

# Algebraic $p$ -Adic $L$ -Functions in Non-Commutative Iwasawa Theory

By

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## Abstract

We define canonical algebraic  $p$ -adic  $L$ -functions in non-commutative Iwasawa theory and establish some of their basic properties.

## §1. Introduction

In this note we use a result of Schneider and Venjakob in [8] to define canonical ‘algebraic  $p$ -adic  $L$ -functions’ in non-commutative Iwasawa theory. This construction refines both the notion of ‘characteristic element’ introduced by Venjakob in [10] and the ‘Akashi series’ introduced by Coates, Schneider and Sujatha in [5] and also plays a key role in descent theory in non-commutative Iwasawa theory. Indeed, in joint work with Venjakob [3], the results proved here are used to establish a general descent theory that, for example, clarifies the precise connection between main conjectures of non-commutative Iwasawa theory in the spirit of Coates, Fukaya, Kato, Sujatha and Venjakob [4] and the relevant cases of the equivariant Tamagawa number conjecture of Flach and the present author [2].

The main contents of this note is as follows. In §2 we define a natural notion of algebraic  $p$ -adic  $L$ -function. In §3 we prove some of the basic functorial properties of these elements and show that they refine the ‘Akashi series’ introduced in [5]. In §4 we prove that algebraic  $p$ -adic  $L$ -functions are ‘characteristic elements’ in the sense of [10] (and [4]).

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## §2. Algebraic $p$ -Adic $L$ -Functions

### §2.1. Preliminaries

In the sequel ‘module’ means ‘left module’. For any ring  $R$  we write  $D(R)$  for the derived category of  $R$ -modules. We also write  $D^{\text{fg}}(R)$ , resp.  $D^{\text{p}}(R)$ , for the full triangulated subcategory of  $D(R)$  comprising complexes that are isomorphic to a bounded complex of finitely generated  $R$ -modules, resp. to an object of the category  $C^{\text{p}}(R)$  of bounded complexes of finitely generated projective  $R$ -modules.

We fix a prime  $p$ . For any  $\mathbb{Z}_p$ -module  $M$  we write  $M_{\text{tor}}$  for its torsion submodule and set  $M_{\text{tf}} := M/M_{\text{tor}}$ . For any profinite group  $J$  we write  $\Lambda(J)$  for the ‘Iwasawa algebra’  $\varprojlim_U \mathbb{Z}_p[J/U]$  where  $U$  runs over all open normal subgroups of  $J$  and the limit is taken with respect to the natural projection maps  $\mathbb{Z}_p[J/U] \rightarrow \mathbb{Z}_p[J/U']$  for  $U \subseteq U'$ . If  $J$  is a compact  $p$ -adic Lie group, then  $\Lambda(J)$  is a noetherian ring and we write  $Q(J)$  for its total quotient ring.

### §2.2. Canonical Ore sets

We assume to be given a compact  $p$ -adic Lie group  $G$  and a normal subgroup  $H$  of  $G$  with the property that the quotient group  $\Gamma := G/H$  is isomorphic (topologically) to the additive group of  $\mathbb{Z}_p$ . We fix a topological generator  $\gamma$  of  $\Gamma$ . We recall from [4, §2-§3] that there are canonical left and right denominator sets  $S(G, H)$  and  $S(G, H)^*$  of  $\Lambda(G)$  where

$$S(G, H) := \{\lambda \in \Lambda(G) : \Lambda(G)/(\Lambda(G) \cdot \lambda) \text{ is a finitely generated } \Lambda(H)\text{-module}\}$$

and

$$S(G, H)^* := \bigcup_{i \geq 0} p^i S(G, H).$$

When  $G$  and  $H$  are clear from context we abbreviate  $S(G, H)$  to  $S$ . We also write  $\mathfrak{M}_S(G)$  and  $\mathfrak{M}_{S^*}(G)$  for the categories of finitely generated  $\Lambda(G)$ -modules  $M$  that satisfy  $\Lambda(G)_S \otimes_{\Lambda(G)} M = 0$  and  $\Lambda(G)_{S^*} \otimes_{\Lambda(G)} M = 0$  respectively. We further recall from [4, Prop. 2.3] that a finitely generated  $\Lambda(G)$ -module  $M$  belongs to  $\mathfrak{M}_S(G)$ , resp. to  $\mathfrak{M}_{S^*}(G)$ , if and only if  $M$ , resp.  $M_{\text{tf}}$ , is a finitely generated  $\Lambda(H)$ -module (by restriction). Thus  $\mathfrak{M}_{S^*}(G)$  coincides with the category  $\mathfrak{M}_H(G)$  used in [4].

### §2.3. Canonical automorphisms

If  $J$  is any profinite group, then we write  $M \hat{\otimes}_{\Lambda(J)} N$  for the completed tensor product of compact  $\Lambda(J)$ -modules  $M$  and  $N$  (cf. [7, p. 230]). In particular, if  $M$  is any compact  $\Lambda(G)$ -module, then

$$I_H^G(M) := \Lambda(G) \hat{\otimes}_{\Lambda(H)} \text{Res}_H^G(M)$$

has a natural structure as a (compact)  $\Lambda(G)$ -module via left multiplication. The functor  $M \mapsto I_H^G(M)$  is exact on the category of compact  $\Lambda(G)$ -modules and if  $M$  belongs to  $\mathfrak{M}_S(G)$ , then  $I_H^G(M)$  identifies with the usual tensor product  $\Lambda(G) \otimes_{\Lambda(H)} M$  (cf. [7, p. 241, Ex. 1]).

We define an endomorphism  $\Delta_{G,\gamma,M}$  of  $I_H^G(M)$  by setting

$$\Delta_{G,\gamma,M}(x \otimes_{\Lambda(H)} y) := x \tilde{\gamma}^{-1} \otimes_{\Lambda(H)} \tilde{\gamma}(y)$$

for each  $x \in \Lambda(G)$  and  $y \in M$ , where  $\tilde{\gamma}$  is any lift of  $\gamma$  through the projection  $G \rightarrow \Gamma$ . We also set

$$\delta_{G,\gamma,M} := \text{id}_{I_H^G(M)} - \Delta_{G,\gamma,M}.$$

Then  $\Delta_{G,\gamma,M}$ , and hence also  $\delta_{G,\gamma,M}$ , is a well-defined endomorphism of the  $\Lambda(G)$ -module  $I_H^G(M)$  that is independent of the precise choice of  $\tilde{\gamma}$ . When  $G$  and  $M$  are both clear from context we usually abbreviate  $\Delta_{G,\gamma,M}$  and  $\delta_{G,\gamma,M}$  to  $\Delta_\gamma$  and  $\delta_\gamma$  respectively.

