\tau(A) = \frac{1}{2}\int_{(0,1)} \mathrm{Tr}_2(A(x))\,d\nu(x), \quad A\in L^\infty((0,1),\nu;M_2(\mathbb{C}))\cong E\{p,q\}''E$$

(hence $\nu((0,1)) = \tau(E)$). Under this isomorphism, EpE and EqE are represented as

$$(EpE)(x) = \begin{bmatrix}1&0\\0&0\end{bmatrix} \quad\text{and}\quad (EqE)(x) = \begin{bmatrix} x & \sqrt{x(1-x)}\\ \sqrt{x(1-x)} & 1-x\end{bmatrix} \quad \text{for } x\in(0,1).$$

In this way, the mixed moments of (p,q) with respect to τ are determined by ν and $\{\tau(E_{ij})\}_{i,j=0}^1$. Although ν is not necessarily a probability measure, we define the free entropy $\Sigma(\nu)$ by

$$\Sigma(\nu) := \iint_{(0,1)^2} \log|x-y|\,d\nu(x)\,d\nu(y)$$

in the same way as in [20]. Furthermore, we set

$$\rho := \min\{\alpha, \beta, 1-\alpha, 1-\beta\}, \tag{2.1}$$

$$C := \rho^2 B\biggl(\frac{|\alpha-\beta|}{\rho}, \frac{|\alpha+\beta-1|}{\rho}\biggr) \tag{2.2}$$

(meant zero if $\rho = 0$), where

$$\begin{aligned} B(s,t) := \frac{(1+s)^2}{2}\log(1+s) - \frac{s^2}{2}\log s + \frac{(1+t)^2}{2}\log(1+t) - \frac{t^2}{2}\log t \\ - \frac{(2+s+t)^2}{2}\log(2+s+t) + \frac{(1+s+t)^2}{2}\log(1+s+t) \end{aligned}$$

for $s,t \geq 0$. With these definitions, the following formula of $\chi_{\mathrm{proj}}(p,q)$ was obtained in [12] as a consequence of the large deviation principle for an independent pair of random projection matrices.

Proposition 2.1 ([12, Theorem 3.2, Proposition 3.3]). *If $\tau(E_{00})\tau(E_{11}) = \tau(E_{01})\tau(E_{10}) = 0$, then*

$$\begin{aligned} \chi_{\mathrm{proj}}(p,q) = \frac{1}{4}\Sigma(\nu) + \frac{|\alpha-\beta|}{2}\int_{(0,1)} \log x\,d\nu(x) \\ + \frac{|\alpha+\beta-1|}{2}\int_{(0,1)} \log(1-x)\,d\nu(x) - C, \end{aligned}$$

and otherwise $\chi_{\mathrm{proj}}(p,q) = -\infty$. Moreover, $\chi_{\mathrm{proj}}(p,q) = 0$ if and only if p and q are free.

Note that the condition $\tau(E_{00})\tau(E_{11}) = \tau(E_{01})\tau(E_{10}) = 0$ is equivalent to

$$\begin{cases} \tau(E_{11}) = \max\{\alpha+\beta-1, 0\}, \\ \tau(E_{00}) = \max\{1-\alpha-\beta, 0\}, \\ \tau(E_{10}) = \max\{\alpha-\beta, 0\}, \\ \tau(E_{01}) = \max\{\beta-\alpha, 0\}. \end{cases} \tag{2.3}$$

When this is the case, the following must hold:

$$\tau(E_{01}) + \tau(E_{10}) = |\alpha - \beta|, \quad \tau(E_{00}) + \tau(E_{11}) = |\alpha + \beta - 1|, \quad \tau(E) = 2\rho.$$

In the case where $\chi_{\mathrm{proj}}(p, q) = 0$ (equivalently, p and q are free), the measure ν was computed in [27] as

$$\frac{\sqrt{(x-\xi)(\eta-x)}}{2\pi x(1-x)} \mathbf{1}_{(\xi,\eta)}(x)\, dx \tag{2.4}$$

with $\xi, \eta := \alpha + \beta - 2\alpha\beta \pm \sqrt{4\alpha\beta(1-\alpha)(1-\beta)}$. It is also worthwhile to note [12] that lim sup in definition (1.4) can be replaced by lim in the case of two projections due to the large deviation result mentioned above.

In §4 the equivalence between the additivity of χ_{proj} and the freeness of projection pairs will be generalized to the "pair-block freeness result" for more than two projections. To do this, we need a kind of separate change of variable formula for χ_{proj}, which will be established in the next section.

§3. Separate Change of Variable Formula

Let $N \in \mathbb{N}$ and $k, l \in \{0, 1, \dots, N\}$. Assume that $0 < k \le l$ and $k + l \le N$. Consider a pair (P, Q) of $N \times N$ projection random matrices with $\mathrm{rank}(P) = k$ and $\mathrm{rank}(Q) = l$, which is assumed to be distributed under the measure $\gamma_{G(N,k)} \otimes \gamma_{G(N,l)}$ on $G(N,k) \times G(N,l)$. Then, by means of the so-called sine-cosine decomposition of two projections, we can represent such (P, Q) as follows:

$$P = U \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \oplus 0 \oplus 0 \right) U^*, \tag{3.1}$$

$$Q = U \left(\begin{bmatrix} X & \sqrt{X(I-X)} \\ \sqrt{X(I-X)} & I - X \end{bmatrix} \oplus I \oplus 0 \right) U^* \tag{3.2}$$

in $\mathbb{C}^N = (\mathbb{C}^k \otimes \mathbb{C}^2) \oplus \mathbb{C}^{l-k} \oplus \mathbb{C}^{N-k-l}$, where U is an $N \times N$ unitary matrix and X is a $k \times k$ diagonal matrix with the diagonal entries $0 \le x_1 \le x_2 \le \cdots \le x_k \le 1$. When $x_1, \dots, x_k$ are in $(0,1)$ and mutually distinct, it is easy to see that U is uniquely determined up to the right multiplication of unitary matrices of the form

$$\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} \oplus V_1 \oplus V_2, \quad T \in \mathbb{T}^k,\ V_1 \in \mathrm{U}(l-k),\ V_2 \in \mathrm{U}(N-k-l).$$

We denote by $V(N,k,l)$ the subgroup of $\mathrm{U}(N)$ consisting of all unitary matrices of the above form so that $\mathrm{U}(N)/V(N,k,l)$ becomes a homogeneous space. Also, let $[0,1]^k_{\le}$ and $(0,1)^k_{<}$ denote the sets of $(x_1,\dots,x_k)$ satisfying $0 \le x_1 \le \cdots \le x_k \le 1$ and $0 < x_1 < \cdots < x_k < 1$, respectively. We then consider the continuous map $\Xi_{N,k,l} : \mathrm{U}(N)/V(N,k,\ell) \times [0,1]^k_{\le} \to G(N,k) \times G(N,l)$ defined by (3.1) and (3.2), that is,

$$\Xi_{N,k,l}([U],X) := \left(U \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \oplus 0 \oplus 0 \right) U^*, U \left(\begin{bmatrix} X & \sqrt{X(I-X)} \\ \sqrt{X(I-X)} & I-X \end{bmatrix} \oplus I \oplus 0 \right) U \right),$$

where $X = (x_1,\dots,x_k)$ in the right-hand side is regarded as a diagonal matrix as above. The set

$$(G(N,k) \times G(N,l))_0 := \Xi_{N,k,l}\left(\mathrm{U}(N)/V(N,k,l) \times (0,1)^k_{<}\right)$$

is open and co-negligible with respect to $\gamma_{G(N,k)} \otimes \gamma_{G(N,l)}$ in $G(N,k) \times G(N,l)$ thanks to [5, Theorem 2.2] (or [12, Lemma 1.1]) and moreover $\Xi_{N,k,l}$ gives a smooth diffeomorphism between $\mathrm{U}(N)/V(N,k,l) \times (0,1)^k_{<}$ and $(G(N,k) \times G(N,l))_0$. Then we show the next lemma for later use.

Lemma 3.1. *The measure* $(\gamma_{G(N,k)} \otimes \gamma_{G(N,l)}) \circ \Xi_{N,k,l}$ *coincides with*

$$\gamma_{N,k,l} \otimes \left(\frac{1}{Z_{N,k,l}} \prod_{i=1}^k x_i^{l-k}(1-x_i)^{N-k-l} \prod_{1\le i<j\le k} (x_i - x_j)^2 \prod_{i=1}^k dx_i \right),$$

where $\gamma_{N,k,l}$ *is the* (*unique*) *probability measure on* $\mathrm{U}(N)/V(N,k,l)$ *induced by the Haar probability measure on* $\mathrm{U}(N)$ *and* $Z_{N,k,l}$ *is a normalization constant.*

Proof. Let λ be the measure on $\mathrm{U}(N)/V(N,k,l) \times (0,1)^k_{<}$ transformed from the restriction of $\gamma_{G(N,k)} \otimes \gamma_{G(N,l)}$ to $(G(N,k) \times G(N,l))_0$ by the inverse of $\Xi_{N,k,l}$, and μ be its image measure by the projection map $([U],X) \mapsto X$. The disintegration theorem (see e.g. [16, Chapter IV, §6.5]) ensures that there is a μ-a.e. unique Borel map $\lambda_{(\cdot)}$ from $(0,1)^k_{<}$ to the probability measures on $\mathrm{U}(N)/V(N,k,l)$ such that $\lambda = \int_{(0,1)^k_{<}} \lambda_X \, d\mu(X)$. Note that $([U],X) \mapsto X$ splits into $\Xi_{N,k,l}$, $(P,Q) \mapsto PQP$ and the map sending PQP to the eigenvalues in increasing order. Hence μ coincides with the eigenvalue distribution of PQP arranged in increasing order, which is known to be equal to the second component given in the lemma by [5, Theorem 2.2]. Therefore, it suffices to show that λ_X coincides with $\gamma_{N,k,l}$ for μ-a.e. $X \in (0,1)^k_{<}$. For each $V \in \mathrm{U}(N)$, the unitary conjugation $\mathrm{Ad}\,V \times \mathrm{Ad}\,V : (P,Q) \mapsto (VPV^*, VQV^*)$ on $G(N,k) \times G(N,l)$ and the left-translation $L_V : [U] \mapsto V[U] := [VU]$ on $\mathrm{U}(N)/V(N,k,l)$ satisfy the relation $\Xi_{N,k,l} \circ (L_V \times \mathrm{id}) = (\mathrm{Ad}\, V \times \mathrm{Ad}\, V) \circ \Xi_{N,k,l}$; hence, in particular, $(G(N,k) \times G(N,l))_0$ is invariant under the action $\mathrm{Ad}\, V \times \mathrm{Ad}\, V$ for every $V \in \mathrm{U}(N)$. Then one can easily verify that

$$\int_{(0,1)^k_<} \left(\int_{\mathrm{U}(N)/V(N,k,l)} f([U], X)\, d(\lambda_X \circ L_V)([U]) \right) d\mu(X)$$
$$= \int_{\mathrm{U}(N)/V(N,k,l) \times (0,1)^k_<} f([U], X)\, d\lambda([U], X)$$

for any bounded Borel function f on $\mathrm{U}(N)/V(N,k,l) \times (0,1)^k$. This means that λ enjoys a new disintegration $\lambda = \int_{(0,1)^k_<} \lambda_X \circ L_V\, d\mu(X)$. The uniqueness of the disintegration says that for μ-a.e. $X \in (0,1)^k_<$ one has $\lambda_X = \lambda_X \circ L_V$ for all $V \in \mathrm{U}(N)$. Since $\gamma_{N,k,l}$ is a unique probability measure on $\mathrm{U}(N)/V(N,k,l)$ invariant under all L_V, it follows that $\lambda_X = \gamma_{N,k,l}$ for μ-a.e. $X \in (0,1)^k_<$ so that

$$\lambda = \int_{(0,1)^k_<} \gamma_{N,k,l}\, d\mu(X) = \gamma_{N,k,l} \otimes \mu,$$

as required. □

For a pair (p, q) of projections in $(\mathcal{M}, \tau)$ we introduce a sort of functional calculus via the representation explained in §2. Let ψ be a continuous increasing function from $(0,1)$ into itself. With the notations in §2 we define a new projection $q(\psi; p)$ in $\{p, q\}''$ by

$$q(\psi; p) := Eq(\psi; p)E + E_{00} + E_{01} + E_{10} + E_{11},$$
$$(Eq(\psi; p)E)(x) := \begin{bmatrix} \psi(x) & \sqrt{\psi(x)(1-\psi(x))} \\ \sqrt{\psi(x)(1-\psi(x))} & 1-\psi(x) \end{bmatrix} \quad \text{for } x \in (0,1).$$

It is obvious that $\tau(q(\psi; p)) = \tau(q)$. (The definition itself is possible for general Borel function from $(0,1)$ into $[0,1]$ but the above case is enough for our purpose.) The aim of this section is to prove the following change of variable formula for free entropy of projections.

Theorem 3.2. *Let $p_1, q_1, \dots, p_n, q_n, r_1, \dots, r_{n'}$ be projections in $(\mathcal{M}, \tau)$ and assume that $\chi_{\mathrm{proj}}(p_i, q_i) > -\infty$ for $1 \le i \le n$. Let $\psi_1, \dots, \psi_n$ be continuous increasing functions from $(0,1)$ into itself, and $q_i(\psi_i; p_i)$ be the projection defined from p_i, q_i and ψ_i in the above manner for $1 \le i \le n$. Then we have*

$$\chi_{\mathrm{proj}}(p_1, q_1(\psi_1; p_1), \dots, p_n, q_n(\psi_n; p_n), r_1, \dots, r_{n'})$$
$$\ge \chi_{\mathrm{proj}}(p_1, q_1, \dots, p_n, q_n, r_1, \dots, r_{n'}) + \sum_{i=1}^{n} \left\{ \chi_{\mathrm{proj}}(p_i, q_i(\psi_i; p_i)) - \chi_{\mathrm{proj}}(p_i, q_i) \right\}.$$

Moreover, if $\psi_1, \dots, \psi_n$ are strictly increasing, then equality holds true in the above inequality.

The proof goes on the essentially same lines as in [22] and it is divided into two steps; one is to analyze the case when $\psi_1, \dots, \psi_n$ are all extended to C^1-diffeomorphisms from $[0,1]$ onto itself and the other is to approximate, in two stages, the given $\psi_1, \dots, \psi_n$ by C^∞-diffeomorphisms from $[0,1]$ onto itself in such a way that the corresponding free entropies converge to those in question. As the first step let us prove the following special case of the theorem.

