Towards a Categorical Construction of Lie Theory

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To the memory of Nguyen Huu Duc (13 August 1950 - 7 June 2007)

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Preface

This is an introduction to the program, we call "towards a categorical construction of Lie Theory". That is, from a datum of a system of 4 integers $W := (a, b, c; h)$ (called a *regular system of weights*), we want to construct a generalization of a simple Lie algebra.[1] However, the program is still in its early stages, and most parts are either history or work in progress. Thus the paper is neither logically complete nor comprehensive, but is a collection of topics, which are rather loosely related to each other. Therefore, the reader is suggested to look at the references for more details or to skip a topic according to his interest.

Let us explain briefly the contents of the paper. In §1-9, we explain the motivation and background from the classics on simple and simply elliptic singularities, where simple or elliptic Lie algebras play roles to construct primitive forms. We are interested in their possible generalizations. As the means to finding new interesting objects, in §10, we introduce the *regular systems of weights* as a system of 4 integers $W := (a, b, c; h)$ satisfying a simple arithmetic condition. They are concisely classified by a numerical invariant $\varepsilon_W \in \mathbb{Z}_{\leq 1}$, called the *smallest exponent*, where weight systems with $\varepsilon_W = 1$ or 0 correspond to simple or simply elliptic singularities, respectively. Then, for the next stage, we obtain 14+8+9 regular weight systems with $\varepsilon_W = -1$. They are the main object of the present paper. Before decoding Lie theoretic information, §11-15 are devoted to describe their geometric aspects: we study a weighted projective plane curve C_W of coordinate degrees a, b, c and of total degree h. In fact, C_W is a smooth orbifold equipped with a fractional $(= \varepsilon_W^{-1})$ power of the canonical bundle, whose zero-section blows down to an isolated singular point (§11).

In order to read Lie theoretic information from such geometry, historically, as described in §3-7, there are two approaches: the algebraic one, using a resolution of the singularity, and the topological one, using the lattice of vanishing cycles in a smoothing (Milnor fiber) of the singularity. In the case of ε_W=1 (the simple singularity case), the resolution describes a Dynkin diagram (§3) and the set of vanishing cycles (§5) gives rise a finite root system. Both give the same type Lie algebra, due to the existence of the simultaneous resolution of the singularity due to Brieskorn (§4). Furthermore, Brieskorn's description of the universal family of the simple singularity enable to describe the primitive form for them by the Kostant-Kirirov form on the co-adjoint orbit of a simple Lie group.

For the case of $\varepsilon_W = 0$ (the simply elliptic singularity case), through the topological approach,i.e. the set of vanishing cycles for the elliptic singularities, we obtain elliptic root systems, which give rise to the elliptic Lie algebras (§6). The symplectic structures on the coadjoint orbits of the elliptic Lie group are expected to form primitive forms (the verification of this is still a work in progress).

[1] This is a part of the long program to construct primitive forms categorically, where a primitive form [Mat] [Od1] [Sa6] is a certain differential form defined on the total space of a universal deformation of a singularity, which is to define the period integrals (see Footnote 6 for a description). However, we restrict our attention only to some algebraic aspects of the program and the present paper is readable without the knowledge of primitive forms.

The present paper may be considered as a continuation of the other overview articles [Sa16] [Sa20]. Knowledge of them is not assumed and unnecessary, but are helpful and recommended.

The singularities associated with the 14 weight systems with $\varepsilon_W = -1$ are called exceptional uni-modular singularities by Arnold. Certain distinguished basis of the lattices of vanishing cycles for them are calculated by Gabrielov (see Table 12), where the triplet (p,q,r) of lengths of the three branches of the diagram is called the Gabrielov number. On the other hand, the exceptional set of the minimal resolution of the 14 singularities is given by 4 rational curves intersecting in a star-shape (see Table 11, Dolgachev), where the triplet (p,q,r) of the minus of the self-intersection numbers of the three branching curves is called the Dolgachev number. Then Arnold observed that there is an involutive one to one correspondence from the 14 classes of the exceptional uni-modular singularities to itself, which exchange the Gabrielov number and the Dolgachev number. The involution is called *strange duality* (§13).

The strange duality, which is nowadays understood as an appearance of *mirror symmetry* [2], admits several interpretations. Among these, in §14, we explain about $*$*-duality on regular systems of weights*. Namely, we introduce a purely arithmetically defined involutive bijection $*$ on certain sets of regular systems of weights [3], which, in particular, induces self-duality on the weight systems with $\varepsilon_W = 1$ and induces strange duality on the 14 weight systems with $\varepsilon_W = -1$. Therefore, we expect in general that the $*$-duality exchanges the algebraic approach for a weight system W with the topological approach for the weight system W^*. But then, on the algebraic side of W, instead of the naive data of the resolution of the singularity, what should stand for the vanishing cycles in the topological side of W^*? Inspired by the recent studies on mirror symmetry in mathematical physics (see §15), we study *the category of matrix factorizations of the singularity for* W as the algebraic approach,[4] where the K-group of the category should stand for the lattice of vanishing cycles for W^*.

In §16, we give three different descriptions of the derived category of matrix factorizations for a regular system of weights. In §17, we obtain the classical finite root systems in the K-group of the category for $\varepsilon_W = 1$, as is expected. In the case $\varepsilon_W = -1$ in §18, we obtain a category equivalent to that of modules over a wild algebra with a Coxeter element of finite order. In particular, in the 14 uni-modular exceptional cases, K-group is isomorphic to the lattice of vanishing cycles for the $*$-dual weight system, as is expected. Throughout the paper, we use the complex number field $\mathbb{C}$ as the coefficient of the varieties and the Lie algebras.

[2] The reader is referred to [Kon],[Yau] for mirror symmetry in general and to [K-Y][Ta1] for the Landau-Ginzburg orbifold case. Already in case of $\varepsilon_W = 0$, the algebraic data, i.e. the elliptic curve in the exceptional set in the resolution of the singularity is not "mirror dual" to the elliptic root system obtained topologically. In order to get mirror symmetry here, one should think of the elliptic curve with a group action [Ta1]. More comprehensive description is obtained by considering the pairs of a regular weight system and a group action [KST3]. However, in the present paper, we do not get into its details.

[3] $*$-duality is defined not on the full set of regular systems of weights. This is due partly to the missing of a group action explained in the previous footnote and partly because the singularities associated to regular systems of weights are only hypersurfaces. But in fact, we do not yet understand the full reason. It seems quite interesting to find a reasonable collection of objects, which includes all regular systems of weights and which is closed under the $*$-duality.

[4] This is proposed by Takahashi [Ta2] (see also [Orl:1]) answering to a problem posed by the author [Sa16] (5.3) Problem. The §16,17 and 18 are based on the joint works [K-S-T 1-2].

1 Simple Polynomials

There are a finite number of regular polyhedra, namely, the icosahedron, dodecahedron, octahedron, hexahedron and the tetrahedron, known at the time of Platon. The regular dihedron, which has only two faces of the n-gon ($n \geq 3$), is nowadays included in the list of regular polyhedra. The subgroup G of $SO(3)$ consisting of rotations of three dimensional Euclidean space, which moves a regular polyhedron (centered at the origin) to itself, is called the regular polyhedral group. The binary extension $\tilde{G}$ of the regular polyhedral group G is obtained by taking the inverse image of G through the surjective homomorphism $SU(2) \to SO(3)$. It is well-known that the binary regular polyhedral groups (including binary dihedral groups) and the cyclic subgroups $Z_n := \left\langle \begin{pmatrix} \exp\frac{2\pi\sqrt{-1}}{n} & 0 \\ 0 & \exp\left(-\frac{2\pi\sqrt{-1}}{n}\right) \end{pmatrix} \right\rangle$ for $n \in \mathbb{Z}_{>0}$ together form a complete list of finite subgroups of $SU(2)$ up to conjugacy. As an abstract group, all of the groups have a presentation:

$$\langle p, q, r \rangle := \langle\ x, y, z\ \mid\ x^p = y^q = z^r = xyz \rangle$$

for suitable integers $p, q, r \in \mathbb{Z}_{>0}$, given in the next Table 1 (here, x, y and z induces the rotation of the polyhedron centered at an edge, a face and a vertex).

Table 1.

$\langle 1, b, c \rangle$	$\simeq Z_n$	$\simeq$	cyclic group of order $n = b + c$
$\langle 2, 2, n \rangle$	$\simeq \mathcal{D}_{2n}$	$\simeq$	binary dihedral group of n-gon $n \geq 2$
$\langle 2, 3, 3 \rangle$	$\simeq \mathcal{A}_4$	$\simeq$	binary regular tetrahedral group
$\langle 2, 3, 4 \rangle$	$\simeq S_4$	$\simeq$	binary regular octahedral group
$\langle 2, 3, 5 \rangle$	$\simeq \mathcal{A}_5$	$\simeq$	binary regular icosahedral group

In fact, these are the only cases when the group $\langle p, q, r \rangle$ is finite (see [C-M]).

The group is sometimes called the Kleinean group because of the following result due to A. Schwarz [Sc] and F. Klein [Kl1].

Theorem. *Let $\tilde{G} \subset SU(2)$ be a Kleinean group. Let it act linearly on $\mathbb{C}^2$, and, hence, on the ring $\mathbb{C}[u, v]$ of polynomial functions on $\mathbb{C}^2$ (where u, v ares coordinates of $\mathbb{C}^2$). Then the subring $\mathbb{C}[u, v]^{\tilde{G}} := \{P \in \mathbb{C}[u, v] \mid gP = P\ {}^\forall g \in \tilde{G}\}$ of invariants is generated by 3-homogeneous elements, say x, y and z, which satisfy a single relation, say $f_{\tilde{G}} = f(x, y, z)$. That is:*

$$\mathbb{C}[u, v]^{\tilde{G}} \simeq \mathbb{C}[x, y, z]/(f_{\tilde{G}}).$$

The polynomial $f_{\tilde{G}}$ is called a *simple polynomial*, which is listed below.

Table 2.

Type	$f_{\tilde{G}}$	Kleinean group
A_l	$x^{l+1} + yz$	Z_n
D_l	$x^2y + y^{1-1} + z^2$	$\langle 2, 2, n \rangle$
E_6	$x^4 + y^3 + z^2$	$\langle 2, 3, 3 \rangle$
E_7	$x^4 + xy^3 + z^2$	$\langle 2, 3, 4 \rangle$
E_8	$x^5 + y^3 + z^2$	$\langle 2, 3, 5 \rangle$

The Types in the left-side shall be explained in §3.

Note. From the knowledge of $f_{\tilde{G}}$, one can recover $\tilde{G}$. See Appendix 3.

F. Klein, in the introduction to his lecture notes on the icosahedron [Kl1], described the time when he and Lie studied together in Berlin and Paris during the years 1869-70: "At that time we jointly conceived the scheme of investigating geometric or analytic forms susceptible of transformation by means of groups of changes. This purpose has been of directing influence in our subsequent labours, though these may have appeared to lie far asunder. Whilst I primary directed my attention to groups of discrete operations, and was thus led to the investigation of regular solids and their relations to the theory of equations, Professor Lie attacked the more recondite theory of continued groups of transformations, and therewith of differential equations".

2 Simple Lie algebras and root systems

Let us explain another stream of mathematics started from Lie and Killing.

The Lie algebras describe "the infinitesimal structure of continuous groups". The series of works [Ki] by Killing starting from the year 1888, determining the structure of *simple Lie algebras*, (which was completed by E. Cartan [Ca]) has introduced a new mathematical structure (see [Ha]) which goes far beyond the class of simple Lie algebras, and is strongly influential on the present program.

Killing looked at the adjoint action of the abelian (Cartan) subalgebra of a simple Lie algebra and decomposed the algebra into a direct sum of equi-eigenspaces of the action. Since an equi-eigenvalue (as an element of the dual space of the Cartan subalgebra) is a root of the characteristic equation, he called it a *root* (*Wurzel*), and showed that the system of roots for a simple Lie algebra satisfies some properties, which are now known as the axioms for a finite root system (see ([Bou]§6n^{o}1)). The classification of simple Lie algebras is reduced to the classification of finite root systems. In fact, it is achieved by determining the matrix $(2I(\alpha,\beta)/I(\alpha,\alpha))_{\alpha,\beta\in\Gamma}$ (called the Cartan matrix), where I is Killing form and Γ is a simple basis of the root system[5].

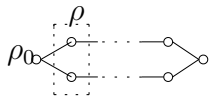
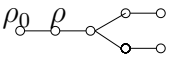
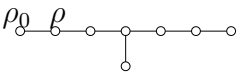
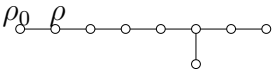
3 Du Val diagrams and Coxeter diagrams

Let us see how the two streams of mathematics, one starting with Klein and the other with Lie-Killing, meet again in the year 1934, when Du Val and Coxeter were together at Trinity college in Cambridge. At that time, the concept of the Weyl group, generated by reflections s_α for all roots α of the Lie algebra, was established in connection with the representation theory of simple Lie algebras (Weyl [We] (1925-6) and Cartan [Ca]). The classification of root systems is

[5]Recall [Bou](chap.6 §1 5.) that a simple basis of a (finite) root system is characterized as a system of linear forms on the Cartan algebra, whose zeros define the system of walls (oriented to the inside) of a Weyl chamber. It is admirable that, even at such an early stage (1888) of the study of simple Lie algebras, Killing ([Ki]S12,13) began to study root basis Γ, the product $\prod_{\alpha\in\Gamma} s_\alpha$ of the reflections s_α associated to the basis (presently known as the Coxeter-Killing transformation) and its eigenvalues (which presently defines the exponents). However, for their geometric significance in terms of the Weyl group and chambers, one must wait until Weyl's work [We]. As we shall see, finding generalizations of the simple root basis, Coxeter- Killing transformations and the exponents are central problem in the present paper.

reduced to the classification of the Weyl group [Wae]. Then Coxeter, by use of the fundamental domain (=Weyl chamber) of the Weyl group, classified all finite reflection groups acting on Euclidean space. Namely, he gave an explicit presentation of the Weyl group in terms of generators and relations, known as the Coxeter relations [Co1].[6] He introduced a diagram (tree) Γ, where the vertices correspond to the generators and an edge is drawn between two vertices which are non-commutative (see [Bou] for more details on reflection groups). In Table 3, the Coxeter's diagram for the Weyl groups of types A_l, D_l, or E_l are given by removing i) the vertex ρ_0 of the diagram and ii) the "tilde $\tilde{}$ " from the types in RHS of table (see Appendix for more details on the table).

Table 3.

Kleinean group	Diagram	Type
Z_n		$\tilde{A}_{n-1}$
$\langle 2,2,n\rangle$		$\tilde{D}_{n+2}$
$\langle 2,3,3\rangle$		$\tilde{E}_6$
$\langle 2,3,4\rangle$		$\tilde{E}_7$
$\langle 2,3,5\rangle$		$\tilde{E}_8$

The complex hypersurface X_0 in $\mathbb{C}^3$ defined by the zero-loci of a simple polynomial in the list of Klein (Table 2) has an isolated singular point at the origin 0 (cf. §11 Fact4.), called a *simple singularity* [Dur]. In the year 1934, Du Val [Du] studied the (minimal) resolution $\pi : \tilde{X}_0 \to X_0$ of the simple singularity. He associated a diagram Γ to the resolution: decompose $E := \pi^{-1}(0)$ into irreducible components $\cup_{i=1}^{l} E_i$, then, vertices x_i of the diagram are in one to one correspondence with components E_i and an edge is drawn between x_i and x_j if $E_i \cap E_j \neq \emptyset$. He observed that for each Kleinean group on the LHS of Table 3, the diagram he obtained is exactly the one given in the middle of the Table after deleting the vertex ρ_0. In the introduction of [Du], he wrote "It may be noted that the "trees" of curves which we have had to consider bear a strict formal resemblance to the spherical simplices whose submultiple of π, considered by Coxeter". In the same volume of the London Journal, Coxeter [Co1] listed diagrams for reflection groups following a request of Du Val.

[6]The generators are given by the reflections attached to the walls of the chamber (which is bijective to the set Γ of simple basis of Killing) and the relations are given by the dihedral group relations for every pair of generators along 2-codimensional facets of the chamber. The higher codimensional facets of the chamber do not play a role in determining the group.

4 Brieskorn's Description of the universal unfolding

We observed in §3 that *there is a one to one correspondence between the diagrams of Du Val associated to simple polynomials and those of Coxeter in the classification of simple Lie algebras* (recall Table 3). However, at this stage, their relation remained a "strict resemblance", as Du Val wrote. A more direct and decisive relationship was found 40years later in the work of Brieskorn and Grothendieck. In ICM Nice 1970, Brieskorn [Br4] reported the following result.

Theorem. (Brieskorn [Br4]) *Let $X \to S$ be the universal unfolding* [7] *of a simple singularity, and let $\mathfrak{g}$ be the corresponding simple Lie algebra. Then, one has a commutative diagram:*

$$\begin{array}{ccccc} X & \subset & \mathfrak{g} & & \\ \downarrow & & \downarrow & & \\ S & \simeq & \mathfrak{g}/\!\!/Ad(\mathfrak{g}) & \simeq & \mathfrak{h}/\!\!/W \end{array}$$

where i) the vertical arrow in right side of the diagram is the adjoint quotient morphism due to Chevalley's theorem, and ii) $X \subset \mathfrak{g}$ is an embedding of X onto a transversal slice to the nilpotent subvariety of $\mathfrak{g}$ at a subregular element.

Brieskorn further described the simultaneous resolution (c.f. [Br1,2]) of the universal family.[8] He wrote "Maybe the two theories do not lie so far asunder".

Remark 1. The Brieskorn's description of the universal unfolding $X \to S$ of a simple singularity by use of a simple Lie algebra has the advantage in determining certain global differential geometric structures on the family, since, in the Lie algebra, the integrability conditions are already built in. For instance, the primitive form of the family $X \to S$ [9], which is defined by an infinite system of non-linear equations, for the simple singularity is described by the Kostant-Kirirov symplectic form [Sa6] [Yah] [Ya1] [Yo]. The flat structure (Frobenius mfd structure) on the deformation parameter space S is described by the Coxeter-Killing transformation of the Weyl group [Sa17] [He] [Sab].