For any compact  $\Lambda(G)$ -module  $M$  we let  $M_S$ , resp.  $M_{S^*}$ , denote the induced  $\Lambda(G)_S$ -module  $\Lambda(G)_S \hat{\otimes}_{\Lambda(G)} M$ , resp.  $\Lambda(G)_{S^*}$ -module  $\Lambda(G)_{S^*} \hat{\otimes}_{\Lambda(G)} M$ .

**Lemma 2.1.** *If  $\Sigma \in \{S, S^*\}$  and  $M \in \mathfrak{M}_\Sigma(G)$ , then  $I_H^G(M)_\Sigma$  is a finitely generated  $\Lambda(G)_\Sigma$ -module and  $\delta_{G,\gamma,M}$  induces an automorphism of  $I_H^G(M)_\Sigma$ .*

*Proof.* We assume first that  $\Sigma = S$  and fix  $M$  in  $\mathfrak{M}_S(G)$ . Then  $I_H^G(M)$  is a finitely generated  $\Lambda(G)$ -module and hence  $I_H^G(M)_S$  is a finitely generated  $\Lambda(G)_S$ -module. There is also a short exact sequence of (finitely generated)  $\Lambda(G)$ -modules

$$(1) \quad 0 \rightarrow I_H^G(M) \xrightarrow{\delta_\gamma} I_H^G(M) \rightarrow M \rightarrow 0$$

in which the third arrow is induced by  $x \otimes_{\Lambda(H)} m \mapsto x(m)$  for each  $x \in \Lambda(G)$  and  $m \in M$ . Indeed, the exactness of this sequence has been proved by Schneider and Venjakob [8, Prop. 2.2, Rem. 2.3]. Thus, by applying the (exact)

scalar extension functor  $\Lambda(G)_S \hat{\otimes}_{\Lambda(G)} -$  to (1) and noting that  $M_S = 0$  (by assumption), we may deduce that  $\delta_\gamma$  induces an automorphism of  $I_H^G(M)_S$ .

From here the analogous results for modules  $M$  in  $\mathfrak{M}_{S^*}(G)$  follow easily from the fact that  $M_{\text{tf}}$  belongs to  $\mathfrak{M}_S(G)$  and  $I_H^G(M)_{S^*} = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} I_H^G(M_{\text{tf}})_S$ .  $\square$

### §2.4. Algebraic $p$ -adic $L$ -functions

The ring  $\Lambda(G)_{S^*}$  is both noetherian and regular [6, Prop. 4.3.4]. Hence there is a natural group isomorphism between the algebraic  $K$ -group  $K_1(\Lambda(G)_{S^*})$  and the group  $G_1(\Lambda(G)_{S^*})$  that is generated (multiplicatively) by symbols of the form  $\langle \alpha \mid N \rangle$  where  $\alpha$  is an automorphism of a finitely generated  $\Lambda(G)_{S^*}$ -module  $N$  (cf. [9, Th. 16.11]).

**Definition 2.2.** *Let  $A^\bullet$  be a complex in  $D^{\text{fg}}(\Lambda(G))$  such that  $\Lambda(G)_S \hat{\otimes}_{\Lambda(G)} A^\bullet$  is acyclic. Then (following Lemma 2.1) we define the Algebraic  $p$ -adic  $L$ -function of  $A^\bullet$  (relative to  $G$  and  $\gamma$ ) to be the element of  $K_1(\Lambda(G)_{S^*})$  obtained by setting*

$$\mathcal{L}_{G,\gamma}^{\text{alg}}(A^\bullet) := \prod_{i \in \mathbb{Z}} \langle \delta_{G,\gamma,H^i(A^\bullet)} \mid I_H^G(H^i(A^\bullet))_{S^*} \rangle^{(-1)^i}.$$

The following result explicates this definition in a classical setting.

**Lemma 2.3.** *If  $G = \Gamma$ , then  $\Lambda(G)_{S^*}$  is the total quotient ring  $Q(\Gamma)$  of  $\Lambda(\Gamma)$  and there is a natural isomorphism of groups  $\iota : K_1(\Lambda(G)_{S^*}) \cong Q(\Gamma)^\times$ . If  $M$  is any  $\Lambda(\Gamma)$ -module that is finitely generated over  $\mathbb{Z}_p$ , then*

$$\iota(\mathcal{L}_{G,\gamma}^{\text{alg}}(M[0])) = (1+T)^{-\lambda(M)} \text{char}_T(M)$$

where  $\lambda(M)$  is the Iwasawa  $\lambda$ -invariant of  $M$  and  $\text{char}_T(M)$  the characteristic polynomial of  $M$  with respect to the variable  $T = \gamma - 1$ .

*Proof.* The first sentence is well-known (with the isomorphism  $\iota$  induced by taking determinants over  $Q(\Gamma)$ ). Further, by the structure theory of finitely generated  $\Lambda(\Gamma)$ -modules, it suffices to prove the second sentence in the case that  $M = \Lambda(\Gamma)/(f)$  where  $f$  is a distinguished polynomial. Now  $\mathcal{L}_{G,\gamma}^{\text{alg}}(M[0]) = \langle \delta_\gamma \mid Q(\Gamma) \otimes_{\mathbb{Z}_p} M \rangle$  and so, if we consider  $N := \mathbb{Z}_p[[T]] \otimes_{\mathbb{Z}_p} (\mathbb{Z}_p[[T]]/(f(T)))$  as a (free)  $\mathbb{Z}_p[[T]]$ -module via left multiplication, then  $\iota(\mathcal{L}_{G,\gamma}^{\text{alg}}(M[0])) = \det_{\mathbb{Z}_p[[T]]}(\alpha)$  where  $\alpha$  is the endomorphism of  $N$  given by multiplication by

$$\text{id} - (1+T)^{-1} \otimes (1+T) = ((1+T)^{-1} \otimes \text{id})(T \otimes \text{id} - \text{id} \otimes T).$$

But, by using the  $\mathbb{Z}_p[[T]]$ -basis  $\{1 \otimes (T^i \bmod (f(T))) : 0 \leq i < \deg(f)\}$  of  $N$ , one computes that  $\det_{\mathbb{Z}_p[[T]]}((1+T)^{-1} \otimes \text{id} \mid N) = (1+T)^{-\text{rank}_{\mathbb{Z}_p[[T]]}(N)} = (1+T)^{-\deg(f)} = (1+T)^{-\lambda(M)}$  and  $\det_{\mathbb{Z}_p[[T]]}(T \otimes \text{id} - \text{id} \otimes T \mid N) = f(T) = \text{char}_T(M)$ . The displayed equality is therefore clear.  $\square$

*Remark 2.4.* In joint work with Venjakob [3] we reinterpret Definition 2.2 in terms of the localized  $K_1$ -groups introduced by Fukaya and Kato in [6]. If  $G$  has no element of order  $p$ , then in [3] we also extend Definition 2.2 (and the results proved in §3 and §4 below) to the case of complexes  $A^\bullet$  in  $D^{\text{fg}}(\Lambda(G))$  for which only  $\Lambda(G)_{S^*} \hat{\otimes}_{\Lambda(G)} A^\bullet$  is assumed to be acyclic.

