Proceedings of
THE 37TH TANIGUCHI SYMPOSIUM ON
TOPOLOGY AND TEICHMÜLLER SPACES
held in Finland, July 1995
ed. by Sadayoshi KOJIMA *et al.*

pp. 253–264

CHARACTER VARIETY OF REPRESENTATIONS OF A FINITELY GENERATED GROUP IN SL_2

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(Received January 10, 1996)

This is a partial exposition of [S4, S5, S6], which study the space of representations of a (finitely generated) group Γ in SL_2 and GL_2 in an attempt of its application in geometry: Teichmüller spaces, knot theory, hyperbolic manifolds, moduli spaces, $\cdots$, etc. (see for instance, [A], [Be], [Br], [C-C-G-L-S], [C-S], [F-K], [G], [H], [H-L-M], [H-K], [J-W], [K], [Kj], [Ko], [Kr], [Ma], [Mu], [N-Z], [O], [S], [S-S], [T], [W], [We], [Wo], [Y], $\cdots$, etc). For simplicity, we omit the case for GL_2 in the present exposition. Let us explain the main result of the present paper.

Let Γ be a group. The purpose of the present paper is to introduce the character variety $\mathrm{Ch}(\Gamma, \mathrm{SL}_2)$ in order to parameterize conjugacy classes of representations of Γ in SL_2 in a functorial way. At first, the character variety is introduced as a scheme over $\mathbb{Z}$, independent of the coefficient ring of representations in question. Then, the scaler field is specialized to $\mathbb{R}$ to obtain results on representations in $\mathrm{SL}_2(\mathbb{R})$ and in $\mathrm{SU}(2)$ with respect to the classical topology. Let us explain this briefly.

Let $\mathrm{Hom}(\Gamma, \mathrm{SL}_n)$ be the functor $R \in \{\text{commutative rings with } 1\} \mapsto \mathrm{Hom}(\Gamma, \mathrm{SL}_n(R)) \in \{\text{sets}\}$. The functor is representable (see §1.3 Lemma) and so, for an abuse of notation, we denote by the same $\mathrm{Hom}(\Gamma, \mathrm{SL}_n)$ the scheme over $\mathbb{Z}$ representing the functor. The group scheme PGL_n acts on $\mathrm{Hom}(\Gamma, \mathrm{SL}_n)$. Whether the universal categorical quotient $\mathrm{Hom}(\Gamma, \mathrm{SL}_n)//\,\mathrm{PGL}_n$ (Mumford [Mu1]) defined over $\mathbb{Z}$ exists or not seems to be a hard and unsolved question. Instead of asking directly for the quotient space, we introduce *i*) the *character variety* $\mathrm{Ch}(\Gamma, \mathrm{SL}_2)$ together with its *discriminant subvariety* D_Γ as schemes over $\mathbb{Z}$ abstractly, and *ii*) the PGL_2-*invariant morphism* π_Γ: $\mathrm{Hom}(\Gamma, \mathrm{SL}_2) \to \mathrm{Ch}(\Gamma, \mathrm{SL}_2)$, for which we prove that *i*) *the restriction of* π_Γ *on the complement of* $\pi_\Gamma^{-1}(D_\Gamma)$ *is a principal* PGL_2-*bundle with respect to the etal topology, and ii*) *the inverse image* $\pi_\Gamma^{-1}(D_\Gamma)$ *is a subfunctor of* $\mathrm{Hom}(\Gamma, \mathrm{SL}_2)$ *consisting of abelian or reducible representations.* This implies that the complement $\mathrm{Hom}^*(\Gamma, \mathrm{SL}_2) := \mathrm{Hom}(\Gamma, \mathrm{SL}_2) \backslash \pi^{-1}(D_\Gamma)$ consists of absolutely irreducible representations, and that $\mathrm{Hom}^*(\Gamma, \mathrm{SL}_2)$ has the universal categorical quotient space $\mathrm{Ch}^*(\Gamma, \mathrm{SL}_2) := \mathrm{Ch}(\Gamma, \mathrm{SL}_2) \backslash D_\Gamma$ defined over $\mathbb{Z}$. Then the result is specialized to $\mathbb{R}$-coefficient. Namely, *the complement* $\mathrm{Ch}^*(\Gamma,\mathrm{SL}_2)(\mathbb{R}) := \mathrm{Ch}(\Gamma,\mathrm{SL}_2)(\mathbb{R})\backslash D_\Gamma(\mathbb{R})$ *of the real discriminant decomposes into a disjoint union of two semialgebraic open components* $H_\Gamma(\mathbb{R})$ *and* $T_\Gamma(\mathbb{R})$ *such that i)* $\mathrm{Hom}^*(\Gamma,\mathrm{SL}_2)(\mathbb{R})$ *is a principal* $\mathrm{PGL}_2(\mathbb{R})$ *bundle over* $H_\Gamma(\mathbb{R})$*, and ii)* $\mathrm{Hom}^*(\Gamma,\mathrm{SU}_2(\mathbb{C}))$ *is a principal* $\mathrm{PU}(2)$ *bundle over* $T_\Gamma(\mathbb{R})$*, respectively.* This fact has an application: *the Teichmüller space* $\mathcal{T}_{g,n}$ *carries a natural semi-algebraic structure defined over* $\mathbb{Z}$ [S5].

The §1 prepares notations of a representation variety $\mathrm{Hom}(\Gamma,\mathrm{SL}_2)$ defined over $\mathbb{Z}$. The §2 studies the PGL_2-invariants of $M_2 \times M_2$ as the building block. The universal character ring $R(\Gamma,\mathrm{SL}_2)$ is introduced in §3, and we put $\mathrm{Ch}(\Gamma,\mathrm{SL}_2) := \mathrm{Spec}(R(\Gamma,\mathrm{SL}_2))$. The principal PGL_2-bundle structure on $\mathrm{Hom}^*(\Gamma,\mathrm{SL}_2)$ over $\mathrm{Ch}^*(\Gamma, SL_2)$ with respect to the etal topology is formulated in §4 Theorem A, and that with respect to the classical topology is formulated in §5 Theorem C.

§1. Universal representation of a group Γ in SL_n

This section is devoted for a preparation of notion and terminology for the representation varieties. One is referred to [Pr1, Pr2] [L-M] etc.

1.1. Let Γ be a group. As in the introduction, for a fixed $n \in \mathbb{Z}_{>0}$, $\mathrm{Hom}(\Gamma,\mathrm{SL}_n)$ is the functor: $R \in \{\text{commutative rings with } 1\} \mapsto \mathrm{Hom}(\Gamma,\mathrm{SL}_n(R)) \in \{\text{sets}\}$. In order to fix the notation for the present paper, we state precisely the representability of the functor in the next lemma.

Lemma (the representability of $\mathrm{Hom}(\Gamma,\mathrm{SL}_n)$).