These facts motivated the author to convince the following: *for a further class of singularities, using suitable Lie algebras, construct primitive forms and flat structures globally.* However, the list of regular polyhedral groups and that of the simple Lie algebras have already been used up. Are these the only cases where singularity theory and Lie theory come happily together?

[7] The concept of an unfolding of a singularity of a function f is due to R. Thom [Th]. We shall give in §5 and in Footnote 10. a brief description of them. From an algebraic geometric view point, it is essentially the same concept as a semi-universal deformation of the hypersurface defined by $f = 0$ near at the singular point ([**?**], [Sch] and [Tu]).

[8] This was reproven by a use of representation of quivers [Kr] (see the works by H. Nakajima for further studies on the relationship between Lie algebras and representations of quivers).

[9] For a primitive form, see [Mat][Od1][Sa6][Sa20]. It is a relative de-Rham cohomology class $\zeta \in \mathbf{H}_{DR}(X/S)$ which 1) generates all the other de-Rham cohomology classes as a $\mathcal{D}_S$-module, and 2) satisfies an infinite system of bi-linear differential equation (by means of residue pairings). Its local existence on S is known [Sai]. Global existence on S is known only for simple or simply elliptic singularities. It is believable that $\mathfrak{g}$ is the Cartan prolongation of X with respect to the primitive form. Such global construction of primitive forms by means of globally defined integrable systems (such as Lie algebras) is the basic motivation in the present paper. However, we shall not discuss the primitive form itself in any further detail.

5 Universal unfolding of a Hypersurface Singularity

Before we go further, we prepare some terminologies on vanishing cycles of a hypersurface isolated singular point studied by authors [Br3] [Le1] [Gab1] [Eb1].

Let $f(\underline{x})$ with $\underline{x}:=(x_0,\cdots,x_n)$ $(n\geq 0)$ be a holomorphic function defined in a neighborhood $\mathcal{U}$ of the origin 0 of $\mathbb{C}^{n+1}$ with the coordinate $\underline{x}$. Assume that the hypersurface $X_0:=\{(\underline{x})\in\mathcal{U} \mid f(\underline{x})=0\}$ has an isolated singular point at the origin $0\in X_0$. This is equivalent to that $J_f:=\mathbb{C}\{\underline{x}\}/\left(\frac{\partial f(\underline{x})}{\partial x_0},\cdots,\frac{\partial f(\underline{x})}{\partial x_n}\right)$ is of finite rank over $\mathbb{C}$, where $\mathbb{C}\{\underline{x}\}$ is the local ring of all convergent series in $\underline{x}$.

Theorem. (Milnor [Mi]) *Consider a map $f:X\to D_\varepsilon$ where $X:=\{\underline{x}\in\mathcal{U} \mid |\underline{x}|<\delta\}\cap f^{-1}(D_\varepsilon)$ and $D_\varepsilon:=\{t\in\mathbb{C} \mid |t|<\varepsilon\}$ for positive real numbers δ,ε such that $0<\varepsilon\ll\delta\ll 1$. Then, $f|_{X\setminus f^{-1}(0)}: X\setminus f^{-1}(0)\to D_\varepsilon\setminus\{0\}$ is a locally trivial topological fibration such that the general fiber is homotopic to a bouquet of μ_f-copies of n-sphere S^n, where $\mu_f:=\dim_{\mathbb{C}} J_f$ is called the Milnor number.*

The fibration is called the Milnor fibration, whose general fiber, denoted by X_1, is called the *Milnor fiber*. *If f is globally defined weighted homogeneous polynomial of positive weights (c.f.), then we may choose $\delta=\varepsilon=\infty$.*

As a consequence of this result, the (reduced) homology group of the Milnor fiber is non-trivial only in dimension n, and we have $\tilde{\mathrm{H}}_n(X_1,\mathbb{Z})\simeq\mathbb{Z}^{\mu_W}$.

Let us introduce particular elements of $\tilde{\mathrm{H}}_n(X_1,\mathbb{Z})$, called *vanishing cycles*: let us consider a *universal unfolding* of f (Thom [Th]), which is a function $F(\underline{x},\underline{t})$ in $\underline{x}\in\mathbb{C}^{n+1}$ and $\underline{t}=(t_1,\cdots,t_{\mu_f})\in\mathbb{C}^{\mu_f}$ defined in a neighborhood of the origin $(\underline{0},\underline{0})\in\mathbb{C}^{n+1}\times\mathbb{C}^{\mu_f}$ satisfying i) $F(\underline{x},\underline{0})=f(\underline{x})$, and

ii) $\frac{\partial F(\underline{x},0)}{\partial t_i}$ $(i=1,\cdots,\mu_f)$ span the $\mathbb{C}$-vector space J_f.

For a small value of $\underline{t}$, again by choosing δ and ε suitably for $f_{\underline{t}}(\underline{x})=F(\underline{x},\underline{t})$, we consider the map $f_{\underline{t}}:X\to D_\varepsilon$ such that, excluding finite number of its fivers over the critical values, it gives a locally trivial fibration, whose general fiber is homeomorphic to the Milnor fiber. If $\underline{t}$ is general, then $f_{\underline{t}}|_X$ has exactly μ_f-number of non-degenerate critical points and the (critical) values are distinct (i.e. $f_{\underline{t}}$ is a Morsification of f). We may choose the "base point" 1 whose fiber $f_{\underline{t}}^{-1}(1)$ is the Milnor fiber X_1 on the boundary of the disc D_ε. Let $g:[0,1]\to D_\varepsilon$ be any continuous path starting at 1 and ending at a critical value c, without passing any critical points on $[0,1)$. Then the pull-back $X_{[0,1]}$ of the fibration $X\to D_\epsilon$ over the interval $[0,1]$ retracts to X_c. Thus, the natural inclusion $X_1\subset X_{[0,1]}$ induces a homomorphism $\iota:\tilde{\mathrm{H}}_n(X_1,\mathbb{Z})\to\tilde{\mathrm{H}}_n(X_c,\mathbb{Z})$ whose kernel $\ker(\iota)$ is rank 1 (since the Hessian of $f_{\underline{t}}$ at the critical point is non-degenerate).

Definition Let the setting be as above. Any base $e\in\ker(\iota)$ (up to sign) of the kernel in $\tilde{\mathrm{H}}_n(X_1,\mathbb{Z})$ is called a *vanishing cycle* along the path g. We denote by R_f the set of all vanishing cycles running the path g and the critical values c.

Let γ be a path in D_ε which starts at 1 and move along g close to the critical value c and then turns once around c counter-clockwisely, and then return to 1 along g. This path induces the monodromy $\rho(\gamma)\in\mathrm{Aut}(\tilde{\mathrm{H}}_n(X_1,\mathbb{Z}))$, whose action on $u\in\tilde{\mathrm{H}}_n(X_1,\mathbb{Z})$ is described by the following Picard-Lefschetz formula:

$$\rho(\gamma)(u) \;=\; u-(-1)^{\frac{n(n-1)}{2}}I(u,e)e$$

where $I:\tilde{\mathrm{H}}_n(X_1,\mathbb{Z})\times\tilde{\mathrm{H}}_n(X_1,\mathbb{Z})\to\mathbb{Z}$ is the intersection form on the middle ho-

mology group (Footnote 33). If n is even, I is symmetric and $I(e,e)=(-1)^{n/2}2$ so that $\rho(\gamma)$ is a *reflection action* with respect to the vector e, denoted by w_e.

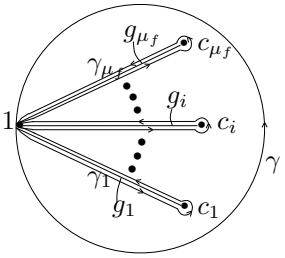


Table 4.

Now, we describe the *distinguished basis* of the middle homology group $\tilde{\mathrm{H}}_n(X_1,\mathbb{Z})$, depending on two choices: i) to give a numbering of the critical values, say $c_1,\cdots,c_{\mu_f}$, of $f_{\underline{t}}$, ii) to choose μ_f paths $g_1,\cdots,g_\mu$ in D_ε such that a) each g_i is a path connecting 1 with c_i as above, which is not self-intersecting, b) distinct paths g_i and g_j are intersecting only at 1, and c) the passes $g_1,\cdots,g_\mu$ are starting at the point 1 in the linear order $1,\ldots,\mu_f$ counter-clock wisely (Table 4).

Fact-Definition. Under the above the setting, the set $e_1,\cdots,e_{\mu_f}$ of vanishing cycles (up to choices of sign) associated to the paths $g_1,\cdots,g_{\mu_f}$ form an ordered basis of $\tilde{\mathrm{H}}_n(X_1,\mathbb{Z})$, called a *distinguished basis* ([Br3], [Le1], [Gab1], [Eb1])

Monodromy. Let γ be the path starting at 1 turning once around the boundary of D_ε counter-clock wisely and comes back to 1. The monodromy of this path $c:=\rho(\gamma)\in\mathrm{Auto}(\tilde{\mathrm{H}}_n(X_1,\mathbb{Z}))$ is called the *Milnor monodromy*. Since γ is homotopic to the product $\gamma_1\cdots\gamma_{\mu_f}$ of paths γ_i (see Table 4), we express c:

$$c \;=\; w_{e_1}\cdots w_{e_{\mu_f}}$$

as a product of reflections associated to a distinguished basis $e_1,\cdots,e_{\mu_f}$.

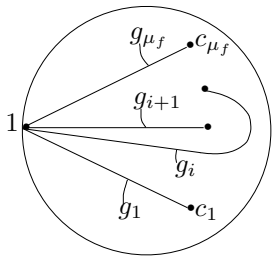


Table 5.

Braid group B_{μ_f} action on distinguished basis: First, we remark that the homotopy classes of the paths $\gamma_1,\cdots,\gamma_{\mu_f}$ give a free generator system of the group $\pi_1(D_\varepsilon\setminus\{c_1,\cdots,c_{\mu_f}\},1)$. Thus the choice of the paths $g_1,\cdots,g_{\mu_f}$, up to homotopy, corresponds to a choice of the generator system of the free group. On the other hand, the braid group B_{μ_f} acts on the set of free generator systems, as usual: for $1\le i<\mu_f$, define an action $\sigma_i:\gamma_1,\cdots,\gamma_{\mu_f}\mapsto\gamma_1,\cdots,\gamma_{i-1},\gamma_i\gamma_{i+1}\gamma_i^{-1},\gamma_i,\gamma_{i+2},\cdots,\gamma_{\mu_f}$. This causes an action of σ_i on paths $g_1,\cdots,g_{\mu_f}$ to those given in Table 5. and on the distinguished basis $e_1,\cdots,e_{\mu_f}$ to $e_1,\cdots,e_{i-1},w_{\gamma_i}(e_{i+1}),e_i,e_{i+2},\cdots,e_{\mu_f}$. One can immediately verify that σ_i $(1\le i<\mu_f-1)$ satisfy Artin braid relations (see [Ar]) so that we obtain a braid group action on the set of distinguished basis.

Remark 2. Even if we start with a globally defined weighted homogeneous polynomial f of positive weights, in order to construct the fibration $f_{\underline{t}}:X\to D_\varepsilon$ above, we need to shrink the domain of $f_{\underline{t}}$ suitably by a use of δ and ε as above. In fact, the embedding of a Milnor fiber $X_{\underline{t}}$ into the global affine surface $\hat{X}_{\underline{t}}:=\{\underline{x}\in\mathbb{C}^{n+1}\,|\,F(\underline{x},\underline{t})=0\}$ induces a non-trivial extension $\tilde{\mathrm{H}}_n(X_{\underline{t}},\mathbb{Z})\subset\tilde{\mathrm{H}}_n(\hat{X}_{\underline{t}},\mathbb{Z})$, if some of the coordinate t_i has negative weight (c.f. §11,b),4)). This extension is generated by vanishing cycles "coming from ∞" and shall play important role in analytic theory of primitive forms (see [Sa20]§6 Conjecture and Problem I').

Remark 3. In mathematical physics, hypersurface singularity is studied under the name Landau-Ginzburg model.

6 Simply Elliptic Singularities

We return to the main stream of our considerations in the present paper: to seek for a connection of primitive forms with Lie theory.

In the year 1974, the author [Sa1] came up with a new class of normal surface singularities, which are "located on the boundary" of the deformation space of simple singularities. They are called the *simply elliptic singularities*, which include the following three types of hypersurfaces:

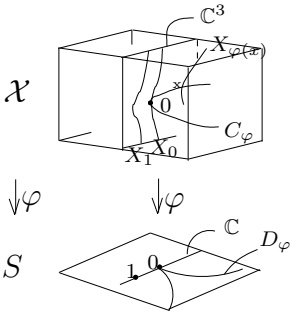
Table 6

Type	equation f_W	$E \cdot E$	(μ_+, μ_0, μ_-)
$\tilde{E}_6$ or $E_6^{(1,1)}$	$x^3+y^3+z^3+\lambda xyz$	-3	0, 2, 6
$\tilde{E}_7$ or $E_7^{(1,1)}$	$x^4+y^4+z^2+\lambda xyz$	-2	0, 2, 7
$\tilde{E}_8$ or $E_8^{(1,1)}$	$x^6+y^3+z^2+\lambda xyz$	-1	0, 2, 8

The simple elliptic singularities X_0 are characterized from two different view points: a) by the resolution of the singularity X_0: *a normal singular point 0 of a surface X_0 is simply elliptic if and only if,* by definition, *the minimal resolution $\pi : \tilde{X}_0 \to X_0$ of the singularity contains only a single elliptic curve $E = \pi^{-1}(0)$,* and b) by deformation of the singularity: *a singular point 0 of a hypersurface surface X_0 is either simple or simply elliptic if and only if any singularity in a local deformation of X_0 admits a weighted homogeneous structure.*[10]

Here, a) the resolution diagram in the sense of Du Val consists only of a single elliptic curve E and it is hard to find Lie theoretic information, in contrast with

[10]Let us explain what do we mean by 1. "singularity in a local deformation of X_0", and 2. "weighted homogeneous structure" on a singularity X_0.



Local deformation of X_0

1. Recall §5 the universal unfolding $F(\underline{x}, \underline{t})$ defined in a neighborhood $\tilde{\mathcal{U}}$ of the origin of $\mathbb{C}^{n+1} \times \mathbb{C}^{\mu_f}$. Then, it defines a local analytic flat family of analytic varieties $\varphi : \mathcal{X} \to S$ where $\mathcal{X} := \{(\underline{x}, \underline{t}) \in \tilde{\mathcal{U}} \mid F(\underline{x}, \underline{t}) = 0\}$, $S := \mathbb{C}^{\mu_f}$ and φ is the projection to the second factor. The fiber $\varphi^{-1}(0)$ over 0 is nothing but the original singular surface X_0 so that the family $\{X_{\underline{t}} := \varphi^{-1}(\underline{t})\}_{\underline{t} \in S}$ is called the semi-universal deformation of the singularity in X_0 ([K-S],[Sch]). One can show that the critical set C_φ of the map φ is (locally near at 0) a smooth subvariety of dimension $\mu_f - 1$, which is finite over S so that the image $D_\varphi := \varphi(C_\varphi)$ is (locally near at 0) a hypersurface in S, called the discriminant of φ. Then, for any point $x \in C_\varphi$, the variety $X_{\varphi(x)} = \varphi^{-1}(\varphi(x))$ is singular at the point x. This is a singularity in a local deformation of X_0. As we saw already, for a generic point $x \in C_\varphi$, $(X_{\varphi(x)}, x)$ is an ordinary double point (i.e. Morse singularity).

2. Let X_0 be a hypersurface in a neighborhood of the origin 0 of $\mathbb{C}^{n+1}$ defined by an analytic equation $f(\underline{x}) = 0$ with an isolated singular point at 0. We say that X_0 admits a weighted homogeneous structure at 0 if there is a local analytic coordinate change at 0 such that the defining equation $f(\underline{x})$ is transformed to a weighted homogeneous polynomial $P(\underline{x})$ (i.e. $P(\underline{x}) = \sum_{a_0 i_0 + \cdots a_n i_n = h} c_{i_0 \cdots i_n} x_0^{i_0} \cdots x_n^{i_n}$ for some positive integers $a_0, \cdots, a_n$ and h). Then, the following i), ii) and iii) are equivalent [Sa0]: i) X_0 admits an weighted homogeneous structure, ii) The sequence: $0 \to \mathbb{C} \to \mathcal{O}_{X_0,0} \xrightarrow{d} \Omega^1_{X_0,0} \to \cdots \xrightarrow{d} \Omega^{n+1}_{X_0,0} \to 0$ is exact, where $(\Omega^\cdot_{X_0,0}, d)$ is the de Rham complex over X_0 at 0, and iii) f belongs to the ideal $\left(\frac{\partial f(\underline{x})}{\partial x_0}, \cdots, \frac{\partial f(\underline{x})}{\partial x_n}\right)$ in the local ring $\mathbb{C}\{\underline{x}\}$.

the case of the simple singularity. However, b) they show a new and immediate relation (in a symbolical level) with Lie theory through deformation theory as follows: *in the local deformation (see* 1. *of* Footnote 10.*) of an elliptic singularity of type* $\tilde{\Gamma} \in \{\tilde{E}_6, \tilde{E}_7, \tilde{E}_8\}$[11]*, only an elliptic singularity of the same type* $\tilde{\Gamma}$ *or a simple singularity can appear. The simple singularity of type* Γ *can appear if and only if* Γ *is a subdiagram of* $\tilde{\Gamma}$. This fact was explained soon after by use of the lattice $(\tilde{\mathrm{H}}_2(X_1, \mathbb{Z}), I)$ (see Footnote 31).[12] Thus, for a simply elliptic singularity X_0, a symbolical relationship with Lie theory begun to appear from the lattice $\mathrm{H}_2(X_1, \mathbb{Z})$ of the smoothing X_1, instead of the resolution $\tilde{X}_0$. Do we need to change our view point? [13] We shall come back again to this question of "change of view-points" later when we discuss $*$-duality in §14 and 15.