### §3. Basic Properties

If  $\Sigma$  denotes either  $S$  or  $S^*$ , then we write  $D_\Sigma^{\text{p}}(\Lambda(G))$  for the full triangulated subcategory of  $D^{\text{p}}(\Lambda(G))$  comprising complexes  $A^\bullet$  in  $D^{\text{p}}(\Lambda(G))$  for which  $\Lambda(G)_{\Sigma} \hat{\otimes}_{\Lambda(G)} A^\bullet$  is acyclic. For each  $M$  in  $\mathfrak{M}_{S^*}(G)$  we also set

$$\mathcal{E}_{G,\gamma}(M) := \langle \delta_{G,\gamma,M} \mid \mathbb{I}_H^G(M)_{S^*} \rangle \in K_1(\Lambda(G)_{S^*}).$$

**Proposition 3.1.** (*Additivity*) *If  $A^\bullet \rightarrow B^\bullet \rightarrow C^\bullet \rightarrow A^\bullet[1]$  is an exact triangle in  $D_S^{\text{p}}(\Lambda(G))$ , then  $\mathcal{L}_{G,\gamma}^{\text{alg}}(B^\bullet) = \mathcal{L}_{G,\gamma}^{\text{alg}}(A^\bullet) \mathcal{L}_{G,\gamma}^{\text{alg}}(C^\bullet)$ .*

*Proof.* Applying the exact functor  $\mathbb{I}_H^G(-)_{S^*}$  to the long exact cohomology sequence of the given triangle gives a commutative diagram of finitely generated  $\Lambda(G)_{S^*}$ -modules

$$\begin{array}{ccccccc} \rightarrow & \mathbb{I}_H^G(H^i(A^\bullet))_{S^*} & \rightarrow & \mathbb{I}_H^G(H^i(B^\bullet))_{S^*} & \rightarrow & \mathbb{I}_H^G(H^i(C^\bullet))_{S^*} & \rightarrow & \mathbb{I}_H^G(H^{i+1}(A^\bullet))_{S^*} & \rightarrow \\ & \delta_{G,\gamma,H^i(A^\bullet)} \downarrow & & \delta_{G,\gamma,H^i(B^\bullet)} \downarrow & & \delta_{G,\gamma,H^i(C^\bullet)} \downarrow & & \delta_{G,\gamma,H^{i+1}(A^\bullet)} \downarrow & \\ \rightarrow & \mathbb{I}_H^G(H^i(A^\bullet))_{S^*} & \rightarrow & \mathbb{I}_H^G(H^i(B^\bullet))_{S^*} & \rightarrow & \mathbb{I}_H^G(H^i(C^\bullet))_{S^*} & \rightarrow & \mathbb{I}_H^G(H^{i+1}(A^\bullet))_{S^*} & \rightarrow \end{array}$$

in which both rows are exact. Taken in conjunction with the defining relations of  $K_1(\Lambda(G)_{S^*})$  this diagram implies the required equality

$$\begin{aligned} \mathcal{L}_{G,\gamma}^{\text{alg}}(B^\bullet) &= \prod_{i \in \mathbb{Z}} \mathcal{E}_{G,\gamma}(H^i(B^\bullet))^{(-1)^i} \\ &= \prod_{i \in \mathbb{Z}} \mathcal{E}_{G,\gamma}(H^i(A^\bullet))^{(-1)^i} \prod_{i \in \mathbb{Z}} \mathcal{E}_{G,\gamma}(H^i(C^\bullet))^{(-1)^i} = \mathcal{L}_{G,\gamma}^{\text{alg}}(A^\bullet) \mathcal{L}_{G,\gamma}^{\text{alg}}(C^\bullet). \end{aligned}$$

$\square$

Now let  $U$  be a closed subgroup of  $H$  that is normal in  $G$  and set  $\overline{G} := G/U$ ,  $\overline{H} := H/U$  and  $\overline{S} := S(\overline{G}, \overline{H})$ . Then the natural projection  $\Lambda(G) \rightarrow \Lambda(\overline{G})$  extends to a ring homomorphism  $\Lambda(G)_{S^*} \rightarrow \Lambda(\overline{G})_{\overline{S}^*}$  and hence induces a homomorphism of groups  $\pi_{\overline{G}} : K_1(\Lambda(G)_{S^*}) \rightarrow K_1(\Lambda(\overline{G})_{\overline{S}^*})$ .

**Proposition 3.2.** (*Change of group*) *If  $A^\bullet$  belongs to  $D_S^p(\Lambda(G))$ , then  $\Lambda(\overline{G}) \hat{\otimes}_{\Lambda(G)}^{\mathbb{L}} A^\bullet$  belongs to  $D_{\overline{S}}^p(\Lambda(\overline{G}))$  and  $\pi_{\overline{G}}(\mathcal{L}_{G,\gamma}^{\text{alg}}(A^\bullet)) = \mathcal{L}_{\overline{G},\gamma}^{\text{alg}}(\Lambda(\overline{G}) \hat{\otimes}_{\Lambda(G)}^{\mathbb{L}} A^\bullet)$ .*

*Proof.* The first assertion follows directly from the fact that there is a natural isomorphism in  $D^p(\Lambda(\overline{G})_{\overline{S}})$  of the form

$$\Lambda(\overline{G})_{\overline{S}} \hat{\otimes}_{\Lambda(\overline{G})}^{\mathbb{L}} (\Lambda(\overline{G}) \hat{\otimes}_{\Lambda(G)}^{\mathbb{L}} A^\bullet) \cong \Lambda(\overline{G})_{\overline{S}} \hat{\otimes}_{\Lambda(G)_S}^{\mathbb{L}} (\Lambda(G)_S \hat{\otimes}_{\Lambda(G)} A^\bullet).$$