(1) *For the given* Γ *and* $n \geq 1$*, there exists a pair* $(A(\Gamma,\mathrm{SL}_n),\sigma)$ *of a commutative ring* $A(\Gamma,\mathrm{SL}_n)$ *with 1 and representation* $\sigma\colon \Gamma \to \mathrm{SL}_n(A(\Gamma,\mathrm{SL}_n))$ *such that for any commutative ring R with 1, the correspondence:*

$$(1.1.1) \qquad \varphi \in \mathrm{Hom}^{ring}(A(\Gamma,\mathrm{SL}_n),R) \mapsto \varphi\circ\sigma \in \mathrm{Hom}^{gr}(\Gamma,\mathrm{SL}_n(R))$$

is a bijection.

(2) *The pair* $(A(\Gamma,\mathrm{SL}_n),\sigma)$ *is unique up to an isomorphism of the ring* $A(\Gamma,\mathrm{SL}_n)$ *which commutes with the universal representation* σ*.*

(3) *If* Γ *is a finitely generated group, then* $A(\Gamma,\mathrm{SL}_n)$ *is a finitely generated ring over* $\mathbb{Z}$ *and hence is noetherian.*

The lemma is proven by routine arguments. Here we give an explicit description of $(A(\Gamma,\mathrm{SL}_n),\sigma)$ without a proof. For each $\gamma \in \Gamma$, consider a $n \times n$ matrix:

$$(1.1.2) \qquad \sigma(\gamma) := (a_{ij}(\gamma))_{i,j=1,\cdots,n}$$

Then $A(\Gamma,\mathrm{SL}_n)$, called the *universal representation algebra,* is generated by the entries $a_{ij}(\Gamma)$ $i,j = 1,\cdots,n$ $\gamma \in \Gamma$, and divided by the ideal generated by all entries of the matrices $\sigma(e)-I_n$ ($I_n =$ the $n\times n$ unit matrix) and $\sigma(\gamma\delta)-\sigma(\gamma)\sigma(\delta)$ and $\det(\sigma(\gamma))-1$.

$$(1.1.3) \qquad A(\Gamma,\mathrm{SL}_n) := \mathbb{Z}[a_{ij}(\gamma) \text{ for } \gamma\in\Gamma \text{ and } 1\leq i,j\leq n]/I,$$

where

$$I := \Big(a_{ij}(e) - \delta_{ij},\, a_{ij}(\gamma\delta) - \sum_k a_{ik}(\gamma)a_{kj}(\delta),\, \det(\sigma(\gamma)) - 1 \\ \text{for } 1 \le i,j \le n \text{ and } \gamma,\delta \in \Gamma\Big).$$

By definition, the map $\sigma\colon \gamma \in \Gamma \to \sigma(\gamma) \in \mathrm{SL}_n(A(\Gamma, \mathrm{SL}_n))$ is a representation of Γ, which is called the *universal representation* of Γ in SL_n.

1.2. Let PGL_n be the group scheme ([SGAIII]), whose coordinate ring $A(\mathrm{PGL}_n)$ is given by the subring $A_0(\mathrm{GL}_n)$ of the coordinate ring $A(\mathrm{GL}_n) := \mathbb{Z}[x_{ij}\ 1 \le i,j \le n]_{\det(X)}$ (here $X := (x_{ij})_{i,j=1}^n$) of GL_n consisting of homogeneous elements of degree 0. The adjoint action of PGL_n on $\mathrm{Hom}(\Gamma, \mathrm{SL}_n)$ is written in terms of its dual action:

$$\text{(1.2.1)} \qquad \mathrm{Ad}\colon A(\Gamma, \mathrm{SL}_n) \to A(\Gamma, \mathrm{SL}_n) \otimes_{\mathbb{Z}} A(\mathrm{PGL}_n),$$

sending an entry of $\sigma(\gamma)$ to the same entry of $X^{-1}\sigma(\gamma)X$. Obviously, the coefficients of characteristic polynomial of $\sigma(\gamma)$ for $\gamma \in \Gamma$ are invariants under the adjoint action.

1.3. ¿From now on, we switch to the case $n = 2$. Let us list up some relations among the PGL_2-invariants $\mathrm{tr}(\sigma(\gamma))$ for $\gamma \in \Gamma$. The first one is trivial:

$$\text{(1.3.1)} \qquad \mathrm{tr}(\sigma(e)) = 2.$$

The next one follows from the Cayley-Hamilton relation: $\sigma(\gamma)^2 + \det(\sigma(\gamma)) \cdot I_2 = \mathrm{tr}(\sigma(\gamma)) \cdot \sigma(\gamma)$. Multiply $\sigma(\gamma^{-1}\delta)$ from right and take traces. So we obtain:

$$\text{(1.3.2)} \qquad \mathrm{tr}(\sigma(\gamma\delta)) + \mathrm{tr}(\sigma(\gamma^{-1}\delta)) = \mathrm{tr}(\sigma(\gamma)) \cdot \mathrm{tr}(\sigma(\delta))$$

for γ and $\delta \in \Gamma$ (cf. [F-K, formulas (2), pp.338]).

§2. PGL_2-invariants for pairs of 2×2 matrices

We study the invariants of the diagonal adjoint action of PGL_2 on the space $M_2 \times M_2$ of pair (A, B) of 2×2 matrices. The morphism $\tilde{\pi}\colon M_2 \times M_2 \to \mathbb{A}^5$ given by $(A, B) \mapsto (\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB), \det(A), \det(B))$ is shown to be the universal quotient map (§2.3 Lemma A,B). The discriminant Δ for $\tilde{\pi}$ is introduced (§2.4 Lemma C) so that the $\tilde{\pi}$ is a principal PGL_2-bundle on the complement of the discriminant loci $\Delta = 0$ (§2.5 Lemma D). These are building blocks of character variety defined over $\mathbb{Z}$.

2.1. Let $M_2 \times M_2$ be the space of pairs of 2×2 matrices. The $X \in \mathrm{PGL}_2$ acts on $(A, B) \in M_2 \times M_2$ from the right diagonally by letting $(A, B) \cdot \mathrm{Ad}(X) := (X^{-1}AX, X^{-1}BX)$. So we have the dual action on the coordinate ring:

$$\text{(2.1.1)} \qquad \mathrm{Ad}\colon \mathbb{Z}[M_2 \times M_2] \to \mathbb{Z}[M_2 \times M_2] \otimes_{\mathbb{Z}} A(\mathrm{PGL}_2)$$

sending an entry of (A, B) to the corresponding entry of $(A, B) \cdot \mathrm{Ad}(X)$ where $\mathbb{Z}[M_2 \times M_2]$ is the polynomial ring generated by 8 entries of (A, B).