Lemma 3.3. *Let $p_1, q_1, \dots, p_n, q_n, r_1, \dots, r_{n'}$ be as in Theorem 3.2. If $\psi_1, \dots, \psi_n$ are C^1-diffeomorphisms from $[0,1]$ onto itself with $\psi_i(0) = 0$ and $\psi_i(1) = 1$ and moreover $\psi_i'(x) > 0$ for all $x \in [0,1]$, then the equality assertion of Theorem 3.2 holds true.*

Proof. In the same way as in the proof of [22, Proposition 3.1] it suffices to show when $n = 1$; hence we assume $n = 1$ and write $p = p_1$, $q = q_1$ and $\psi = \psi_1$ for brevity. Let ν and $\{E_{ij}\}_{i,j=0}^1$ be as in §2 for (p,q). By Propositions 1.2 (iv) and 2.1 we may assume that $\tau(p) \le \tau(q) \le 1/2$ so that $E_{11} = E_{10} = 0$ by (2.3). We may further assume that p is non-zero; otherwise there is nothing to do. With the polar decomposition $(\mathbf{1} - p)qp = v_{p,q}\sqrt{pqp(p - pqp)}$, we thus represent p, q and $q(\psi;p)$ as follows:

$$\begin{aligned} p &= v_{p,q}^* v_{p,q}, \\ q &= pqp + v_{p,q}\sqrt{pqp(p-pqp)} + \sqrt{pqp(p-pqp)}\,v_{p,q}^* + v_{p,q}(p-pqp)v_{p,q}^* \\ &\quad + \Big(q - pqp - (\mathbf{1}-p)qp - pq(\mathbf{1}-p) - v_{p,q}(p-pqp)v_{p,q}^*\Big), \\ q(\psi;p) &= \psi(pqp) + v_{p,q}\sqrt{\psi(pqp)(p-\psi(pqp))} \\ &\quad + \sqrt{\psi(pqp)(p-\psi(pqp))}\,v_{p,q}^* + v_{p,q}(p-\psi(pqp))v_{p,q}^* \\ &\quad + \Big(q - pqp - (\mathbf{1}-p)qp - pq(\mathbf{1}-p) - v_{p,q}(p-pqp)v_{p,q}^*\Big), \end{aligned}$$

where $\psi(pqp)$ means the functional calculus of pqp. Choose two sequences $k(N)$, $l(N)$ for $N \ge 2$ in such a way that $0 < k(N) \le l(N) \le N/2$ and $k(N)/N \to \tau(p)$, $l(N)/N \to \tau(q)$ as $N \to \infty$. As explained at the beginning of this section, for each $(P,Q) \in (G(N,k(N)) \times G(N,l(N)))_0$ there is a unitary $U \in \mathrm{U}(N)$, unique up to $V(N,k(N),l(N))$, for which we have (3.1) and (3.2). Then we can define the map $\Phi_{N,\psi}$ on $(G(N,k(N)) \times G(N,l(N)))_0$ by sending (P,Q) to $(P, Q(\psi;P))$ with

$$Q(\psi;P) := U\left(\begin{bmatrix} \psi(X) & \sqrt{\psi(X)(I-\psi(X))} \\ \sqrt{\psi(X)(I-\psi(X))} & I - \psi(X) \end{bmatrix} \oplus I \oplus 0\right)U^*.$$

With the polar decomposition $(I-P)QP = V_{P,Q}\sqrt{PQP(I-PQP)}$ we have the following expressions:

$$\begin{aligned}Q = PQP &+ V_{P,Q}\sqrt{PQP(P-PQP)}\\ &+ \sqrt{PQP(P-PQP)}V_{P,Q}^* + V_{P,Q}(P-PQP)V_{P,Q}^*\\ &+ \Big(Q - PQP - (I-P)QP - PQ(I-P) - V_{P,Q}(P-PQP)V_{P,Q}^*\Big),\\ Q(\psi;P) = \psi(PQP) &+ V_{P,Q}\sqrt{\psi(PQP)(P-\psi(PQP))}\\ &+ \sqrt{\psi(PQP)(P-\psi(PQP))}V_{P,Q}^* + V_{P,Q}(P-\psi(PQP))V_{P,Q}^*\\ &+ \Big(Q - PQP - (I-P)QP - PQ(I-P) - V_{P,Q}(P-PQP)V_{P,Q}^*\Big).\end{aligned}$$

Upon these expressions, what we now need is to approximate $v_{p,q}$ and $V_{P,Q}$ by polynomials of p, q and P, Q, respectively, as stated in the next lemma very similarly to [11, 6.6.4].

Lemma 3.4. *For each $t \ge 1$ and $\varepsilon > 0$ one can find $N_0, m_0 \in \mathbb{N}$, $\varepsilon_0 > 0$ and a real polynomial G in such a way that the following assertions hold*:

- $\|v_{p,q} - (\mathbf{1}-p)qp \cdot G(pqp)\|_t < \varepsilon$.
- *For each $N \ge N_0$, if $(P,Q) \in (G(N,k(N)) \times G(N,l(N)))_0$ satisfies*

$$\left|\frac{1}{N}\mathrm{Tr}_N((PQP)^m) - \tau((pqp)^m)\right| < \varepsilon_0 \quad \textit{for } 1 \le m \le m_0, \tag{3.3}$$

then $\|V_{P,Q} - (\mathbf{1}-P)QP \cdot G(PQP)\|_t < \varepsilon$.

Here, $\|\cdot\|_t$ denotes the Schatten t-norm with respect to τ as well as $N^{-1}\mathrm{Tr}_N$.

The proof of this technical lemma is essentially similar to that of [11, 6.6.4] so that its sketch will be given later.

Proof of Lemma 3.3 (*continued*). Choose $k_1(N), \dots, k_{n'}(N)$ so that $k_i(N)/N \to \tau(r_i)$ as $N \to \infty$, and set

$$\Phi_N := \Phi_{N,\psi} \times \prod_{i=1}^{n'} \mathrm{id}_{G(N,k_i(N))} \ \text{ on } (G(N,k(N)) \times G(N,l(N)))_0 \times \prod_{i=1}^{n'} G(N,k_i(N))$$

and $\gamma_N := \gamma_{G(N,k(N))} \otimes \gamma_{G(N,l(N))} \otimes \bigotimes_{i=1}^{n'} \gamma_{G(N,k_i(N))}$. Let $m \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary. In what follows, for brevity we write $\Gamma_{\mathrm{proj}}(p,q,r_1,\dots,r_{n'}; N, m_0, \varepsilon_0)$ etc. without $k(N), l(N), k_1(N), \dots, k_{n'}(N)$. Thanks to Lemma 3.4 together with the expressions of $q(\psi;p)$ and $Q(\psi;P)$ above, we can choose $N_0, m_0 \in \mathbb{N}$

and $\varepsilon_0 > 0$ with $m_0 \geq m$ and $\varepsilon_0 \leq \varepsilon$ such that, for every $N \geq N_0$, if $(P, Q, R_1, \dots, R_{n'}) \in \Gamma_{\mathrm{proj}}(p, q, r_1, \dots, r_{n'}; N, m_0, \varepsilon_0)$ and $(P, Q) \in (G(N, k(N)) \times G(N, l(N)))_0$, then $\Phi_N(P, Q, R_1, \dots, R_{n'})$ falls into $\Gamma_{\mathrm{proj}}(p, q(\psi; p), r_1, \dots, r_{n'}; N, m, \varepsilon)$. Via $\Xi_{N,k(N),l(N)}$ in the first two coordinates, Lemma 3.1 enables us to estimate the Radon-Nikodym derivative $d\gamma_N \circ \Phi_N / d\gamma_N$ on a co-negligible subset of $\Gamma_{\mathrm{proj}}(p, q, r_1, \dots, r_{n'}; N, m, \varepsilon)$ from below by the infimum value of

(3.4)
$$\prod_{1\leq i<j\leq k(N)} \left(\frac{\psi(\lambda_i(PQP)) - \psi(\lambda_j(PQP))}{\lambda_i(PQP) - \lambda_j(PQP)}\right)^2 \prod_{i=1}^{k(N)} \psi'(\lambda_i(PQP))$$
$$\times \prod_{i=1}^{k(N)} \left(\frac{\psi(\lambda_i(PQP))}{\lambda_i(PQP)}\right)^{l(N)-k(N)} \prod_{i=1}^{k(N)} \left(\frac{1-\psi(\lambda_i(PQP))}{1-\lambda_i(PQP)}\right)^{N-k(N)-l(N)}$$

for all $(P, Q) \in (G(N, k(N)) \times G(N, l(N)))_0 \cap \Gamma_{\mathrm{proj}}(p, q; N, m_0, \varepsilon_0)$ with the eigenvalue list $\lambda_1(PQP), \dots, \lambda_{k(N)}(PQP)$ in increasing order.

Let $\psi^{[1]}(x, y)$ be the so-called divided difference of ψ, i.e.,

$$\psi^{[1]}(x,y) := \begin{cases} \frac{\psi(x)-\psi(y)}{x-y} & (x \neq y), \\ \psi'(x) & (x = y). \end{cases}$$

Then, quantity (3.4) is rewritten in the coordinate (P, Q) as

$$\begin{aligned}
&\det{}_{k(N)^2\times k(N)^2}\left[P\otimes P\cdot\psi^{[1]}(PQP\otimes P, P\otimes PQP)\cdot P\otimes P\right] \\
&\quad\times \left(\det{}_{k(N)\times k(N)}[P(PQP)^{-1}\psi(PQP)P]\right)^{l(N)-k(N)} \\
&\quad\times \left(\det{}_{k(N)\times k(N)}[P(P-PQP)^{-1}(P-\psi(PQP)P]\right)^{N-k(N)-l(N)} \\
&= \exp\left(\mathrm{Tr}_{k(N)}^{\otimes 2}\left(P\otimes P\cdot\log(\psi^{[1]}(PQP\otimes P, P\otimes PQP))\cdot P\otimes P\right)\right) \\
&\quad\times \left(\exp\left(\mathrm{Tr}_{k(N)}\left(P\cdot\log\left((PQP)^{-1}\psi(PQP)\right)\cdot P\right)\right)\right)^{l(N)-k(N)} \\
&\quad\times \left(\exp\left(\mathrm{Tr}_{k(N)}\left(P\cdot\log\left((P-PQP)^{-1}(P-\psi(PQP))\right)\cdot P\right)\right)\right)^{N-k(N)-l(N)},
\end{aligned}$$

where $\psi^{[1]}(PQP \otimes P, P \otimes PQP)$ is defined on $P\mathbb{C}^N \otimes P\mathbb{C}^N$ while $(PQP)^{-1}\psi(PQP)$ and $(P - PQP)^{-1}(P - \psi(PQP))$ are on $P\mathbb{C}^N$. Let $\delta > 0$ be arbitrary. Since ψ is C^1, $\log\psi^{[1]}(x, y)$ is continuous on $[0, 1]^2$ so that there is a real polynomial $L(x, y)$ on $[0, 1]^2$ such that $\|\log\psi^{[1]} - L\|_\infty < \delta$. If $m' \in \mathbb{N}$ is larger than the degree of L, then we have, for each $(P, Q) \in \Gamma_{\mathrm{proj}}(p, q; N, m', \varepsilon')$

with an arbitrary $\varepsilon' > 0$,

$$\begin{aligned}
&\Big|\frac{1}{N^2}\mathrm{Tr}_N^{\otimes 2}\left(P\otimes P\cdot\log\psi^{[1]}(PQP\otimes P, P\otimes PQP)\cdot P\otimes P\right) \\
&\qquad\qquad - \tau^{\otimes 2}(p\otimes p\cdot\log\psi^{[1]}(pqp\otimes p, p\otimes pqp)\cdot p\otimes p)\Big| \\
&\quad\le 2\delta + \Big|\frac{1}{N^2}\mathrm{Tr}_N^{\otimes 2}(P\otimes P\cdot L(PQP\otimes P, P\otimes PQP)\cdot P\otimes P) \\
&\qquad\qquad - \tau^{\otimes 2}(p\otimes p\cdot L(pqp\otimes p, p\otimes pqp)\cdot p\otimes p)\Big| \\
&\quad\le 2\delta + C\varepsilon
\end{aligned}$$

with $C > 0$ depending only on L (hence on δ). Therefore, for each $\eta > 0$ there are $m_1 \in \mathbb{N}$ and $\varepsilon_1 > 0$ such that

$$\begin{aligned}
\text{(3.5)}\quad &\exp\left(\mathrm{Tr}_{k(N)}^{\otimes 2}\left(P\otimes P\cdot\log\left(\psi^{[1]}(PQP\otimes P, P\otimes PQP)\right)\cdot P\otimes P\right)\right) \\
&\quad\ge \exp\left(N^2\left\{\tau^{\otimes 2}(p\otimes p\cdot\log\psi^{[1]}(pqp\otimes p, p\otimes pqp)\cdot p\otimes p) - \eta\right\}\right)
\end{aligned}$$

for all $(P, Q) \in \Gamma_{\mathrm{proj}}(p, q; N, m', \varepsilon')$ as long as $m' \ge m_1$ and $0 < \varepsilon' \le \varepsilon_1$. Since $x^{-1}\psi(x)$ and $(1-x)^{-1}(1-\psi(x))$ are both bounded away from zero on $[0, 1]$ due to the assumption on ψ, the same argument works for the other two terms

$$\begin{gathered}
\exp\left(\mathrm{Tr}_{k(N)}\left(P\cdot\log\left((PQP)^{-1}\psi(PQP)\right)\cdot P\right)\right), \\
\exp\left(\mathrm{Tr}_{k(N)}\left(P\cdot\log\left((P-PQP)^{-1}(P-\psi(PQP))\right)\cdot P\right)\right).
\end{gathered}$$

Therefore, for each $\eta > 0$ there are $m_2 \in \mathbb{N}$ and $\varepsilon_2 > 0$ such that

$$\begin{aligned}
\text{(3.6)}\quad &\exp\left(\mathrm{Tr}_{k(N)}\left(P\cdot\log\left((PQP)^{-1}\psi(PQP)\right)\cdot P\right)\right) \\
&\quad\ge \exp\left(N\left\{\tau(p\cdot\log((pqp)^{-1}\psi(pqp))\cdot p) - \eta\right\}\right), \\
\text{(3.7)}\quad &\exp\left(\mathrm{Tr}_{k(N)}\left(P\cdot\log\left((P-PQP)^{-1}(P-\psi(PQP))\right)\cdot P\right)\right) \\
&\quad\ge \exp\left(N\left\{\tau(p\cdot\log((p-pqp)^{-1}(p-\psi(pqp)))\cdot p) - \eta\right\}\right)
\end{aligned}$$

for all $(P, Q) \in \Gamma_{\mathrm{proj}}(p, q; N, m', \varepsilon')$ as long as $m' \ge m_2$ and $0 < \varepsilon' \le \varepsilon_2$. Hence, whenever $N \ge N_0$, $m' \ge \max\{m_0, m_1, m_2\}$ and $0 < \varepsilon' < \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$, we have

$$\begin{aligned}
&\frac{1}{N^2}\log\gamma_N\left(\Gamma_{\mathrm{proj}}(p, q(\psi; p), r_1, \dots, r_{n'}; N, m, \varepsilon)\right) \\
&\qquad\ge \frac{1}{N^2}\log\gamma_N\left(\Phi_N\left(\Gamma_{\mathrm{proj}}(p, q, r_1, \dots, r_{n'}; N, m', \varepsilon')\right)\right) \\
&\qquad\ge \frac{1}{N^2}\log\gamma_N\left(\Gamma_{\mathrm{proj}}(p, q, r_1, \dots, r_{n'}; N, m', \varepsilon')\right)
\end{aligned}$$

$$
\begin{aligned}
&+ \tau^{\otimes 2}(p \otimes p \cdot \log \psi^{[1]}(pqp \otimes p, p \otimes pqp) \cdot p \otimes p) \\
&+ \left(\frac{l(N)}{N} - \frac{k(N)}{N}\right) \tau(p \cdot \log((pqp)^{-1}\psi(pqp)) \cdot p) \\
&+ \left(1 - \frac{l(N)}{N} - \frac{k(N)}{N}\right) \tau(p \cdot \log((p-pqp)^{-1}(p - \psi(pqp))) \cdot p) - 3\eta \\
= {}& \frac{1}{N^2} \log \gamma_N\left(\Gamma_{\mathrm{proj}}(p, q, r_1, \ldots, r_{n'}; N, m', \varepsilon')\right) \\
&+ \frac{1}{4} \iint_{(0,1)^2} \log \left|\frac{\psi(x) - \psi(y)}{x - y}\right| d\nu(x)\, d\nu(y) \\
&+ \frac{1}{2}\left(\frac{l(N)}{N} - \frac{k(N)}{N}\right) \int_{(0,1)} \log \frac{\psi(x)}{x}\, d\nu(x) \\
&+ \frac{1}{2}\left(1 - \frac{l(N)}{N} - \frac{k(N)}{N}\right) \int_{(0,1)} \log \frac{1 - \psi(x)}{1 - x}\, d\nu(x) - 3\eta.
\end{aligned}
$$