To prove the second assertion we first recall that each term  $\mathcal{E}_{G,\gamma}(H^i(A^\bullet))$  can be computed explicitly as follows. One can fix a complex  $P_i^\bullet$  in  $C^p(\Lambda(G)_{S^*})$ , an isomorphism  $\psi_i : P_i^\bullet \rightarrow I_H^G(H^i(A^\bullet))_{S^*}[0]$  in  $D^p(\Lambda(G)_{S^*})$  and a morphism  $\alpha_i : P_i^\bullet \rightarrow P_i^\bullet$  in  $C^p(\Lambda(G)_{S^*})$  which is bijective in each degree and such that the following diagram commutes in  $D^p(\Lambda(G)_{S^*})$

$$\begin{array}{ccc} P_i^\bullet & \xrightarrow{\psi_i} & I_H^G(H^i(A^\bullet))_{S^*}[0] \\ \alpha_i \downarrow & & \downarrow \delta_{G,\gamma,H^i(A^\bullet)}[0] \\ P_i^\bullet & \xrightarrow{\psi_i} & I_H^G(H^i(A^\bullet))_{S^*}[0]; \end{array}$$

one then has  $\mathcal{E}_{G,\gamma}(H^i(A^\bullet)) = \prod_{j \in \mathbb{Z}} \langle \alpha_i^j \mid P_i^j \rangle^{(-1)^j}$ . Hence

$$\begin{aligned} \pi_{\overline{G}}(\mathcal{L}_{G,\gamma}^{\text{alg}}(A^\bullet)) &= \prod_{i \in \mathbb{Z}} \pi_{\overline{G}}(\mathcal{E}_{G,\gamma}(H^i(A^\bullet)))^{(-1)^i} \\ &= \prod_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} \langle \text{id} \otimes \alpha_i^j \mid \Lambda(\overline{G})_{\overline{S}^*} \otimes_{\Lambda(G)_{S^*}} P_i^j \rangle^{(-1)^{i+j}} \\ (2) \quad &= \prod_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} \langle H^j(\text{id} \otimes \alpha_i) \mid H^j(\Lambda(\overline{G})_{\overline{S}^*} \otimes_{\Lambda(G)_{S^*}} P_i^\bullet) \rangle^{(-1)^{i+j}} \end{aligned}$$

where the last equality follows from the regularity of  $\Lambda(\overline{G})_{\overline{S}^*}$  and the defining relations of  $K_1(\Lambda(\overline{G})_{\overline{S}^*})$ . Now there are natural isomorphisms of  $\Lambda(\overline{G})_{\overline{S}^*}$ -modules

$$H^j(\Lambda(\overline{G})_{\overline{S}^*} \otimes_{\Lambda(G)_{S^*}} P_i^\bullet) \cong I_H^{\overline{G}}(\text{Tor}_j^{\Lambda(G)}(\Lambda(\overline{G}), H^i(A^\bullet)))_{\overline{S}^*}$$

under which  $H^j(\text{id} \otimes \alpha_i)$  corresponds to the endomorphism  $\delta_{\overline{G},\gamma, \text{Tor}_j^{\Lambda(G)}(\Lambda(\overline{G}), H^i(A^\bullet))}$

The product expression (2) is therefore equal to

$$\begin{aligned}
& \prod_{i \in \mathbb{Z}} \prod_{j \in \mathbb{Z}} \langle \delta_{\overline{G}, \gamma, \text{Tor}_j^{\Lambda(G)}(\Lambda(\overline{G}), H^i(A^\bullet))} \mid \mathbb{1}_{\overline{H}}(\text{Tor}_j^{\Lambda(G)}(\Lambda(\overline{G}), H^i(A^\bullet)))_{\overline{S}^*} \rangle^{(-1)^{i+j}} \\
&= \prod_{i \in \mathbb{Z}} \langle \delta_{\overline{G}, \gamma, H^i(\Lambda(\overline{G}) \hat{\otimes}_{\Lambda(G)}^{\mathbb{L}} A^\bullet)} \mid \mathbb{1}_{\overline{H}}(H^i(\Lambda(\overline{G}) \hat{\otimes}_{\Lambda(G)}^{\mathbb{L}} A^\bullet))_{\overline{S}^*} \rangle^{(-1)^i} \\
&= \prod_{i \in \mathbb{Z}} \mathcal{E}_{\overline{G}, \gamma}(H^i(\Lambda(\overline{G}) \hat{\otimes}_{\Lambda(G)}^{\mathbb{L}} A^\bullet))^{(-1)^i} \\
&= \mathcal{L}_{\overline{G}, \gamma}^{\text{alg}}(\Lambda(\overline{G}) \hat{\otimes}_{\Lambda(G)}^{\mathbb{L}} A^\bullet)
\end{aligned}$$

where the first displayed equality is a consequence of the spectral sequence  $E_2^{r,s} = \text{Tor}_r^{\Lambda(G)}(\Lambda(\overline{G}), H^s(A^\bullet)) \implies H^{s-r}(\Lambda(\overline{G}) \hat{\otimes}_{\Lambda(G)}^{\mathbb{L}} A^\bullet)$ .  $\square$

*Remark 3.3.* (Akashi series) If  $G$  has no element of order  $p$ , then for each module  $M$  in  $\mathfrak{M}_S(G)$  the complex  $M[0]$  belongs to  $D_S^{\mathbb{P}}(\Lambda(G))$ . Proposition 3.2 (with  $\overline{G} = \Gamma$ ) thus combines with Lemma 2.3 to imply that the composite homomorphism  $K_1(Q(\Gamma)) \cong Q(\Gamma)^\times \rightarrow Q(\Gamma)^\times / \Lambda(\Gamma)^\times$  sends the element  $\pi_\Gamma(\mathcal{L}_{\overline{G}, \gamma}^{\text{alg}}(M[0])) = \mathcal{L}_{\Gamma, \gamma}^{\text{alg}}(\Lambda(\Gamma) \hat{\otimes}_{\Lambda(G)}^{\mathbb{L}} M[0])$  to the ‘Akashi series’  $f_M$  of  $M$  that is introduced by Coates, Schneider and Sujatha in [5, §4] (and is denoted by  $\text{Ak}(M)$  in [4, §3]).

#### §4. Characteristic Elements

We write  $G_0(\mathfrak{M}_{S^*}(G))$  for the Grothendieck group of the category  $\mathfrak{M}_{S^*}(G)$  and for each module  $M$  in  $\mathfrak{M}_{S^*}(G)$  we write  $[M]$  for the associated element of  $G_0(\mathfrak{M}_{S^*}(G))$ . We also write  $K_0(\Lambda(G), \Lambda(G)_{S^*})$  for the relative algebraic  $K_0$ -group of the natural homomorphism  $\Lambda(G) \rightarrow \Lambda(G)_{S^*}$  and recall that this group is generated by triples of the form  $(P, \kappa, Q)$  where  $P$  and  $Q$  are finitely generated projective  $\Lambda(G)$ -modules and  $\kappa$  is an isomorphism of  $\Lambda(G)_{S^*}$ -modules  $P_{S^*} \cong Q_{S^*}$  (for further details see [9, p. 215]).

If  $G$  has no element of order  $p$ , then  $\Lambda(G)$  is a noetherian regular ring and the groups  $K_0(\Lambda(G), \Lambda(G)_{S^*})$  and  $G_0(\mathfrak{M}_{S^*}(G))$  are naturally isomorphic. We normalise this isomorphism in the following way: if  $g = s^{-1}h$  with  $s \in S^*$  and  $h \in \Lambda(G) \cap \Lambda(G)_{S^*}^\times$ , then the element  $(\Lambda(G), r_g, \Lambda(G))$  of  $K_0(\Lambda(G), \Lambda(G)_{S^*})$  corresponds to the element  $[\text{cok}(r_h)] - [\text{cok}(r_s)]$  of  $G_0(\mathfrak{M}_{S^*}(G))$  where  $r_g, r_h$  and  $r_s$  denote the automorphisms of  $\Lambda(G)_{S^*}$  that are induced by right multiplication by  $g, h$  and  $s$  respectively.