7 Vanishing cycles for simple and simply elliptic singularities

In order to sharpen the new view point, i.e. to study the lattice $\mathrm{H}_2(X_1, \mathbb{Z})$ of the middle homology group of the smoothing X_1 of X_0, we consider a particular subset $R \subset \mathrm{H}_2(X_1, \mathbb{Z})$, the set of *vanishing cycles* introduced in §5 (c.f. [Sa16](5.2),(5.3)). From this view point, let us state some consequences of Brieskorn's description [Br4] on simple singularities:

1) *The minimal resolution* $\tilde{X}_0$ *and the smoothing* X_1 *of a simple singularity* X_0 *of type* Γ *are homeomorphic. Hence one obtains an isomorphism of lattices:*

$$*) \qquad \mathrm{H}_2(X_1, \mathbb{Z}) \quad \simeq \quad \mathrm{H}_2(\tilde{X}_0, \mathbb{Z}) \ .$$

Here, the homotopy type of the homeomoprhims, and hence the isomorphism $*$) depend on the Weyl group of type Γ. In fact, the ambiguity of the isomorphism can be resolved (up to an outer automorphism of the Weyl group) by choosing the base point 1 in the totally real region (see Footnote 14).

2) *The set of vanishing cycles* R *in* $\mathrm{H}_2(X_1, \mathbb{Z})$ *(see* §5*) forms a finite root system of type* Γ*, and* $\mathrm{H}_2(X_1, \mathbb{Z})$ *is identified with the root lattice* Q_Γ *of the root system.*

3) *The homology classes* $[C_i] \in \mathrm{H}_2(\tilde{X}_0, \mathbb{Z})$ $(i = 1, \cdots, l)$ *of the exceptional curves in the resolution* $\tilde{X}_0$ *are mapped by the homomorphism* $*$) *to a simple root basis* Γ *of the root system* R*, which are also distinguished basis in the sense in* §5.[14]

If X_0 is a simply elliptic singularities, none of 1), 2) or 3) holds. However, 2) suggests *to regard the set of vanishing cycles in* $\mathrm{H}_2(X_1, \mathbb{Z})$ *for a Milnor fiber*

[11] The names $\tilde{E}_i$ are taken from that of the affine Coxeter diagrams (Table 3) for the reason explained in this section. They are nowadays called also $E_i^{(1,1)}$ for the reason in the next §7.

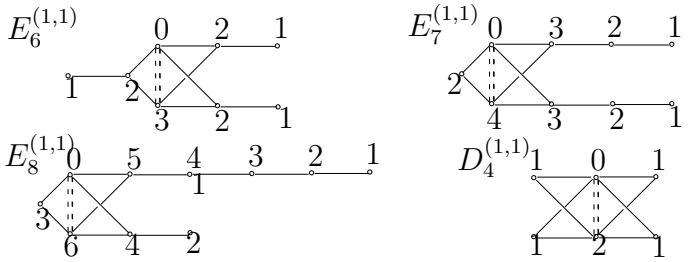
[12] This is shown by the use of the fact that the lattice $\tilde{\mathrm{H}}_2(X_1, \mathbb{Z})$ is isomorphic to $Q_{\tilde{\Gamma}} \oplus \mathbb{Z}$ (see [Ga], [Eb1,2]) where $Q_{\tilde{\Gamma}}$ is the affine root lattice of a type $\tilde{E}_6, \tilde{E}_6$ and $\tilde{E}_8$. See next §7.

[13] This question is supported by the fact that the period domain for the period map $\int \zeta$ of the primitive form is determined from the lattice $\mathrm{H}_2(X_1, \mathbb{Z})$ [Sa14].

[14] The paths $g_1, \cdots, g_{\mu_f}$ in S_φ (Footnote 10), with whom associated distinguished basis $e_1, \cdots, e_{\mu_f}$ is the simple root basis, is given in [Sa21] §4.3 Figure 6. and Theorems 4.1 and 4.2, using semi-algebraic geometry of the real discriminant $D_{\varphi,\mathbb{R}}$ of the universal deformation of the simple singularity. Furthermore, the associated paths γ_i $i = 1, \cdots, \mu$ (Table 4) generate the fundamental group $\pi_1(S_\varphi \setminus D_\varphi, 1)$ and satisfy Artin braid relations of type Γ so that the fundamental group becomes an Artin group ([Br5] [B-S]). Then, *the intersection matrix* $(I(e_i, e_j))_{i,j=1,\cdots,\mu}$ *is shown to become the Cartan matrix of type* Γ *by solving the braid relations where* $\gamma_1, \cdots, \gamma_\mu$ *are substituted by Picard-Lefschetz formula for* $\rho(\gamma_1), \cdots, \rho(\gamma_\mu)$ *in* §5.

X_1 *of an elliptic singularity as a generalization of root systems.* In fact, we can generalize the root systems[15] by removing the finiteness axiom from [Bou] so that the set of vanishing cycles for any even dim. hypersurface isolated singularity becomes a generalized root system. Then the set of vanishing cycles for a simply elliptic singularity is characterized as an *elliptic root system*, that is, a root system belonging in a semipositive lattice with radical of rank 2 ([Sa15]). However, by the lack of 1) and 3), we give up *finding a generalization of the simple root basis for an elliptic root system naively from the geometry of* X_0. There are distinguished basis (§5). However, the braid group B_{μ_f} acts on the set of distinguished basis, and, from a purely topological view point, we cannot choose and separate one from the other.[16] However, we choose some root basis given in Table 7. from an arithmetic consideration[17] such that the elliptic Coxeter-Killing transformation defined as a product of reflections associated to the basis is of finite order and plays a basic role as in the finite root system case.

Table 7. Simply laced Elliptic diagrams of Codim=1 ([Sa15] I, Table 1).



The numbers attached at vertices are the exponents of the root system (see §7).

Further more, from the datum of an elliptic root system, one defines the elliptic Lie algebra. The construction of the primitive forms is a work in progress.[18]

[15] A subset R of a real vector space equipped with a symmetric form I is called a (generalized) root system if $\mathbb{Z}R$ is a full lattice, $2I(\alpha,\beta)/I(\beta,\beta)\in\mathbb{Z}$ and $\alpha-2I(\alpha,\beta)/I(\beta,\beta)\beta\in R$ for $\forall\alpha,\beta\in R$, and irreducible in a suitable sense ([Sa15]). A root system is finite or affine if I is positive definite or semidefinite and rank(radical)$=$1, respectively. A root system is called *elliptic* if I is positive semidefinite and rank(radical)$=$2. The set of vanishing cycles for a simply elliptic singularity of type $\tilde{E}_6, \tilde{E}_7$ or $\tilde{E}_8$ is the elliptic root system of type $E_6^{(1,1)}, E_7^{(1,1)}$ or $E_8^{(1,1)}$.

[16] Gabrielov [Gab2] (Fig. 10 and 11.) obtained the diagrams in Table 7. as one of the possible choices by the braid group action (as the diagrams containing small number of triangles). There seems here a gap between semi-algebraic geometry and topology. For instance, in the simple singularity case, the semialgebraic geometry of the discriminant can yields the distinguished basis which corresponds to the simple root basis of the root system (recall Footnote 14, see also A'Campo's description [AC] the basis of vanishing cycles).

[17] There does not exist elliptic Weyl chambers and, hence, there seemed no a priori definition of a simple basis for an elliptic root system (see [Klu]). However, the elliptic diagram in Table 7. is defined by duplicating the vertex of the affine diagram at the largest exponent (see [Sa15]I(8.6)). We define the elliptic Coxeter-Killing transformation c_e as the product of reflections (acting on $\mathrm{H}_2(X_1,\mathbb{Z})$) attached to the vertices of the elliptic diagram (in a suitable order). Then one has: i) c_e is of finite order h, and the eigenvalues of c_e determine the exponents of the elliptic root system (see §8 and Table 9), ii) the eigenvector of c_e belonging to the eigenvalue 1 is regular in the elliptic Cartan algebra $\mathfrak{h}_e$ with respect to the elliptic Weyl group W_e and iii) the universal central extension $\tilde{W}_e$ of W_e is generated by a lift $\tilde{c}_e^h$. Using i), ii) and iii), a flat structure on the quotient space $\mathfrak{h}_e//\tilde{W}_e$ is constructed ([Sa15]II, [Sat,1,2]).

[18] In [S-Y] the following three algebras are shown to be isomorphic: a) an algebra generated

8 Exponents and Weight Systems

We introduce the *exponents* for a finite or elliptic root system, which play important role in the classical and elliptic Lie theory as well as in the primitive form theory[19]. In this section, we try to continue the definition of exponents for a generalized root system, and meet with a problem of "choice of the phases". This fact leads us to introduce a new concept: the *regular system of weights*.

First, we recall a definition of exponents for a finite or elliptic root system. In both cases, define a Coxeter-Killing transformation as a product c, in a suitable order, of reflection actions on the lattice $\mathrm{H}_2(X_1, \mathbb{Z})$ attached to a simple root basis (recall §5). The c is of finite order h (called the *Coxeter number*, see §19 Remark)[20]. Then the exponents $m_1, \cdots, m_\mu$ are integers such that $\exp(2\pi\sqrt{-1}\frac{m_i}{h})$ $(i = 1, \cdots, \mu)$ are the eigenvalues of c (see [Bou]Ch.v,n^{o}6.2 and [Sa15] I (9.7) Lemma A.iii)). However, this determines only the exponents modulo h. In case of finite root systems and elliptic root systems, one poses further the constraint on the range $0 \leq m_i \leq h$ and on the symmetricity $m_i + m_{\mu-i+1} = h$. Under these constraints, we determine uniquely their exponents as in the next tables.

Table 8.

Type	$(a, b, c; h)$	exponents
A_l $(l \geq 1)$	$(1, b, c; l+1)$	$1, 2, \ldots, l \quad (b{+}c{=}l{+}1)$
D_l $(l \geq 3)$	$(2, l{-}2, 1{-}1; 2(l{-}1))$	$1, 3, 5, \ldots, 2l-3, l-1$
E_6	$(3, 4, 6; 12)$	$1, 4, 5, 7, 8, 11$
E_7	$(4, 6, 9; 18)$	$1, 5, 7, 9, 11, 13, 17$
E_8	$(6, 10, 15; 30)$	$1, 7, 11, 13, 17, 19, 23, 29$

Table 9

Type	$(a, b, c : h)$	exponents
$E_6^{(1,1)}$	$(1, 1, 1 : 3)$	$0, 1, 1, 1, 2, 2, 2, 3$
$E_7^{(1,1)}$	$(1, 1, 2 : 4)$	$0, 1, 1, 2, 2, 2, 3, 3, 4$
$E_8^{(1,1)}$	$(1, 2, 3 : 6)$	$0, 1, 2, 2, 3, 3, 4, 4, 5, 6$

by vertex operators [Bo1] for all elliptic real roots, b) an algebra generated by the Chevalley triplets attached to the elliptic diagram (Table 7) satisfying the generalized Serre relations, and c) an amalgamation of an affine algebra and a Heisenberg algebra. An algebra isomorphic to one of them is called an elliptic algebra. By construction, it is a universal central extension of a 2-toroidal algebra. We remark that the elliptic root systems and the Lie algebras are found also from the representation theory of tubular algebras (see Y. Lin and L. Peng [L-P,1&2]). Some works on highest weight representations and Chevalley type invariants for an elliptic algebra and group are in progress. Due to the above existence of the regular element, several properties similar to classical algebraic groups and its invariant theory hold for the elliptic Lie algebras and its adjoint groups. These facts seem to support the program that the elliptic primitive forms are constructed on the elliptic Lie algebras ([Sa15]V,VI,VII in preparation, see also [Ya3]). However, in the present paper, we shall not go into details on the subject.

[19]The exponents give the degrees of basic $\mathfrak{g}$- or W-invariants and play basic roles in Lie theory (see [Ko],[Sp],[St1]), and also in the study of the flat structures ([Sa17],[Sa15]II,[Sa6]).

[20]The Coxeter-Killing transformation has distinguished properties: i) c is of finite order h, ii) the primitive hth roots of unity (or, 1 for an elliptic root system) are eigenvalues of c, and the eigenvectors of c belonging to them are *regular* (i.e. they are not fixed by the Weyl group and the adjoint group of the Lie algebra, [Col], [Bou] chap.V§6 n^{o}2, [Sa15]II §10 Lemma B). This existence of regular eigenvectors is basic for the construction of the adjoint quotient morphism $\mathfrak{g} \to \mathfrak{g}/\!\!/Ad(\mathfrak{g}) \simeq \mathfrak{h}/\!\!/W$ ([Ko],[Sp],[St1]) and of the flat structure on $\mathfrak{h}/\!\!/W$ ([Sa17],[Sa15]).

We intend further to introduce the exponents through Coxeter-Killing tr. (Milnor Monodromy) for root systems of singularities (since they are necessary data for primitive forms; see discussions below). In fact, we shall obtain in §18 quite interesting class of generalized root systems of Witt index 2 together with some distinguished root basis. However, we meet here a subtle problem: *range of exponents*, which lead the author to introduce the concept of the weight system. To explain the problem concretely, we cite some results from later sections.

1. Consider a polynomial in LHS of Table 10 in §13, whose zero loci defines a hypersurface X_0 in $\mathbb{C}^3$ with an isolated singular point at the origin.

2. The generalized root system (= the set of vanishing cycles) in $\mathrm{H}_2(X_1, \mathbb{Z})$ in the middle homology group of a Milnor fiber X_1 of X_0 has a root basis in the diagram Table 12 (where p, q, r, called the Gabrielov#, are given in Table 13).

3. Define the Coxeter-Killing transformation c as the product of reflection actions on $\mathrm{H}_2(X_1, \mathbb{Z})$ of the vertices of the diagram in a suitable order. Then, c is of finite order h and the characteristic polynomial of c is given in the form (15) for a system of integers m_i called exponents given in Table 10.

4. Observes that m_i's in Table 10 is exceeding the interval $[0, h]$. Thus, *the Coxeter-Killing transformation is unable to determine their phases* ($:= [m_i/h]$) for these new class of root systems. On the other hand, these m_i/h without the ambiguity "modulo 1" are well defined directly from a primitive form. [21]

Concern: The root system with basis may not have sufficient data to determine the phases of exponents and to construct the primitive forms.

We shall discuss again on this issue. See §14 Remark 7. This fact, due to the important role of exponents [Sa6][Sai], leads the author to handle them directly (but not through eigenvalues of Coxeter transform) as follows.

Consider the generating function [22] of exponents in Tables 8 and 9.

$$\chi(T) \ := \ T^{m_1} + T^{m_2} + \cdots + T^{m_\mu}. \tag{1}$$

Then, marvelously, the generating functions of the exponents for the finite and elliptic root systems of types A_l, D_l, E_6, E_7, E_8 and $E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}$ have a decomposition[23] of the form:

$$\chi(T) \ = \ T^{-h} \frac{(T^h - T^a)(T^h - T^b)(T^h - T^c)}{(T^a - 1)(T^b - 1)(T^c - 1)} \tag{2}$$

where a, b, c are integers and h is the order of the Coxeter element such that

$$0 < a, b, c < h \quad \text{and} \quad \gcd(a, b, c) = 1. \tag{3}$$

Note that the weights a, b, c are uniquely determined from the function $\chi(T)$, except for the type A_{h-1}.[24] See Tables 8 and 9 for explicit lists of $(a, b, c; h)$. The

[21] The proportions m_i/h are eigenvalues of an operator N in the flat structure associated to the primitive form, and are called exponents of the flat structure ([Sa3][Sa6] (3.3) Definition).

[22] It is introduced as the Fourier transform of the distribution of the exponents [Sa3] (3.1.1) [Sa6] (3.3.14) in order to study the distribution of the zero-loci of χ.

[23] In the present paper, we are interested in only the cases when all roots of $\chi(T) = 0$ are on the unit circle. But, this is not the case in general for a general primitive form (see [Sa3]).

[24] The characteristic function for the type A_{h-1} is expressed as $\chi_{A_{h-1}}(T) = T + \cdots + T^{h-1} = \frac{T^h - T}{T-1} = T^{-h} \frac{(T^h - T)(T^h - T^b)(T^h - T^c)}{(T-1)(T^b-1)(T^c-1)}$ for $a = 1$ and for any integer b, c with $b + c = h$.

generating function (1) of exponents for a finite or an elliptic root system are characterized by the factorization (2) as follows. Consider abstractly a system:

$$W := (a, b, c; h) \tag{4}$$

of 4 integers satisfying (3) (and additionally, $a=1$ if $b+c=h$ called type A_{h-1}), and call it a *weight system* (a, b, c the *weights* and h the *Coxeter number*).