Take the lim sup as $N \to \infty$ and the limit as $m \to \infty$, $\varepsilon \searrow 0$ in the above inequality. Since $\eta > 0$ is arbitrary, we get

$$
\begin{aligned}
&\chi_{\mathrm{proj}}(p, q(\psi; p), r_1, \ldots, r_{n'}) \\
&\geq \chi_{\mathrm{proj}}(p, q, r_1, \ldots, r_{n'}) + \frac{1}{4} \iint_{(0,1)^2} \log \left|\frac{\psi(x) - \psi(y)}{x - y}\right| d\nu(x) d\nu(y) \\
&+ \frac{\tau(q) - \tau(p)}{2} \int_{(0,1)} \log \frac{\psi(x)}{x}\, d\nu(x) + \frac{1 - \tau(q) - \tau(p)}{2} \int_{(0,1)} \log \frac{1 - \psi(x)}{1 - x}\, d\nu(x) \\
&= \chi_{\mathrm{proj}}(p, q, r_1, \ldots, r_{n'}) + \chi_{\mathrm{proj}}(p, q(\psi; p)) - \chi_{\mathrm{proj}}(p, q)
\end{aligned}
$$

thanks to Proposition 2.1. The reverse inequality can be shown as well if we replace the inequalities (3.5)–(3.7) by their reversed versions. Hence we complete the proof of Lemma 3.3. □

Proof of Lemma 3.4 (*sketch*). Given small $\alpha, \beta > 0$ we can estimate

$$
\begin{aligned}
(3.8)\quad & \|v_{p,q} - ((\mathbf{1} - p)qp)(\sqrt{pqp(p - pqp)} + \alpha\mathbf{1})^{-1}\|_t^t \\
& \leq \frac{1}{2}\left\{\nu((0,\beta)) + \nu((1-\beta,1)) + \nu([\beta, 1-\beta])\left(\frac{\alpha}{\sqrt{\beta(1-\beta)} + \alpha}\right)^t\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
(3.9)\qquad & \|V_{P,Q} - (I - P)QP(\sqrt{PQP(P - PQP)} + \alpha I)^{-1}\|_t^t \\
& \leq \frac{1}{N}\#\{i : \lambda_i(PQP) < \beta\} + \frac{1}{N}\#\{i : \lambda_i(PQP) > 1 - \beta\}
\end{aligned}
$$

$$+\frac{k(N)}{N}\left(\frac{\alpha}{\sqrt{\beta(1-\beta)}+\alpha}\right)^t,$$

where $0<\lambda_1(PQP)<\cdots<\lambda_{k(N)}(PQP)<1$ are the eigenvalues of $PQP|_{P\mathbb{C}^N}$ for $(P,Q)\in(G(N,k(N))\times G(N,l(N)))_0$. For any $\eta>0$ let us choose $\beta>0$ so that $\nu((0,2\beta))+\nu((1-2\beta,1))<\eta^t$. By (3.8) we get

(3.10)
$$\|v_{p,q}-((\mathbf{1}-p)qp)(\sqrt{pqp(p-pqp)}+\alpha\mathbf{1})^{-1}\|_t^t\le\frac{\eta^t}{2}+\frac{\tau(E)}{2}\left(\frac{\alpha}{\sqrt{\beta(1-\beta)}}\right)^t.$$

Note that ν is non-atomic on $(0,1)$ due to the assumption $\chi_{\mathrm{proj}}(p,q)>-\infty$. Set $\xi_{N,i}:=\min\{x\in[0,1]:\nu((0,x))=i\tau(E)/k(N)\}$ for $1\le i\le k(N)$; then we get

$$\tau((pqp)^m)=\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{k(N)}\left(\xi_{N,i}\right)^m\quad\text{for all } m\in\mathbb{N}.$$

Also choose a constant $C>\sup_{N\ge2}N/k(N)$. By [11, 4.3.4] there are $m_0\in\mathbb{N}$ and $\varepsilon_0>0$ such that, for every $N\in\mathbb{N}$ and for every $(\lambda_1,\dots,\lambda_{k(N)})\in(0,1)_<^{k(N)}$,

$$\left|\frac{1}{k(N)}\sum_{i=1}^{k(N)}\lambda_i^m-\frac{1}{k(N)}\sum_{i=1}^{k(N)}\left(\xi_{N,i}\right)^m\right|<2C\varepsilon_0\quad\text{for } 1\le m\le m_0$$

implies

$$\frac{1}{k(N)}\sum_{i=1}^{k(N)}\left|\lambda_i-\xi_{N,i}\right|^m<\beta\eta^t. \tag{3.11}$$

Assume (3.11). Set $i_0:=\#\{i:\lambda_i<\beta\}$ and $i_1:=\#\{i:\xi_{N,i}<2\beta\}$. If $i_1<i\le i_0$, then $\left|\lambda_i-\xi_{N,i}\right|=\xi_{N,i}-\lambda_i\ge\beta$ so that we get $i_0<i_1+k(N)\eta^t$ by (3.11). Since $i_1\tau(E)/k(N)\le\nu((0,2\beta))<\eta^t$, we get $i_0<\tau(E)^{-1}(1+\tau(E))k(N)\eta^t$. If there is no $i_1<i\le i_0$, then $i_0\le i_1<\tau(E)^{-1}k(N)\eta^t$. Therefore, $\#\{i:\lambda_i<\beta\}<\tau(E)^{-1}(1+\tau(E))k(N)\eta^t$. Similarly, we have $\#\{i:\lambda_i>1-\beta\}<\tau(E)^{-1}(1+\tau(E))k(N)\eta^t$. Now, choose $N_0\in\mathbb{N}$ so that

$$\left|\frac{1}{N}\sum_{i=1}^{k(N)}\left(\xi_{N,i}\right)^m-\tau((pqp)^m)\right|<\varepsilon_0$$

for all $1\le m\le m_0$ and $N\ge N_0$. We then conclude that if $N\ge N_0$ and $(P,Q)\in(G(N,k(N))\times G(N,l(N)))_0$ satisfies (3.3), then

$$\#\{i:\lambda_i(PQP)<\beta\}<\frac{1+\tau(E)}{\tau(E)}k(N)\eta^t, \tag{3.12}$$

$$\#\{i : \lambda_i(PQP) > 1-\beta\} < \frac{1+\tau(E)}{\tau(E)} k(N)\eta^t. \tag{3.13}$$

Inserting (3.12) and (3.13) into (3.9) we get

$$\begin{aligned} &\|V_{P,Q} - (I-P)QP(\sqrt{PQP(P-PQP)} + \alpha I)^{-1}\|_t^t \\ &\qquad \le \frac{1+\tau(E)}{\tau(E)}\eta^t + \frac{1}{2}\left(\frac{\alpha}{\sqrt{\beta(1-\beta)}}\right)^t. \end{aligned} \tag{3.14}$$

Finally, let $\alpha > 0$ be so small as $\alpha/\sqrt{\beta(1-\beta)} < \eta$, and choose a real polynomial $G(x)$ such that $|G(x) - (\sqrt{x(1-x)} + \alpha)^{-1}| < \eta$ for all $x \in [0,1]$. Then by (3.10) and (3.14) we obtain

$$\|v_{p,q} - (\mathbf{1}-p)qp \cdot G(pqp)\|_t < 2\eta$$

and

$$\|V_{P,Q} - (I-P)QP \cdot G(PQP)\|_t < \left(\left(\frac{1}{\tau(E)} + \frac{3}{2}\right)^{1/t} + 1\right)\eta.$$

The proof is completed if $\eta > 0$ was chosen so small as $\big((1/\tau(E) + 3/2)^{1/t} + 1\big)\eta < \varepsilon$. □

For the second step we present two more technical lemmas. The proof of the next lemma should be compared with that of [22, Lemma 4.1].

Lemma 3.5. *Let μ be a measure on $[0,1]$ with no atom at 0 and 1, and assume the conditions*

$$\iint_{(0,1)^2} \log|x-y|\, d\mu(x)\, d\mu(y) > -\infty, \tag{3.15}$$

$$\int_{(0,1)} \log x \, d\mu(x) > -\infty, \tag{3.16}$$

$$\int_{(0,1)} \log(1-x)\, d\mu(x) > -\infty. \tag{3.17}$$

If ψ is a continuous increasing function from $[0,1]$ onto itself with $\psi(0) = 0$ and $\psi(1) = 1$, then there exists a sequence of C^∞-diffeomorphisms ψ_j from $[0,1]$ onto itself with $\psi_j(0) = 0$ and $\psi_j(1) = 1$ such that

(i) *$\psi_j'(x) \ge 1/j$ for all $j \in \mathbb{N}$ and $x \in [0,1]$,*

(ii) *$\psi_j \longrightarrow \psi$ uniformly on $[0,1]$,*

(iii) $$\lim_{j\to\infty}\iint_{(0,1)^2}\log|x-y|\,d(\psi_{j*}\mu)(x)\,d(\psi_{j*}\mu)(y) = \iint_{(0,1)^2}\log|x-y|\,d(\psi_*\mu)(x)\,d(\psi_*\mu)(y),$$

(iv) $$\lim_{j\to\infty}\int_{(0,1)}\log x\,d(\psi_{j*}\mu)(x) = \int_{(0,1)}\log x\,d(\psi_*\mu)(x),$$

(v) $$\lim_{j\to\infty}\int_{(0,1)}\log(1-x)\,d(\psi_{j*}\mu)(x) = \int_{(0,1)}\log(1-x)\,d(\psi_*\mu)(x),$$

where $\psi_*\mu$ is the image measure of μ by ψ. Furthermore, even when conditions (3.16) *and/or* (3.17) *for μ are dropped, the conclusion holds without* (iv) *and/or* (v) *correspondingly.*

Proof. Extend ψ to a continuous increasing function on the whole $\mathbb{R}$ periodically, namely, $\psi(x+m)=\psi(x)+m$ for $x\in[0,1]$ and $m\in\mathbb{Z}$. For each $j\in\mathbb{N}$, by (3.15)–(3.17) one can choose $\delta_j\in(0,1/j]$ such that

$$\iint_{\{(x,y)\in(0,1)^2:|x-y|<\delta_j\}}\log|x-y|\,d\mu(x)\,d\mu(y)\ge -1/j, \tag{3.18}$$

$$\iint_{\{(x,y)\in(0,1)^2:|x-y|<\delta_j\}}d\mu(x)\,d\mu(y)\le 1/(j\log j), \tag{3.19}$$

$$\int_{(0,\delta_j)}\log x\,d\mu(x)\ge -1/j, \tag{3.20}$$

$$\mu((0,\delta_j))\le 1/(j\log j), \tag{3.21}$$

$$\int_{(1-\delta_j,1)}\log(1-x)\,d\mu(x)\ge -1/j, \tag{3.22}$$

$$\mu((1-\delta_j,1))\le 1/(j\log j). \tag{3.23}$$

We further choose a C^∞-function $\phi_j\ge 0$ supported in $[-1/j,1/j]$ with $\int\phi_j(x)\,dx=1$ such that $|(\psi*\phi_j)(x)-\psi(x)|\le\delta_j/2j$ for all $x\in[0,1]$, and define

$$\psi_j(x) := \frac{x}{j}+\left(1-\frac{1}{j}\right)((\psi*\phi_j)(x)-(\psi*\phi_j)(0))\quad\text{for } x\in[0,1].$$

Then one can immediately see that ψ_j is C^∞, $\psi_j(0)=0$, $\psi_j(1)=1$ and also (i) and (ii) are satisfied.

Notice that, for $x,y\in[0,1]$ with $|x-y|\ge\delta_j$,

$|\psi_j(x)-\psi_j(y)|$

$$
\begin{aligned}
&= \frac{|x-y|}{j} + \left(1-\frac{1}{j}\right)|(\psi*\phi_j)(x)-(\psi*\phi_j)(y)| \\
&\ge \frac{|x-y|}{j} \\
&+ \left(1-\frac{1}{j}\right)\big\{|\psi(x)-\psi(y)| - |(\psi*\phi_j)(x)-\psi(x)| - |\psi(y)-(\psi*\phi_j)(y)|\big\} \\
&\ge \left(1-\frac{1}{j}\right)|\psi(x)-\psi(y)|,
\end{aligned}
$$

and in particular

$$
\begin{aligned}
\psi_j(x) &\ge \left(1-\frac{1}{j}\right)\psi(x) && \text{for } x\in[\delta_j,1), \\
1-\psi_j(x) &\ge \left(1-\frac{1}{j}\right)(1-\psi(x)) && \text{for } x\in(0,1-\delta_j].
\end{aligned}
$$

Hence we have by (3.18) and (3.19)

$$
\begin{aligned}
&\iint_{(0,1)^2} \log|x-y|\,d(\psi_{j*}\mu)(x)d(\psi_{j*}\mu)(y) \\
&\quad\ge \iint_{\{(x,y)\in(0,1)^2:|x-y|<\delta_j\}} \log\frac{|x-y|}{j}\,d\mu(x)\,d\mu(y) \\
&\qquad + \iint_{\{(x,y)\in(0,1)^2:|x-y|\ge\delta_j\}} \log\left(\left(1-\frac{1}{j}\right)|\psi(x)-\psi(y)|\right)d\mu(x)\,d\mu(y) \\
&\quad\ge -\frac{2}{j} + \log\left(1-\frac{1}{j}\right) + \iint_{(0,1)^2}\log|x-y|\,d(\psi_*\mu)(x)\,d(\psi_*\mu)(y),
\end{aligned}
$$

and also we have by (3.20)–(3.23)

$$
\begin{aligned}
&\int_{(0,1)} \log x\,d(\psi_{j*}\mu)(x) \\
&\qquad\ge \int_{(0,\delta_j)} \log\frac{x}{j}\,d\mu(x) + \int_{[\delta_j,1)} \log\left(\left(1-\frac{1}{j}\right)\psi(x)\right)d\mu(x) \\
&\qquad\ge -\frac{2}{j} + \log\left(1-\frac{1}{j}\right) + \int_{(0,1)}\log\psi(x)\,d\mu(x),
\end{aligned}
$$

$$
\begin{aligned}
&\int_{(0,1)} \log(1-x)\,d(\psi_{j*}\mu)(x) \\
&\quad\ge \int_{(1-\delta_j,1)} \log\frac{1-x}{j}\,d\mu(x) + \int_{(0,1-\delta_j]} \log\left(\left(1-\frac{1}{j}\right)(1-\psi(x))\right)d\mu(x)
\end{aligned}
$$

$$\geq -\frac{2}{j} + \log\biggl(1 - \frac{1}{j}\biggr) + \int_{(0,1)} \log(1 - \psi(x))\, d\mu(x).$$

Therefore,

$$\begin{aligned}\liminf_{j\to\infty} \iint_{(0,1)^2} &\log|x-y|\, d(\psi_{j*}\mu)(x)\, d(\psi_{j*}\mu)(y) \\ &\geq \iint_{(0,1)^2} \log|x-y|\, d(\psi_*\mu)(x)\, d(\psi_*\mu)(y),\end{aligned}$$

$$\liminf_{j\to\infty} \int_{(0,1)} \log x\, d(\psi_{j*}\mu)(x) \geq \int_{(0,1)} \log x\, d(\psi_*\mu)(x),$$

$$\liminf_{j\to\infty} \int_{(0,1)} \log(1-x)\, d(\psi_{j*}\mu)(x) \geq \int_{(0,1)} \log(1-x)\, d(\psi_*\mu)(x).$$