We next note that, irrespective of whether  $G$  has an element of order  $p$ , each complex  $A^\bullet$  in  $D_{S^*}^{\mathbb{P}}(\Lambda(G))$  gives rise to a canonical ‘euler character-

istic' element  $\chi(A^\bullet)$  in  $K_0(\Lambda(G), \Lambda(G)_{S^*})$ . We define this element by identifying  $K_0(\Lambda(G), \Lambda(G)_{S^*})$  with  $\pi_0$  of a natural Picard category that is constructed from the categories of virtual objects  $\mathcal{V}(\Lambda(G))$  and  $\mathcal{V}(\Lambda(G)_{S^*})$  associated to the categories of finitely generated projective  $\Lambda(G)$ -modules and finitely generated projective  $\Lambda(G)_{S^*}$ -modules respectively (for further details see, for example, [1, Lem. 5.1]). Then, with respect to this identification, we let  $\chi(A^\bullet)$  denote the *inverse* of the element of  $K_0(\Lambda(G), \Lambda(G)_{S^*})$  that corresponds to the pair  $([P^\bullet], \iota_{P^\bullet})$  where  $[P^\bullet]$  is the object of  $\mathcal{V}(\Lambda(G))$  associated to any  $P^\bullet$  in  $C^p(\Lambda(G))$  that is isomorphic in  $D^p(\Lambda(G))$  to  $A^\bullet$  and  $\iota_{P^\bullet}$  is the morphism in  $\mathcal{V}(\Lambda(G)_{S^*})$  associated to the isomorphism  $\Lambda(G)_{S^*} \otimes_{\Lambda(G)} P^\bullet \cong \Lambda(G)_{S^*} \hat{\otimes}_{\Lambda(G)} P^\bullet \cong \Lambda(G)_{S^*} \hat{\otimes}_{\Lambda(G)} A^\bullet \cong 0$  in  $D^p(\Lambda(G)_{S^*})$ . This element  $\chi(A^\bullet)$  is the inverse of the element  $\chi_{\Lambda(G), \Lambda(G)_{S^*}}(A^\bullet, t)$  that is defined in [1, Def. 5.5] with  $t$  equal to the isomorphism  $\bigoplus_{i \in \mathbb{Z}} H^{2i}(A^\bullet)_{S^*} \cong 0 \cong \bigoplus_{i \in \mathbb{Z}} H^{2i+1}(A^\bullet)_{S^*}$ . (We prefer to define  $\chi(A^\bullet)$  in terms of the inverse in order to ensure that if  $G$  has no element of order  $p$  and  $M$  belongs to  $\mathfrak{M}_{S^*}(G)$ , then the isomorphism  $K_0(\Lambda(G), \Lambda(G)_{S^*}) \cong G_0(\mathfrak{M}_{S^*}(G))$  described above sends  $\chi[M[0]]$  to  $[M]$ .)

In the next result we use the natural connecting homomorphisms in  $K$ -theory

$$\partial'_G : K_1(\Lambda(G)_{S^*}) \cong G_1(\Lambda(G)_{S^*}) \rightarrow G_0(\mathfrak{M}_{S^*}(G))$$

and

$$\partial_G : K_1(\Lambda(G)_{S^*}) \rightarrow K_0(\Lambda(G), \Lambda(G)_{S^*}).$$

**Theorem 4.1.** *Let  $A^\bullet$  be a complex in  $D_S^p(\Lambda(G))$ .*

- (i)  $\partial'_G(\mathcal{L}_{G,\gamma}^{\text{alg}}(A^\bullet)) = \sum_{i \in \mathbb{Z}} (-1)^i [H^i(A^\bullet)]$ .
- (ii) *If  $G$  has rank one (as a  $p$ -adic Lie group), then  $\partial_G(\mathcal{L}_{G,\gamma}^{\text{alg}}(A^\bullet)) = \chi(A^\bullet)$ .*

*Remark 4.2.* (Characteristic elements) In the setting of Lemma 2.3 claim (i) of this result recovers the fact that  $\text{char}_T(M)$  generates the characteristic ideal of  $M$ . More generally, claim (i) implies that if  $G$  has no element of order  $p$ , then  $\mathcal{L}_{G,\gamma}^{\text{alg}}(A^\bullet)$  is a 'characteristic element of  $A^\bullet$ ' in the sense of [4, (33)]. (If  $G$  has rank one, then) the equality of claim (ii) is in general much finer than that of claim (i) and plays a key role in the descent theory formulated in [3].

#### §4.1. The proof of Theorem 4.1(i)

We write  $\partial''_G : G_1(\Lambda(G)_{S^*}) \rightarrow G_0(\mathfrak{M}_{S^*}(G))$  for the natural connecting homomorphism (that occurs in the above definition of  $\partial'_G$ ) and recall that if  $\nu$

is an automorphism of a finitely generated  $\Lambda(G)_{S^*}$ -module  $N$ , then

$$(3) \quad \partial'_G(\langle \nu, N \rangle) = [N/(\mathcal{N} \cap \nu(\mathcal{N}))] - [\nu(\mathcal{N})/(\mathcal{N} \cap \nu(\mathcal{N}))]$$

where  $\mathcal{N}$  is any finitely generated  $\Lambda(G)$ -submodule of  $N$  with  $\mathcal{N}_{S^*} = N$ .

For each integer  $i$  we set  $M^i := H^i(A^\bullet)$ . Then Lemma 4.3(ii) below (with  $M = M^i$ ) implies that the (finitely generated)  $\Lambda(G)$ -module  $\mathbb{I}_H^G(M_{\text{tf}}^i)$  is isomorphic to its image in  $\mathbb{I}_H^G(M^i)_{S^*}$ . Hence, if we set  $\delta_\gamma^i := \delta_{G, \gamma, M^i}$ , then in  $G_0(\mathfrak{M}_{S^*}(G))$  one has

$$\begin{aligned} \partial'_G(\mathcal{L}_{G, \gamma}^{\text{alg}}(A^\bullet)) &= \sum_{i \in \mathbb{Z}} (-1)^i \partial'_G(\langle \delta_\gamma^i | \mathbb{I}_H^G(M^i)_{S^*} \rangle) = \sum_{i \in \mathbb{Z}} (-1)^i [\mathbb{I}_H^G(M_{\text{tf}}^i) / \delta_\gamma^i(\mathbb{I}_H^G(M_{\text{tf}}^i))] \\ &= \sum_{i \in \mathbb{Z}} (-1)^i [M_{\text{tf}}^i] = \sum_{i \in \mathbb{Z}} (-1)^i ([M^i] - [M_{\text{tor}}^i]). \end{aligned}$$

Here the second equality follows from (3) (with  $\mathcal{N} = \mathbb{I}_H^G(M_{\text{tf}}^i)$  and  $\nu = \delta_\gamma^i$ ) and the fact that  $\delta_\gamma^i(\mathbb{I}_H^G(M_{\text{tf}}^i)) \subseteq \mathbb{I}_H^G(M_{\text{tf}}^i)$  and the third from the isomorphism  $\mathbb{I}_H^G(M_{\text{tf}}^i) / \delta_\gamma^i(\mathbb{I}_H^G(M_{\text{tf}}^i)) \cong M_{\text{tf}}^i$  that is induced by (1) with  $M = M_{\text{tf}}^i$ . Given the last displayed formula, the proof of Theorem 4.1(i) is therefore completed by applying Lemma 4.3(i) below with  $M = M^i$  (for each  $i$ ).