Fact 1. ([Sa11]Theorem 2) *If the function χ_W (2) for W has no poles, then it is equal to a generating function* (1) *of exponents either for a finite root system of type A_l, D_l, E_6, E_7, E_8 or for an elliptic root system of type $E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}$.*

Let us call the rational function $\chi_W := \chi$ in (2) the *characteristic function* associated to W, and call a weight system W is *simple* (resp. *elliptic*) if its characteristic function χ_W is equal to a generating function (1) for a finite root system (resp. elliptic root system) (explicitly, see Table 8 and 9). [25]

Before analyzing the characteristic function χ_W further, we state another fact, which gives a meaning to the weights a, b, c and to the Coxeter number h in case of a simple and elliptic weight system (see Table 2 and 4 for a proof):

Fact 2. *A simple polynomial $f_{\tilde{G}}(x, y, z)$ in* Table 2 *(resp. an equation for an elliptic singularity in* Table 6*) is a weighted homogeneous polynomial of degree h with the weights a, b, c on the variables x, y, z for a simple (resp. elliptic) weight system $(a, b, c; h)$. The simple weight system determines the simple polynomial, up to a homogeneous coordinate change, uniquely. The elliptic weight system determines the equation up to one parameter (=the modulus of elliptic curves).*

9 Triangle Δ of Weight System, Geometry and Algebra

Summarizing the results of previous sections, we obtain the following triangle among three mathematical objects: weight system, geometry and algebra:

$$\begin{array}{ccc} & \{\text{ Simple weight systems }\} & \\ & \swarrow \quad\quad \nwarrow & \\ \{\text{ Kleinean groups }\} & \Rightarrow & \left\{ \begin{array}{c} \text{Simple Lie algebras} \\ \text{with simple root basis} \end{array} \right\} \end{array} . \tag{5}$$

Here, the three arrows are construed as follows.
1) The correspondence $\Rightarrow$ (denoted by $\Phi_{\Rightarrow}$) is given in three different ways (depending on the view points), all of which give the same result:

a) Use the Du Val diagram for the simple singularity (§1 and 2) and obtain the diagram of the simple root basis of the simple Lie algebra,
b) Use the set of vanishing cycles for the singularity (§5) and obtain the set of real roots of the simple Lie algebra,
c) Use the McKay correspondence ([Mc], see Appendix) and obtain the

[25] To be exact, one should add the diagram for $D_4^{(1,1)}$ (recall Table 7.) in the list. A diagram is called simply-laced if it does not contain a multiple edges. Any other diagrams for simple (or, elliptic) root system is obtained by the foldings of these simply-laced diagrams.

Dynkin graph for the simple Lie algebra.

Here, a) and b) are equivalent due to Brieskorn's theorem (recall §7 1),2) and 3)). c) gives the dual basis of the basis given by a) with respect to the Killing form (see Appendix), but is more direct algebraic construction.

2) The correspondence $\nwarrow$ (denoted by $\Phi_{\nwarrow}$) is given by the decomposition (2) of the generating function (1) of the exponents (Table 7) of the root system.

3) The correspondence $\swarrow$ (denoted by $\Phi_{\swarrow}$) is given by the pair of the fundamental group $\pi_1(X_0 \setminus \{0\})$ for the hypersurface X_0 defined by the polynomial in Table 2 and its action to the covering space $\tilde{X}_0$ (use Fact 2, c.f. [Mu1]).

By a direct inspection of the cases, we see that *a composition of the three arrows* $\Phi_{\swarrow}$, $\Phi_{\Rightarrow}$ *and* $\Phi_{\nwarrow}$ *starting at any corner of the triangle* (5) *is an identity.*[26] Here we stress that the key step among the three arrows is the horizontal correspondence $\Phi_{\Rightarrow}$. The others are rather straight forward. As a consequence of this observation, we conclude that

The datum of the set of exponents for a finite root system, which, a priori, is a very small part of the information of the root system, is sufficient to recover the root system and the simple Lie algebra. In the same way, the datum of a system W *of weights* (4) *is sufficient to reconstruct the simple Lie algebra.*

A similar triangle as (5) holds for the triple of elliptic weight systems, Heisenberg groups of rank 2 ([Sa15] II, Appendix) and elliptic Lie algebras ([Sa15] IV). This supports the construction of the elliptic primitive forms and the flat structures from the elliptic Lie algebras (still in progress). This motivates the author to generalize the triangle by starting with a wider class of weight systems.

We propose to use the top corner of the triangle as follows: consider any weight system W (4) of 4 integers. Then, *relaxing the condition on* $\chi_W(T)$ (2) *to be a polynomial to to be a Laurent polynomial, we extend the class of weight systems and then look for some geometric objects in the left corner and some algebras in the right corner, respectively.* That is: we try to recover the triangle:

$$\begin{array}{ccc} & \{\text{ Weight system } W\ \} & \\ (6) \qquad \swarrow & & \nwarrow \\ \{\text{Geometry of } X_W\} & \Longrightarrow & \{\text{Algebra } \mathfrak{g}_W\} \end{array}$$

where *the goal of the triangle is to construct primitive forms and their associated period mappings and automorphic forms* (see [Sa20] for the details on the goal). Actually, without this setting of the goal, the objects and the correspondences in the triangle (6) are ambiguous (see §12). Note that each corner of the triangle is not a category and the correspondences $\Rightarrow, \swarrow$ and $\nwarrow$ are not functors. However, *we expect a sort of "functoriarity" (yet to be defined) due to the deformation relations among* X_W*'s.*

[26] A similar triangle can be obtained by replacing the three corners by {Heisenberg groups of rank 2 with the extension classes -3,-2,-1}, {Elliptic root systems of type $E_6^{(1,1)}$,$E_7^{(1,1)}$, $E_8^{(1,1)}$ with their simple basis}, {elliptic weight systems}, where we choose b) as for the arrow $\Rightarrow$.

10 Top corner of the triangle: regular systems of weights

We start with recalling the definition [Sa11] of a regular system of weights.[27]

Definition. A weight system $W=(a,b,c;h)$ (4) is called *regular* if the characteristic function $\chi_W(T):=T^{-h}\frac{(T^h-T^a)(T^h-T^b)(T^h-T^c)}{(T^a-1)(T^b-1)(T^c-1)}$ belongs to $\mathbb{Z}[T,T^{-1}]$.

In the following Fact 3 and in Fact 4 in the next section, we give two basic properties of a regular system of weights. The two properties are equivalent to the definition of the regular systems of weights, and they already attribute to the properties in the right and left corners of the triangle (6), respectively.

We, first, discuss about the new definition of exponents.

Fact 3.([Sa11]Theorem 1) *A weight system W (4) is regular, if and only if there exist integers $m_1,\cdots,m_\mu$ with $\mu=\mu_W=\frac{(h-a)(h-b)(h-c)}{abc}$ called the rank of W, such that χ_W (2) is developed into the sum of monomials of the form (1).*

We call $m_1,\cdots,m_\mu$ the *exponents* of W[28], which we order: $m_1\leq\cdots\leq m_\mu$ linearly. By use of the functional equality $T^h\chi_W(T^{-1})=\chi_W(T)$, one has the duality of exponents:

$$m_i+m_{\mu-i+1} \;=\; h \qquad\qquad (i=1,\cdots,\mu). \tag{7}$$

A fact which is not used in the present paper but shall be of basic importance (see Footnote 28), is that there exists always an exponent prime to h [Sa,13,18].

The advantage to start from a weight system is that the exponents are a priori defined without an ambiguity of their phases (i.e. $[m_i/h]\in\mathbb{Z}$). The smallest exponent $\min\{m_1,\cdots,m_\mu\}$ is given and denoted by

$$\varepsilon_W \;:=\; a+b+c-h. \tag{8}$$

Actually, §8 Fact 1. implies that if $\varepsilon_W>0$ (resp. $=0$), then automatically one has $\varepsilon_W=1$ and W is a simple (resp. elliptic) weight system, whose exponents coincide with the exponents of the corresponding finite or elliptic root system.

For each negative integer $\varepsilon<0$, there always exist a finite number of regular systems of weights having ε as the smallest exponent (e.g. $W=(1,1,1;3-\varepsilon)$, see [Sa12, Sa18] Appendix 1,2. for many interesting examples of W with $\varepsilon_W<0$). In particular, there exist 14+8 regular systems of weights for the case $\varepsilon_W=-1$ having no 0 exponents (see Table 10), on which we shall discuss more in details in the present paper.

We are now to analyze the other corners of the triangle (6). Recall that the finite or elliptic root system cannot be directly constructed from the weight system, but we needed to turn the triangle (5) counter-clockwisely. Similarly, we start with analyzing the left corner of (6) in the next section.

[27]This is slightly modified ([Yas]) from the original definition [Sa11]: $\chi_W(T)$ has a pole at most at $T=0$. Using a relation: $T^h\chi_W(T^{-1})=\chi_W(T)$, the two definitions are equivalent.

[28]Here, in order to agree with the classical convention in Lie theory (e.g. [Bou]), we have called the integers m_i exponents. However, from a view point of the flat structure on S (recall Footnote 19), one should better call the rational numbers m_i/h exponents. This view point becomes important again, when we consider the category of graded matrix factorizations §16.

11 Left corner of the triangle: a geometry of X_W

Finding the objects in the left corner and $\Phi_{\not\parallel}$ of the triangle (6) follows from the following characterization Fact 4. of the regularity of a weight system W.

For any given $W=(a,b,c;h)$, consider a weighted homogeneous polynomial

$$f_W(x,y,z) \ := \ \textstyle\sum_{ai+bj+ck=h} c_{ijk}x^iy^jz^k. \tag{9}$$

Fact 4. ([Sa11]Theorem 3) *The weight system W (4) is regular, if and only if there exists a polynomial f_W of the form (9) such that the quotient ring:*

$$J_W \ := \ \mathbb{C}[x,y,z]/\left(\tfrac{\partial f_W}{\partial x}, \tfrac{\partial f_W}{\partial y}, \tfrac{\partial f_W}{\partial z}\right), \tag{10}$$

called the Jacobi ring of f_W, is of finite rank μ_W over $\mathbb{C}$ ("if" part is trivial).

In fact, Fact 4. is trivially equivalent to that the hypersurface

$$X_{W,0} \ := \ \{(x,y,z)\in\mathbb{C}^3 \mid f_W(x,y,z)=0\} \tag{11}$$

has an isolated singular point at the origin, i.e. $X_{W,0}$ is smooth except at the origin $0\in X_{W,0}$, due to the Nullstellensatz of Hilbert. Let us call f_W in Fact 4. a *polynomial of type W*. We employ the polynomial f_W, or the hypersurface $X_{W,0}$ (11) with the isolated singular point at 0 and admitting a good $\mathbb{C}^\times$-action[29] $\lambda\in\mathbb{C}^\times : (x,y,z)\mapsto(\lambda^a x,\lambda^b y,\lambda^c z)$ on it as for the object in the left corner of the triangle (6). Following the history described in §2-7, we analyze $X_{W,0}$ from two: a) algebraic and b) topological view points.

a) **Orbi-bundle $K_{C_W}^{\frac{1}{\varepsilon_W}}$ over the curve C_W.**

There are many studies on surface singularities with a good $\mathbb{C}^\times$-action (e.g. [Dol1,2,3,4], [Pin4,5], [Sa11,12,16], [Wa,1,2]). We recall a few results of them, which are necessary in our purpose. First, we remark that the smoothness of $X_0\setminus\{0\}$ implies that the quotient variety

$$C_W \ := \ (X_{W,0}\setminus\{0\})/\mathbb{C}^\times \ = \ \mathrm{Proj}(\mathbb{C}[x,y,z]/(f_W(x,y,z)) \tag{12}$$

is a smooth curve. However, the $\mathbb{C}^\times$-bundle $X_{W,0}\setminus\{0\}\stackrel{\mathbb{C}^\times}{\to}C_W$ has some finite number of singular fibers (i.e. fixed by some non-trivial finite subgroups, called isotropy groups, of $\mathbb{C}^\times$). In this sense, C_W carries also a structure of an *orbifold curve* (to be precise, an *algebraic stack*). The pair $(g:p_1,\cdots,p_r)$ of the genus g of the curve C_W and the set of the orders of the isotropy groups:

$$A(W) \ = \ \{p_1,\cdots,p_r\} \tag{13}$$

is called the *signature* of the orbifold ([F-K]pp.182-190). In fact, we have

Fact 5. ([Sa11]Theo.6) *The genus g of the curve C_W is equal to the multiplicity $a_0:=\#\{1\le i\le\mu \mid m_i=0\}$ of exponents equal to 0. The set $A(W)$, up to some $p_i=1$, is explicitly determined from the weights W arithmetically.*[30]

[29] The action is said good since the the exponents of the action a,b,c are positive (or, equivalently, the coordinate ring $R_W:=\mathbb{C}[x,y,z]/(f_W)$ is positively graded.

[30] The genus and the signature of the orbi-curve C_W is explicitly give as follows.

The orbifold Euler number: $2-2g+\sum(1/p_i-1)$ is positive, 0 or negative according to whether ε_W is positive, 0 or negative. Accordingly, the orbifold universal covering of C_W is either $\mathbb{P}_1$, the complex plane $\mathbb{C}$ or the complex upper half plane $\mathfrak{H}:=\{z\in\mathbb{C}\mid \mathrm{Im}(z)>0\}$. Then, for W with $\varepsilon_W\neq 0$, we have the description of the $\mathbb{C}^\times$-bundle: $X_{W,0}\backslash\{0\}\to C_W$ ([Dol3]Prop.1, [Sa11](5.5)Lemma)

Fact 6. *Let W be a regular system of weights. According as $\varepsilon_W>0$ or <0, one has the following natural commutative diagrams, respectively.*

$$(14)\qquad \begin{array}{ccc} K_{\mathbb{P}^1}^{\frac{1}{\varepsilon_W}}\setminus 0\text{-section} & \underset{/\tilde\Gamma_W}{\longrightarrow} & X_{W,0}\setminus\{0\} \\ \downarrow/\mathbb{C}^\times & & \downarrow/\mathbb{C}^\times \\ \mathbb{P}^1 & \underset{/\tilde\Gamma_W}{\longrightarrow} & C_W \end{array} \quad\text{and}\quad \begin{array}{ccc} K_{\mathfrak{H}}^{\frac{1}{\varepsilon_W}}\setminus 0\text{-section} & \underset{/\tilde\Gamma_W}{\longrightarrow} & X_{W,0}\setminus\{0\} \\ \downarrow/\mathbb{C}^\times & & \downarrow/\mathbb{C}^\times \\ \mathfrak{H} & \underset{/\tilde\Gamma_W}{\longrightarrow} & C_W. \end{array}$$

Here, 1) $K_{\mathbb{P}^1}^{\frac{1}{\varepsilon_W}}$ and $K_{\mathfrak{H}}^{\frac{1}{\varepsilon_W}}$ is a ε_Wth root of the canonical bundle of $\mathbb{P}_1$ or $\mathfrak{H}$, respectively, and 2) $\tilde\Gamma_W$ is a co-compact discrete subgroup of $SU(2)$ or $PSL(2,\mathbb{R})$, whose actions on $\mathbb{P}^1$ or $\mathfrak{H}$ are liftable to the bundles (Footnote 31), respectively.

The action of $\tilde\Gamma_W$ on $\mathbb{P}_1$ or $\mathfrak{H}$ may have fixed points such that the quotient map $/\tilde\Gamma_W$ gives the orbifold universal covering of C_W. That is: the signature of the group $\tilde\Gamma_W$ ([Ma]) coincides with that $(a_0:A(W))$ of the orbifold curve C_W.[31]

These gives the "algebraic data" of the geometry of $X_{W,0}$ for $\varepsilon_W\neq 0$.[32]

Example. Case $\varepsilon_W>0$ (i.e. W is a simple weight system in Table 8). Then, we naturally have $K_{\mathbb{P}^1}^{\frac{1}{\varepsilon_W}}\setminus\{0\}\simeq\mathbb{C}^2\setminus\{0\}$, and the $\tilde\Gamma_W$ action in LHS is identified with the Kleinean group $\tilde G$-action in RHS (recall §1). I.e. the liftablity condition in Fact 6. is automatically satisfied). The induced action of $\tilde\Gamma_W$ on $\mathbb{P}_1=(\mathbb{C}^2\setminus\{0\})/\mathbb{C}^\times$ is identified with the regular polyhedral group G-action on S^2. So, there are three singular orbits {centers of faces of the polyhedron}, {centers of edges of the polyhedron} and {vertices of the polyhedron} of fixed points on $\mathbb{P}_1$. Therefore, the signature set $A(W)$ (13) consists of the three numbers p,q,r in Table 1. Similarly, in case $\varepsilon_W<0$, the Fuchsian group $\tilde\Gamma_W$ has elliptic fixed points in $\mathfrak{H}$, whose orbits correspond in 1:1 to the elements of $A(W)$.

$$a_0:=\#\{(i,j,k)\in\mathbb{Z}^3_{\geq 0}\mid ai+bj+ck=h\}=\#\{1\leq i\leq\mu\mid m_i=0\},$$
$$A(W):=\{a_i\mid\ a_i\not| h, 1\leq i\leq 3\}\amalg\{\ \gcd(a_i,a_j)*(m(a_i,a_j;h)-1),\ 1\leq i<j\leq 3\}$$

where $\{a_1,a_2,a_3\}=\{a,b,c\}$ and $m(a,b;h)=\#\{(u,v)\in\mathbb{Z}^2_{\geq 0}\mid au+bv=h\}$.

[31] We have a similar geometry for $\varepsilon_W=0$. Namely, the three simply elliptic singularities of types $\tilde E_6,\tilde E_7$ and $\tilde E_8$ are quotients of the trivial $\mathbb{C}^\times$ bundle over $\mathbb{C}$ by an action of a Heisenberg group of rank 2 of characteristic class -3,-2 and -1, respectively ([Sa1],[Sa15]II Appendix).

[32] To be exact, there remains still the problem to characterize (or, to list up) the pair $(\tilde\Gamma_W,\varepsilon)$ of a number $\varepsilon\in\mathbb{Z}_{<0}$ and a co-compact Fuchsian group $\tilde\Gamma_W\subset PSL(2,\mathbb{R})$ such that the action of $\tilde\Gamma_W$ on $\mathfrak{H}$ is liftable to that on $K_{\mathfrak{H}}^{1/\varepsilon}$. This condition on $(\tilde\Gamma_W,\varepsilon)$ (in order to obtain a Gorenstein normal surface singularity $K_{\mathfrak{H}}^{1/\varepsilon}//\tilde\Gamma_W^*$) is equivalent to finding a splitting factor $\tilde\Gamma_W^*$ of the central extension $1\to\mathbb{Z}/\varepsilon\mathbb{Z}\to\tilde\Gamma_d\to\tilde\Gamma_W\to 1$ (c.f. [Sa12] (5.2)(5.3) and (5.4)). To list the cases when $K_{\mathfrak{H}}^{1/\varepsilon}//\tilde\Gamma_W^*$ is a hypersurface requires further works (e.g. [Dol2] [Wa1]) which is unsolved yet. To remain in the category of Gorenstein singularities seems theoretically easier and natural.

b) **Generalized root system and Coxeter-Killing transformation.**

We recollect and discuss about some of the topological data obtained from the semi-universal deformation (also, called universal unfolding) of $X_{W,0}$.