On the other hand, Fatou's lemma says that the reverse inequalities of the above three ones with lim sup in place of lim inf actually hold true. Hence we have (iii)–(v). Finally, the above proof shows the last statement as well. □

Lemma 3.6. *Let μ be a measure on $[0,1]$ with no atom at 0 and 1, and let ψ be a continuous increasing function from $[0,1]$ into itself. Assume that μ satisfies conditions* (3.15)–(3.17) *in Lemma* 3.5 *and also $\psi_*\mu$ does* (3.16) *and* (3.17). *Then, there exists a sequence of continuous increasing functions ψ_m from $[0,1]$ onto itself with $\psi_m(0)=0$ and $\psi_m(1)=1$ such that*

(i) $\int_{(0,1)} |\psi_m(x) - \psi(x)|^2\, d\mu(x) \longrightarrow 0$,

(ii) $$\begin{aligned}\lim_{m\to\infty} \iint_{(0,1)^2} &\log|x-y|\, d(\psi_{m*}\mu)(x)\, d(\psi_{m*}\mu)(y) \\ &= \iint_{(0,1)^2} \log|x-y|\, d(\psi_*\mu)(x)\, d(\psi_*\mu)(y),\end{aligned}$$

(iii) $$\lim_{m\to\infty} \int_{(0,1)} \log x\, d(\psi_{m*}\mu)(x) = \int_{(0,1)} \log x\, d(\psi_*\mu)(x),$$

(iv) $$\lim_{m\to\infty} \int_{(0,1)} \log(1-x)\, d(\psi_{m*}\mu)(x) = \int_{(0,1)} \log(1-x)\, d(\psi_*\mu)(x).$$

Furthermore, even when conditions (3.16) *and/or* (3.17) *for μ and $\psi_*\mu$ are dropped, the conclusion holds without* (iii) *and/or* (iv) *correspondingly.*

Proof. We assume that both $\psi(0) > 0$ and $\psi(1) < 1$; the other cases can be handled easier. Condition (3.15) implies

$$(-\log m)\nu((0,1/m))^2 \geq \iint_{(0,1/m)^2} \log|x-y|\, d\nu(x)\, d\nu(y) \longrightarrow 0, \tag{3.24}$$

$$(3.25)\quad (-\log m)\nu((1-1/m,1))^2 \ge \iint_{(1-1/m,1)^2} \log|x-y|\,d\nu(x)\,d\nu(y) \longrightarrow 0$$

as $m \to \infty$. On the other hand, (3.16) and (3.17) imply

$$(3.26)\qquad (-\log m)\nu((0,1/m)) \ge \int_{(0,1/m)} \log x\,d\mu(x) \longrightarrow 0,$$

$$(3.27)\qquad (-\log m)\nu((1-1/m,1)) \ge \int_{(1-1/m,1)} \log(1-x)\,d\mu(x) \longrightarrow 0,$$

respectively. For each $m \ge 2$ define a function ψ_m on $[0,1]$ by

$$\psi_m(x) := \begin{cases} mx\psi(x) & (0 \le x < 1/m), \\ \psi(x) & (1/m \le x \le 1-1/m), \\ 1-m(1-x)(1-\psi(x)) & (1-1/m < x \le 1), \end{cases}$$

which is clearly continuous and increasing with $\psi_m(0) = 0$ and $\psi_m(1) = 1$. Then (i) immediately follows. It is easy to check the following:

$$|\psi_m(x)-\psi_m(y)| \ge \begin{cases} m\psi(0)|x-y| & \text{for } x,y \in (0,1/m), \\ m(1-\psi(1))|x-y| & \text{for } x,y \in (1-1/m,1), \\ |\psi(x)-\psi(y)| & \text{for other } x,y \in (0,1). \end{cases}$$

Hence we have

$$\begin{aligned}
&\iint_{(0,1)^2} \log|x-y|\,d(\psi_{m*}\mu)(x)\,d(\psi_{m*}\mu)(y) \\
&\quad \ge \iint_{(0,1/m)^2} \log(m\psi(0)|x-y|)\,d\mu(x)\,d\mu(y) \\
&\qquad + \iint_{(1-1/m,1)^2} \log(m(1-\psi(1))|x-y|)\,d\mu(x)\,d\mu(y) \\
&\qquad + \iint_{(0,1)^2} \log|\psi(x)-\psi(y)|\,d\mu(x)\,d\mu(y) \\
&\quad = (\log m + \log\psi(0))\mu((0,1/m))^2 + \iint_{(0,1/m)^2} \log|x-y|\,d\mu(x)\,d\mu(y) \\
&\qquad + (\log m + \log(1-\psi(1)))\mu((1-1/m,1))^2 \\
&\qquad + \iint_{(1-1/m,1)^2} \log|x-y|\,d\mu(x)\,d\mu(y) \\
&\qquad + \iint_{(0,1)^2} \log|\psi(x)-\psi(y)|\,d\mu(x)\,d\mu(y)
\end{aligned}$$

$$\longrightarrow \iint_{(0,1)^2} \log|\psi(x)-\psi(y)|\,d\mu(x)\,d\mu(y)$$

as $m\to\infty$ by (3.24), (3.25) and (3.15). Therefore,

$$\begin{aligned}\liminf_{m\to\infty} &\iint_{(0,1)^2} \log|x-y|\,d(\psi_{m*}\mu)(x)\,d(\psi_{m*}\mu)(y)\\ &\ge \iint_{(0,1)^2} \log|x-y|\,d(\psi_{m*}\mu)(x)\,d(\psi_{m*}\mu)(y).\end{aligned}$$

This together with Fatou's lemma implies (ii). On the other hand, by (3.16) for μ and $\psi_*\mu$ we have

$$\begin{aligned}&\int_{(0,1/m)} \log(mx\psi(x))\,d\mu(x)\\ &\quad= (\log m)\nu((0,1/m)) + \int_{(0,1/m)} \log x\,d\mu(x) + \int_{(0,1/m)} \log\psi(x)\,d\mu(x) \longrightarrow 0\end{aligned}$$

thanks to (3.26). Furthermore,

$$\begin{aligned}0 &\ge \int_{(1-1/m,1)} \log(1-m(1-x)(1-\psi(x)))\,d\mu(x)\\ &\ge \log\psi(1-1/m)\cdot\mu((1-1/m,1)) \longrightarrow 0.\end{aligned}$$

These imply (iii). Similarly, (iv) follows from (3.17) for μ and $\psi_*\mu$ thanks to (3.27). □

We are now in the final position to prove Theorem 3.2 in full generality.

Proof of Theorem 3.2. As mentioned before we may assume $n=1$, and write $p=p_1$, $q=q_1$ and $\psi=\psi_1$. We may further assume that $\chi_{\mathrm{proj}}(p,q(\psi;p)) > -\infty$ as well as $\chi_{\mathrm{proj}}(p,q) > -\infty$; otherwise, both sides of the inequality are $-\infty$ thanks to Proposition 1.2 (ii). By Proposition 2.1 both ν and $\psi_*\nu$ satisfy condition (3.15); moreover they satisfy (3.16) unless $\tau(p)=\tau(q)$ and also (3.17) unless $\tau(p)=\tau(\mathbf{1}-q)$. In each case where those equalities of traces occur or not, we choose a sequence ψ_m correspondingly as mentioned in Lemma 3.6. Since

$$\|p\psi_m(pqp)p - p\psi(pqp)p\|_2^2 = \int_{(0,1)} |\psi_m(x)-\psi(x)|^2\,d\nu(x) \longrightarrow 0,$$

we get $p\psi_m(pqp)p \to p\psi(pqp)p$ strongly so that $q(\psi_m;p)\to q(\psi;p)$ strongly as $m\to\infty$ due to the definition of $q(\psi;p)$. By Propositions 1.2 (iii) and 2.1 we see that it suffices to prove the inequality in the case where $\psi(0)=0$ and $\psi(1)=1$.

The same argument by using Lemma 3.5 in turn enables us to reduce the proof to Lemma 3.3, and the proof of the inequality is now completed.

To prove the equality of the last statement, let ψ be a strictly increasing function on $(0,1)$ and define $\tilde{\psi}$ on $[0,1]$ by

$$\tilde{\psi}(x) := \begin{cases} 0 & (0 \le x \le \psi(0+)), \\ \psi^{-1}(x) & (\psi(0+) < x < \psi(1-)), \\ 1 & (\psi(1-) \le x \le 1). \end{cases}$$

Furthermore, set $\tilde{q} := q(\psi; p)$ and $\tilde{\nu} := \psi_* \nu$. Then it is clear that $\tilde{\nu}$ is the measure corresponding to the pair $(p, \tilde{q})$ so that $\nu = \tilde{\psi}_* \tilde{\nu}$ and $q = \tilde{q}(\tilde{\psi}; p)$. Hence the inequality established above can be applied to $(p, \tilde{q})$ and $\tilde{\psi}$ too, and we have the reversed inequality as well. □

§4. Additivity and Freeness

In this section, we prove the next additivity theorem asserting that the pair-block freeness of projections is characterized by the additivity of their free entropy. For the projection version of free entropy we have no counterpart of the so-called infinitesimal change of variable formula in [22, Proposition 1.3], and hence we need to find another route to proving that the additivity implies the freeness.

Theorem 4.1. *Let $p_1, q_1, \dots, p_n, q_n, r_1, \dots, r_{n'}$ be projections in $(\mathcal{M}, \tau)$.*

(1) *If $\{p_1, q_1\}, \dots, \{p_n, q_n\}, \{r_1\}, \dots, \{r_{n'}\}$ are free, then*

$$\chi_{\mathrm{proj}}(p_1, q_1, \dots, p_n, q_n, r_1, \dots, r_{n'}) = \chi_{\mathrm{proj}}(p_1, q_1) + \cdots + \chi_{\mathrm{proj}}(p_n, q_n).$$

(2) *Conversely, if $\chi_{\mathrm{proj}}(p_i, q_i) > -\infty$ for $1 \le i \le n$ and equality holds in (1), then $\{p_1, q_1\}, \dots, \{p_n, q_n\}, \{r_1\}, \dots, \{r_{n'}\}$ are free.*

(3) *In particular, $\chi_{\mathrm{proj}}(p_1, \dots, p_n) = 0$ if and only if $p_1, \dots, p_n$ are free.*

Proof. (1) It suffices to prove the following two assertions:

(a) If $\{p, q\}$ and $\{p_1, \dots, p_n\}$ are free, then

$$\chi_{\mathrm{proj}}(p, q, p_1, \dots, p_n) = \chi_{\mathrm{proj}}(p, q) + \chi_{\mathrm{proj}}(p_1, \dots, p_n).$$

(b) If $\{p\}$ and $\{p_1, \dots, p_n\}$ are free, then

$$\chi_{\mathrm{proj}}(p, p_1, \dots, p_n) = \chi_{\mathrm{proj}}(p_1, \dots, p_n).$$

Since the proofs of these are identical, we give only that of (a), which is essentially same as in [21, 24] (see also [11, pp. 269–272]).

To prove (a), we may and do assume that $\chi_{\mathrm{proj}}(p,q) > -\infty$ and $\chi_{\mathrm{proj}}(p_1,\dots,p_n) > -\infty$. Choose $k(N), l(N), k_i(N) \in \{0,1,\dots,N\}$ for each $N \in \mathbb{N}$ and $1 \le i \le n$ so that $k(N)/N \to \tau(p)$, $l(N)/N \to \tau(q)$ and $k_i(N)/N \to \tau(p_i)$ as $N \to \infty$. We set

$$\begin{aligned}\Omega_N(m,\varepsilon) &:= \Gamma_{\mathrm{proj}}(p,q;k(N),l(N);N,m,\varepsilon)\\ &\qquad \times \Gamma_{\mathrm{proj}}(p_1,\dots,p_n;k_1(N),\dots,k_n(N);N,m,\varepsilon),\\ \Theta_N(m,\varepsilon) &:= \Gamma_{\mathrm{proj}}(p,q,p_1,\dots,p_n;k(N),l(N),k_1(N),\dots,k_n(N);N,m,\varepsilon).\end{aligned}$$

Given $m \in \mathbb{N}$ and $\varepsilon > 0$ one can show as in [11, 6.4.3] that there exists $\varepsilon_1 > 0$ such that

$$\lim_{N\to\infty} \frac{\gamma_N(\Omega_N(m,\varepsilon_1)\cap\Theta_N(m,\varepsilon))}{\gamma_N(\Omega_N(m,\varepsilon_1))} = 1,$$

where $\gamma_N := \gamma_{G(N,k(N))} \otimes \gamma_{G(N,l(N))} \otimes \gamma_{\vec{k}(N)}$ and $\gamma_{\vec{k}(N)} := \bigotimes_{i=1}^n \gamma_{G(N,k_i(N))}$. Hence we have

$$\begin{aligned}&\limsup_{N\to\infty} \frac{1}{N^2}\log\gamma_N(\Theta_N(m,\varepsilon))\\ &\quad\ge \limsup_{N\to\infty}\frac{1}{N^2}\log\gamma_N(\Omega_N(m,\varepsilon_1))\\ &\quad= \lim_{N\to\infty}\frac{1}{N^2}\log\bigl(\gamma_{G(N,k(N))}\otimes\gamma_{G(N,l(N))}\bigr)\bigl(\Gamma_{\mathrm{proj}}(p,q;k(N),l(N);N,m,\varepsilon_1)\bigr)\\ &\qquad\quad + \limsup_{N\to\infty}\frac{1}{N^2}\log\gamma_{\vec{k}(N)}\bigl(\Gamma_{\mathrm{proj}}(p_1,\dots,p_n;k_1(N),\dots,k_n(N);N,m,\varepsilon_1)\bigr)\\ &\quad\ge \chi_{\mathrm{proj}}(p,q) + \chi_{\mathrm{proj}}(p_1,\dots,p_n).\end{aligned}$$

Here the above equality is due to [12, Proposition 3.3]. Therefore,

$$\chi_{\mathrm{proj}}(p,q,p_1,\dots,p_n) \ge \chi_{\mathrm{proj}}(p,q) + \chi_{\mathrm{proj}}(p_1,\dots,p_n),$$

and the reverse inequality is Proposition 1.2 (ii).

(3) will be proven in Corollary 5.7 of the next section as a consequence of a transportation cost inequality for projection multi-variables.

(2) We may assume that $p_1, q_1, \dots, p_n, q_n$ are all non-zero. For $1 \le i \le n$ let ν_i be the measure on $(0,1)$ corresponding to the pair (p_i, q_i) (see §2). Since ν_i is non-atomic by the assumption $\chi_{\mathrm{proj}}(p_i,q_i) > -\infty$, one can choose a continuous increasing function ψ_i from $(0,1)$ into itself such that $\psi_{i*}\nu_i$ is equal to (2.4) with $\alpha = \tau(p_i)$ and $\beta = \tau(q_i)$. Consider $q_i(\psi_i;p_i)$ constructed

from (p_i, q_i) and ψ_i (see §3). Since $\psi_{i*}\nu_i$ corresponds to the pair $(p_i, q_i(\psi_i; p_i))$, we get $\chi_{\rm proj}(p_i, q_i(\psi_i; p_i)) = 0$. Therefore, by Theorem 3.2 and the additivity assumption, we have

$$\begin{aligned}&\chi_{\rm proj}(p_1, q_1(\psi_1; p_1), \dots, p_n, q_n(\psi_n; p_n), r_1, \dots, r_{n'})\\ &\qquad \geq \chi_{\rm proj}(p_1, q_1, \dots, p_n, q_n, r_1, \dots, r_{n'}) - \sum_{i=1}^{n} \chi_{\rm proj}(p_i, q_i) = 0.\end{aligned}$$

This implies by (3) that $p_1, q_1(\psi_1; p_1), \dots, p_n, q_n(\psi_n; p_n), r_1, \dots, r_{n'}$ are free. Since ν_i and $\psi_{i*}\nu_i$ are non-atomic, it is plain to see that $\{p_i, q_i\}'' = \{p_i, q_i(\psi_i; p_i)\}''$ for $1 \leq i \leq n$. Hence the freeness of $\{p_1, q_1\}, \dots, \{p_n, q_n\}, \{r_1\}, \dots, \{r_{n'}\}$ is obtained. □

§5. Asymptotic Freeness and Free Transportation Cost Inequality

The aim of this section is to prove a transportation cost inequality for tracial distributions of projection multi-variables. To do so, we first present an asymptotic freeness result for random projection matrices generalizing Voiculescu's result in [19].