2.2. Let us consider the morphism

$$\tilde{\pi}\colon M_2 \times M_2 \to \mathbb{A}^5 := \mathrm{Spec}(\mathbb{Z}[\underline{T}, \underline{D}]), \tag{2.2.1}$$

where $\mathbb{Z}[\underline{T}, \underline{D}]$ denotes the polynomial ring $\mathbb{Z}[T_1, T_2, T_3, D_1, D_2]$ of the 5 indeterminates and $\tilde{\pi}$ is associated to the ring homomorphism:

$$\iota\colon \mathbb{Z}[\underline{T}, \underline{D}] \to \mathbb{Z}[M_2 \times M_2], \tag{2.2.2}$$

given by $\iota(T_1) := \mathrm{tr}(A)$, $\iota(T_2) := \mathrm{tr}(B)$, $\iota(T_3) := \mathrm{tr}(AB)$, $\iota(D_1) := \det(A)$ and $\iota(D_2) := \det(B)$. The morphism $\tilde{\pi}$ is $\mathbb{G}_m$-equivariant and PGL_2-invariant, since ι is homogeneous w.r.t the weights $\deg(T_1) = \deg(T_2) = 1$ and $\deg(T_3) = \deg(D_1) = \deg(D_2) = 2$, and the Image($\iota$) is fixed by the PGL_2-action pointwisely.

2.3. The next lemma is easily proven by a use of Euclid division algorithms.

Lemma A. *$\mathbb{Z}[M_2 \times M_2]$ is a free module over $\mathbb{Z}[\underline{T}, \underline{D}]$.*

As a consequence of the Lemma A, $\mathbb{Z}[M_2 \times M_2]$ *is faithfully flat over* $\mathbb{Z}[\underline{T}, \underline{D}]$. In particular, the map ι (2.2.2) is injective, and we regard $\mathbb{Z}[\underline{T}, \underline{D}]$ as a subring of $\mathbb{Z}[M_2 \times M_2]$. More strongly, the next lemma says that it is the universal invariant subring with respect to the PGL-action.

Lemma B. *Let R be any $\mathbb{Z}[\underline{T}, \underline{D}]$-algebra with 1. Then*

$$R \simeq \left(R \otimes_{\mathbb{Z}[\underline{T},\underline{D}]} \mathbb{Z}[M_2 \times M_2]\right)^{\mathrm{PGL}_2}. \tag{2.3.1}$$

Here, the PGL_2 *action on* $\mathbb{Z}[M_2 \times M_2]$ *is extended to the tensor product by letting* PGL_2 *act trivially on R.*

Remark. (1) It is classically well known that $\mathbb{Q}[M_2 \times M_2]^{\mathrm{PGL}_2}$ is generated by traces $\mathrm{tr}(W)$ for $W \in \{$ the monoid generated by the A and $B\}$ ([G-Y], [Pr1, Pr2]). This is not true for the $\mathbb{Z}$-coefficient. For instance, the relation $2\det(A) = \mathrm{tr}(A)^2 - \mathrm{tr}(A^2)$ implies the algebraic dependence of $\mathrm{tr}(A^2)$ and $\mathrm{tr}(A)$ if char $= 2$, whereas $\det(A)$ and $\mathrm{tr}(A)$ are universally algebraically independent as shown in Lemma B.

(2) Donkin [D] has shown that $\mathbb{Z}[M_n \times \cdots \times M_n]^{\mathrm{PGL}_n}$ is generated by $\mathrm{tr}(\overset{i}{\wedge} W)$ for $i = 1, \cdots, n$ and $W \in \{$ the monoid generated by $A_1, \cdots, A_m\}$, where we denote by $M_n \times \cdots \times M_n$ the space of m-tuple $n \times n$ matrices $(A_1, \cdots, A_m)$.

2.4. For a matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the adjoint matrix A^* is the matrix $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ so that $A^*A = AA^* = det(A)I_2$ and $A + A^* = \mathrm{tr}(A)I_2$.

Definition. The *discriminant* for $\tilde{\pi}$ is the polynomial in $\mathbb{Z}[M_2 \times M_2]$ given by:

$$\begin{aligned} \Delta(A, B) &:= \mathrm{tr}(ABA^*B^*) - \mathrm{tr}(AA^*BB^*) \\ &= T_1^2 D_2 + T_2^2 D_1 + T_3^2 - T_1 T_2 T_3 - 4 D_1 D_2. \end{aligned} \tag{2.4.1}$$

A justification for this name is given in the next Lemma C. Let J be the ideal of $\mathbb{Z}[M_2 \times M_2]$ generated by all 5×5 minors of the Jacobian matrix of $\tilde{\pi}$:

$$\frac{\partial(T_1, T_2, T_3, D_1, D_2)}{\partial(a, b, c, d, e, f, g, h)} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ e & g & f & h & a & c & b & d \\ d & -c & -b & a & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & h & -g & -f & e \end{bmatrix}.$$

Lemma C. *The intersection of the Jacobian ideal J with the invariant subring $\mathbb{Z}[\underline{T}, \underline{D}]$ is a principal ideal generated by the discriminant.*

$$J \cap \mathbb{Z}[\underline{T}, \underline{D}] = (\Delta). \tag{2.4.2}$$

2.5. Let us denote by D_Δ the divisor in $\mathbb{A}^5 = \mathrm{Spec}(\mathbb{Z}[\underline{T}, \underline{D}])$ defined by the principal ideal (Δ). Owing to the (2.4.2), the Jacobian criterion implies that the restriction of the morphism $\tilde{\pi}$ (2.2.1) on the complement of $\tilde{\pi}^{-1}(D_\Delta)$

$$\tilde{\pi}_\Delta \colon M_2 \times M_2 \setminus \pi^{-1}(D_\Delta) \to \mathbb{A}^5 \setminus D_\Delta \tag{2.5.1}$$

is smooth. More strongly, we show in the next lemma, to which the proof of main theorem A in §4 is reduced.

Lemma D. *The morphism (*2.5.1*) is a principal* PGL_2*-bundle with respect to the etal topology.*

§3. The universal character ring $R(\Gamma, \mathrm{SL}_2)$

The universal character ring $R(\Gamma, \mathrm{SL}_2)$ is introduced in this § by generators and relations in terms of Γ. The goal of this § is the formula (3.4.3), which is used in the proof of the main theorem B in §4 essentially.

3.1.