1) Generalized root system

Let us denote by Q_W the lattice $(\mathrm{H}_2(X_{W,1},\mathbb{Z}),(-1)^{\frac{n(n-1)}{2}}I)$ of vanishing cycles[33] and by R_W the set of vanishing cycles for f_W (which depend only on W but on a choice of f_W). As is explained already, it is easy to see that R_W *satisfies the axiom of generalized root system having the lattice* Q_W *as its root lattice in the sense* [Sa15] I. The following 2) and 3) describe some strong properties carried by R_W. However, we do not know a characterization of a root system which arises as the set of vanishing cycles associated to a singular point.

2) Coxeter-Killing element

The Milnor monodromy induces an automorphism c of finite order of the lattice Q_W, which we shall call also the Coxeter-Killing element of the root system R_W. Using the weighted homogeneity of f_W of the defining equation, it is easy to see that the characteristic polynomial $\det(\lambda\cdot \mathrm{id}_Q - c)$ is given by

$$\varphi_W(\lambda) \ := \ \textstyle\prod_{i=1}^{\mu}(\lambda - \exp(2\pi\sqrt{-1}\frac{m_i}{h})) \quad \in \quad \mathbb{Z}[\lambda]. \tag{15}$$

The set $\{\exp(2\pi\sqrt{-1}m_i/h) \mid i=1,\cdots,\mu\}$ is Galois-closed over $\mathbb{Q}$.[34] Recall that:

Fact 7. ([Sa, 13(Theorem 1), 18(Theorem 5.1)]) Let us denote by $e_W(h)$ the multiplicity of the hth primitive roots of unity in the roots of the equation $\varphi_W(\lambda)=0$. Then, for any regular system of weights W, one has $e_W(h)>0$.

Remark 4. In the classical simple Lie algebra case, the eigenvector of the Coxeter-Killing element belonging in to the hth primitive root of unity (in the Cartan subalgebra of $\mathfrak{g}_W$) is regular with respect to the adjoint action of the simple Lie group and that of the Weyl group. This gives a special (key) role to the vector in the invariant theory by Kostant [Ko], Springer [Sp], Steinberg [St1] as well as in the construction of the primitive form and the flat structure [Sa19].

3) Root basis

Any distinguished basis $(e_1,\cdots,e_{\mu_W})$ (recall §5) gives a root basis of the root system R_W in the following sense: i) $R_W = \cup_{i=1}^{\mu_W}\langle w_{e_1},\cdots w_{e_{\mu_W}}\rangle\cdot e_i$, and ii) the Coxeter-Killing element is given by the product $w_{e_1}\cdots w_{e_{\mu_W}}$. This implies: iii) $Q_W = \oplus_{i=1}^{\mu_W}\mathbb{Z}e_i$ and iv) $\langle w_{e_1},\cdots w_{e_{\mu_W}}\rangle$ coincides with the group generated by reflections for all $e\in R_W$ (=the Weyl group of the root system R_W).

[33]The middle homology group $\mathrm{H}_2(X_{W,1},\mathbb{Z})$ admits the symmetric bilinear form, called the intersection form, $I(u,v):=\langle u,P(v)\rangle$ obtained from the Poincare duality $P:\mathrm{H}_2(X_{W,1},\mathbb{Z})\to \mathrm{H}^2(X_{W,1},\mathbb{Z})$. The signature (μ_-,μ_0,μ_+) of the form I is given by a use of exponents: $\mu_- = \#\{1\le i\le\mu \mid m_i<0 \text{ or } h<m_i\}$, $\mu_0=\#\{1\le i\le\mu \mid m_i=0 \text{ or } h\}=2a_0$, $\mu_+=\#\{1\le i\le\mu \mid 0<m_i<h\}$, see [Sai]. Then the *Witt index* (=the maximal rank of totally isotropic subspace) of $H_2(X_{W,1},\mathbb{Z})$=$\mu_0+\mu_-=\#\{$exponents exceeding the interval $(0,h)\}$ is always *even*. This fact supports the existence of the Coxeter element of finite order (see [**?**]) and to ask for Chevalley type invariant theory to the algebra $\mathfrak{g}_W$ in §12 iv).

[34]Substitute any power of $\exp(2\pi\sqrt{-1}/h)$ in (1). (2) implies that it is a rational number.

As we saw already in §5, the braid group of μ_W-strings acts on the set of distinguished basis. It is desirable to find some "simple" basis for the root system R_W by the use of the action, and there are several works in the direction by Gabrielov [Gab 1,2], Ebeling [Eb 1,2], Kluitman [Klu] and others. However, purely topological data of the braid group action alone seems insufficient to choose some distinguished ones. On the other hand, one may still have a hope to choose some particular basis, either by a use of semi-algebraic geometry of the discriminant of the family $\mathcal{X} \to S$ (see Footnote 14 and [Sa21]), or by the algebraic approach a) by a use of the orbifold structure on C_W given in the first half of the present §. The study of these things belongs still to a future work.

4) Cycles from ∞.

We already discussed about the cycles from infinity in §5 Remark2. Under the setting of a regular system of weights W, let us discuss about it again.

Let us define (again) a universal unfolding:

$$F(\underline{x}, \underline{t}) \ := \ f_W(\underline{x}) + \varphi_1(\underline{x}) \cdot t_1 + \varphi_2(\underline{x}) \cdot t_2 + \cdots + 1 \cdot t_{\mu_W}$$

where $\varphi_1, \varphi_2, \cdots, \varphi_\mu \equiv 1$ are weighted homogenous polynomials in $\mathbb{C}[x, y, z]$ (with respect to the weights (a, b, c)) such that their images in the Jacobi ring J_W (10) gives its $\mathbb{C}$-basis. Clearly, the function in a neighborhood of origin gives the universal unfolding in the sense explained in §5. However, we remark that $F(\underline{x}, \underline{t})$ is affine globally defined, where, by putting $\deg(t_i) := h - \deg(\varphi_i) = m_i + \varepsilon_W$ $(1 \leq i \leq \mu_W)$, it is a weighted homogeneous polynomial. The lowest degree coordinate is t_1 and its degree is equal to $2\varepsilon_W$. That is, the unfolding parameter $\underline{t}$ gets negative weights if (and only if) $\varepsilon_W < 0$. Consider the affine gloval family of affine surfaces: $\hat{\varphi}_W : \hat{\mathcal{X}}_W \to S_W$, where $\hat{\mathcal{X}}_W := \{(\underline{x}, \underline{t}) \in \mathbb{C}^3 \times \mathbb{C}^{\mu_W} \mid F(\underline{x}, \underline{t}) = 0\}$, $S_W := \mathbb{C}^{\mu_W}$ and $\hat{\varphi}$ is the projection to the second factor. The discriminants of $\hat{\varphi}_W$ is a divisor of S_W and decomposes into a union of $D_{W,+}$, $D_{W,0}$ and $D_{W,-}$ (see [Sa20]§6). Then, as was shown in §5, the lattice Q_W of middle homology group of the Milnor fiber is generated by the vanishing cycles which are degenerating to the discriminant $D_{W,+}$. Then, the extension $\hat{Q}_W := (\mathrm{H}_2(\hat{X}_{W,\underline{t}}, \mathbb{Z}), -I)$ for a generic parameter value $\underline{t}$ such that the coordinate component $t_1 \neq 0$ is a orthogonal direct sum of the lattice Q_W with the lattice Q_W^∞ generated by the vanishing cycles which are degenerating to the discriminant $D_{W,-}$. It is expected that the periods of the cycles in Q_W^∞ give the denominators for primitive forms ([Sa20] §6 Conjecture and Problem). However, the understanding of the general nature of the part Q_W^∞ is still left to the future work.

Remark 5. The concept of the generalized root system of vanishing cycles and the braid group action on its basis may better be lifted to a categorical level due to the recent developments of the study of Floer homology groups of Lagrangean subvarieties in symplectic varieties [Sei] and [F-O-O-O].

Remark 6. As we shall see in §16, for weight systems W having its $*$-dual, the lattices Q_W and Q_W^∞ are expected to have a categorical construction as the K-groups of the category of the graded and un-graded matrix factorizations, respectively, where the Coxeter element is defined as the A-R translation.

12 Right corner of the triangle: an algebra $\mathfrak{g}_W$

We now come to the main question of the present paper:

Question. *For any regular system of weights W, define the correspondence $\Phi_{\Rightarrow}$ which make the triangle* (6) *commutative. Precisely, construct the algebra $\mathfrak{g}_W$ from the data of the geometry of X_W, satisfying the following conditions* i) - vi).

We explain below what we mean by "the algebra $\mathfrak{g}_W$" by posing working hypothetical conditions i)-vi), otherwise the question may have ambiguous answers. Under the constraints, we expect a sort of functionality for the correspondence $\Phi_{\Rightarrow}$ (recall §9).[35]

i) The algebra $\mathfrak{g}_W$ should be the simple Lie algebra for the case $\varepsilon_W > 0$ and the elliptic Lie algebra for the case $\varepsilon_W = 0$.

ii) The algebra $\mathfrak{g}_W$ should carry an integrability structure, generalizing the Jacobi identity for Lie algebra (i.e. $\mathfrak{g}_W$ should be the prolongation of X_W with respect to the equations for a primitive form; see the last paragraph in §4).

iii) The $\mathfrak{g}_W$ should contain an abelian subalgebra $\mathfrak{h}_W$ isomorphic to $\mathrm{Hom}(Q_W, \mathbb{C})$ (which we may call the Cartan-Killing subalgebra of $\mathfrak{g}_W$ or, simply, the Killing subalgebra of $\mathfrak{g}_W$). The adjoint action of $\mathfrak{h}_W$ on $\mathfrak{g}_W$ induces the root space decomposition of $\mathfrak{g}_W$ so that R_W should be the set of real roots, whose multiplicities are equal to 1.

iv) Depending on a choice (*Note* 3. below), one should have a family of Chevalley type invariant theory for the adjoint group G_W action on $\mathfrak{g}_W$ and the adjoint quotient morphism with the identification of the quotient varieties $\tilde{\mathfrak{g}}_W//\mathrm{Ad}(\tilde{\mathfrak{g}}_W) \simeq \tilde{\mathfrak{h}}_W//\tilde{W}_W$. Here, $\tilde{\mathfrak{g}}_W$, $\tilde{\mathfrak{h}}_W$ and $\tilde{W}_W$ are suitable cone extensions of $\mathfrak{g}_W$, $\mathfrak{h}_W$ and $\tilde{W}_W$, if the Killing form I has a degeneration .[36]

v) The universal unfolding $\mathcal{X}_W \to S_W$ of the singularity $X_{W,0}$ should be embedded into the adjoint quotient map $\mathfrak{g}_W \to \tilde{\mathfrak{h}}_W//\tilde{W}_W$ (c.f. §4 when $\varepsilon_W > 0$). The relative (with respect to the adjoint quotient map) symplectic form on $\mathfrak{g}_W$ (Kostant-Kirirov form when $\varepsilon_W > 0$) induced from the involutive structure given in ii) should (up to a unit factor) induce a primitive form on the family $\mathcal{X}_W \to S_W$, whose exponents (recall foot note 18) coincide with the exponents of the weight system W (up to the factor h).

vi) The flat structure on the quotient variety $\tilde{\mathfrak{h}}_W//\tilde{W}_W$ (c.f. [Sa17],[Sa15] II) and the flat structure on S_W defined from the theory of primitive forms [Sa6]

[35] Owing to recent developments in several categorical approaches, there are several important and interesting algebra structures related to our subjects (e.g. [Bo1], [P-X],). However, in connection with our final goal (the construction of primitive forms) stated in §9, we would like to be cautious before we come to something which exactly fits with our purpose.

[36] One supporting reason for this question is the following fact ([Sa13](2.2) Theorem1, [Sa18] Theorem5.1 and (5.6)): *for any regular system of weights, there always exists an exponent which is prime to the Coxeter number h.* This is a generalization of the existence of an eigenvalue of a primitive hth root of unity of the Coxeter-Killing transformation in the classical case [Col] [Bou]Ch.v§6 Theorem 1. This is a key fact for a construction of the adjoint quotient morphism and for a global construction of a flat structure (see Footnotes 16,17).

should be identified by the isomorphism in iv). This, in particular, asks that the set of exponents for the primitive form on $X_{W,0}$ should coincides with the set of exponents associated to the flat structure of the algebra $\mathfrak{g}_W$.

The last condition vi) implies that the generating function (1) ([Sa6] (3.3.14)) of the exponents for the flat structures of the algebra $\mathfrak{g}_W$ decomposes as in (2), and defines the weight system $W = (a, b, c; h)$, which was at the starting. That is, the correspondence $\Phi_{\nwarrow}$ of the triangle (6) is defined by the use of decomposition of the generating function of the exponents of the algebra, and, then, the composition $\Phi_{\nwarrow} \circ \Phi_{\Rightarrow} \circ \Phi_{\swarrow}$ is the identity on the top of the triangle (6). In particular, we have constructed the family of primitive forms having the exponents given at the beginning by a regular system of weights.

Note. 1. If the Killing form on the root lattice $H_2(X_{W,1}, \mathbb{Z})$ degenerates ($\Leftrightarrow$ the genus a_0 of the curve C_W is positive, see Fact 5. and Footnote 30.), then the algebra $\mathfrak{g}_W$ may have the "radicals" (corresponding to the moduli parameter of the curve C_W). In that case, as for $\mathfrak{g}_W$, we assign the universal algebra (i.e. the one having the largest radicals) for the unicity of the notation $\mathfrak{g}_W$.

2. The other problem in answering the **Question** is that, which view-point a) or b) in §9 do we generalize? It seems likely that, in the above iii), the two view points a) and b) give two different root systems and two different algebras. Let us, tentatively, denote by $\Phi^{a)}_{\Rightarrow}$ the correspondence using the algebraic geometric data of the singularity $X_{W,0}$ and by $\Phi^{b)}_{\Rightarrow}$ the correspondence using the topological data of the deformations of $X_{W,0}$. In fact, these two different view points are, nowadays, called mirror symmetric to each other (see [Kon1], [Yau] for mirror symmetry in general). Marvelously, in our program on the regular systems of weights, we have a duality operation on the set of weight systems, called the $*$-duality, which, conjecturally, exchanges the two approaches (see §14 **Addition to Question**). Then, the conditions iv) and v) on the period map seems to choose $\Phi^{b)}_{\Rightarrow}$ as for the correspondence $X_W \Rightarrow \mathfrak{g}_W$.

3. Recall that an elliptic primitive form depends on a choice of a primitive element in the radical of the root lattice ([Sa6]), which determine the polarization (marking [Sa15]) of the elliptic root system. Similar structure appears also for the primitive forms for the 14 exceptional unimodular singularities, where the primitive form is a proportion of a form with its integral over the cycle coming from infinity (see §11b) 4) and Footnote 43).

13 Strange Duality of Arnold

In order to get an insight to the Question in §12 and also to sharpen it in the next §14, Addition to Question, we look closely at the case $\varepsilon_W = -1$ in this section. We recall the *strange duality* among the 14 cases due to Arnold [Ar4], where the singularities are called exceptional unimodular singularties.

There are $14+8+9$ regular systems of weights of $\varepsilon_W = -1$, where the first $14+8$ cases are genus $a_0 = 0$ and the remaining 9 cases are positive genus $a_0 > 0$. The multiplicity $e_W(h)$ of the first 14 weight systems is equal to 1 and that of

the next 8 weight systems are either equal to 2 or 3. Accordingly, the signature set $A(W)$ (Footnote 28) consists of 3 elements for the first 14 cases, and of 4 or 5 elements for the 8 cases (so that f_W depends on parameter(s)). (see [Sa11, Tables 3,4 and 5] for details on the geometry of them in the sense of §11). In the present paper, we study only the 14+8 cases of genus a_0 is zero.