§5.1. Asymptotic freeness for random projection matrices

Let $\big(\{P(s,N), Q(s,N)\}\big)_{s\in S}$ be an independent family of pairs of $N \times N$ random projection matrices, and let $k(s,N)$, $l(s,N)$, $n_{11}(s,N)$, $n_{10}(s,N)$, $n_{01}(s,N)$ and $n_{00}(s,N)$ denote the ranks of $P(s,N)$, $Q(s,N)$, $P(s,N)\wedge Q(s,N)$, $P(s,N) \wedge Q(s,N)^\perp$, $P(s,N)^\perp \wedge Q(s,N)$, $P(s,N)^\perp \wedge Q(s,N)^\perp$, respectively. For each $s \in S$ we assume the following:

(1) $k(s,N)$, $l(s,N)$ and $n_{ij}(s,N)$'s are constant almost surely and $k(s,N)/N$, $l(s,N)/N$ and $n_{ij}(s,N)/N$ converge as $N \to \infty$.

(2) The joint distribution of $(P(s,N), Q(s,N))$ is invariant under unitary conjugation $(P,Q) \mapsto (UPU^*, UQU^*)$ for $U \in \mathrm{U}(N)$.

(3) For each $s \in S$ the distribution measure of $P(s,N)Q(s,N)P(s,N)$ with respect to $N^{-1}\mathrm{Tr}_N$ converges almost surely to a (non-random) measure on $[0,1]$ in the weak topology as $N \to \infty$.

Let $(R(s',N))_{s'\in S'}$ be an independent family of $N \times N$ random projection matrices, also independent of $\big(\{P(s,N), Q(s,N)\}\big)_{s\in S}$, and assume that each $R(s',N)$ is distributed under the measure $\gamma_{G(N,k(s',N))}$ on $G(N, k(s',N))$ with

$0 \le k(s', N) \le N$ such that $k(s', N)/N$ converges. Finally, let $(D(t, N))_{t\in T}$ be a family of $N \times N$ constant matrices such that $\sup_N \|D(t, N)\|_\infty < +\infty$ for each $t \in T$ and $(D(t, N), D(t, N)^*)_{t\in T}$ has the limit distribution. In this setup, we have the following asymptotic freeness result for random projection matrices generalizing [19, Theorem 3.11].

Theorem 5.1. *With the above notations and assumptions the family*

$$\Big(\big(\{P(s,N), Q(s,N)\}\big)_{s\in S}, \big(R(s',N)\big)_{s'\in S'}, \big\{D(t,N), D(t,N)^* : t \in T\big\}\Big)$$

is asymptotically free almost surely as $N \to \infty$.

Proof. Set $n(s, N) := \big(N - \sum_{i,j=0}^{1} n_{ij}(s, N)\big)/2$. By assumption (1), $n(s, N)$ is constant almost surely and $n(s, N)/N$ converges as $N \to \infty$. As before, the sine-cosine decomposition of two projections enables us to represent

$$P(s,N) = U(s,N)\left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \oplus I \oplus I \oplus 0 \oplus 0\right) U(s,N)^*,$$

$$Q(s,N) = U(s,N)\left(\begin{bmatrix} X & \sqrt{X(I-X)} \\ \sqrt{X(I-X)} & I - X \end{bmatrix} \oplus I \oplus 0 \oplus I \oplus 0\right) U(s,N)^*$$

in $\mathbb{C}^N = (\mathbb{C}^{n(s,N)} \otimes \mathbb{C}^2) \oplus \mathbb{C}^{n_{11}(s,N)} \oplus \mathbb{C}^{n_{10}(s,N)} \oplus \mathbb{C}^{n_{01}(s,N)} \oplus \mathbb{C}^{n_{00}(s,N)}$, where $U(s, N)$ is a random unitary matrix and $X = X(s, N)$ is a diagonal matrix whose diagonal entries are $0 \le x_1(s, N) \le x_2(s, N) \le \cdots \le x_{n(s,N)}(s, N) \le 1$. Also, we can represent

$$R(s', N) = U(s', N) P_{k(s',N)} U(s', N)^*$$

for each $s' \in S'$, where $U(s', N)$ is a unitary random matrix and $P_{k(s',N)}$ the diagonal matrix whose first $k(s', N)$ entries are 1 and the others 0. As in the proof of [11, 4.3.5] we can assume that $(U(s, N))_{s\in S} \sqcup (U(s', N))_{s'\in S'}$ forms an independent family of standard unitary matrices thanks to independence and assumption (2). We fix $s \in S$ and assume $\lim_{N\to\infty} n_0(s, N)/N > 0$. (When $n(s, N)/N \to 0$ the discussion below becomes rather trivial.) Write $A(s, N)$ and $B(s, N)$ for the matrices appearing inside $\mathrm{Ad}\, U(s, N)$ in the above representation of $P(s, N)$ and $Q(s, N)$, that is, $A(s, N) = U(s, N)^* P(s, N) U(s, N)$ and $B(s, N) = U(s, N)^* Q(s, N) U(s, N)$. By assumption (3) one observes that the empirical distribution $n(s, N)^{-1} \sum_{i=1}^{n(s,N)} \delta_{x_i(s,N)}$ converges to a measure ρ_s on $[0, 1]$ weakly in the almost sure sense as $N \to \infty$. Choose (non-random) $0 \le \xi_1(s, N) \le \cdots \le \xi_{n(s,N)}(s, N) \le 1$ in such a way that $n(s, N)^{-1} \sum_{i=1}^{n(s,N)} \delta_{\xi_i(s,N)}$

converges to ρ_s weakly as $N \to \infty$. Let $\Xi(s,N)$ be the diagonal matrix with diagonal entries $\xi_1(s,N), \dots, \xi_{n(s,N)}(s,N)$ and define

$$C(s,N) := \begin{bmatrix} \Xi(s,N) & \sqrt{\Xi(s,N)(I-\Xi(s,N))} \\ \sqrt{\Xi(s,N)(I-\Xi(s,N))} & I-\Xi(s,N) \end{bmatrix} \oplus I \oplus 0 \oplus I \oplus 0.$$

By [11, 4.3.4] we then have

$$\lim_{N\to\infty} \|X(s,N) - \Xi(s,N)\|_{p,n(s,N)^{-1}\mathrm{Tr}_{n(s,N)}} = 0 \text{ almost surely for all } p \ge 1$$

so that for any polynomial F

$$\lim_{N\to\infty} \|F(B(s,N)) - F(C(s,N))\|_{p,N^{-1}\mathrm{Tr}_N} = 0 \text{ almost surely for all } p \ge 1.$$

Moreover, note that $(A(s,N), C(s,N))$ has the limit distribution for each $s \in S$.

Under the above preparations, the remaining proof is similar to that of [11, 4.3.5] as sketched below. We may assume that $\{(D(t,N))_{N\in\mathbb{N}} : t \in T\}$ forms a $*$-subalgebra of $\prod_{N\in\mathbb{N}} M_N(\mathbb{C})$. What we have to prove is

$$\lim_{N\to\infty} \frac{1}{N}\mathrm{Tr}_N\big(Y_1(N)Y_2(N)\cdots Y_m(N)\big) = 0 \text{ almost surely}$$

if $(Y_i(N))_{N\in\mathbb{N}}$, $1 \le i \le m$, are one of the following:

(a) $Y_i(N) = F_i(P(s_i,N), Q(s_i,N)) = U(s_i,N)F_i(A(s_i,N), B(s_i,N))U(s_i,N)^*$ with $s_i \in S$ and a noncommutative polynomial F_i such that $N^{-1}\mathrm{Tr}_N\big(F_i(A(s_i,N), B(s_i,N))\big) \to 0$ almost surely,

(b) $Y_i(N) = \alpha_i R(s_i',N) + \beta_i I = U(s_i',N)(\alpha_i P_{k(s_i',N)} + \beta_i I)U(s_i',N)^*$ with $s_i' \in S'$ and $\alpha_i, \beta_i \in \mathbb{C}$ such that $\alpha_i N^{-1}k(s_i',N) + \beta_i \to 0$,

(c) $Y_i(N) = D(t_i,N)$ with $t_i \in T$ and $N^{-1}\mathrm{Tr}_N(D(t_i,N)) \to 0$,

where each neighboring $Y_i(N)$ and $Y_{i+1}(N)$ are of the different forms (a)–(c) or of the same form (a) with $s_i \ne s_{i+1}$ or of the same form (b) with $s_i' \ne s_{i+1}'$. Set $Z_i(N) := U(s_i,N)F_i(A(s_i,N), C(s_i,N))U(s_i,N)^*$ if $Y_i(N)$ is of the form (a) (that is, $B(s_i,N)$ in the form (a) is replaced by the corresponding $C(s_i,N)$ contructed above), and otherwise $Z_i(N) := Y_i(N)$. By the Hölder inequality one has

$$\left|\frac{1}{N}\mathrm{Tr}_N(Y_1(N)\cdots Y_m(N)) - \frac{1}{N}\mathrm{Tr}_N(Z_1(N)\cdots Z_m(N))\right| \longrightarrow 0 \text{ almost surely.}$$

Then one gets

$$\lim_{N\to\infty} \frac{1}{N}\mathrm{Tr}_N(Z_1(N)\cdots Z_m(N)) = 0 \text{ almost surely}$$

(see [11, 4.3.1]). Here the existence of the limit joint distribution of three different families $\{(A(s,N),C(s,N))\}_{s\in S}$, $\{P_{k(s',N)}\}_{s'\in S'}$ and $\{D(t,N)\}_{t\in T}$ together is unnecessary (and not assumed here), because any two constant matrices from those different families appearing in the string $Z_1(N)\cdots Z_m(N)$ are separated by one of $U(s,N)$, $U(s,N)^*$, $U(s',N)$ and $U(s',N)^*$. The proof is now completed. □

§5.2. Free transportation cost inequality for projections

Let $\mathcal{A}_{\mathrm{proj}}^{(2n+n')}$ be the universal free product C^*-algebra of $2n+n'$ copies of $C^*(\mathbb{Z}_2)=\mathbb{C}\oplus\mathbb{C}$, and denote the canonical $2n+n'$ generators of projections by $e_1,f_1,\dots,e_n,f_n,e'_1,\dots,e'_{n'}$. For a given $2n+n'$-tuple $\vec{P}=(P_1,Q_1,\dots,P_n,Q_n,R_1,\dots,R_{n'})$ of projections in $M_N(\mathbb{C})$, there is a unique $*$-homomorphism from $\mathcal{A}_{\mathrm{proj}}^{(2n+n')}$ into $M_N(\mathbb{C})$ sending e_i,f_i,e'_j to P_i,Q_i,R_j, respectively, which we denote by $h\in\mathcal{A}_{\mathrm{proj}}^{(2n+n')}\mapsto h(\vec{P})\in M_N(\mathbb{C})$. For $\vec{k}:=(k_1,l_1,\dots,k_n,l_n,k'_1,\dots,k'_{n'})\in\{0,1,\dots,N\}^{2n+n'}$, denote by $G(N,\vec{k})$ the product $\prod_{i=1}^n\big(G(N,k_i)\times G(N,l_i)\big)\times\prod_{j=1}^{n'}G(N,k'_j)$ of Grassmannian manifolds, and by $\mathcal{P}\big(G(N,\vec{k})\big)$ the set of Borel probability measures on $G(N,\vec{k})$. Note that each $\lambda\in\mathcal{P}\big(G(N,\vec{k})\big)$ clearly gives rise to the unique tracial state $\hat{\lambda}$ on $\mathcal{A}_{\mathrm{proj}}^{(2n+n')}$ defined by

$$\hat{\lambda}(h):=\int\frac{1}{N}\mathrm{Tr}_N\big(h(\vec{P})\big)\,d\lambda(\vec{P})\quad\text{for } h\in\mathcal{A}_{\mathrm{proj}}^{(2n+n')}.$$

Let us denote by $TS\big(\mathcal{A}_{\mathrm{proj}}^{(2n+n')}\big)$ the set of tracial states on $\mathcal{A}_{\mathrm{proj}}^{(2n+n')}$. Moreover, we denote by $TS_{\vec{\alpha}}(\mathcal{A}_{\mathrm{proj}}^{(2n+n')})$ with $\vec{\alpha}:=(\alpha_1,\beta_1,\dots,\alpha_n,\beta_n,\alpha'_1,\dots,\alpha'_{n'})\in[0,1]^{2n+n'}$ the set of $\tau\in TS\big(\mathcal{A}_{\mathrm{proj}}^{(2n+n')}\big)$ such that $\tau(e_i)=\alpha_i$, $\tau(f_i)=\beta_i$ and $\tau(e'_j)=\alpha'_j$. Following [4], we define the (free probabilistic) Wasserstein distance $W_{2,\mathrm{free}}(\tau_1,\tau_2)$ between $\tau_1,\tau_2\in TS\big(\mathcal{A}_{\mathrm{proj}}^{(2n+n')}\big)$ to be the infimum of

$$\sqrt{\tau\left(\sum_{i=1}^n\big(|\sigma_1(e_i)-\sigma_2(e_i)|^2+|\sigma_1(f_i)-\sigma_2(f_i)|^2\big)+\sum_{j=1}^{n'}|\sigma_1(e'_j)-\sigma_2(e'_j)|^2\right)}$$

over all $\tau\in TS\big(\mathcal{A}_{\mathrm{proj}}^{(2n+n')}\bigstar\mathcal{A}_{\mathrm{proj}}^{(2n+n')}\big)$ with $\tau\circ\sigma_1=\tau_1$, $\tau\circ\sigma_2=\tau_2$, where σ_1 and σ_2 stand for the canonical embedding maps of $\mathcal{A}_{\mathrm{proj}}^{(2n+n')}$ into the left and right copies in $\mathcal{A}_{\mathrm{proj}}^{(2n+n')}\bigstar\mathcal{A}_{\mathrm{proj}}^{(2n+n')}$, respectively. The next lemma will be one of the keys in proving a free transportation cost inequality.

Lemma 5.2. *For each pair $\lambda_1, \lambda_2 \in \mathcal{P}\big(G(N,\vec{k})\big)$ we have*

$$W_{2,\mathrm{free}}(\hat{\lambda}_1, \hat{\lambda}_2) \le \frac{1}{\sqrt{N}} W_{2,HS}(\lambda_1, \lambda_2) \le \frac{1}{\sqrt{N}} W_{2,d}(\lambda_1, \lambda_2).$$

Here, $W_{2,HS}$ and $W_{2,d}$ are the usual Wasserstein distances determined by the Hilbert-Schmidt norm $\|P-Q\|_{HS}$ and the geodesic distance $d(P,Q)$ with respect to the Riemannian metric induced from Tr_N, respectively.