Definition. The *universal character ring* $R(\Gamma, \mathrm{SL}_2)$ *of representations of* Γ *in* SL_2 is generated by the indeterminates $s(\gamma)$ for $\gamma \in \Gamma$ and divided by the ideal generated by $s(e) - 2$ ($e =$ the unit of Γ) and by $s(\gamma)s(\delta) - s(\gamma\delta) - s(\gamma^{-1}\delta)$ for all $\gamma, \delta \in \Gamma$.

$$R(\Gamma, \mathrm{SL}_2) := \mathbb{Z}[s(\gamma), \gamma \in \Gamma] \Big/ \Big(s(e) - 2, s(\gamma)s(\delta) - s(\gamma\delta) - s(\gamma^{-1}\delta)\Big). \tag{3.1.1}$$

3.2. The ring $R(\Gamma, \mathrm{SL}_2)$ is finitely generated over $\mathbb{Z}$, if Γ is finitely generated as shown in the next lemma. Such finiteness was asserted for the ring of traces of SL_2 by Fricke [F-K] and proven in [H1, H2, (2.f)], [Ho1, Ho2, Theorem 3.1] and [C-S]. We formulate the finiteness in terms of the universal character ring.

Proposition. *Let A be a linearly ordered subset of Γ, which generates Γ. Then $R(\Gamma, \mathrm{SL}_2)$ is generated by $G := \bigcup_{m \in \mathbb{N}} \{s(\alpha_1 \cdots \alpha_m) | \alpha_i \in A, \alpha_1 < \cdots < \alpha_m\}$ over $\mathbb{Z}$.*

Corollary 1. *If the group Γ is finitely generated, then the $R(\Gamma, \mathrm{SL}_2)$ is a finitely generated ring over $\mathbb{Z}$. Hence it is noetherian.*

3.3. For α and $\beta \in \Gamma$, define the *discriminant* $\Delta(\alpha,\beta) \in R(\Gamma, \mathrm{SL}_2)$ by

$$\begin{aligned}(3.3.1)\qquad \Delta(\alpha,\beta) :=& s(\alpha\beta\alpha^{-1}\beta^{-1}) - 2\\ =& s(\alpha)^2 + s(\beta)^2 + s(\alpha\beta)^2 - s(\alpha)s(\beta)s(\alpha\beta) - 4.\end{aligned}$$

This definition of Δ is parallel to that in (2.4.1). In fact, we shall confuse them in a proof of main theorem B in §4. The polynomial has been studied extensively by R. Fricke [F] and others.

3.4. An importance of the discriminant is explained in the next lemma, which plays an essential role in a proof of the main theorem B in §4.

Definition. Let M be a $R(\Gamma, \mathrm{SL}_2)$-module. A map $h\colon \Gamma \to M$ is called a *form* with values in M, if for any γ and $\delta \in \Gamma$ one has a relation:

$$(3.4.1)\qquad h(\gamma\delta) + h(\gamma^{-1}\delta) = s(\gamma)h(\delta).$$

Lemma. *Let h be a form with values in M. Suppose $h(e) = h(\alpha) = h(\beta) = h(\alpha\beta) = 0$ for some α and $\beta \in \Gamma$. Then for any $\gamma \in \Gamma$ one has*

$$(3.4.2)\qquad \Delta(\alpha,\beta)h(\gamma) = 0.$$

Therefore, the images of the values of h by the localization $M \to M_{\Delta(\alpha,\beta)}$ are zero.

Corollary. *For any α, β, γ and $\delta \in \Gamma$, one has the bilinear expression of $s(\gamma\delta)$*

(3.4.3)

$$s(\gamma\delta) = -\frac{1}{\Delta(\alpha\beta)}(s(\gamma), s(\gamma\alpha), s(\gamma\beta), s(\gamma\alpha\beta)) \cdot T \cdot {}^t(s(\delta), s(\alpha\delta), s(\beta\delta), s(\alpha\beta\delta))$$

in the localization $R(\Gamma, \mathrm{SL}_2)_{\Delta(\alpha,\beta)}$, where $T \in M_4(R(\Gamma, \mathrm{SL}_2))$ is a 4×4 matrix such that $T \cdot (s(\xi_i\xi_j))_{i,j=1}^4 = -\Delta(\alpha,\beta)I_4$ for $\xi_1 = e, \xi_2 = \alpha, \xi_3 = \beta$ and $\xi_4 = \alpha\beta$.

This corollary is the key step to obtain the universality of the character ring, since it implies that the system $(s(\gamma), \gamma \in \Gamma)$ satisfies any algebraic relations which is satisfied by the system $(\mathrm{tr}(\sigma(\gamma)), \gamma \in \Gamma)$ of characters, so far as $\Delta(\alpha,\beta)$ is invertible for some $\alpha, \beta \in \Gamma$.

3.5.

Remark. The study of the algebra of traces (=characters) of representations of a group Γ into SL_2 started by Voigt and Fricke and is developed by many authors Helling, Horowitz, Magnus, Bass, Lubotzky, Procesi, Platonov and others. See references of the quoted papers.

· Let F_n be a free group generated by n elements. Magnus called the homomorphic image of the ring $R(F_n, \mathrm{SL}_2)$ in the ring of functions on the representation space $\mathrm{Hom}(F_n, \mathrm{SL}_2(\mathbb{C}))$ the ring of Fricke characters [Ma]. We do not know whether the homomorphism has non trivial kernel or not.

§4. The invariant morphism π_Γ

In the present §4.1, we introduce the character variety $\mathrm{Ch}(\Gamma, \mathrm{SL}_2)$, the PGL_2-invariant morphism $\pi_\Gamma\colon \mathrm{Hom}(\Gamma, \mathrm{SL}_2) \to \mathrm{Ch}(\Gamma, \mathrm{SL}_2)$ and the discriminant D_Γ. The inverse image $\pi_\Gamma^{-1}(D_\Gamma)$ consist of the representations ρ such that $\rho(\Gamma)$ is either abelian or reducible (§4.2 Assertion). Then the first main results of the present paper are formulated in 4.3 theorem A and 4.4 theorem B, which say that the complement $\mathrm{Ch}^*(\Gamma, \mathrm{SL}_2) := \mathrm{Ch}(\Gamma, \mathrm{SL}_2)\backslash D_\Gamma$ of the discriminant is the regular orbit space of PGL_2-action on absolutely irreducible representations $\mathrm{Hom}^*(\Gamma, \mathrm{SL}_2) := \mathrm{Hom}(\Gamma, \mathrm{SL}_2)\backslash \pi^{-1}(D_\Gamma)$.