Table 10. 14+8 regular systems of weights of genus $a_0 = 0$ and $\varepsilon_W = -1$

Polynomial f_W	$(a, b, c; h)$	exponents
$x^7 + y^3 + z^2$:	$(6, 14, 21; 42)$	$-1, 5, 11, 13, 17, 19, 23, 25, 29, 31, 37, 43$
$yx^5 + y^3 + z^2$:	$(4, 10, 15; 30)$	$-1, 3, 7, 9, 11, 13, 15, 17, 19, 21, 23, 27, 31$
$x^4z + y^3 + z^2$:	$(3, 8, 12; 24)$	$-1, 2, 5, 7, 8, 10, 11, 13, 14, 16, 17, 19, 22, 25$
$x^5 + xy^3 + z^2$:	$(6, 8, 15; 30)$	$-1, 5, 7, 11, 13, 15, 17, 19, 23, 25, 31$
$yx^4 + xy^3 + z^2$:	$(4, 6, 11; 22)$	$-1, 3, 5, 7, 9, 11, 11, 13, 15, 17, 19, 23$
$x^3z + xy^3 + z^2$:	$(3, 5, 9; 18)$	$-1, 2, 4, 5, 7, 8, 9, 10, 11, 13, 14, 16, 19$
$x^5 + y^2z + z^2$:	$(4, 5, 10; 20)$	$-1, 3, 4, 7, 8, 9, 11, 12, 13, 16, 17, 21$
$yx^4 + y^2z + z^2$:	$(3, 4, 8; 16)$	$-1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 13, 14, 17$
$x^4 + y^3 + xz^2$:	$(6, 8, 9; 24)$	$-1, 5, 7, 8, 11, 13, 16, 17, 19, 25$
$x^3y + y^3 + xz^2$:	$(4, 6, 7; 18)$	$-1, 3, 5, 6, 7, 9, 11, 12, 13, 15, 19$
$x^3z + y^3 + xz^2$:	$(3, 5, 6; 15)$	$-1, 2, 4, 5, 5, 7, 8, 10, 10, 11, 13, 16$
$x^4 + y^2z + z^2x$:	$(4, 5, 6; 16)$	$-1, 3, 4, 5, 7, 8, 9, 11, 12, 13, 17$
$x^3y + y^2z + z^2x$:	$(3, 4, 5; 13)$	$-1, 2.3.4.5.6.7.8.9.10.11.14$
$x^4 + yz(y - z)$:	$(3, 4, 4; 12)$	$-1, 2, 3, 3, 5, 6, 6, 7, 9, 9, 10, 13$
$y(y - x^3)(y - \lambda x^3)$:	$(2, 6, 9; 18)$	$-1, 1, 3, 5, 5, 7, 7, 9, 9, 11, 11, 13, 15, 17, 19$
$xy(y - x^2)(y - \lambda x^2) + z^2$:	$(2, 4, 7; 14)$	$-1, 1, 3, 3, 5, 5, 7, 7, 7, 9, 9, 11, 11, 13, 15$
$y(y - x^2)(y - \lambda x^2) + xz^2$:	$(2, 4, 5; 12)$	$-1, 1, 3, 3, 4, 5, 5, 7, 7, 8, 9, 9, 11, 13$
$(y^2 - x^3)(y^2 - \lambda x^3) + z^2$:	$(2, 3, 6; 12)$	$-1, 1, 2, 3, 4, 5, 5, 6, 7, 7, 8, 9, 10, 11, 13$
$x(z - x^2)(z - \lambda x^2) + y^2z$:	$(2, 3, 4; 10)$	$-1, 1, 2, 3, 3, 4, 55, 6, 7, 7, 8, 9, 11$
$x^3y + z(z - y)(z - \lambda y)$:	$(2, 3, 3; 9)$	$-1, 1, 2, 2, 3, 4, 4, 5, 5, 6, 7, 7, 8, 10$
$xy(x - y)(y - \lambda_1 x)(y - \lambda_2 x) + z^2$:	$(2, 2, 5; 10)$	$-1, 1, 1, 3, 3, 3, 5, 5, 5, 5, 7, 7, 7, 9, 9, 11$
$y(y - x)(y - \lambda_1 x)(y - \lambda_2 x) + xz^2$:	$(2, 2, 3; 8)$	$-1, 1, 1, 2, 3, 3, 3, 4, 5, 5, 5, 6, 7, 7, 9$

Here λ, λ_1 and λ_2 are parameters $\neq 0, 1$ and $\lambda_1 \neq \lambda_2$.

In order to construct $\Phi_{\Rightarrow}$, remember a) and b) of §9, 1) for the case $\varepsilon_W = 1$. An immediate analogy of $\Phi^{a)}_{\Rightarrow}$ is to study the resolution of the singularity $X_{W,0}$ (actually, we modify this later in §15). The minimal resolution $\pi : \tilde{X}_{W,0} \to X_{W,0}$ of the singularity is determined by [Dol1] as follows: the exceptional set $\pi^{-1}(0) \subset \widetilde{X}_0$ of the minimal resolution is a union of 4-rational curves E_0, E_1, E_2 and E_4, which are intersecting transversally as in Table 11 and are self-intersecting as

$$-1 = E_0^2, \quad -p = E_1^2, \quad -q = E_2^2, \quad -r = E_3^2$$

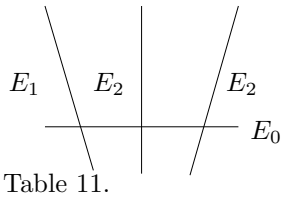


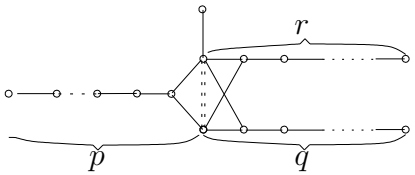
Table 11.

where p, q, r are positive integers such that $(0 : p, q, r)$ is the signature of the curve C_W (§11 Fact 5., [Dol 1,2,3,4],[Pin4,5],[Sa 11,12]). The triplet $A(W) = \{p, q, r\}$ (Footnote 28) is called the *Dolgachev numbers* [Dol1] (see Table 13).

On the other hand, as for the correspondence $\Phi^{b)}_{\Rightarrow}$, one should describe the set of vanishing cycles in the lattice $H_2(X_{W,1}, \mathbb{Z})$, whose signature is $(2, 0, *)$

(Footnote 31.). The *distinguished basis* of the lattices were studied by the authors A.M. Gabrielov [Gab1,2] and W. Ebeling [Eb1,2]. In particular, they found the following "simple"[37] basis for the 14 exceptional singularities:

Table 12.



where the length p, q, r of the three branches is called the *Gabrielov numbers.*[38]

Then, Arnold [Ar3] observed the following duality and called it the strange duality: *there exists an involutive bijection σ on the set of 14 exceptional singularities, by which Dolgachev numbers and Gabrielov numbers interchange.*

In the next table, we indicate the involution σ by the two-sided arrows $\leftrightarrow$ between the weight systems corresponding to the singularities.

Table 13. The strange duality and the $*$-duality.

	Weights	$A(W)$=Dolgachev#	Gabrielov#	$\varphi_W(\lambda)$
$\circlearrowleft$	(3, 4, 4; 12)	4, 4, 4	4, 4, 4	$\frac{(\lambda^{12}-1)(\lambda^4-1)}{(\lambda^3-1)(\lambda-1)}$
$\circlearrowleft$	(3, 4, 5; 13)	3, 4, 5	3, 4, 5	$\frac{(\lambda^{13}-1)}{(\lambda-1)}$
	(4, 5, 6; 16)	2, 5, 6	3, 4, 4	$\frac{(\lambda^{16}-1)(\lambda^4-1)}{(\lambda^8-1)(\lambda-1)}$
$\circlearrowleft$	(3, 5, 6; 15)	3, 3, 6	3, 3, 6	$\frac{(\lambda^{15}-1)(\lambda^3-1)}{(\lambda^5-1)(\lambda-1)}$
	(4, 6, 7; 18)	2, 4, 7	3, 3, 5	$\frac{(\lambda^{18}-1)(\lambda^3-1)}{(\lambda^9-1)(\lambda-1)}$
	(6, 8, 9; 24)	2, 3, 9	3, 3, 4	$\frac{(\lambda^{24}-1)(\lambda^4-1)(\lambda^3-1)}{(\lambda^{12}-1)(\lambda^8-1)(\lambda-1)}$
	(3, 4, 8; 16)	3, 4, 4	2, 5, 6	$\frac{(\lambda^{16}-1)(\lambda^2-1)}{(\lambda^4-1)(\lambda-1)}$
$\circlearrowleft$	(4, 5, 10; 20)	2, 5, 5	2, 5, 5	$\frac{(\lambda^{20}-1)(\lambda^5-1)(\lambda^2-1)}{(\lambda^{10}-1)(\lambda^4-1)(\lambda-1)}$
	(3, 5, 9; 18)	3, 3, 5	2, 4, 7	$\frac{(\lambda^{18}-1)(\lambda^2-1)}{(\lambda^6-1)(\lambda-1)}$
$\circlearrowleft$	(4, 6, 11; 22)	2, 4, 6	2, 4, 6	$\frac{(\lambda^{22}-1)(\lambda^2-1)}{(\lambda^{11}-1)(\lambda-1)}$
	(6, 8, 15; 30)	2, 3, 8	2, 4, 5	$\frac{(\lambda^{30}-1)(\lambda^5-1)(\lambda^2-1)}{(\lambda^{15}-1)(\lambda^{10}-1)(\lambda-1)}$
	(3, 8, 12; 24)	3, 3, 4	2, 3, 9	$\frac{(\lambda^{24}-1)(\lambda^3-1)(\lambda^2-1)}{(\lambda^8-1)(\lambda^6-1)(\lambda-1)}$
	(4, 10, 15; 30)	2, 4, 5	2, 3, 8	$\frac{(\lambda^{30}-1)(\lambda^3-1)(\lambda^2-1)}{(\lambda^{15}-1)(\lambda^6-1)(\lambda-1)}$
$\circlearrowleft$	(6, 14, 21; 42)	2, 3, 7	2, 3, 7	$\frac{(\lambda^{42}-1)(\lambda^7-1)(\lambda^3-1)(\lambda^2-1)}{(\lambda^{21}-1)(\lambda^{14}-1)(\lambda^6-1)(\lambda-1)}$

The strange duality captured the attention of many authors and was interpreted by Dolgachev, Nikulin and Pikham in terms of duality between algebraic

[37]Here, we mean by "simple" the following: 1) the vertices of the diagram is a $\mathbb{Z}$-basis of the lattice $H_2(X, \mathbb{Z})$, 2) a product in suitable order of the reflections on the lattice attached to the vertices of the diagram (i.e. a Coxeter-Killing transformation) is of finite order h and its eigenvalues are $\exp(2\pi\sqrt{-1}m_i/h)$ for the exponents m_i, 3) consider the group W acting on the lattice generated by the reflections attached to the vertices of the diagram. Then the set of the vanishing cycles is equal to the union of the W-orbits of the vertices of the diagram.

[38]There is a strong reason why we suspect that the diagram should be (a part of) the correspondence $\Phi_{\Rightarrow}$ for the 14 weight systems, since it partially answers iv) and v) in §12 as follows.. Let $\xi, \overline{\xi}$ be the eigenvectors of the Coxeter-Killing transformation belonging to the eigenvalues $\exp(\pm 2\pi\sqrt{-1}/h)$. Then each belongs to the two connected component of the cone $\{x \in Q_W \otimes_{\mathbb{Z}} \mathbb{C} \mid I(x, \overline{x}) < 0, I(x, x) = 0\}$ over a symmetric domain of type IV and is regular w.r.t. the Weyl group (i.e. does not belong to any reflection hyperplane ([Sa16] (5.6) Lemma 2). We remark also that the diagram defines a splitting hyperbolic plane of the lattice Q_W.

cycles and transcendental cycles on certain K3 surfaces [Pin1,2]. Further generalizations of the duality were studied by several authors [N-G][Pin4,5][Lo4][E-W].

In §14, we induce arithmetically the strange duality from the $*$-duality of weight systems [Sa18], which is interpreted as a mirror symmetry [Ta1].

14 $*$-duality of regular systems of weights

We introduce one key operation $*$ of the present paper: the $*$-duality on regular systems of weights [Sa18]. It induces the strange duality in the arithmetical level, and induces, much further, wide class of dualities among weight systems.

Recall the *characteristic polynomial* $\varphi_W(\lambda)$ (15) of a regular system of weights W. Since $\varphi_W \in \mathbb{Z}[\lambda]$ (recall §11 b)) and is a cyclotomic polynomial, by solving Möbius inversion formula, there exists a unique expression:

$$\varphi_W(\lambda) \;=\; \prod_{i|h}(\lambda^i-1)^{e_W(i)} \tag{16}$$

for some integer $e_W(i) \in \mathbb{Z}$ for all $i \in \mathbb{Z}_{>0}$ with $i|h$, where h is the Coxeter number of W.

Definition. A regular system of weights W' is called $*$-*dual* to W if 1) its Coxeter number coincides with that h of W, and 2) one has the duality relation:

$$e_W(i)+e_{W'}(h/i) \;=\; 0 \qquad \text{for } {}^{\forall}i \in \mathbb{Z}_{>0}. \tag{17}$$

Here, we put $e_W(i)=e_{W'}(i)=0$ for an integer i with $i \not| h$.

Example. 1. The characteristic polynomial for the type E_8 decomposes as

$$\varphi_{E_8}(\lambda) \;=\; \frac{(\lambda^{30}-1)(\lambda^5-1)(\lambda^3-1)(\lambda^2-1)}{(\lambda-1)(\lambda^6-1)(\lambda^{10}-1)(\lambda^{15}-1)}.$$

Then $e_{E_8}(30)+e_{E_8}(1)=e_{E_8}(5)+e_{E_8}(6)=e_{E_8}(3)+e_{E_8}(10)=e_{E_8}(2)+e_{E_8}(15)=0$. This implies E_8 is selfdual. This is a special case of the next 2.

2. All regular weight systems W with $\varepsilon_W > 1$ (i.e. simple weight systems) are selfdual ([Sa18] Theo.7.10.1). This fact is resonant with the result of Brieskorn in §7, 1). However, the $*$-duality, in general, implies neither the homeomorphism: $\tilde{X}_{W,0} \simeq X_{W^*,1}$ nor $\tilde{X}_{W^*,0} \simeq X_{W,1}$ (see the examples below). Therefore, it seems interesting to ask what is a natural generalization of [O-O] for the $*$-dual pair?

3. Any regular system of weights W of rank μ_W equal to 24 is selfdual. It is curious to observe that there are 11 such weight system with $\varepsilon_W < 0$, and the set of their characteristic polynomials is exactly equal to the set of all, except for the four 6A, 10A, 15D and 18A, self-dual characteristic polynomials of the conjugacy classes of the Conway group $\cdot 0$ ([ibid] Appendix 1).

We have the following uniqueness of the $*$-dual weight system.

Theorem. ([Sa18] The.7.8) 1. *For a regular system of weights W if there exists a $*$-dual weight system, then it is uniquely determined from W, which we denote by W^*. By definition, we have $(W^*)^* = W$.*

2. *The smallest exponent of W^* is equal to that of W.*
3. *The multiplicities $e_W(h)$ and $e_{W^*}(h)$ are equal to 1.*

In general, there does not exist a dual weight system W^* for a given regular system of weights W (eg. if the multiplicity $e_W(h)$ is larger or equal than 2, then W cannot have the dual weight system), but if the $*$-dual for W exists, then it is purely arithmetically determined from W. In fact, we have a complete list of dual pair of regular systems of weights ([Sa18] The.7.9). As a consequence, we can prove the following.

Fact 7. ([Sa18] Theo.7.10.2 & §12) *Any of the 14 regular systems of weights W in* Table 10 *(i.e. $\varepsilon_W = -1$, $a_0 = 0$ and $e_W(h) = 1$) is $*$-dual to a weight system in the same* Table. *Further, if W and W^* are dual, then $\mu_W + \mu_{W^*} = 24$*[39] *and*

$$\text{Dolgachev \# of } W = \text{Gabrielov \# of } W^*$$
$$\text{Dolgachev \# of } W^* = \text{Gabrielov \# of } W$$

That is: the strange duality of Arnold is induced from the $*$-duality.

Remark 7. Whether W^* is dual to W or not is determined only by the characteristic polynomials φ_{W^*} and φ_W, and hence, in view of (15), it is determined only by the exponential: $\exp(2\pi\sqrt{-1}m_i/h)$ $(i = 1,\cdots,\mu)$ and $\exp(2\pi\sqrt{-1}m_i^*/h)$ $(i = 1,\cdots,\mu^*)$. That is, *the information of the phases $[m_i/h]$, $[m_i^*/h]$ of the exponents are unnecessary to determine the $*$-duality.*

This causes a puzzle: we had a concern (§8 Concern) that the eigenvalues $\exp(2\pi\sqrt{-1}m_i/h)$ of a Coxeter element may not be sufficient to recover the phases of the exponents. Therefore, in §10, we settled a regular system of weights (but not a root system with a simple basis) as our starting point, since a regular system of weights carries the full data of the set of exponents. Then, we now arrive at a result that the phase is unnecessary for the duality among them. Is this not a puzzle?

The author does not have a good explanation to this puzzle. The only fact, we can mention here is that *a regular system of weights W, which admits the dual W^*, has a particularity such that the datum of the set of exponentials $\{\exp(2\pi\sqrt{-1}m_i/h) \mid i = 1,\cdots,\mu\}$ is sufficient to recover $\{m_i \mid i = 1,\cdots,\mu\}$.*

Namely, the uniqueness of the dual weight system in Theorem is shown briefly as follows: if a weight system W admits a dual weight system, then the characteristic polynomial $\varphi_W(\lambda)$ is reduced (i.e. $e_W(i) \in \{0, \pm 1\}$ for $i \in \mathbb{Z}_{>0}$). This is equivalent to $e_W(h) = 1$ and we call such W *simple*. On the other hand, a simple weight system W is arithmetically reconstructed from its characteristic polynomial φ_W ([ibid] The.6.3). This proves the uniqueness of W (and W^*).

Perhaps, the above puzzle is closely related to another puzzle: the $*$-duality is described purely in terms of arithmetics whereas the strange duality exchanges the algebraic and the transcendental structures in geometry.

[39] In the original proof [Sa18] Theorem 7.10, 2., the rank relation was not stated explicitly.

Remark 8. It seems an interesting and important problem to find a reasonable extension of regular systems of weights, which is closed under the $*$-duality. For instance, the end remark of Footnote 29 suggests that Gorenstein surface singularities with good $G \times \mathbf{A}^{\times}$-action should be included (see [Ta1]). However, in the present paper, we do not go into any details of the question. Instead, we proceed here as if we were already in the extended category which is closed under the $*$-duality, and ask the following follow-up to the question in §12.

For two decades, inspired from mathematical physics, one observes "symmetry relations" called *mirror symmetry* between some symplectic topological varieties, called A-model side and some algebraic (or complex analytic) varieties, called B-model side. The mirror symmetry is formulated in different levels from identities of numerical invariants of the varieties to the equivalence of categories associated to the varieties, and, in the present paper, we do not go into any details of the subject but just refer the reader to some of the literature (see for instance [Kon1],[Yau],[H-V]). It is expected that the models in both side finally should induce an isomorphic flat structures (recall the condition vi) in §12). The mirror symmetry in topological Landau-Ginzburg orbifold model (which corresponds to the singularity theory in mathematics) is described by Kawai-Yang [K-Y] in terms of the duality of orbifoldized Poincaré polynomials. Therefore, it is natural to ask whether and, actually proven by A. Takahashi [Ta1] that *the $*$-duality of weight systems is equivalent to the mirror symmetry in the Landau-Ginzburg orbifold model in mathematical physics.*[40]

Having these background, we ask the following mathematical question.