Proof. The first inequality is shown in the same way as in [14, Lemma 1.3], while the second immediately follows from the inequality $\|P-Q\|_{HS} \le d(P,Q)$ (see e.g. [9, Appendix B]). □

Let $\vec{\alpha} \in [0,1]^{2n+n'}$ and $\vec{k}(N) = (k_1(N), l_1(N), \dots, k_n(N), l_n(N), k'_1(N), \dots, k'_{n'}(N)) \in \{0,1,\dots,N\}^{2n+n'}$ for $N \in \mathbb{N}$ be given so that $\vec{k}(N)/N \to \vec{\alpha}$ as $N \to \infty$. The free entropy $\chi_{\mathrm{proj}}(\tau)$ for $\tau \in TS_{\vec{\alpha}}\big(\mathcal{A}_{\mathrm{proj}}^{(2n+n')}\big)$ is defined as follows. We denote by $\Gamma_{\mathrm{proj}}(\tau; \vec{k}(N); N, m, \varepsilon)$ the set of all $2n+n'$-tuples $\vec{P} \in G(N, \vec{k}(N))$ such that

$$\left| \frac{1}{N} \mathrm{Tr}_N\big(h(\vec{P})\big) - \tau(h) \right| < \varepsilon$$

for all monomials $h \in \mathcal{A}_{\mathrm{proj}}^{(2n+n')}$ in $e_1, f_1, \dots, e_n, f_n, e'_1, \dots, e'_{n'}$ of degree at most m. We then define

$$\chi_{\mathrm{proj}}(\tau) := \lim_{\substack{m\to\infty \\ \varepsilon\searrow 0}} \limsup_{N\to\infty} \frac{1}{N^2} \log \gamma_{\vec{k}(N)}\big(\Gamma_{\mathrm{proj}}(\tau; \vec{k}(N); N, m, \varepsilon)\big),$$

where $\gamma_{\vec{k}(N)} := \bigotimes_{i=1}^{n} \big(\gamma_{G(N,k_i(N))} \otimes \gamma_{G(N,l_i(N))}\big) \otimes \bigotimes_{j=1}^{n'} \gamma_{G(N,k'_j(N))}$ on $G(N, \vec{k}(N))$. Note that the quantity $\chi_{\mathrm{proj}}(\tau)$ coincides with $\chi_{\mathrm{proj}}(p_1, q_1, \dots, p_n, q_n, r_1, \dots, r_{n'})$ when $p_i := \pi_\tau(e_i)$, $q_i := \pi_\tau(f_i)$ and $r_j := \pi_\tau(e'_j)$ in the GNS representation of $\mathcal{A}_{\mathrm{proj}}^{(2n+n')}$ associated with τ; hence it is independent of the particular choice of $\vec{k}(N)$ due to Proposition 1.1.

In what follows, let $\tau \in TS_{\vec{\alpha}}\big(\mathcal{A}_{\mathrm{proj}}^{(2n+n')}\big)$ be arbitrarily fixed. Then one can choose a subsequence $N_1 < N_2 < \cdots$ so that

$$\chi_{\mathrm{proj}}(\tau) = \lim_{m\to\infty} \frac{1}{N_m^2} \log \gamma_{\vec{k}(N_m)}\big(\Gamma_{\mathrm{proj}}(\tau; \vec{k}(N_m); N_m, m, 1/m)\big). \tag{5.1}$$

Set $\Gamma_{N_m} := \Gamma_{\mathrm{proj}}(\tau; \vec{k}(N_m); N_m, m, 1/m)$ and define $\lambda_{N_m}^{\tau} \in \mathcal{P}\big(G(N_m, \vec{k}(N_m))\big)$ by

$$d\lambda_{N_m}^{\tau}(\vec{P}) := \frac{1}{\gamma_{\vec{k}(N_m)}(\Gamma_{N_m})} \mathbf{1}_{\Gamma_{N_m}}(\vec{P})\, d\gamma_{\vec{k}(N_m)}(\vec{P}).$$

Here the next lemma can be easily proven as in the proof of [14, Eq. (2.5)].

Lemma 5.3. $\lim_{m\to\infty} \hat{\lambda}^{\tau}_{N_m} = \tau$ *in the weak* topology.*

For $1 \le i \le n$ the C^*-subalgebra generated by e_i, f_i (obviously identified with $\mathcal{A}^{(2)}_{\mathrm{proj}} = C^*(\mathbb{Z}_2 \bigstar \mathbb{Z}_2)$) is isomorphic to

$$\mathcal{A} := \left\{ a(\cdot) = [a_{ij}(\cdot)]^2_{i,j=1} \in C([0,1]; M_2(\mathbb{C})) : a(0),\, a(1) \text{ are diagonals} \right\}$$

by

$$e_i \mapsto e(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad f_i \mapsto f(t) = \begin{bmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{bmatrix}.$$

Under this isomorphism, any tracial state on $\mathcal{A}$ is written as

$$\begin{aligned} \tau_{\nu,\{\alpha_{ij}\}}(a) := {} & \alpha_{10} a_{11}(0) + \alpha_{01} a_{22}(0) + \alpha_{11} a_{11}(1) + \alpha_{00} a_{22}(1) \\ & + \int_{(0,1)} \frac{1}{2} \mathrm{Tr}_2(a(t))\, d\nu(t), \end{aligned}$$

where $\alpha_{ij} \ge 0$, $\sum_{i,j=0}^1 \alpha_{ij} \le 1$ and ν is a measure on $(0,1)$ with $\nu((0,1)) = 1 - \sum_{i,j=0}^1 \alpha_{ij}$. Let $\vec{\psi} = (\psi_1, \dots, \psi_n)$ be an n-tuple of continuous functions on $[0,1]$, and define the probability distribution $\lambda^{\psi_i}_N$ on $G(N, k_i(N)) \times G(N, l_i(N))$ by

$$d\lambda^{\psi_i}_N(P,Q) := \frac{1}{Z^{\psi_i}_N} \exp\bigl(-N \mathrm{Tr}_N(\psi_i(PQP))\bigr)\, d(\gamma_{G(N,k_i(N))} \otimes \gamma_{G(N,l_i(N))})(P,Q)$$

with the normalization constant $Z^{\psi_i}_N$. For $\tau_{\nu,\{\alpha_{ij}\}} \in TS_{(\alpha_i,\beta_i)}\bigl(\mathcal{A}^{(2)}_{\mathrm{proj}}\bigr)$ one has

$$\begin{aligned} & \chi_{\mathrm{proj}}(\tau_{\nu,\{\alpha_{ij}\}}) - \tau_{\nu,\{\alpha_{ij}\}}(\psi_i(efe)) \\ & = \frac{1}{4}\Sigma(\nu) + \frac{1}{2} \int_{(0,1)} \bigl((\alpha_{01} + \alpha_{10}) \log t + (\alpha_{00} + \alpha_{11}) \log(1-t) - \psi_i(t)\bigr)\, d\nu(t) - C \end{aligned}$$

with some constant C if $\alpha_{00}\alpha_{11} = \alpha_{01}\alpha_{10} = 0$, and otherwise $-\infty$. Thus, a general result on weighted logarithmic energy (see [18]) ensures that there is a unique maximizer $\tau^{\psi_i}_{(\alpha_i,\beta_i)} \in TS_{(\alpha_i,\beta_i)}\bigl(\mathcal{A}^{(2)}_{\mathrm{proj}}\bigr)$ of the functional $\tau \in TS_{(\alpha_i,\beta_i)}\bigl(\mathcal{A}^{(2)}_{\mathrm{proj}}\bigr) \mapsto \chi_{\mathrm{proj}}(\tau) - \tau(\psi_i(efe))$. Then, we define the tracial state $\tau^{\vec{\psi}}_{\vec{\alpha}} \in TS\bigl(\mathcal{A}^{(2n+n')}_{\mathrm{proj}}\bigr)$ by

$$\tau^{\vec{\psi}}_{\vec{\alpha}} := \left(\bigstar_{i=1}^n \tau^{\psi_i}_{(\alpha_i,\beta_i)} \right) \bigstar \tau_{(\alpha'_1,\dots,\alpha'_{n'})}, \quad \tau_{(\alpha'_1,\dots,\alpha'_{n'})} := \bigstar_{j=1}^{n'} \bigl(\alpha'_j \delta_0 + (1-\alpha'_j)\delta_1\bigr)$$

in the natural identification $\mathcal{A}^{(2n+n')}_{\mathrm{proj}} = \left(\bigstar_{i=1}^n \mathcal{A}^{(2)}_{\mathrm{proj}} \right) \bigstar \left(\bigstar_{j=1}^{n'} C^*(\mathbb{Z}_2) \right)$. Furthermore, we define the joint distribution

$$\lambda^{\vec{\psi}}_N := \left(\bigotimes_{i=1}^n \lambda^{\psi_i}_N \right) \otimes \left(\bigotimes_{j=1}^{n'} \gamma_{G(N,k'_j(N))} \right) \quad \text{on } G(N, \vec{k}(N))$$

(also considered as a $2n+n'$-tuple of random projection matrices). The next lemma follows from a large deviation result for two projection matrices as in [12] and Theorem 5.1.

Lemma 5.4.

(1) $B_{\psi_i} := \lim_{N\to\infty} \frac{1}{N^2} \log Z_N^{\psi_i}$ *exists for every* $1 \le i \le n$.

(2) $\lim_{N\to\infty} \hat{\lambda}_N^{\vec{\psi}} = \tau_{\vec{\alpha}}^{\vec{\psi}}$ *in the weak* topology.*

Proof. When $e_1, f_1, \dots, e_n, f_n$ disappear and only $e'_1, \dots, e'_{n'}$ appear, we have nothing to do for (1) and moreover (2) immediately follows from Voiculescu's original result [19, Theorem 3.11] rather than Theorem 5.1 as follows. Let $R_1(N), \dots, R_{n'}(N)$ be an independent family of random projection matrices of ranks $k'_j(N)$ distributed under $\gamma_{G(N,k'_j(N))}$, respectively. Note that $\lambda_N^{\vec{\psi}}$ in this case coincides with $\gamma_N := \bigotimes_{j=1}^{n'} \gamma_{G(N,k'_j(N))}$. For a monomial $h = r_{j_1} \cdots r_{j_m} \in \mathcal{A}_{\mathrm{proj}}^{(n')}$ one has

$$\hat{\gamma}_N(h) = \mathbb{E} \circ \left(\frac{1}{N} \mathrm{Tr}_N \right) (R_{j_1}(N) \cdots R_{j_m}(N)),$$

which converges to $\tau_{(\alpha'_1,\dots,\alpha'_{n'})}(r_{j_1} \cdots r_{j_m})$ thanks to [19, Theorem 3.11]. This immediately implies (2) in this special case.

For the general case, i.e., when $e_1, f_1, \dots, e_n, f_n$ really appear, we need to show (1) and $\lim_{N\to\infty} \hat{\lambda}_N^{\psi_i} = \tau_{\vec{\alpha}}^{\psi_i}$ weakly* for each $1 \le i \le n$. Both are simple applications of the large deviation result for the empirical eigenvalue distribution of two random projection matrix pair $(P(N), Q(N))$ distributed under $\lambda_N^{\psi_i}$, whose proof is essentially same as in [12, Proposition 2.1] (or in the proof of [15, Theorem 2.1]). Once the latter convergence was established, the above argument would equally work well even in the general setting when [19, Theorem 3.11] is replaced by Theorem 5.1. □

With the above lemmas we can now prove the following transportation cost inequality in the essentially same manner as in [14].

Theorem 5.5. *Assume that* ψ_i*'s are* C^2*-functions and* $\rho := \min\{1 - c_1\|\psi'_i\|_\infty - c_2\|\psi''_i\|_\infty : 1 \le i \le n\} > 0$ *for some universal constants* $c_1, c_2 > 0$. *For every* $\tau \in TS_{\vec{\alpha}}(\mathcal{A}_{\mathrm{proj}}^{(2n+n')})$ *we have*

$$W_{2,\mathrm{free}}\left(\tau, \tau_{\vec{\alpha}}^{\vec{\psi}}\right) \le \sqrt{\frac{2}{\rho}\left(-\chi_{\mathrm{proj}}(\tau) + \tau\left(\sum_{i=1}^n \psi_i(pqp)\right) + B_{\vec{\psi}}\right)} \tag{5.2}$$

with $B_{\vec{\psi}} := \sum_{i=1}^n B_{\psi_i}$*. In particular, when* $e_1, f_1, \dots, e_n, f_n$ *disappear and only* $e'_1, \dots, e'_{n'}$ *appear,* (5.2) *simply becomes*

$$W_{2,\mathrm{free}}\big(\tau, \tau_{(\alpha'_1,\dots,\alpha'_{n'})}\big) \le \sqrt{-2\chi_{\mathrm{proj}}(\tau)}. \tag{5.3}$$

Proof. Since $W_{2,\mathrm{free}}$ is lower semi-continuous in the weak* topology, we have by Lemmas 5.3 and 5.4 (2)

$$W_{2,\mathrm{free}}\Big(\tau, \tau_{\vec{\alpha}}^{\vec{\psi}}\Big) \le \liminf_{m\to\infty} W_{2,\mathrm{free}}\Big(\hat{\lambda}_{N_m}^{\tau}, \hat{\lambda}_{N_m}^{\vec{\psi}}\Big).$$

By Lemma 5.2 we also have

$$W_{2,\mathrm{free}}\Big(\hat{\lambda}_{N_m}^{\tau}, \hat{\lambda}_{N_m}^{\vec{\psi}}\Big) \le \frac{1}{\sqrt{N_m}} W_{2,d}\Big(\lambda_{N_m}^{\tau}, \lambda_{N_m}^{\vec{\psi}}\Big).$$

We then need to confirm Bakry and Emery's Γ_2-criterion [1] for $\lambda_N^{\vec{\psi}}$ with the constant ρN, that is,

$$\mathrm{Ric}\big(G(N, \vec{k}(N))\big) + \mathrm{Hess}(\Psi_N) \ge \rho N I_{d(N)}, \tag{5.4}$$

where $\mathrm{Ric}\big(G(N, \vec{k}(N))\big)$ is the Ricci curvature tensor of $G(N, \vec{k}(N))$, $\mathrm{Hess}(\Psi_N)$ is the Hessian of the trace function

$$\Psi_N(P_1, Q_1, \dots, P_n, Q_n, R_1, \dots, R_{n'}) := N\mathrm{Tr}_N\left(\sum_{i=1}^n \psi_i(P_i Q_i P_i)\right),$$

and $d(N)$ is the dimension of $G(N, \vec{k}(N))$, i.e.,

$$d(N) := 2\sum_{i=1}^n \big(k_i(N)(N - k_i(N)) + l_i(N)(N - l_i(N))\big) + 2\sum_{j=1}^{n'} k'_j(N)(N - k'_j(N)).$$

It is known (see [15, Eq. (2.2)]) that $\mathrm{Ric}(G(N,k)) = N I_{2k(N-k)}$ so that we need only to estimate the Hessian $\mathrm{Hess}(\Psi_N^{(i)})$ of the trace function $\Psi_N^{(i)} : (P, Q) \in G(N, k_i(N)) \times G(N, l_i(N)) \mapsto N\mathrm{Tr}_N(\psi_i(PQP))$ from below. This can be done by computing $\mathrm{Hess}(\Psi_N^{(i)})$ explicitly, and consequently we can find two universal constants $c_1, c_2 > 0$ so that

$$\mathrm{Hess}(\Psi_N^{(i)}) \ge -N(c_1\|\psi'_i\|_\infty + c_2\|\psi''_i\|_\infty) I_{2k_i(N)(N-k_i(N)) + 2l_i(N)(N-l_i(N))}.$$