4.1. Let Γ be a group. Put

$$\mathrm{Ch}(\Gamma, \mathrm{SL}_2) := \mathrm{Spec}(R(\Gamma, \mathrm{SL}_2)), \tag{4.1.1}$$

where $R(\Gamma, \mathrm{SL}_2)$ is defined in §3.1. Recall the fact that $\mathrm{Hom}(\Gamma, \mathrm{SL}_2)$ is identified with the affine scheme for the universal representation ring $A(\Gamma, \mathrm{SL}_2)$ (cf. §1.3 Lemma). Then the invariant morphism

$$\pi_\Gamma\colon \mathrm{Hom}(\Gamma, \mathrm{SL}_2) \to \mathrm{Ch}(\Gamma, \mathrm{SL}_2) \tag{4.1.2}$$

is defined through the ring homomorphism:

$$\begin{array}{rccc} \Phi\colon & R(\Gamma, \mathrm{SL}_2) & \to & A(\Gamma, \mathrm{SL}_2) \\ & s(\gamma) & \mapsto & \mathrm{tr}(\sigma(\gamma)) \end{array}. \tag{4.1.3}$$

Comparing relations (3.1.1) with (1.3.1) and (1.3.2), Φ is well defined. The *discriminant subvariety* in $\mathrm{Ch}(\Gamma, \mathrm{SL}_2)$ of the morphism π_Γ is defined as

$$D_\Gamma := \bigcap_{\alpha,\beta\in\Gamma} V(\Delta(\alpha, \beta)) \tag{4.1.4}$$

where the discriminant $\Delta(\alpha, \beta) \in R(\Gamma, \mathrm{SL}_2)$ is defined in (3.3.1).

4.2. First, in the next lemma we characterize the inverse image $\pi_\Gamma^{-1}(D_\Gamma)$ of the discriminant loci in terms of representation of Γ.

Assertion. *Let $p \in \mathrm{Spec}(A(\Gamma, \mathrm{SL}_2))$. Then p belongs to $\pi_\Gamma^{-1}(D_\Gamma)$, if and only if the image $\sigma_p(\Gamma)$ in $\mathrm{SL}_2(k_p)$ of the group Γ is either abelian or reducible, where k_p is the fraction field of the integral domain $A(\Gamma, \mathrm{SL}_2)/p$ and $\sigma_p\colon \Gamma \to \mathrm{SL}_2(k_p)$ is a representation obtained by a specialization of the universal σ at p.*

4.3. Let us state the first main result of the present paper.

Theorem A. *The restriction of the invariant morphism π_Γ to the complement of the inverse image $\pi_\Gamma^{-1}(D_\Gamma)$ of the discriminant is a principal PGL_2-bundle with respect to the etal topology defined over $\mathbb{Z}$.*

Proof. By definition of D_Γ, one has an affine open covering $\mathrm{Ch}(\Gamma, \mathrm{SL}_2) \setminus D_\Gamma = \bigcup_{\alpha,\beta\in\Gamma} \mathrm{Spec}(R(\Gamma, \mathrm{SL}_2)_{\Delta(\alpha,\beta)})$. Then the proof is reduced to each affine open piece, stated as a consequence of the next Theorem B. $\square$

4.4. For any fixed pair α and β of Γ, consider the PGL_2-equivariant morphism $\mathrm{Hom}(\Gamma, \mathrm{SL}_2) \to M_2 \times M_2$ and a morphism $h_{\alpha\beta}\colon \mathrm{Ch}(\Gamma, \mathrm{SL}_2) \to \mathbb{A}^5 = \mathrm{Spec}(\mathbb{Z}[\underline{T}, \underline{D}])$ defined by the coordinate ring homomorphisms given by

$$A \mapsto \sigma(\alpha) \text{ and } B \mapsto \sigma(\beta), \tag{4.4.1}$$

$$T_1 \mapsto s(\alpha), T_2 \mapsto s(\beta), T_3 \mapsto s(\alpha\beta), D_1 \mapsto 1 \text{ and } D_2 \mapsto 1. \tag{4.4.2}$$

Then the next diagram becomes commutative

$$\begin{array}{ccc} \mathrm{Hom}(\Gamma, \mathrm{SL}_2) & \longrightarrow & M_2 \times M_2 \\ \Big\downarrow \pi_\Gamma & & \Big\downarrow \tilde{\pi} \\ \mathrm{Ch}(\Gamma, \mathrm{SL}_2) & \longrightarrow & \mathbb{A}^5 = \mathrm{Spec}(\mathbb{Z}[\underline{T}, \underline{D}]) \end{array} \tag{4.4.3}$$

Here we remark that the discriminant $\Delta(\alpha, \beta)$ (3.3.1) is the pull back of $\Delta(A, B)$ (2.4.1). For an abuse of notation, we shall denote both of them by Δ.

Theorem B. *The diagram (4.4.3) is Cartesian on the complement of the loci $\Delta = 0$. That is: the localization by Δ of the homomorphism*

$$\Psi_{\alpha,\beta}\colon R(\Gamma, \mathrm{SL}_2) \otimes_{\mathbb{Z}[\underline{T},\underline{D}]} \mathbb{Z}[M_2 \times M_2] \to A(\Gamma, \mathrm{SL}_2) \tag{4.4.4}$$

(obtained from (4.1.3), (4.4.1) and (4.4.2)) is an isomorphism.

Theorem A follows from the theorem B applied with §2.5 Lemma D. The theorem B is proved by a construction of the inverse homomorphism for the localization $(\Psi_{\alpha,\beta})_\Delta$ of (4.4.4) by Δ. This is equivalent to the *construction of a representation σ^* for a prescribed system of "characters" $s(\gamma)$ $\gamma \in \Gamma$ and a pair of matrix $(A, B) \in M_2 \times M_2$ (over the same point of $\mathbb{A}^5 \setminus \{\Delta = 0\}$) such that $\mathrm{tr}(\sigma^*(\gamma)) = s(\gamma)$ and $(\sigma^*(\alpha), \sigma^*(\beta)) = (A, B)$.* Actually, this is achieved by the formula:

$$\sigma^*(\gamma) := -\frac{1}{\Delta(\alpha\beta)}(I_2, A, B, AB) \cdot T \cdot {}^t(s(\gamma), s(\alpha\gamma), s(\beta\gamma), s(\alpha\beta\gamma)),$$

where T is the same matrix in $M_4(R(\Gamma, \mathrm{SL}_2))$ as in (3.4.3). The multiplicativity $\sigma^*(\gamma\delta) = \sigma^*(\gamma)\sigma^*(\delta)$ is shown by an essential use of the formula (3.4.3).

§5. Representation variety with real coefficients

We specialize the result in the previous § to $\mathbb{R}$-coefficient case. The second main result of the present paper is formulated in 5.5 theorem C, which says that the complement $\mathrm{Ch}(\Gamma, \mathrm{SL}_2)(\mathbb{R})$ of real discriminant $D_\Gamma(\mathbb{R})$ decomposes into two semialgebraic sets, which are regular orbit spaces of $\mathrm{PGL}_2(\mathbb{R})$ action on $\mathrm{Hom}^*(\Gamma, \mathrm{SL}_2(\mathbb{R}))$ and $\mathrm{PU}(2)$ action on $\mathrm{Hom}^*(\Gamma, \mathrm{SU}_2(\mathbb{C}))$, respectively. For the purpose we analyze the discriminant Δ over $\mathbb{R}$ (§5.3 Lemma E). See also [K1, K2, K3, K4], [G1], [H1] and [Ko] for the geometry of the discriminant.