**Lemma 4.3.**

- (i) *If  $M$  belongs to  $\mathfrak{M}_S(G)$ , then  $[M_{\text{tor}}] = 0$  in  $G_0(\mathfrak{M}_{S^*}(G))$ .*
- (ii) *If  $M$  belongs to  $\mathfrak{M}_{S^*}(G)$ , then the natural map  $\mathbb{I}_H^G(M_{\text{tf}}) \rightarrow \mathbb{I}_H^G(M)_{S^*}$  is injective.*

*Proof.* Claim (i) follows from the fact that if  $M$  is in  $\mathfrak{M}_S(G)$ , then (1) with  $M = M_{\text{tor}}$  is a short exact sequence of objects of  $\mathfrak{M}_{S^*}(G)$  and hence implies that  $[M_{\text{tor}}] = [\mathbb{I}_H^G(M_{\text{tor}})] - [\mathbb{I}_H^G(M_{\text{tor}})] = 0$  in  $G_0(\mathfrak{M}_{S^*}(G))$ .

To prove claim (ii) we note that  $\mathbb{I}_H^G(M_{\text{tf}})$  is a finitely generated  $\Lambda(G)$ -module that is  $\mathbb{Z}_p$ -torsion-free (and hence that a  $\Lambda(G)$ -submodule of  $\mathbb{I}_H^G(M_{\text{tf}})$  belongs to  $\mathfrak{M}_{S^*}(G)$  if and only if it belongs to  $\mathfrak{M}_S(G)$ ). It therefore suffices to prove that if  $M$  is any  $\mathbb{Z}_p$ -torsion-free module in  $\mathfrak{M}_S(G)$  and  $N$  a  $\Lambda(G)$ -submodule of  $\mathbb{I}_H^G(M)$  that is finitely generated over  $\Lambda(H)$ , then  $N = 0$ . To do this we fix a pro- $p$  open subgroup  $J$  of  $H$  that is normal in  $G$ . We write  $I(J)$  for the kernel of the projection map  $\Lambda(G) \rightarrow \Lambda(G/J)$  (so  $M_J \cong M/I(J)M$ ) and let  $\overline{N}$  denote the image of  $N$  under the canonical projection  $\mathbb{I}_H^G(M) \rightarrow \mathbb{I}_H^G(M_J) \cong \mathbb{I}_{H/J}^{G/J}(M_J)$ .

Now if  $\tilde{\Gamma}$  denotes the subgroup of  $G/J$  that is generated topologically by a choice of pre-image  $\tilde{\gamma}$  of  $\gamma$  under the surjection  $G/J \rightarrow \Gamma$ , then  $\mathbb{I}_{H/J}^{G/J}(M_J)$  is

isomorphic as a  $\Lambda(\tilde{\Gamma})$ -module to  $\Lambda(\tilde{\Gamma}) \otimes_{\mathbb{Z}_p} M_J$  and  $\overline{N}$  identifies with a  $\Lambda(\tilde{\Gamma})$ -submodule of  $\Lambda(\tilde{\Gamma}) \otimes_{\mathbb{Z}_p} M_J$  that is finitely generated over  $\Lambda(H/J) = \mathbb{Z}_p[H/J]$  and hence also over  $\mathbb{Z}_p$ . We set  $t := \tilde{\gamma} - 1 \in \Lambda(\tilde{\Gamma})$ . Now  $\overline{N}_{\text{tor}}$  is a finite  $\Lambda(\tilde{\Gamma})$ -submodule of  $\Lambda(\tilde{\Gamma}) \otimes_{\mathbb{Z}_p} M_{J,\text{tor}} \subseteq \Lambda(\tilde{\Gamma}) \otimes_{\mathbb{Z}_p} M_J$  and so if  $x \in \overline{N}_{\text{tor}}$ , then there exist integers  $r > s > 0$  such that  $t^r x = t^s x$ . But every element of  $\Lambda(\tilde{\Gamma}) \otimes_{\mathbb{Z}_p} M_{J,\text{tor}}$  can be written uniquely in the form  $\sum_{a \geq 0} t^a \otimes y_a$  with  $y_a \in M_{J,\text{tor}}$  and so  $t^r x = t^s x$  implies that  $x = 0$ . It follows that  $\overline{N}_{\text{tor}} = 0$  and hence we can regard  $\overline{N}$  as a  $\Lambda(\tilde{\Gamma})$ -submodule of  $\Lambda(\tilde{\Gamma}) \otimes_{\mathbb{Z}_p} M_{J,\text{tf}}$  which is itself finitely generated over  $\mathbb{Z}_p$ . But  $M_{J,\text{tf}}$  is a free  $\mathbb{Z}_p$ -module so  $\Lambda(\tilde{\Gamma}) \otimes_{\mathbb{Z}_p} M_{J,\text{tf}}$  is a free  $\Lambda(\tilde{\Gamma})$ -module and hence cannot contain any non-zero  $\Lambda(\tilde{\Gamma})$ -submodule which is finitely generated over  $\mathbb{Z}_p$ . Hence  $\overline{N} = 0$ . From the exact sequence  $0 \rightarrow \mathbb{I}_H^G(I(J)M) \rightarrow \mathbb{I}_H^G(M) \rightarrow \mathbb{I}_H^G(M_J) \rightarrow 0$  we therefore deduce that  $N \subseteq \mathbb{I}_H^G(I(J)M)$ .