Hence (5.4) is confirmed. See Remark 5.6 below for more details on this estimate. Thus, by the transportation cost inequality in the Riemannian manifold setting due to Otto and Villani [17] we obtain

$$W_{2,d}\Big(\lambda_{N_m}^{\tau}, \lambda_{N_m}^{\vec{\psi}}\Big) \le \sqrt{\frac{2}{\rho N_m} S\Big(\lambda_{N_m}^{\tau}, \lambda_{N_m}^{\vec{\psi}}\Big)}, \tag{5.5}$$

where $S\Big(\lambda^{\tau}_{N_m}, \lambda^{\vec\psi}_{N_m}\Big)$ stands for the usual relative entropy. We compute

$$\begin{aligned} S\Big(\lambda^{\tau}_{N_m}, \lambda^{\vec\psi}_{N_m}\Big) &= \int \log \frac{d\lambda^{\tau}_{N_m}}{d\lambda^{\vec\psi}_{N_m}}\, d\lambda^{\tau}_{N_m} \\ &= -\log \gamma_{\vec k(N_m)}(\Gamma_{N_m}) + N_m^2 \hat\lambda^{\tau}_{N_m}\left(\sum_{i=1}^n \psi_i(e_if_ie_i)\right) + \sum_{i=1}^n \log Z^{\psi_i}_{N_m}. \end{aligned}$$

Consequently, we obtain the desired inequality (5.2) by taking the limit of (5.5) as $m \to \infty$ after divided by N_m^2 due to (5.1), Lemmas 5.3 and 5.4 (1). Finally, we should remark that if $e_1, f_1, \dots, e_n, f_n$ disappeared, then the argument would become simpler without estimating the Hessian of Ψ_N. □

Remark 5.6. The computation of $\mathrm{Hess}(\Psi_N^{(i)})$ mentioned in the above proof is outlined here. The tangent space $T_PG(N,k)$ at $P \in G(N,k)$ is identified with the set of $X \in M_N(\mathbb{C})^{sa}$ satisfying $X = PX + XP$, on which our Riemannian metric is given by $\langle X|Y\rangle := \mathrm{Re}\,\mathrm{Tr}_N(YX)$ (this is inherited from that on the Euclidean space $M_N(\mathbb{C})^{sa}$). Moreover, the geodesic curve started at P with tangent vector X is given by $C(t) := \exp(t[X,P])P\exp(-t[X,P])$ for $t \in \mathbb{R}$. (See e.g. [6, §2] for a brief summary and references therein.) Since

$$\begin{aligned} &\Big\langle \mathrm{Hess}(\Psi_N^{(i)})((C_1(0), C_2(0))(C_1'(0) \oplus C_2'(0)|C_1'(0) \oplus C_2'(0))\Big\rangle \\ &= \frac{d^2}{dt^2}\bigg|_{t=0} N\mathrm{Tr}_N(\psi_i(C_1(t)C_2(t)C_1(t))) \end{aligned}$$

for geodesic curves $C_1(t) \in G(N, k_i(N))$ and $C_2(t) \in G(N, l_i(N))$, it suffices (for getting the desired inequality in the above proof) to estimate, at $t = 0$, the second derivative of the composition of $\phi(t) := C_1(t)C_2(t)C_1(t) \in M_N(\mathbb{C})^{sa}$ and $X \in M_N(\mathbb{C})^{sa} \mapsto \Phi(X) := N\mathrm{Tr}_N(\psi_i(X))$ with the usual Euclidean structure on $M_N(\mathbb{C})^{sa}$. Passing once to the identification $M_N(\mathbb{C})^{sa} = \mathbb{R}^{N^2}$, we observe that

$$(\Phi\circ\phi)''(0) = \Big\langle (\nabla^2\Phi)(\phi(0))\phi'(0)|\phi'(0)\Big\rangle + \Big\langle (\nabla\Phi)(\phi(0))|\phi''(0)\Big\rangle$$

thanks to the usual chain rule. By [13, Lemma 1.2] we can estimate the operator norms $\|(\nabla^2\Phi)(\phi(0))\|_\infty$ (for linear operators on $(M_N(\mathbb{C})^{sa}, \langle\cdot|\cdot\rangle)$) and $\|(\nabla\Phi)(\phi(0))\|_\infty$ (for elements in $M_N(\mathbb{C})^{sa}$) by $N\|\psi_i''\|_\infty$ and $N\|\psi_i'\|_\infty$, respectively, from the above. As mentioned above the tangent vector $C_i'(0) \in M_N(\mathbb{C})^{sa}$ satisfies $C_i'(0) = C_i(0)C_i'(0) + C_i'(0)C_i(0)$ and the geodesic curve $C_i(t)$ must be

$$C_i(t) = \exp(t[C_i'(0), C_i(0)])C_i(0)\exp(-t[C_i'(0), C_i(0)]).$$

It follows from these facts that

$$\begin{aligned}\phi'(0) &= C_1'(0)C_2(0)C_1(0) + C_1(0)C_2'(0)C_1(0) + C_1(0)C_2(0)C_1'(0),\\ \phi''(0) &= [[C_1'(0), C_1(0)], C_1'(0)]C_2(0)C_1(0)\\ &\quad + C_1(0)[[C_2'(0), C_2(0)], C_2'(0)]C_1(0)\\ &\quad\quad + C_1(0)C_2(0)[[C_1'(0), C_1(0)], C_1'(0)]\\ &\quad\quad\quad + 2\big\{C_1'(0)C_2'(0)C_1(0) + C_1'(0)C_2(0)C_1'(0) + C_1(0)C_2'(0)C_1'(0)\big\}.\end{aligned}$$

Hence we get the rough estimates

$$\begin{aligned}\|\phi'(0)\|_{HS}^2 &\le 6\|C_1'(0)\|_{HS}^2 + 3\|C_2'(0)\|_{HS}^2,\\ \|\phi''(0)\|_{1,\mathrm{Tr}_N} &\le 8\|C_1'(0)\|_{HS}^2 + 4\|C_2'(0)\|_{HS}^2\end{aligned}$$

(we used $2C_i'(0)^2 = |[[C_i'(0), C_i(0)], C_i'(0)]|$, $i = 1, 2$, for the latter). Therefore,

$$\begin{aligned}&(\Phi\circ\phi)''(0)\\ &\quad\le N\big\{(8\|\psi_i'\|_\infty + 6\|\psi_i''\|_\infty)\|C_1'(0)\|_{HS}^2 + (4\|\psi_i'\|_\infty + 3\|\psi_i''\|_\infty)\|C_2'(0)\|_{HS}^2\big\}.\end{aligned}$$

Since $\Phi\circ\phi(t)$ does not change when $C_1(t), C_2(t)$ are interchanged, one finally finds two universal constants $c_1 = 6 > 0$, $c_2 = 9/2 > 0$ so that

$$|(\Phi\circ\phi)''(0)| \le N(c_1\|\psi_i'\|_\infty + c_2\|\psi_i''\|_\infty)(\|C_1'(0)\|_{HS}^2 + \|C_2'(0)\|_{HS}^2),$$

which immediately implies the desired inequality.

Corollary 5.7. *If $p_1, \dots, p_n$ are projections in $(\mathcal{M}, \tau)$ and $\chi_{\mathrm{proj}}(p_1, \dots, p_n) = 0$, then $p_1, \dots, p_n$ are free.*

Proof. This follows from (5.3) and the fact that $W_{2,\mathrm{free}}$ is a metric on $TS_{\vec\alpha}\big(\mathcal{A}_{\mathrm{proj}}^{(n)}\big)$, where $\vec\alpha := (\tau(p_1), \dots, \tau(p_n))$. □

The corollary was an essential ingredient of the proof of Theorem 4.1. In the self-adjoint case, the free transportation cost inequality [14, Theorem 2.2 or Corollary 2.3] provides a new proof of the fact that $X_1, \dots, X_n$ form a free semicircular system if $\chi(X_1, \dots, X_n)$ attains the maximum under the restriction $\tau(X_i^2) = 1$, while Voiculescu's original proof in [22] is based on the infinitesimal change of variable formula.

§6. Free Pressure

Let $(\mathcal{A}^{(n)}_{\mathrm{proj}})^{sa}$ denote the space of self-adjoint elements in the universal C^*-algebra $\mathcal{A}^{(n)}_{\mathrm{proj}}$ with the canonical projection generators $e_1,\dots,e_n$ as in the previous section. Elements in $(\mathcal{A}^{(n)}_{\mathrm{proj}})^{sa}$ are considered as "free probabilistic hamiltonians on $\mathbb{Z}_2^{\star n}$." Motivated from the viewpoint of statistical mechanics, we introduce the free pressure for those free hamiltonians, and its Legendre transform with respect to the duality between $(\mathcal{A}^{(n)}_{\mathrm{proj}})^{sa}$ and $TS(\mathcal{A}^{(n)}_{\mathrm{proj}})$ is compared with χ_{proj}.

Let $\vec{\alpha}=(\alpha_1,\dots,\alpha_n)\in[0,1]^n$ and $\vec{k}(N)=(k_1(N),\dots,k_n(N))\in\{0,1,\dots,N\}$ for $N\in\mathbb{N}$ be given so that $\vec{k}(N)/N\to\vec{\alpha}$ as $N\to\infty$. As before we write $G(N,\vec{k}(N)):=\prod_{i=1}^n G(N,k_i(N))$ and $\gamma_{\vec{k}(N)}:=\bigotimes_{i=1}^n\gamma_{G(N,k_i(N))}$ for short. For $\vec{P}=(P_1,\dots,P_n)\in G(N)^n$ we have the $*$-homomorphism $h\in\mathcal{A}^{(n)}_{\mathrm{proj}}\mapsto h(\vec{P})\in M_N(\mathbb{C})$ sending e_i to P_i, $1\le i\le n$. For each $h\in(\mathcal{A}^{(n)}_{\mathrm{proj}})^{sa}$ define

$$\pi_{\vec{\alpha}}(h):=\limsup_{N\to\infty}\frac{1}{N^2}\log\int_{G(N,\vec{k}(N))}\exp\Big(-N\mathrm{Tr}_N\big(h(\vec{P})\big)\Big)\,d\gamma_{\vec{k}(N)}(\vec{P}), \tag{6.1}$$

which we call the *free pressure* of h under the trace values $(\alpha_1,\dots,\alpha_n)$.

Proposition 6.1. *The above definition of $\pi_{\vec{\alpha}}(h)$ is independent of the choices of $\vec{k}(N)$ with $\vec{k}(N)/N\to\vec{\alpha}$.*

Proof. Let $\vec{l}(N)=(l_1(N),\dots,l_n(N)))$, $N\in\mathbb{N}$, be another sequence such that $\vec{l}(N)/N\to\vec{\alpha}$ as $N\to\infty$. We set

$$\delta_N(h):=\max_{\vec{U}\in\mathrm{U}(N)^n}\left|\frac{1}{N}\mathrm{Tr}_N\big(h(\xi_{\vec{k}(N)}(\vec{U})\big)-\frac{1}{N}\mathrm{Tr}_N\big(h(\xi_{\vec{l}(N)}(\vec{U})\big)\right|$$

for $h\in\mathcal{A}^{(n)}_{\mathrm{proj}}$ and $N\in\mathbb{N}$, where $\xi_{\vec{k}(N)}(\vec{U}):=(\xi_{N,k_1(N)}(U_1),\dots,\xi_{N,k_n(N)}(U_n))$ for $\vec{U}=(U_1,\dots,U_n)$ (see §1). Thanks to (1.2) we get

$$\begin{aligned}&\Bigg|\frac{1}{N^2}\log\int_{G(N,\vec{k}(N))}\exp\Big(-N\mathrm{Tr}_N\big(h(\vec{P})\big)\Big)\,d\gamma_{\vec{k}(N)}(\vec{P})\\ &\qquad-\frac{1}{N^2}\log\int_{G(N,\vec{k}(N))}\exp\Big(-N\mathrm{Tr}_N\big(h(\vec{P})\big)\Big)\,d\gamma_{\vec{k}(N)}(\vec{P})\Bigg|\\ &=\Bigg|\frac{1}{N^2}\log\int_{\mathrm{U}(N)^n}\exp\Big(-N\mathrm{Tr}_N\big(h(\xi_{\vec{k}(N)}(\vec{U})\big)\Big)\,d\big(\gamma_{\mathrm{U}(N)}\big)^{\otimes n}(\vec{U})\\ &\qquad-\frac{1}{N^2}\log\int_{\mathrm{U}(N)^n}\exp\Big(-N\mathrm{Tr}_N\big(h(\xi_{\vec{l}(N)}(\vec{U})\big)\Big)\,d\big(\gamma_{\mathrm{U}(N)}\big)^{\otimes n}(\vec{U})\Bigg|\end{aligned}$$

$$\leq \delta_N(h)$$

for each $h \in \big(\mathcal{A}_{\mathrm{proj}}^{(n)}\big)^{sa}$. Hence, it suffices to prove that $\delta_N(h) \to 0$ as $N \to \infty$. Since $|\delta_N(h_1) - \delta_N(h_2)| \leq \|h_1 - h_2\|$ for all $h_1, h_2 \in \mathcal{A}_{\mathrm{proj}}^{(n)}$, we may show that $\delta_N(h) \to 0$ for $h = e_{i_1} \cdots e_{i_r}$ with $1 \leq i_1, \dots, i_r \leq n$. As in the proof of Proposition 1.1 we have, for such h,

$$\begin{aligned} &\left| \frac{1}{N}\mathrm{Tr}_N\big(h(\xi_{\vec{k}(N)}(\vec{U})\big) - \frac{1}{N}\mathrm{Tr}_N\big(h(\xi_{\vec{l}(N)}(\vec{U})\big) \right| \\ &\qquad \leq \sum_{j=1}^r \|\xi_{N,k_{i_j}(N)}(U_{i_j}) - \xi_{N,l_{i_j}(N)}(U_{i_j})\|_1 \\ &\qquad \leq \sum_{j=1}^r \frac{|k_{i_j}(N) - l_{i_j}(N)|}{N} \longrightarrow 0 \end{aligned}$$

as $N \to \infty$, and the conclusion follows. □

The following are basic properties of $\pi_{\vec{\alpha}}(h)$; we omit the proofs very similar to those of [10, Proposition 2.3] but note that the last assertion of (iv) follows from (6.2) and Proposition 6.4 (1) below.

Proposition 6.2.

(i) *$\pi_{\vec{\alpha}}(h)$ is convex on $\big(\mathcal{A}_{\mathrm{proj}}^{(n)}\big)^{sa}$.*

(ii) *If $h_1, h_2 \in \big(\mathcal{A}_{\mathrm{proj}}^{(n)}\big)^{sa}$ and $h_1 \leq h_2$, then $\pi_{\vec{\alpha}}(h_1) \geq \pi_{\vec{\alpha}}(h_2)$.*

(iii) *$|\pi_{\vec{\alpha}}(h_1) - \pi_{\vec{\alpha}}(h_2)| \leq \|h_1 - h_2\|$ for all $h_1, h_2 \in \big(\mathcal{A}_{\mathrm{proj}}^{(n)}\big)^{sa}$.*

(iv) *If $h_1 \in \big(\mathcal{A}_{\mathrm{proj}}^{(j)}\big)^{sa}$ and $h_2 \in \big(\mathcal{A}_{\mathrm{proj}}^{(n-j)}\big)^{sa}$ with $1 \leq j < n$, then*

$$\pi_{\vec{\alpha}}(h_1 + h_2) \leq \pi_{(\alpha_1,\dots,\alpha_j)}(h_1) + \pi_{(\alpha_{j+1},\dots,\alpha_n)}(h_2),$$

where $h_1 + h_2$ is the sum as an element of $\mathcal{A}_{\mathrm{proj}}^{(n)} = \mathcal{A}_{\mathrm{proj}}^{(j)} \bigstar \mathcal{A}_{\mathrm{proj}}^{(n-j)}$. In particular when $j = 1$ or $j = 2$, equality holds in the above inequality.