5.1. Consider the invariant map $\tilde{\pi}$ (2.2.1) over the real number field $\mathbb{R}$.

$$\tilde{\pi}(\mathbb{R})\colon M_2(\mathbb{R}) \times M_2(\mathbb{R}) \to \mathbb{A}^5(\mathbb{R}) := \mathrm{Hom}(\mathbb{Z}[\underline{T}, \underline{D}], \mathbb{R}) \tag{5.1.1}$$

given by $\tilde{\pi}(A, B) := (\mathrm{tr}(A), \mathrm{tr}(B), \mathrm{tr}(AB), \det(A), \det(B))$. Consider also the real loci of the discriminant Δ (§2.4).

5.2. Let us introduce an open semialgebraic subset of $\mathbb{A}^5(\mathbb{R})$:

$$\widetilde{D_\Delta}(\mathbb{R}) := \left\{\varphi \in \mathbb{A}^5(\mathbb{R}) : \Delta(\varphi) := \varphi(\Delta) = 0\right\}. \tag{5.2.1}$$

where

$$\delta_1 = T_1^2 - 4D_1,\ \delta_2 = T_2^2 - 4D_2 \text{ and } \delta_3 = T_3^2 - 4D_1D_2. \tag{5.2.2}$$

are the discriminants for the characteristic polynomials of A, B and AB respectively. The following facts are shown by direct elementary calculations.

Lemma E. *i)* $\widetilde{T_\Delta}$ *is a convex connected component of* $\mathbb{A}^5(\mathbb{R}) \setminus \widetilde{D_\Delta}$.
ii) $\mathbb{A}^5(\mathbb{R}) \setminus \widetilde{T_\Delta} = \tilde{\pi}(\mathbb{R})\left(M_2(\mathbb{R}) \times M_2(\mathbb{R})\right)$.

Let us decompose the space $\mathbb{A}^5(\mathbb{R})$ into semialgebraic sets:

$$\mathbb{A}^5(\mathbb{R}) = \widetilde{D_\Delta} \amalg \widetilde{H_\Delta} \amalg \widetilde{T_\Delta}, \tag{5.2.3}$$

where $\widetilde{H_\Delta} := \mathbb{A}(\mathbb{R}) \setminus (\widetilde{D_\Delta} \amalg \widetilde{T_\Delta})$ is a union of connected components of $\mathbb{A}(\mathbb{R}) \setminus \widetilde{D_\Delta}$. Then the lemma E E can be paraphrased as: $\mathrm{Image}(\tilde{\pi}(\mathbb{R})) = \widetilde{D_\Delta} \amalg \widetilde{H_\Delta}$.

5.3. For any given $\alpha, \beta \in \Gamma$, consider the homomorphism $F_2 \to \Gamma$ associating the two generators of F_2 to α and β. This induces the diagram (4.4.3) over $\mathbb{R}$:

$$\begin{array}{ccc} \mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{R})) & \longrightarrow & \mathrm{Hom}(F_2, \mathrm{SL}_2(\mathbb{R})) \subset M_2(\mathbb{R}) \times M_2(\mathbb{R}) \\ \Big\downarrow{\scriptstyle \pi_\Gamma(\mathbb{R})} & & \Big\downarrow{\scriptstyle \tilde{\pi}(\mathbb{R})} \\ \mathrm{Ch}(\Gamma, \mathrm{SL}_2)(\mathbb{R}) & \underset{h_{\alpha,\beta}}{\longrightarrow} & \mathbb{A}^5(\mathbb{R}) \end{array} \tag{5.3.1}$$

where the morphism $h_{\alpha,\beta}$ is defined by (4.4.2) and $\mathrm{Ch}(\Gamma, \mathrm{SL}_2)(\mathbb{R}) = \mathrm{Hom}(R(\Gamma, \mathrm{SL}_2), \mathbb{R})$ is the real character variety. The fact that the diagram (5.3.1) is Cartesian outside of the loci $\Delta = 0$ (§4.4 Theorem B) together with Lemma E implies the following disjoint decomposition:

$$\mathrm{Ch}(\Gamma, \mathrm{SL}_2)(\mathbb{R}) = D_\Gamma(\mathbb{R}) \amalg H_\Gamma(\mathbb{R}) \amalg T_\Gamma(\mathbb{R}), \tag{5.3.2}$$

where

(5.3.3)

$$D_\Gamma(\mathbb{R}) := \left\{t \in \mathrm{Hom}(R(\Gamma, \mathrm{SL}_2), \mathbb{R}) \mid h_{\alpha,\beta}(t) \in D_\Delta(\mathbb{R}) \text{ for } {}^\forall\alpha, \beta \in \Gamma\right\}$$
$$H_\Gamma(\mathbb{R}) := \left\{t \in \mathrm{Hom}(R(\Gamma, \mathrm{SL}_2), \mathbb{R}) \mid {}^\forall\alpha, \beta \in \Gamma \text{ such that } h_{\alpha,\beta}(t) \in \widetilde{H_\Delta}\right\}$$
$$T_\Gamma(\mathbb{R}) := \left\{t \in \mathrm{Hom}(R(\Gamma, \mathrm{SL}_2), \mathbb{R}) \mid {}^\forall\alpha, \beta \in \Gamma \text{ such that } h_{\alpha,\beta}(t) \in \widetilde{T_\Delta}\right\}.$$

By definition, $D_\Gamma(\mathbb{R})$ is Zariski closed. $H_\Gamma(\mathbb{R})$ and $T_\Gamma(\mathbb{R})$ are open semialgebraic (Due to the basis theorem of Hilbert, one can find a finite system $\{(\alpha_i, \beta_i)\}_{i\in I}$ such that $D_\Gamma(\mathbb{R}) = \bigcap_{i\in I}\{\Delta(\alpha_i,\beta_i)=0\}$. Then for the same index set I, one can show the equalities: $H_\Gamma(\mathbb{R}) = \bigcup_{i\in I} h^{-1}_{\alpha_i\beta_i}(\widetilde{H_\Delta})$, $T_\Gamma(\mathbb{R}) = \bigcup_{i\in I} h^{-1}_{\alpha_i\beta_i}(\widetilde{T_\Delta})$.

5.4. Let us determine the image set of $\pi_\Gamma(\mathbb{R})$ as a semialgebraic set (up to the discriminant) in the next lemma. It is proven by a use of the fact that (5.3.1) is Cartesian (§4.4 Theorem B) together with the lemma E.