By successively repeating the above argument with  $M$  replaced by  $I(J)M$ , then  $I(J)^2M$  etc., we deduce that  $N$  is contained in  $\bigcap_{k \geq 0} \mathbb{I}_H^G(I(J)^k M)$ . Now  $I(J)^{k'} M \subseteq I(J)^k M$  and hence also  $\mathbb{I}_H^G(I(J)^{k'} M) \subseteq \mathbb{I}_H^G(I(J)^k M)$  for each  $k' \geq k$  and so the intersections  $\bigcap_{k \geq 0} I(J)^k M$  and  $\bigcap_{k \geq 0} \mathbb{I}_H^G(I(J)^k M)$  can both be computed as inverse limits. Since completed tensor products commute with inverse limits it follows that  $\bigcap_{k \geq 0} \mathbb{I}_H^G(I(J)^k M) = \mathbb{I}_H^G(\bigcap_{k \geq 0} I(J)^k M)$ . But  $I(J)$  is contained in the radical of  $\Lambda(G)$  (since  $J$  is pro- $p$ ) and so  $\bigcap_{k \geq 0} I(J)^k M = 0$  (cf. [7, Prop. (5.2.17)]). Hence  $N = 0$ , as required.  $\square$

#### §4.2. The proof of Theorem 4.1(ii)

In this subsection we assume that  $G$  has rank one as a  $p$ -adic Lie group and hence that  $\Lambda(G)_{S^*}$  is equal to the semisimple artinian ring  $Q(G)$ . We note first that since  $A^\bullet$  belongs to  $D_S^p(\Lambda(G))$  the complex  $\mathbb{I}_H^G(A^\bullet)$  belongs to  $D^p(\Lambda(G))$ . We may thus choose a complex  $P^\bullet$  in  $C^p(\Lambda(G))$  for which there exists an isomorphism  $\psi : P^\bullet \xrightarrow{\sim} \mathbb{I}_H^G(A^\bullet)$  in  $D^p(\Lambda(G))$  and a morphism  $\alpha : P^\bullet \rightarrow P^\bullet$  in  $C^p(\Lambda(G))$  such that the following diagram commutes in  $D^p(\Lambda(G))$

$$(4) \quad \begin{array}{ccc} P^\bullet & \xrightarrow{\alpha} & P^\bullet \\ \psi \downarrow & & \downarrow \psi \\ \mathbb{I}_H^G(A^\bullet) & \xrightarrow{\delta_\gamma^\bullet} & \mathbb{I}_H^G(A^\bullet), \end{array}$$

where  $\delta_\gamma^\bullet$  is the morphism with  $\delta_\gamma^i = \delta_{G,\gamma,A^i}$  in each degree  $i$ . Now  $H^i(\mathbb{I}_H^G(A^\bullet)) = \mathbb{I}_H^G(H^i(A^\bullet))$  and so (1) with  $M = H^i(A^\bullet)$  implies that  $H^i(\delta_\gamma^\bullet)$  (and therefore also  $H^i(\alpha)$ ) is injective in each degree  $i$ . Hence, by using Lemma 4.4 below, we

may change  $\alpha$  by a homotopy in order to assume that  $\alpha^i$  is itself injective in each degree  $i$ . Thus there exists a short exact sequence of (bounded) complexes of finitely generated  $\Lambda(G)$ -modules

$$(5) \quad 0 \rightarrow P^\bullet \xrightarrow{\alpha} P^\bullet \rightarrow \text{cok}(\alpha)^\bullet \rightarrow 0$$

where  $\text{cok}(\alpha)^i = \text{cok}(\alpha^i)$  in each degree  $i$  and the differentials of  $\text{cok}(\alpha)^\bullet$  are induced by those of  $P^\bullet$ . This sequence implies that for each  $i$  the complex  $\text{cok}(\alpha)^i[-i]$  is naturally quasi-isomorphic to  $P^i \xrightarrow{\alpha^i} P^i$ , where the first term occurs in degree  $i-1$ , and hence both belongs to  $D_{S^*}^p(\Lambda(G))$  and also satisfies

$$(6) \quad \chi(\text{cok}(\alpha)^i[-i]) = (-1)^i \langle P^i, \alpha^i, P^i \rangle = \partial_G(\langle \alpha^i | P_{S^*}^i \rangle^{(-1)^i}),$$

where the first equality is a consequence of our chosen normalisation of  $\chi(-)$  and the second a consequence of the definition of  $\partial_G$ .

Next we combine the exact sequence (5) with the commutativity of (4) to deduce that  $\text{cok}(\alpha)^\bullet$  is isomorphic in  $D^p(\Lambda(G))$  to the mapping cone  $\text{cone}(\delta_\gamma^\bullet)$  of  $\delta_\gamma^\bullet$ . On the other hand, by using the exact sequences (1) with  $M = H^i(A^\bullet)$  for each  $i$ , it is straightforward to show that the morphism  $\text{cone}(\delta_\gamma^\bullet) \rightarrow A^\bullet$  which, in each degree  $i$ , sends  $(x^i, x^{i+1}) \in \text{I}_H^G(A^i) \oplus \text{I}_H^G(A^{i+1}) = \text{cone}(\delta_\gamma^\bullet)^i$  to the image of  $x^i$  under the natural map  $\text{I}_H^G(A^i) \rightarrow A^i$  is a quasi-isomorphism. It follows that  $\text{cok}(\alpha)^\bullet$  is isomorphic in  $D^p(\Lambda(G))$  to  $A^\bullet$  and hence that  $\chi(A^\bullet) = \chi(\text{cok}(\alpha)^\bullet)$  by [1, Prop. 5.6]. Now in each degree  $i$  there is an exact sequence of complexes  $0 \rightarrow \text{cok}(\alpha)^i[-i] \rightarrow \tau_{\leq i}(\text{cok}(\alpha)^\bullet) \rightarrow \tau_{\leq i-1}(\text{cok}(\alpha)^\bullet) \rightarrow 0$  where  $\tau_{\leq d}$  denotes naive truncation in degree  $d$ . By applying [1, Th. 5.7] to each of these exact sequences we obtain an equality  $\chi(\text{cok}(\alpha)^\bullet) = \sum_{i \in \mathbb{Z}} \chi(\text{cok}(\alpha)^i[-i])$  and hence

$$\begin{aligned} \partial_G(\mathcal{L}_{G,\gamma}^{\text{alg}}(A^\bullet)) &= \partial_G \left( \prod_{i \in \mathbb{Z}} \langle \delta_{G,\gamma,H^i(A^\bullet)} | \text{I}_H^G(H^i(A^\bullet))_{S^*} \rangle^{(-1)^i} \right) \\ &= \partial_G \left( \prod_{i \in \mathbb{Z}} \langle H^i(\alpha) | H^i(P^\bullet)_{S^*} \rangle^{(-1)^i} \right) \\ &= \partial_G \left( \prod_{i \in \mathbb{Z}} \langle \alpha^i | P_{S^*}^i \rangle^{(-1)^i} \right) \\ &= \sum_{i \in \mathbb{Z}} \chi(\text{cok}(\alpha)^i[-i]) \\ &= \chi(A^\bullet) \end{aligned}$$

where the second equality follows from the commutativity of (4), the third from the defining relations of  $K_1(\Lambda(G)_{S^*})$  and the fourth from (6). This completes our proof of Theorem 4.1(ii).