Remark 6.3. Another possible definition of free pressure is to use the probability measure $\gamma_{G(N)}^{(1)}$ or $\gamma_{G(N)}^{(2)}$ on $G(N)$ given in Remark 1.3. For $h \in \big(\mathcal{A}_{\mathrm{proj}}^{(n)}\big)^{sa}$ define

$$\pi^{(j)}(h) := \limsup_{N\to\infty} \frac{1}{N^2} \log \int_{G(N)^n} \exp\Big(-N\mathrm{Tr}_N\big(h(\vec{P})\big)\Big)\, d\big(\gamma_{G(N)}^{(j)}\big)^{\otimes n}(\vec{P})$$

for $j = 1, 2$. It is not difficult to show that

$$\pi^{(1)}(h) = \pi^{(2)}(h) = \max\{\pi_{\vec{\alpha}}(h) : \vec{\alpha} \in [0,1]^n\}$$

for every $h \in \left(\mathcal{A}_{\mathrm{proj}}^{(n)}\right)^{sa}$. We simply write $\pi(h)$ for these equal quantities; then $\pi(h)$ has the same properties as in Proposition 6.2. However, unlike the free entropy quantities $\chi_{\mathrm{proj}}^{(j)}$ discussed in Remark 1.3, $\pi(h)$ does not coincide with $\pi_{\vec{\alpha}}(h)$; the latter actually depends on $\vec{\alpha}$.

In the single projection case, $\mathcal{A}_{\mathrm{proj}}^{(1)} = \mathbb{C}^2$, $\left(\mathcal{A}_{\mathrm{proj}}^{(1)}\right)^{sa} = \mathbb{R}^2$ and $TS\left(\mathcal{A}_{\mathrm{proj}}^{(1)}\right) = \{\tau_\alpha : 0 \le \alpha \le 1\}$ where $\tau_\alpha(\zeta_1, \zeta_2) = \alpha\zeta_1 + (1-\alpha)\zeta_2$ for $(\zeta_1, \zeta_2) \in \mathbb{C}^2$. Let $0 \le \alpha \le 1$ and choose $k(N)$ such that $k(N)/N \to \alpha$. For each $h = (h_1, h_2) \in \mathbb{R}^2$, it is straightforward to check that

$$\begin{aligned} \pi_\alpha(h) &= \lim_{N\to\infty} \frac{1}{N^2} \log \int_{G(N,k(N))} \exp\left(-N\mathrm{Tr}_N(h(P))\right) d\gamma_{G(N,k(N))}(P) \\ &= -\alpha h_1 - (1-\alpha)h_2 = -\tau_\alpha(h) \end{aligned} \tag{6.2}$$

and hence $\chi_{\mathrm{proj}}(\tau_\alpha) = 0 = \tau_\alpha(h) + \pi_\alpha(h)$. Moreover,

$$\pi(h) = -\min\{h_1, h_2\} = \min\{-\tau_\alpha(h) + \chi_{\mathrm{proj}}(\tau_\alpha) : 0 \le \alpha \le 1\}.$$

In the case of two projections, $\mathcal{A}_{\mathrm{proj}}^{(2)} = C^*(\mathbb{Z}_2 \bigstar \mathbb{Z}_2)$ with the canonical projection generators e, f. Let $\alpha, \beta \in [0,1]$. The next theorem says that the free entropy $\chi_{\mathrm{proj}}(\tau)$ for $\tau \in TS_{(\alpha,\beta)}\left(\mathcal{A}_{\mathrm{proj}}^{(2)}\right)$ and the free pressure $\pi_{(\alpha,\beta)}(h)$ for $h \in \left(\mathcal{A}_{\mathrm{proj}}^{(2)}\right)^{sa}$ are the Legendre transforms of each other.

Proposition 6.4.

(1) *In the definition of* $\pi_{(\alpha,\beta)}(h)$ *in* (6.1) $\limsup$ *can be replaced by* $\lim$.

(2) $\pi_{(\alpha,\beta)}(h) = \max\left\{-\tau(h) + \chi_{\mathrm{proj}}(\tau) : \tau \in TS_{(\alpha,\beta)}\left(\mathcal{A}_{\mathrm{proj}}^{(2)}\right)\right\}$ *for every* $h \in \left(\mathcal{A}_{\mathrm{proj}}^{(2)}\right)^{sa}$.

(3) $\chi_{\mathrm{proj}}(\tau) = \inf\left\{\tau(h) + \pi_{(\alpha,\beta)}(h) : h \in \left(\mathcal{A}_{\mathrm{proj}}^{(2)}\right)^{sa}\right\}$ *for every* $\tau \in TS_{(\alpha,\beta)}\left(\mathcal{A}_{\mathrm{proj}}^{(2)}\right)$.

(4) $\pi(h) = \max\left\{-\tau(h) + \chi_{\mathrm{proj}}(\tau) : \tau \in TS\left(\mathcal{A}_{\mathrm{proj}}^{(2)}\right)\right\}$ *for every* $h \in \left(\mathcal{A}_{\mathrm{proj}}^{(2)}\right)^{sa}$.

Proof. Thanks to the Lipschitz continuity in h of the quantity inside $\limsup$ in (6.1) as well as both sides of the equality in (2), to prove (1) and (2), we may assume that h is a self-adjoint polynomial of e, f written as

$$h = C\mathbf{1} + Ae + Bf + \sum_{k=1}^m A_j(efe)^j + \sum_{j=1}^m B_j(fef)^j + \sum_{j=1}^m D_j((ef)^j + (fe)^j)$$

with $A, B, C, A_j, B_j, D_j \in \mathbb{R}$. Set

$$h_0 := Ae + Bf + \sum_{k=0}^{m} C_k (efe)^k$$

with $C_0 := C$, $C_j := A_j + B_j + D_j$, $1 \le j \le m$. We then get $\tau(h) = \tau(h_0)$ and $\mathrm{Tr}_N(h(P,Q)) = \mathrm{Tr}_N(h_0(P,Q))$ for $P, Q \in G(N)$ so that $\pi_{(\alpha,\beta)}(h) = \pi_{(\alpha,\beta)}(h_0)$. Hence it is enough to prove (1) and (2) for h_0 above. A bit more generally, let $h \in \big(\mathcal{A}^{(2)}_{\mathrm{proj}}\big)^{sa}$ be of the form

$$h = Ae + Bf + \psi(efe),$$

where ψ is a real continuous function on $[0,1]$. Choosing $k(N), l(N)$ such that $k(N)/N \to \alpha$ and $l(N)/N \to \beta$, we have

(6.3)

$$\begin{aligned}
&\frac{1}{N^2}\log\int_{G(N,k(N))\times G(N,l(N))} \exp\big(-N\mathrm{Tr}_N(\psi(P,Q))\big) \\
&\qquad\qquad\qquad\qquad\qquad d\big(\gamma_{G(N,k(N))}\otimes\gamma_{G(N,l(N))}\big)(P,Q) \\
&= -A\frac{k(N)}{N} - B\frac{l(N)}{N} + \frac{1}{N^2}\log\int_{[0,1]^n}\exp\left(-N\sum_{i=1}^{N}\psi(x_i)\right)d\lambda_N(x_1,\dots,x_N),
\end{aligned}$$

where λ_N is the empirical eigenvalue distribution of PQP when (P,Q) is distributed under $\gamma_{G(N,k(N))}\otimes\gamma_{G(N,l(N))}$. By applying Varadhan's integral lemma (see [7, 4.3.1]) to the large deviation in [12, Theorem 2.2] we have

(6.4)

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N^2}\log\int_{[0,1]^n}\exp\left(-N\sum_{i=1}^{N}\psi(x_i)\right)d\lambda_N(x_1,\dots,x_N) \\
&= \sup_{\nu}\Bigg\{-(1-\min\{\alpha,\beta\})\psi(0) - \max\{\alpha+\beta-1,0\}\psi(1) - \frac{1}{2}\int_{[0,1]}\psi(x)\,d\nu(x) \\
&\qquad + \frac{1}{4}\Sigma(\nu) + \frac{|\alpha-\beta|}{2}\int_{[0,1]}\log x\,d\nu(x) \\
&\qquad\qquad + \frac{|\alpha+\beta-1|}{2}\int_{[0,1]}\log(1-x)\,d\nu(x) - C\Bigg\},
\end{aligned}$$

where ν runs over all measures on $(0,1)$ with $\nu((0,1)) = 2\rho$. Here, ρ is in (2.1) and C in (2.2). Let $\tau = \tau_{\nu,\{\alpha_{ij}\}} \in TS_{(\alpha,\beta)}\big(\mathcal{A}^{(2)}_{\mathrm{proj}}\big)$ (see §2 and §5). When

$\alpha_{00}\alpha_{11} = \alpha_{01}\alpha_{10} = 0$ (this is necessary for $\chi_{\mathrm{proj}}(\tau) > -\infty$), $\chi_{\mathrm{proj}}(\tau)$ is given as in Proposition 2.1 and moreover we get

$$\begin{aligned}\tau(h) &= A\alpha + B\beta + (\alpha_{10} + \alpha_{01} + \alpha_{00})\psi(0) + \alpha_{11}\psi(1)\\ &\quad + \frac{1}{2}\int_{(0,1)} (\psi(x) + \psi(0))\, d\nu(x)\\ &= A\alpha + B\beta + (1 - \min\{\alpha, \beta\})\psi(0) + \max\{\alpha + \beta - 1, 0\}\psi(1)\\ &\quad + \frac{1}{2}\int_{(0,1)} \psi(x)\, d\nu(x)\end{aligned}$$

thanks to (2.3). Furthermore, Proposition 2.1 implies that $\chi_{\mathrm{proj}}(\tau)$ is concave and weakly* upper semi-continuous restricted on $TS_{(\alpha,\beta)}\big(\mathcal{A}^{(2)}_{\mathrm{proj}}\big)$. Hence we obtain (1) and (2) by (6.3) and (6.4) together with the formulas of $\chi_{\mathrm{proj}}(\tau)$ and $\tau(h)$. Moreover, (3) follows from (2) due to the duality for conjugate functions (or Legendre transforms). Finally, (4) is obvious from (2) and Remark 6.3. □

Now, we introduce a free entropy-like quantity for tracial states on $\mathcal{A}^{(n)}_{\mathrm{proj}}$ (or for n-tuples of projections) via the (minus) Legendre transform of free pressure. Define

$$\eta_{\mathrm{proj}}(\tau) := \inf\Big\{\tau(h) + \pi_{\vec{\alpha}}(h) : h \in \big(\mathcal{A}^{(n)}_{\mathrm{proj}}\big)^{sa}\Big\}$$

for $\vec{\alpha} \in [0,1]^n$ and $\tau \in TS_{\vec{\alpha}}\big(\mathcal{A}^{(n)}_{\mathrm{proj}}\big)$. Since $\pi_{\vec{\alpha}}$ is a convex and continuous function on $\big(\mathcal{A}^{(n)}_{\mathrm{proj}}\big)^{sa}$ by Proposition 6.2, the above Legendre transform is reversed so that we have

$$\pi_{\vec{\alpha}}(h) = \sup\Big\{-\tau(h) + \eta_{\mathrm{proj}}(\tau) : \tau \in TS_{\vec{\alpha}}\big(\mathcal{A}^{(n)}_{\mathrm{proj}}\big)\Big\}$$

for $h \in \big(\mathcal{A}^{(n)}_{\mathrm{proj}}\big)^{sa}$. For each $h \in \big(\mathcal{A}^{(n)}_{\mathrm{proj}}\big)^{sa}$ there exists a $\tau \in TS_{\vec{\alpha}}\big(\mathcal{A}^{(n)}_{\mathrm{proj}}\big)$ such that

$$\pi_{\vec{\alpha}}(h) = -\tau(h) + \eta_{\mathrm{proj}}(\tau).$$

This equality condition is a kind of variational principle and such τ may be called an equilibrium tracial state associated with h (and $\vec{\alpha}$).

Moreover, for each n-tuple $(p_1, \dots, p_n)$ of projections in $(\mathcal{M}, \tau)$, we have $\tau_{(p_1,\dots,p_n)} \in TS\big(\mathcal{A}^{(n)}_{\mathrm{proj}}\big)$ defined by $\tau_{(p_1,\dots,p_n)}(h) := \tau(h(p_1, \dots, p_n))$, where $h \in \mathcal{A}^{(n)}_{\mathrm{proj}} \mapsto h(p_1, \dots, p_n) \in \mathcal{M}$ is the $*$-homomorphism sending e_i to p_i, $1 \le i \le n$. We define

$$\eta_{\mathrm{proj}}(p_1, \dots, p_n) := \eta_{\mathrm{proj}}(\tau_{(p_1,\dots,p_n)}).$$

It is obvious by definition that the quantity $\eta_{\mathrm{proj}}(p_1, \dots, p_n)$ has the same properties as $\chi_{\mathrm{proj}}(p_1, \dots, p_n)$ given in Proposition 1.2.

Theorem 6.5. *Let $p_1, q_1, \ldots, p_n, q_n, r_1, \ldots, r_{n'}$ be projections in $(\mathcal{M}, \tau)$.*

(1) $\eta_{\mathrm{proj}}(p_1, \ldots, p_n) \ge \chi_{\mathrm{proj}}(p_1, \ldots, p_n)$.

(2) *If $\{p_1, q_1\}, \ldots, \{p_n, q_n\}, \{r_1\}, \ldots, \{r_{n'}\}$ are free, then*

$$\eta_{\mathrm{proj}}(p_1, q_1, \ldots, p_n, q_n, r_1, \ldots, r_{n'}) = \chi_{\mathrm{proj}}(p_1, q_1, \ldots, p_n, q_n, r_1, \ldots, r_{n'}).$$

Proof. The proof of (1) is similar to that of [10, Theorem 4.5 (1)]. By (6.2) and Proposition 6.4 (3), $\eta_{\mathrm{proj}} = \chi_{\mathrm{proj}}$ holds when $n = 1$ or $n = 2$. Hence (2) is seen from (1) together with the subadditivity of η_{proj} and the additivity of χ_{proj} in Theorem 4.1 (1). □

Remark 6.6. It is known that η_{proj} and χ_{proj} are not equal in general. This can be seen as in [10, Remark 4.6] by proving the so-called degenerate convexity [26, χ.8] for χ_{proj}. In fact, Miyamoto [28] proved that if $\chi_{\mathrm{proj}}(p_1, \ldots, p_n) > -\infty$ and $\sum_{i=1}^n \min\{\tau(p_i), \tau(\mathbf{1} - p_i)\} > 1$ (this forces $n \ge 3$), then $\{p_1, \ldots, p_n\}''$ is a non-Γ II_1 factor.

Finally, we note that the definition (6.1) is slightly modified in such a way that the modified free pressure $\pi_{\vec{\alpha}}^{(2)}(g)$ is defined for self-adjoint elements of the minimal C^*-tensor product $\mathcal{A}_{\mathrm{proj}}^{(n)} \otimes_{\min} \mathcal{A}_{\mathrm{proj}}^{(n)}$ and the modified quantity $\tilde{\eta}_{\mathrm{proj}}(p_1, \ldots, p_n)$ induced from $\pi_{\vec{\alpha}}^{(2)}$ via Legendre transform is always equal to $\chi_{\mathrm{proj}}(p_1, \ldots, p_n)$. The procedure of this modification is essentially same as [10, §6] so that we omit the details.

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