Lemma F. *Let $\pi_\Gamma(\mathbb{R})$ be the invariant morphism (4.1.2) defined over $\mathbb{R}$. Then*

$$\mathrm{Image}(\pi_\Gamma(\mathbb{R})) \setminus D_\Gamma(\mathbb{R}) = H_\Gamma(\mathbb{R}). \tag{5.4.1}$$

A meaning of the set $T_\Gamma(\mathbb{R})$ is given by the next lemma. Since $\mathrm{SU}(2) \subset \mathrm{SL}_2(\mathbb{C})$, the restriction of $\pi_\Gamma(\mathbb{C})$ induces the following map:

$$\begin{array}{llll} u_\Gamma & \mathrm{Hom}(\Gamma,\mathrm{SU}(2)) & \to & \mathrm{Ch}(\Gamma,\mathrm{SL}_2)(\mathbb{R}). \\ & \rho & \mapsto & s(\gamma)\mapsto \mathrm{tr}(\rho(\gamma)) \end{array} \tag{5.4.2}$$

Lemma G. *The image set of the morphism u_Γ is given by*

$$\mathrm{Image}(u_\Gamma) \setminus D_\Gamma(\mathbb{R}) = T_\Gamma(\mathbb{R}). \tag{5.4.3}$$

5.5. This is the goal of the present paper. Combining the above (5.4.1), (5.4.2) with the §4 Theorem A, we obtain principal bundles by the natural adjoint action with respect to the classical topology.

Theorem C. *The restrictions of the maps $\pi_\Gamma(\mathbb{R})$ (4.1.2) and u_Γ (5.4.2) to the subset of absolutely irreducible representations $\mathrm{Hom}^*(\Gamma,\mathrm{SL}_2(\mathbb{R})) := \mathrm{Hom}(\Gamma,\mathrm{SL}_2(\mathbb{R})) \setminus \pi_\Gamma^{-1}(D_\Gamma(\mathbb{R}))$ and $\mathrm{Hom}^*(\Gamma,\mathrm{SU}(2)) := \mathrm{Hom}(\Gamma,\mathrm{SU}(2)) \setminus u_\Gamma^{-1}(D_\Gamma(\mathbb{R}))$ are a principal $\mathrm{PGL}_2(\mathbb{R})$-bundle and a principal $\mathrm{U}(2)/\mathrm{U}(1)$-bundle over the open semialgebraic sets $H_\Gamma(\mathbb{R})$ and $T_\Gamma(\mathbb{R})$ (5.3.2) of the real character variety $\mathrm{Ch}(\Gamma,\mathrm{SL}_2)(\mathbb{R})$ respectively.*

$$\pi_\Gamma(\mathbb{R})\colon\ \mathrm{Hom}^*(\Gamma,\mathrm{SL}_2(\mathbb{R})) \xrightarrow{\mathrm{PGL}_2(\mathbb{R})} H_\Gamma(\mathbb{R}),$$
$$u_\Gamma\colon\ \mathrm{Hom}^*(\Gamma,\mathrm{SU}(2)) \xrightarrow{\mathrm{U}(2)/\mathrm{U}(1)} T_\Gamma(\mathbb{R}).$$

The value of a coordinate $s(\gamma) \in R(\Gamma,\mathrm{SL}_2)$ at a point of the base space is equal to the character $\mathrm{tr}(\rho(\gamma))$ for any representation ρ in its fiber.

References

[A] W. Abikoff, *The real analytic theory of Teichmüller space*, Lect. Notes in Math. **820**, Springer, (1980).

[B-L] Hyman Bass and Alexander Lubotzky, *Automorphism of groups and of schemes of finite type*, Israel J. of Math. **44 No. 1** (1983), 1–22.

[Be] Alan F. Beardon, *The geometry of discrete groups*, GTM, Springer Verlag, 197.

[Br] G. W. Brumfiel, *The real spectrum compactification of Teichmüller space*, Contem. Math. **74** (1988), 51–75.

[C-C-G-L-S] D. Cooper, M. Culler, H. Gillet, D. D. Long and P. Shalen, *Plane curves associated to character varieties of 3-manifolds*, Invent. math. **118** (1994), 47–84.

[C-W] Paula Cohen and Jürgen Wolfart, *Modular Embedding for some nonarithmetic Fuchsian groups*, preprint, IHES/M/88/57, Nov. 1988.

[C-S] Marc Culler and Peter B. Shalen, *Varieties of group representations and splittings of* 3-*manifolds*, Annals of Math. **117** (1983), 109–146.

[D] Stephen Donkin, *Invariants of several matrices*, Invent. Math. **110** (1992), 389–401.

[F] G. Faltings, *Real projective structures on Riemann surfaces*, Compos. Math. **48** (1983), 223–369.

[F-K] Robert Fricke and Felix Klein, *Vorlesungen uber die Theorie der Automorphen Funktionen*, Vol. 1, pp. 365–370, Leipzig: B.G. Teubner 1987, Reprint: New York Johnson Reprint Corporation (Academic Press) 1965.

[G1] William Goldman, *Topological components of spaces of representations*, Invent. Math. **93** (1988), 557–607.

[G2] William Goldman, *Geometric structures on manifolds and varieties of representations*, Contemporary Math. **74** (1988), 169–198.

[G-M] F. González-Acuña and José María Montesinos-Amilibia, *On the character variety of group representations in* $SL(2, \mathbb{C})$ *and* $PSL(2, \mathbb{C})$, Math. Z. **214** (1993), 627–652.

[G-Y] J. H. Grace and A. Young, *The ALGEBRA of INVARIANTS*, Cambridge Univ. Press., New York, 1903.

[H1] Heinz Helling, *Diskrete Untergruppen von* $SL_2(\mathbb{R})$, Inventiones math. **17** (1972), 217–229.

[H2] Heinz Helling, *Ueber den Raum der kompakten Riemannschen flächen vom geschlecht 2*, J. reine angew. Math. **268**/**269**:286–293, (1974).

[H-L-M] M. Hilden, M. Lozano and J. Montesinos-Amilibia, *Volumes and Chern-Simons invariant of cyclic coverings over rational knots*, These Proceedings, 31–55.

[H-K] C. Hodgson and S. Kerckhoff, *Rigidity of hyperbolic cone-manifolds and and hyperbolic Dehn surgery*, preprint.

[Ho1] Robert Horowitz, *Characters of free groups represented in the two dimensional linear group*, Comm. Pure Appl. Math. **25** (1972), 635–649.

[Ho2] Robert Horowitz, *Induced automorphisms of Fricke characters of free groups*, Trans. Am. Math. Soc. **208** (1975), 41–50.

[J-W] Lisa Jeffrey and Jonathan Weitsman, *Bohr-Sommerfeld orbits in the moduli space of flat connections and the Verlinde dimension formula*, Comm. Math. Phys. **150** (1992), 593–630.

[J] T. Jørgensen, *On discrete groups of Möbius transformations*, Amer. J. Math. **98** (1976), 739–749.