**Lemma 4.4.** *Let  $P^\bullet$  in  $C^p(\Lambda(G))$  be as in (4) and assume that  $G$  has rank one. If  $\alpha : P^\bullet \rightarrow P^\bullet$  is any morphism of complexes for which  $H^i(\alpha)$  is injective in each degree  $i$ , then there is a morphism of complexes  $\hat{\alpha} : P^\bullet \rightarrow P^\bullet$  that is homotopic to  $\alpha$  and injective in each degree.*

*Proof.* In each degree  $i$  there are tautological exact sequences

$$(7) \quad \begin{array}{ccccccc} 0 & \longrightarrow & Z^i & \longrightarrow & P^i & \xrightarrow{d^i} & B^{i+1} \longrightarrow 0 \\ & & & & & & \\ 0 & \longrightarrow & B^i & \longrightarrow & Z^i & \longrightarrow & H^i \longrightarrow 0 \end{array}$$

where  $B^i$ ,  $Z^i$  and  $H^i$  are the coboundaries, cocycles and cohomology of  $P^\bullet$ .

Now every finitely generated  $\Lambda(G)[\frac{1}{p}]$ -module is of projective dimension at most one. (Indeed, by [7, Prop. (5.3.19)(i)], this claim is true if we replace  $G$  by any (open) subgroup that is topologically isomorphic to  $\mathbb{Z}_p$  and then the result for  $G$  itself follows by the argument of [6, Prop. 4.3.4].) Thus, the (image under  $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} -$  of the) exact sequences (7) allow one to prove, by descending induction on  $i$ , that each  $\Lambda(G)[1/p]$ -module  $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} B^i$  is projective. The upper sequence of (7) therefore splits after applying  $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} -$  and so in each degree  $i$  we can choose a  $\Lambda(G)$ -submodule  $\hat{B}^{i+1}$  of  $P^i$  such that  $\hat{B}^{i+1} \cap Z^i = 0$  and  $B^{i+1}/d^i(\hat{B}^{i+1})$  is  $p$ -torsion.

We next construct a homomorphism of  $\Lambda(G)$ -modules  $k^i : P^i \rightarrow \hat{B}^i$  with

$$(8) \quad k^i(\hat{B}^{i+1}) = 0$$

and such that the quotient module

$$(9) \quad B^i / (d^{i-1} \circ (\alpha^{i-1} + k^i \circ d^{i-1}))(\hat{B}^i)$$

is  $p$ -torsion. To do this we claim first that the cokernel of the map

$$\eta^i : \text{Hom}_{\Lambda(G)}(P^i / \hat{B}^{i+1}, \hat{B}^i) \rightarrow \text{Hom}_{\Lambda(G)}(B^i, \hat{B}^i)$$

that is induced by the composite  $B^i \subset P^i \rightarrow P^i / \hat{B}^{i+1}$  is  $p$ -torsion. Indeed, the natural map  $Z^i \rightarrow P^i / \hat{B}^{i+1}$  is injective with  $p$ -torsion cokernel and so there is a natural complex  $\text{Hom}_{\Lambda(G)}(P^i / \hat{B}^{i+1}, \hat{B}^i) \rightarrow \text{Hom}_{\Lambda(G)}(B^i, \hat{B}^i) \rightarrow \text{Ext}_{\Lambda(G)}^1(H^i(P^\bullet), \hat{B}^i)$  which has  $p$ -torsion cohomology in the central degree. Thus one has  $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \text{cok}(\eta^i) = 0$  if  $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \text{Ext}_{\Lambda(G)}^1(H^i(P^\bullet), \hat{B}^i) = 0$ . But  $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \hat{B}^i$  is a projective  $\Lambda(G)[1/p]$ -module and so  $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \text{Ext}_{\Lambda(G)}^1(H^i(P^\bullet), \hat{B}^i) = 0$  if  $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \text{Ext}_{\Lambda(G)}^1(H^i(P^\bullet), \Lambda(G)) = 0$ . Moreover, the module  $H^i(P^\bullet)$  is isomorphic to  $\Lambda(G) \otimes_{\Lambda(H)} H^i(A^\bullet)$  and so  $\text{Ext}_{\Lambda(G)}^1(H^i(P^\bullet), \Lambda(G))$  is isomorphic to  $\text{Ext}_{\Lambda(H)}^1(H^i(A^\bullet), \Lambda(H)) \otimes_{\Lambda(H)} \Lambda(G)$  (cf. the discussion following [8,

Prop. 3.1]). To prove that  $\text{cok}(\eta^i)$  is  $p$ -torsion it is thus enough to note that  $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \text{Ext}_{\Lambda(H)}^1(H^i(A^\bullet), \Lambda(H)) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \text{Ext}_{\mathbb{Z}_p[H]}^1(H^i(A^\bullet), \mathbb{Z}_p[H])$  vanishes because  $H$  is finite.

Next we note that the vanishing of  $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} (B^{i+1}/d^i(\hat{B}^{i+1}))$  implies that the kernel and cokernel of the homomorphism  $\text{Hom}_{\Lambda(G)}(B^i, \hat{B}^i) \rightarrow \text{Hom}_{\Lambda(G)}(\hat{B}^i, B^i)$  that sends  $k$  to  $d^{i-1} \circ k \circ d^{i-1}$  are both  $p$ -torsion. Since  $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \text{cok}(\eta^a) = 0$  in each degree  $a$  this allows us to choose  $k^i$  in  $\text{Hom}_{\Lambda(G)}(P^i/\hat{B}^{i+1}, \hat{B}^i)$  so that  $d^{i-1} \circ \alpha^i + d^{i-1} \circ k^i \circ d^{i-1} \in \text{Hom}_{\Lambda(G)}(\hat{B}^i, B^i)$  has  $p$ -torsion kernel and  $p$ -torsion cokernel. Then these homomorphisms  $k^i$  satisfy both (8) and (9) and we set  $\hat{\alpha}^i := \alpha^i - (d^{i-1} \circ k^i + k^{i+1} \circ d^i)$ . We thereby obtain a morphism of complexes  $\hat{\alpha}$  that is homotopic to  $\alpha$  via the homotopy  $\{-k^i\}_{i \in \mathbb{Z}}$ . Also (8) and (9) combine to imply that  $B^{i+1}/d^i(\hat{\alpha}^i(\hat{B}^{i+1}))$  is  $p$ -torsion in each degree  $i$ . But each map  $d^i : \hat{B}^{i+1} \rightarrow B^i$  is injective with  $p$ -torsion cokernel and hence  $\hat{\alpha}^{i+1} : B^{i+1} \rightarrow B^{i+1}$  is also injective with  $p$ -torsion cokernel in each degree  $i+1$ . In addition  $H^i(\hat{\alpha}) = H^i(\alpha)$  is assumed to be injective in each degree  $i$  and hence, by an easy exercise involving the exact sequences (7), we may deduce that each homomorphism  $\hat{\alpha}^i$  is itself injective, as required.  $\square$

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