Addition to §12 Question. Whether there exists an involutive correspondence $\mathfrak{g} \mapsto \mathfrak{g}^*$ on the set (objects of a small category?) of algebras in the right corner of the triangle (6) so that one has the isomorphism: $\mathfrak{g}_{W^*} \simeq (\mathfrak{g}_W)^*$? That is: whether there exists a $*$-duality making the following diagram commutative?

$$
\begin{array}{ccc}
\{\text{Regular systems of weights } W\} & \Leftrightarrow & \{\text{Algebras } \mathfrak{g}_W\} \\
\Updownarrow\ * & & \Updownarrow\ * \\
\{\text{Regular systems of weights } W^*\} & \Leftrightarrow & \{\text{Algebras } \mathfrak{g}_{W^*} \simeq (\mathfrak{g}_W)^*\}\ .
\end{array}
\tag{18}
$$

This means that we ask duality operations in each corner of the triangle 6) (i.e. the $*$-duality on the top, the mirror symmetry in the left and the new conjectural $*$-duality in the right) so that the arrows are compatible with them. Likewise that the domain of the definition of the $*$-duality in the top is restricted, it is likely that similar constraint on the domain of the definition in the RHS might exist. Note also that the $*$-duality in the RHS does not keep the rank μ_W (of the Killing-subalgebra $\mathfrak{h}_W$, recall §12 iii)), but rather supplementary to each other in the sense that $\mu_W + \mu_{W^*} = 24$ in case of $\varepsilon_W = -1$.

What seems remarkable here is the fact that the $*$-duality in the RHS exchanges the algebras which is constructed from algebraic data with that from transcendental data, whereas the $*$-duality in the LHS is purely arithmetically defined. In §15, we shall discuss on the duality from a categorical level.

[40] Accordingly, the definiton of the $*$-duality for the weight systems of type V is modified.

Remark 14.1. In the rest of this section, we give a digression on the reformulation of the $*$-duality in terms of eta products ([Sa18] §13).

1. Let $\eta(\tau) := q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$ (where $q^a := \exp(2\pi\sqrt{-1}a\tau)$ be the eta function of Dedekind. To the polynomial (16), we associate a product

$$\eta_W(\tau) \ := \ \prod_{i|h}\eta(i\tau)^{e_W(i)}. \tag{19}$$

Assertion ([Sa18](13.3)) *Two weight systems W and W^* are dual to each other if and only if one of the following (equivalent) relations holds:*

$$\begin{aligned}\eta_W(-1/h\tau)\cdot\eta_{W^*}(\tau) &= (\tau/\sqrt{-1})^{a_0}/\sqrt{d_{W^*}},\\ \eta_{W^*}(-1/h\tau)\cdot\eta_W(\tau) &= (\sqrt{-1}/\tau)^{a_0}/\sqrt{d_W},\end{aligned}$$

where d_W is the discriminant defined by $\prod_{i|h} i^{e_W(i)}$.

2. The coefficients of the expansion of $\eta_W(\tau)$ in the powers of q (called the Fourier coefficients of $\eta_W(\tau)$ at ∞) behaves very characteristically. Namely,

i) Fourier coefficients of the eta-products of type A_l $(l \geq 1)$, D_l $(l \geq 4)$ and E_l $(l = 6, 7, 8)$ are positive and are exponentially growing.

ii) Fourier coefficients of the eta-products of type $D_4^{(1,1)}, E_6^{(1,1)}, E_7^{(1,1)}$ or $E_8^{(1,1)}$ are non-negative and are polynomially growing. For example:

$$\eta_{E_8^{(1,1)}}(12\tau) \ = \ q^5+q^{17}+q^{29}+q^{41}+q^{53}+2q^{65}+q^{89}+q^{101}+q^{113}+2q^{125}+q^{137}+q^{149}+\cdots.$$

Here, i) is trivially checked but ii) requires some work [Sa15]IV, where the Melin transform $L_W(s)$ of η_W is expressed by the L-functions of the cyclotomic field $\mathbb{Q}(\sqrt{-1}), \mathbb{Q}(\sqrt{-3}), \mathbb{Q}(\sqrt{-1},\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-1},\sqrt{-3})$ according as W is of type $D_4^{(1,1)}, E_6^{(1,1)}, E_7^{(1,1)}$ or $E_8^{(1,1)}$. Then, the expression implies the non-negativity. For example: choosing suitable Dirichlet characters ε, χ on $\mathbb{Z}/12\mathbb{Z}$, one has

$$L_{E_8^{(1,1)}}(s) = \frac{1}{4}\prod_{p\in P_1}\frac{1}{(1-p^{-s})^2}\prod_{p\in P_{\varepsilon}\cup P_{\varepsilon\chi}}\frac{1}{1-p^{-2s}}\left\{\prod_{p\in P_{\chi}}\frac{1}{(1-p^{-1})^2}-\prod_{p\in P_{\chi}}\frac{1}{(1+p^{-1})^2}\right\}.$$

Remarkable, *the non-negativity of the Fourier coefficients characterizes the elliptic root systems having the flat structure* in the following sense: for each elliptic root system, we associate the eta-product (19) using the decomposition (16) of the characteristic polynomial of its Coxeter element. Then,

Fact8. *The Fourier coefficients are non-negative if and only if the root system is of type $D_4^{(1,1)}, E_6^{(1,1)}, E_7^{(1,1)}$ or $E_8^{(1,1)}$, which are exactly the types of simply-laced elliptic root systems admitting the flat structure* (see [Sa15]II, Footnote 14.) [41]

The author do not know where in the triangle (6) does the eta product η_W and its Fourier coefficients locate. So, we ask

[41] Including i) and ii), the author gave a conjecture [Sa15]: *an eta product $\eta(h\tau)^{\nu}\eta_W(\tau)$ has non-negative Fourier coefficients if and only if $\nu_W \geq \nu$ where ν_W is the dual rank of W defined by $-\sum_{i|h}(h/i)e_W(i)$.* Recently, S. Yasuda [Yas] solved the conjecture affirmatively.

Question: Interpret the Fourier series of η_W as the generating function of counting of some objects either from the geometry X_W or from the algebra $\mathfrak{g}_W$.

Finally, we remark that a stronger form of the following statement was conjectured in [Sa18](Conjecture 13.5) and is proven by S. Yasuda [Yas].

Fact 9. Let us define the dual rank ν_W of W by $\nu_W := -\sum e_W(h/i)i$ (actually, $\nu_W = \mu_{W^*}$ if the $*$-dual of W exists). Then, all Fourier coefficients of $\eta_W(\tau)$ at ∞ are non-negative integers if and only if $\nu_W \geq 0$.

15 Towards algebraic construction of the correspondence $\Phi_{\Rightarrow}$

We return to the question of constructing the correspondence $\Phi_{\Rightarrow}$ posed in §12. According the program §12 iii), the algebra $\mathfrak{g}_W$ should have the root space decomposition with respect to the adjoint action of the Cartan-Killing subalgebra $\mathfrak{h} \simeq \mathrm{H}^2(X_1, \mathbb{C})$ with the generalized root system R_W of the vanishing cycles in $\mathrm{H}_2(X_1, \mathbb{Z})$ (§5,7). So, the first task should be a good description of the set of vanishing cycles R_W and to find a good basis for that.

For the 14 cases of $\varepsilon_W = -1$, we explained already that the diagram in Table 12. due to Gabrielov is a good candidate for the simple root basis for the generalized root system (recall Footnotes 35 and 36). However, the diagram was obtained after several braid actions on the basis of the lattice of vanishing cycles ([Gab2], see also [Eb1]), and hardly foreseen. It seems also hard to explain the naturality (or, the necessity) of the diagram and the Gabrielov numbers (see Table 12.) in this formulation.

On the other hand, once we admitt to use $*$-duality in §14, the situation changes drastically. Namely, owing to §14 Fact., the Gabrielov numbers of the diagram of the 14 weight systems W with $\varepsilon_W = -1$ is given by the signature set $A(W^*)$ (recall §11 Fact 5. and Footnote 28) of the $*$-dual weight system W^*. That is, the Gabrielov number for W is determined "intrinsically" by two arithmetic steps: step 1. calculate the $*$-dual weight system W^* from W ([Sa18] The.7.9) and step 2. calculator the signature set $A(W^*)$ of W^* ([Sa11] The.6) without any ambiguity. That is: the diagram in Table 12. for W is, at least in its numerical level, determined from the algebraic poach through the $*$-dual W^*. Actually, the same phenomenon occured already for the simple singularities, where $\varepsilon_W > 0$ and the weight system W is self-dual (§14 Example 2) and then the signature set $A(W^*)$ $(= A(W))$ gives the branch length-es of the diagram of the simple basis of the finite root system R_W (recall §7, 3), §11 a) Example and Table 3).

These facts led the author to ask a problem: [42]

Problem: *Construct the root system R_W and its basis through the algebraic Apache $\Phi^{a)}_{\Rightarrow}(X_{W^*})$ instead of the topological Apache $\Phi^{b)}_{\Rightarrow}(X_W)$.*

[42] It was asked in [Sa16]. Here, we quote from the Englisher translation p.124): Problem. Construct directly from the system of weights $(a, b, c; h)$, without pathing through the homology group of the Milnor fiber, arithmetically or gubernatorial, the generalized root system $(Q, -I, R, c)$ given above. That is to say, give a basis $\alpha_1, \cdots, \alpha_\mu$ and their inner products $I(\alpha_i, \alpha_j)$ $(1 \leq i \leq j \leq \mu)$ directly from the data $(a, b, c; h)$.

The numerical data $A(W^*)$ alone is not sufficient, and we need to find a structural approach to reconstruct the root system R_W and its basis. Based on the recent development's in mathematical physics[43], A. Takahashi [Ta2] gave an answer to the first part of the problem (i.e. the construction of the root system R_W). He conjectured that the K-group of the category of graded matrix factorizations for f_{W^*} should be isomorphic to the lattice of vanishing cycles $(H_2(X_1, \mathbb{Z}), I)$. He has shown that the category for the case of the polynomial $f_{A_l} = x^{l+1}$ of type A_l is derived equivalent to the category of modules over the path algebra for the Dynkin quiver of type A_l so that the set of indecomposable objects in the K-group gives the set of roots of type A_l and conjectured that this should holds for all the other simple polynomials.

In the following three sections, we report the results of some joint works of H. Kajiura, A. Takahashi and the author along this line. We introduce in §16 the homotopy category $\mathrm{HMF}^{gr}_{A_W}(f_W)$ of graded matrix factorizations for f_W, in three different formulations.

In §17, we study the category for a simple weight system W, and show that it has strongly exceptional collections (see §16, 4. for a definition) whose associated quiver is the classical Dynkin quiver of the type of $W = W^*$ [K-S-T 1]. Then, due to a classical result by Gabriel [Ga], the set of the indecomposable objects in the category form the classical finite root system in the associated Grothendieck group (= K-group), as was expected.

In §18, we study the category for a weight system W of $14 + 8$ weight systems of $\varepsilon_W = -1$ with $a_0 = 0$. We show that it has the strongly exceptional collections whose associated quiver is of the form Table 12 (and its generalization, see Table 14) where the set of lengths of branches of the quiver is given by the signature set $A(W)$ of the weight system W [K-S-T 2]. Then, owing to a result of Bondal-Kapranov, the category is again derived equivalent to the category of modules over a path algebra, however, for a wild type quiver with relations. In particular, in case 14 exceptional module cases, *the Grothendieck group* $\mathrm{K}_0(\mathrm{HMF}^{gr}_{A_W}(f_W))$ *is isomorphic to the lattice of vanishing cycles* $\mathrm{H}_2(X_{W^*,1}, \mathbb{Z})$ *for the* $*$*-dual weight system* W^* *of* W. *"Whether the generalized root system* R_W *in* $\mathrm{H}_2(X_{W^*,1}, \mathbb{Z})$ *(i.e. the set of vanishing cycles) is exactly the image of the set of exceptional indecomposable objects in* $\mathrm{HMF}^{gr}_{A_W}(f_W)$ *or not*" is still an open and interesting question, which we would like to ask.[44]

The above results on the category of matrix factorization for $\varepsilon_W = \pm 1$ seem to suggest that the category $\mathrm{HMF}^{gr}_{A_W}(f_W)$ for W with $a_0 = 0$ may potentially have

[43] A hint was given by the Gonzalez-Verdier interpretation of c) ([G-V], see Appendix), where the dual basis of the simple root basis was constructed by certain vector bundles on $\tilde{X}_{W,0}$. Furthermore, the derived of the abelian category of coherent sheaves was acknowledged in the recent development in mirror symmetry D-branes ([K-L 1,2],[H-W],[Wal]).

Category of graded D-branes of type B in Landau-Ginzburg models was formulated D. Orlov [Orl2] as the triangulated category of the singularity X.

[44] We know that the root system R_W is irreducible (see §11,b), 1)) due to the irreducibility of the discriminant D_φ (see Footnote 10). Therefore, R_W is a union of the Weyl group orbits of the distinguished basis. Since the distinguished basis are exactly the image of the exceptional collections, the question may be reformulated "whether all exceptional indecompsable objects in $\mathrm{HMF}^{gr}_{A_W}(f_W)$ are obtained by mutation-actions on the exceptional collections.

certain canonical strongly exceptional collection, which should be an answer to the latter half of Problem (at least) for the $*$-dual weight system W^*.

It seems likely that these objects give geometric supports to the Lie algebras studied abstractly in [S-Y], which is a study still in progress.

16 The category of graded matrix factorizations

In this section, we introduce the triangulated category $\mathcal{T}_W$ associated with a regular system of weights W in three equivalent forms, by the homotopy category of the graded matrix factorizations,[45] by the stable category of maximal Cohen-Macaulay modules, and by the category of singularities ([Bu],[Orl1],[Ta2]). We discuss some basic properties of the category such as Serre duality, the generation of the category, exceptional collection and Auslander-Reiten translation. For basic terminology and concepts in category theory, one is referred to [Ke].

Let $W=(a,b,c;h)$ be a regular system of weights. We regard the polynomial ring

$$A_W \ := \ \mathbb{C}[x,y,z]$$

to be graded by the weight $\deg(x)=2a/h$, $\deg(y)=2b/h, \deg(z)=2c/h$.[46] Fix a polynomial $f_W \in A_W$ of type W (9), which is of degree 2. Put

$$R_W \ := \ A_W/(f_W) \ = \ \mathbb{C}[x,y,z]/(f_W).$$

An obvious remark is that A_W is a regular ring and R_W is a Gorenstein ring. By definition, both A_W and R_W are graded rings whose gradings are in $\frac{2}{h}\mathbb{Z}_{\geq 0}$. In the present paper, by a graded module M over A_W or R_W, we always mean a module which is graded by $\frac{2}{h}\mathbb{Z}$, i.e. $M=\oplus_{d\in\frac{2}{h}\mathbb{Z}}M_d$. Graded homomorphisms $f : M \to N$ of degree a between graded modules is defined as usual a homomorphism with $f(M_d)\subset N_{d+a}$ for any $d \in \frac{2}{h}\mathbb{Z}$. We denote by gr-$A_W$ or gr-R_W the category of finitely generated graded A_W or R_W-modules, respectively, whose morphisms are homogenous of degree 0. We denote by τ the degree shift orator on the set of graded modules to itself defined by $(\tau M)_d = M_{d+\frac{2}{h}}$. For a morphism f, we associate the same morphism $\tau(f) : \tau M \to \tau N$. For $M, N \in$ gr-A_W, letting $\mathrm{Hom}_{A_W}(M,N)_{\frac{2n}{h}} := \mathrm{Hom}_{\text{gr-}A_W}(M,\tau^n N)$, $\mathrm{Hom}_{A_W}(M,N)$ naturally belongs to gr-A_W. The same statement replacing A_W by R_W holds also.

[45] The concept of a matrix factorization is introduced by D. Eisenbud [Ei] in order to describe the *two periodic resolutions of maximal Cohen-Macaulay modules*. It was applied in the study of hyper-surface isolated angularity's ([Kn1,2],[Gr],[Sch]). It obtained a new impetus through mathematical physics ([Kon2], [K-L], [H-W]) and found new application to the clarification of link invariant's ([K-R]).
From a graded matrix factorization, forgetting about its grading one obtains a non-graded Marx factorization. This induces a comparison of the categories of graded and non-graded Matrix factorizations. This forgetful fornicator induces the embedding of the corresponding K-groups, which should conjecture corresponds to the embedding of the lattice of vanishing cycles to that of *cycles coming from infinity* and an analytic expression of a primitive form ([SS20] §6) in $\Phi^{b)}_{\Rightarrow}$ side of the construction (see §11 b) 4) and §12 Note 3.). However, in the present paper, we shall not treat the subject further.

[46] In order to adjust with the convention in matrix factorizations, we have to duplicate the grading compared with that for the flat structure. Hence, one should note that $\deg(f_W)=2$.