[K1] Linda Keen, *Intrinsic Moduli*, Ann. of Math. **84** (1966), 404–420.

[K2] Linda Keen, *On Fricke Moduli, Advances in the theory of Riemann surfaces*, Ann. of Math. Studies **66** (1971), 205–2024.

[K3] Linda Keen, *A correction to "On Fricke Moduli"*, Proc. Amer. Math. Soc. **40** (1973), 60–62.

[K4] Linda Keen, *A rough fundamental domain for Teichmüller spaces*, Bull. Amer. Math. Soc. **83** (1977), 1199–1226.

[Kj] S. Kojima *Deformations of hyperbolic 3-cone-manifolds*, preprint.

[Ko] Yohei Komori, *Semialgebraic description of Teichmüller space*, Thesis, Rims, Jan. 1994.

[Kr1] Irvin Kra, *Automorphic Forms and Kleinian Groups*, W. A. Benjamin Inc., (1972).

[Kr2] Irvin Kra, *On ligring of Kleinian groups to* $SL(2, \mathbb{C})$ *in Differential Geometry and Complex Analysis*, (H. E. Rouch, Memorial Volume)., Springer, Berlin, Heidelberg, New York, 1985, 181–193.

[Kr3] Irvin Kra, *Maskit coordinate*, Holom. Funct. & Moduli 2, Math. Sci. Research Inst. Publication **11**, Springer, 1992.

[L-M] Alexander Lubotzky and Andy R. Magid, *Varieties of representations of finitely generated groups*, Memoirs of the AMS, Vol. 58 No. 336, (1985).

[Ma] William Magnus, *Rings of Fricke characters and automorphism groups of free groups*, Math. Z. **170** (1980), 91–103.

[Mu1] David Mumford, *Geometric Invariant Theory*, Springer Verlag, Berlin, Heidelberg, 1965, Library of Congress Catalog Card Number 65–16690.

[Mu2] David Mumford, *Curves and their Jacobians*, ©by University of Michigan, 1975, All right reserved, ISBN 0–472–66000–4, Library of Congress Catalog Card No. 75–14899.

[N-Z] W. Neumann and D. Zagier, *Volumes of hyperbolic 3-manifolds*, Topology **24** (1985), 307–332.

[O1] Yoshihide Okumura, *Fricke moduli and Keen moduli for Fuchsian groups and a certain* *class of quasi-Fuchsian groups*, Master Thesis, Shizuoka Univ., 1986.

[O2] Yoshihide Okumura, *Global real analytic coordinates for Teichmüller spaces*, Thesis, Kanazawa Univ., 1989.

[O3] Yoshihide Okumura, *On lifting problem of Kleinian group into* $\mathrm{SL}(2,\mathbb{C})$, Summer Seminar on Function Theory, July 1992.

[O4] Yoshihide Okumura, *On the global real analytic coordinates for Teichmüller spaces*, J. Math. Soc. Japan **42** (1990), 91–101.

[P-B-K] V. P . Platonov and V. V. Benyash-Krivets, *Characters of representations of finitely generated groups*, Proc. of Steklov Inst. of Math. (1991), 203–213.

[Pr1] Claudio Procesi, *Finite dimensional representations of algebras*, Israel J. of Math. **19** (1974), 169–182.

[Pr2] Claudio Procesi, *Invariant theory of* $n \times n$ *matrices*, Adv. Math. **19** (1976), 306–381.

[S1] Kyoji Saito, *Moduli Space for Fuchsian Groups, Algebraic Analysis*, Vol. II, Academic Press, (1988), 735–787

[S2] Kyoji Saito, *The Limit Element in the Configuration Algebra for a Discrete Group*, A précis: Proc. Int. Congr. Math., Kyoto 1990, 931–942, Preprint: RIMS–726, Nov. 1990.

[S3] Kyoji Saito, *The Teichmüller space and a certain modular function from a view point of group representations*, Alg. Geom. and related Topics, Proc. Int. Symp., Inchoen, Republic of Korea, 1992.

[S4] Kyoji Saito, *Representation variety of a finitely generated group into* SL_2 *or* GL_2, Preprint RIMS–958, Dec. 1993.

[S5] Kyoji Saito, *Algebraic Representation of the Teichmüller spaces*, LMS. Lecture Note Series **200** (1994), 255–288.

[S6] Kyoji Saito, *Algebraic Representation of the Teichmüller spaces*, Kodai Math. J. **17** (1994), 609–626.

[S-S1] Mika Seppälä and Tuomas Sorvali, *Parametrization of Möbius groups acting in a disk*, Comment. Math. Helvetici **61** (1986), 149–160.

[S-S2] Mika Seppälä and Tuomas Sorvali, *Affine coordinates for Teichmüller spaces*, Math. Ann. **284** (1989), 169–176.

[S-S3] Mika Seppälä and Tuomas Sorvali, *Trace commutators of Möbius transformations*, Math. Scand. **68** (1991), 53–58.

[S-S4] Mika Seppälä and Tuomas Sorvali, *Geometry of Riemann Surfaces and Teichmüller Spaces*, North-Holland Mathematics Studies **169** (1992).

[T] W. Thurston, *The geometry and topology of 3-manifolds*, Lecture Notes, Princeton Univ., 1977/78.

[V] H. Vogt, *Sur les invariants, fondamentaux des équations différentielles linéaires du second ordre*, Ann. Sci. Ecole Nrom. sup. **6(3)**, Suppl.3–72 (1889)(Thèse, Paris).

[W1] Andre Weil, *On discrete subgroups of Lie Groups I*, Ann. Math. **72** (1960), 369–384,

[W2] Andre Weil, *On discrete subgroups of Lie Groups II*, Ann. Math. **75** (1962), 578–602,

[We] Jonathan Weitsman, *Geometry of the intersection ring of the moduli space of flat connections and the conjectures of Newstead and Witten*, preprint 1993.

[Wf] Jürgen Wolfart, *Eine arithmetische Eigenschaft automorpher Formen zu gewiisen nichtarithmetischen Gruppen*, Math. Ann. **62** (1983), 1–21.

[Wo1] Scott Wolpert, *The Fenchel-Nielsen deformation*, Ann. of Math. **115** (1982), 501–528.

[Wo2] Scott Wolpert, *On the Symplectic Geometry of deformation of a hyperbolic surface*, Ann. of Math. **117** (1983), 207–234.

[Wo3] Scott Wolpert, *Geodesic length functions and the Nielsen problem*, J. Diff. Geom. **25** (1987), 275–296.

[Y] T. Yoshida, *On ideal points of deformation curves of hyperbolic 3-manifold with one cusp*, Topology **30** (1991), 155–170.

[Z-V-C] Ziechang, Vogt and Colderway, Lect. Notes in Math. **122, 835** and **875.**

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