1. The homotopy category of graded matrix factorizations for f_W.

Definition. i) A graded matrix factorization for f_W is a system

$$M \ := \ (P_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} P_1)$$

where $P_1,\ P_2$ are graded free A_W-modules of finite rank and $p_0,\ p_1$ are graded A_W-homomorphisms such that $p_0p_1 = f_W \cdot id_{P_0},\ p_1p_0 = f_W \cdot id_{P_1}$ and $\deg(p_0) = 0,\ \deg(p_1) = 2$. The set of all graded matrix factorizations for f_W is denoted by

$$\mathrm{MF}^{gr}_{A_W}(f_W) \ := \ \{\text{graded matrix factorizations for } f_W\}.$$

ii) A graded homomorphism from $M = (P_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} P_1)$ to $M' = (P'_0 \underset{p'_1}{\overset{p'_0}{\rightleftarrows}} P'_1)$ is a pair $\Phi = (\phi_0, \phi_1) : (P_0, P_1) \to (P'_0, P'_1)$ of graded A_W-homomorphisms homogeneous of degree 0 making the following diagram commutative.

$$\begin{array}{ccccccccccc}
 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & & \\
\cdots & & \downarrow{\scriptstyle \phi_0} & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} & & \cdots & \\
 & \xrightarrow{p'_1} & P'_0 & \xrightarrow{p'_0} & P'_1 & \xrightarrow{p'_1} & P'_0 & \xrightarrow{p'_0} & &
\end{array}$$

The set of all graded homomorphisms is denoted by $\mathrm{Hom}_{\mathrm{MF}^{gr}_{A_W}(f_W)}(M, M')$.

iii) We denote also by $\mathrm{MF}^{gr}_{A_W}(f_W)$ the additive category of matrix factorizations with respect to above defined homomorphisms.

Definition. We denote by $\mathrm{HMF}^{gr}_{A_W}(f_W)$ the homotopy category of $\mathrm{MF}^{gr}_{A_W}(f_W)$. That is, the objects of $\mathrm{HMF}^{gr}_{A_W}(f_W)$ are the same as $\mathrm{MF}^{gr}_{A_W}(f_W)$. The module of homomorphisms is defined as the quotient space by the homotopy equivalence

$$\mathrm{Hom}_{\mathrm{HMF}^{gr}_{A_W}(f_W)}(M, M') := \mathrm{Hom}_{\mathrm{MF}^{gr}_{A_W}(f_W)}(M, M')/\sim$$

where a morphism $\Phi = (\phi_0, \phi_1)$ is homotopic to zero, denoted by $\Phi \sim 0$, if there exists A_W-homomorphisms $h_0 : P_0 \to P'_1$ and $h_1 : P_1 \to P'_0$ with $\deg(h_0) = -2$ and $\deg(h_1) = 0$ such that $\phi_0 = p'_1h_0 + h_1p_0$ and $\phi_1 = p'_0h_1 + h_0p_1$.

$$\begin{array}{ccccccccccc}
 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & P_1 & \xrightarrow{p_1} & P_0 & \xrightarrow{p_0} & & \\
\cdots & \swarrow & \phi_0\downarrow & \swarrow{\scriptstyle h_1} & \phi_1\downarrow & \swarrow{\scriptstyle h_0} & \phi_0\downarrow & \swarrow & \cdots & \\
 & \xrightarrow{p'_1} & P'_0 & \xrightarrow{p'_0} & P'_1 & \xrightarrow{p'_1} & P'_0 & \xrightarrow{p'_0} & &
\end{array}$$

Example. The $A_W \underset{f}{\overset{1}{\rightleftarrows}} A_W$ and $A_W \underset{1}{\overset{f}{\rightleftarrows}} \tau^h A_W$ are matrix factorizations which are homotopic to 0, since we have the following commutative diagram:

$$\begin{array}{ccccccccc}
 & \to & A_W & \xrightarrow{f} & A_W & \xrightarrow{1} & A_W & \xrightarrow{f} & \\
\swarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1} & \swarrow{\scriptstyle 0} & \downarrow{\scriptstyle 1} & \swarrow{\scriptstyle 1} & \downarrow{\scriptstyle 1} & \swarrow{\scriptstyle 0} & \\
 & \to & A_W & \xrightarrow{f} & A_W & \xrightarrow{1} & A_W & \to &
\end{array}$$

Any 0-object M (i.e. $1_M \sim 0$) in the category $\mathrm{HMF}^{gr}_{A_W}(f_W)$ is a direct sum of copies of some τ-powers shifts of $(A_W \underset{f}{\overset{1}{\rightleftarrows}} A_W)$ and $(A_W \underset{1}{\overset{f}{\rightleftarrows}} A_W)$.

Definition. (Shift functors) We introduce two auto-equivalence functors:

$$\tau(P_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} P_1) := (\tau P_0 \underset{\tau p_1}{\overset{\tau p_0}{\rightleftarrows}} \tau P_1), \qquad \tau(\phi_0, \phi_1) = (\tau\phi_0, \tau\phi_1),$$

$$T(P_0 \underset{p_1}{\overset{p_0}{\rightleftarrows}} P_1) := (P_1 \underset{-\tau^h p_0}{\overset{-p_1}{\rightleftarrows}} \tau^h P_0), \quad T(\phi_0, \phi_1) = (-\phi_1, -\tau^h \phi_0),$$

satisfying the relation:

(20) $$T^2 = \tau^h$$

as an immediate consequence of the definition.

The direct sum $\oplus_{n\in\mathbb{Z}}\mathrm{Hom}_{\mathrm{HMF}^{gr}_A(f_W)}(M, \tau^n M')$ is a finitely generated A_W-module. Since the sum is annihilated by the multiplications of $\partial_x f_W, \partial_y f_W, \partial_z f_W$, it is a finite module over $J_W := A_W/(\partial_x f_W, \partial_y f_W, \partial_z f_W)$, and, in view of §11 Fact 4., it is of finite rank over $\mathbb{C}$. From this, one deduces some consequences:

i) $\mathrm{HMF}^{gr}_A(f_W)$ *is a Krull-Schmidt category*: i.e. if $e \in End_{\mathrm{HMF}^{gr}_R(f)}(M)$ for an object M is idempotent $e^2 = e$, then there exist an object M' and morphisms $\Phi' : M' \to M$ and $\Phi : M \to M'$ such that $\Phi' \circ \Phi = e$ and $\Phi \circ \Phi' = \mathrm{id}_{M'}$,

ii) $\mathrm{HMF}^{gr}_A(f_W)$ *is of finite type*: i.e. $\oplus_{n\in\mathbb{Z}}\mathrm{Hom}(M, T^n N)$ is finite dimensional vector space for all objects M and N of the category.

Definition. (Mapping cone) For any morphism $\Phi = (\phi_0, \phi_1) \in \mathrm{Hom}_{\mathrm{MF}^{gr}_R(f)}(M, M')$, we introduce the *mapping cone* $C(\Phi) \in \mathrm{MF}^{gr}_R(f)$ as follows.

$$\begin{pmatrix} P_1 & \begin{pmatrix} -p_1 & 0 \\ \phi_1 & p_0' \end{pmatrix} & \tau^h P_0 \\ \oplus & \rightleftarrows & \oplus \\ P_0' & \begin{pmatrix} -\tau^h p_0 & 0 \\ \tau^h \phi_0 & p_1' \end{pmatrix} & P_1' \end{pmatrix}$$

and obtains a sequence: $*)$ $\quad M \xrightarrow{\Phi} M' \xrightarrow{inclusion} C(\Phi) \xrightarrow{-projection} TM.$

Then, we have the following general fact (c.f. [G-M], [K-S], [B-K2], [Ta2]).

Theorem. *The additive category* $\mathrm{HMF}^{gr}_{A_W}(f_W)$ *endowed with the shift function* T *and distinguished triangles isomorphic to* $*)$ *for all morphisms* Φ *forms an enhanced triangulated category of finite type.*

See [B-K2] for a definition of the *enhanced category.*

2. The stable category of maximal Cohen-Macaulay modules over R_W.

Definition. A graded module $M \in$ gr-R_W is a maximal Cohen-Macaulay module over R_W if $\mathrm{depth}(M) = \dim(R_W)$ $(=: d = 2)$ $(\Leftrightarrow \mathrm{Ext}^i_R(R_W/(x, y, z), M) = 0$ for $i < d = 2)$. The full subcategory of gr-R_W consisting of all graded maximal Cohen-Macaulay modules over R_W is denoted by $\mathrm{CM}^{\mathrm{gr}}(R_W)$.

Syzygy $syz^n(M)$ of any element M of gr-R_W for $n \geq d$ up to a free module factor, becomes maximal Cohen-Macaulay module and then doubly periodic [Yos]. Hence, one sees $\mathrm{CM}^{\mathrm{gr}}(R_W)$ is a Frobenius category (i.e. it has enough injective and projective objects which coincide to each other). Then, *the stable*

category $\underline{\mathrm{CM}}^{\mathrm{gr}}(R_W)$*, defined below, becomes a triangulated category* (c.f. [Ke]): the objects of the stable category $\underline{\mathrm{CM}}^{\mathrm{gr}}(R_W)$ is the same as $\mathrm{CM}^{\mathrm{gr}}(R_W)$ and the space of morphisms is given by

$$\underline{\mathrm{Hom}}_{\text{gr-}R_W}(M,N) := \mathrm{Hom}_{\text{gr-}R_W}(M,N)/\mathcal{P}(M,N),$$

where $\mathcal{P}(M,N)$ is the subspace of $\mathrm{Hom}_{\text{gr-}R_W}(M,N)$ consisting of morphisms which factor through projective modules.

Fact 10. *For a graded matrix factorization* $M \in \mathrm{MF}^{\mathrm{gr}}_{A_W}(f_W)$*, we associate a maximal Cohen-Macaulay module* $\mathrm{coker}(P_1 \xrightarrow{p_1} P_0) \in \mathrm{CM}^{\mathrm{gr}}(R_W)$ *over* R_W*. This correspondence induces an equivalence of the triangulated categories:*

$$\mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W) \quad \simeq \quad \underline{\mathrm{CM}}^{\mathrm{gr}}(R_W). \tag{21}$$

The advantage of the category $\underline{\mathrm{CM}}^{\mathrm{gr}}(R_W)$ is that it easily admits the concepts: *Auslander-Reiten triangles* and *Serre duality*, which we explain below. For details, the reader is referred to some text books on the subject, e.g. [Yos].

We, first, define the Auslander transpose $\mathrm{tr}(M)$ (up to free module factor) of $M \in$gr-R_W by putting $\mathrm{tr}(M) := \mathrm{Coker}({}^t f)$ where $F_1 \xrightarrow{f} F_0 \to M \to 0$ is a finite presentation of M and ${}^t f$ is the contragradient homomorphism of f. Let us denote by $\mathrm{syz}^d(\mathrm{tr}(M))$ the reduced dth syzygy of $\mathrm{tr}(M)$ obtained by avoiding all graded free summands from a dth syzygy of $\mathrm{tr}(M)$. Then, the Auslander-Reiten translation, or A-R translation, $\tau_{AR}(M) \in$ gr-R_W is defined by

$$\textbf{A-R translation}: \qquad \tau_{AR}(M) \ := \ \mathrm{Hom}_{R_W}(\mathrm{syz}^d(\mathrm{tr}(M)), \mathrm{K}_{R_W})$$

where $\mathrm{K}_{R_W} = \mathrm{Res}_{X_{W,0}} \begin{bmatrix} A_W & dxdydz \\ & f_W \end{bmatrix} = \tau^{-\varepsilon_W} R_W$ is the canonical module of $R_W = A_W/(f_W)$. If M is a maximal Cohen-Macaulay module without a free direct summand, then we easily see that $\mathrm{syz}^2(\mathrm{tr}(M)) \simeq \mathrm{Hom}_{R_W}(M, R_W)$, and, hence,

$$\tau_{AR}(M) \ \simeq \ \tau^{-\varepsilon_W} M. \tag{22}$$

The auto-equivalence of the category $\underline{\mathrm{CM}}^{\mathrm{gr}}(R_W)$ induced by τ_{AR} is denoted again by τ_{AR}. In view of the relation (20), we have the following relation:

$$\tau_{AR}^h \ = \ (T^2)^{-\varepsilon_W}.$$

The following duality was shown by Auslander and Reiten [A-R3]:

$$\mathrm{Ext}^d_{\text{gr-}R_W}(\underline{\mathrm{Hom}}_{R_W}(M,N), \mathrm{K}_{R_W}) \ \simeq \ \mathrm{Ext}^1_{\text{gr-}R_W}(N, \tau_{AR}(M))$$

for $M, N \in \underline{\mathrm{CM}}^{\mathrm{gr}}(R_W)$. This, in particular, implies the following

$$\textbf{Serre duality}: \qquad \mathrm{Hom}_{\mathbb{C}}(\underline{\mathrm{Hom}}_{\text{gr-}R_W}(M,N), \mathbb{C}) \ \simeq \ \underline{\mathrm{Hom}}_{\text{gr-}R_W}(N, SM)$$

as a bi-functorial isomorphism of vector spaces for $M, N \in \underline{\mathrm{CM}}^{\mathrm{gr}}(R_W)$, where S is an auto-equivalence of the category $\underline{\mathrm{CM}}^{\mathrm{gr}}(R_W)$ defined by

$$S := T\tau_{AR}. \tag{23}$$

As a consequence of Serre duality, one can show that, for any indecomposable object Z of $\underline{\mathrm{CM}}^{\mathrm{gr}}(R_W)$, there exists the *AR-triangle of* Z in the following sense: let $Z \xrightarrow{w} T\tau_{AR}(Z)$ be the morphism, which, by Serre duality, corresponds to the dual of the identity element in $\mathrm{Hom}_{\mathbb{C}}(\underline{\mathrm{Hom}}_{\text{gr-}R_W}(Z,Z),\mathbb{C})$. Then, *there exists an object* $AR(Z)$ *and the triangle, called A-R triangle, in* $\underline{\mathrm{CM}}^{\mathrm{gr}}(R_W)$*:*

A-R triangle : $$\tau_{AR}(Z) \xrightarrow{u} AR(Z) \xrightarrow{v} Z \xrightarrow{w} T\tau_{AR}(Z)$$

such that, for any morphism $g : W \to Z$ *in* $\underline{\mathrm{CM}}^{\mathrm{gr}}(R_W)$ *which is not a split epimorphism, there exists* $h : W \to AR(Z)$ *with* $vh = g$. If W is indecomposable,

$$\underline{\mathrm{hom}}_{\text{gr-}R_W}(W, AR(Z)) = \underline{\mathrm{hom}}_{\text{gr-}R_W}(W,Z) + \underline{\mathrm{hom}}_{\text{gr-}R_W}(W,\tau_{AR}) - \delta_{W,Z} - \delta_{W,T^{-1}Z},$$
$$\underline{\mathrm{hom}}_{\text{gr-}R_W}(AR(Z), W) = \underline{\mathrm{hom}}_{\text{gr-}R_W}(Z,W) + \underline{\mathrm{hom}}_{\text{gr-}R_W}(\tau_{AR}, W) - \delta_{W,Z} - \delta_{W,T^{-1}Z}.$$

where $\underline{\mathrm{hom}}_{\text{gr-}R_W}$ means $\dim_{\mathbb{C}} \underline{\mathrm{Hom}}_{\text{gr-}R_W}$, and $\delta_{A,B} = 1$ only when $A \simeq B$ else 0.

3. The category of the singularity $X_{W,0} := \mathrm{Spec}(R_W)$**.**

Definition. ([Orl1]) The triangulated category of the singularity $X_{W,0}$ is

$$D^{\mathrm{gr}}_{Sg}(R_W) := D^b(\text{gr-}R_W)/D^b(\text{gr proj-}R_W)$$

where $D^b(\text{gr-}R_W)$ is the bounded derived category of the abelian category gr-R_W with the natural triangulated structure and $D^b(\text{gr } \textit{proj-}R_W)$ is its full triangulated subcategory consisting of objects which are isomorphic to bounded complexes of projectives. Actually, the subcategory is the derived category of the exact category of graded projective modules [Ke], and is called the subcategory of perfect complexes.

Since R_W defines the hypersurface $X_{W,0}$ and is Gorenstein, we have

Theorem. (Buchweitz [Bu], Orlov [Orl2]§1.3) *The natural inclusion map* $\mathrm{CM}^{\mathrm{gr}}(R_W) \to \text{gr-}R_W$ *induces the equivalence of the triangulated categories:*

$$\underline{\mathrm{CM}}^{\mathrm{gr}}(R_W) \quad \simeq \quad D^{\mathrm{gr}}_{Sg}(R_W). \tag{24}$$

4. The triangulated category $\mathcal{T}_W$ **associated with a regular system of weights** W (category generation theorem and strongly exceptional collections). Owing to (21) and (24), we introduce a triangulated category

$$\mathcal{T}_W \;:=\; \mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W) \;\simeq\; \underline{\mathrm{CM}}^{\mathrm{gr}}(R_W) \;\simeq\; D^{\mathrm{gr}}_{Sg}(R_W) \tag{25}$$

associated to a regular system of weights W up to equivalences. The advantage of the third expression is that we have the following generation theorem ([K-S-T 2]), which we shall use in the proof of the main theorem in §18.

Theorem. *Let* $\mathcal{T}$ *be a right-admissible full triangulated subcategory of* $\mathcal{T}_W$ *satisfying: i) The shift functor* τ *induces an auto-equivalence of* $\mathcal{T}$*.*
ii) There is an object of $\mathcal{T}$ *which is isomorphic to the pure complex of the torsion module* $R_W/(x,y,z)$ *in* $\mathcal{T}_W$*.*
Then $\mathcal{T}$ *is equivalent to* $\mathcal{T}_W$*.*

Here, a subcategory $\mathcal{T}'$ of a triangulated category $\mathcal{T}$ is called *right-admissible* if, for any object X of $\mathcal{T}$, there exist $N \in \mathcal{T}'$, $M \in \mathcal{T}'^{\perp} := \{M \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(N, M) = 0 \ \forall N \in \mathcal{T}'\}$ and a triangle: $N \to X \to M \to TN$ in $\mathcal{T}$. For a proof of the generating theorem, we use a result due to Bondal which we recall below. Before stating that, for a later use, we recall some terminologies from [Bon] and [Orl2].

Definition. i) An object E in a triangulated category $\mathcal{T}$ over $\mathbb{C}$ is called *exceptional* if $\mathrm{Hom}_{\mathcal{T}}(E, T^p E) \simeq \begin{cases} \mathbb{C} \text{ if } p = 0, \\ 0 \text{ if } p \neq 0. \end{cases}$

ii) An *exceptional collection* is a sequence $(E_1, \cdots, E_\mu)$ of exceptional objects satisfying $\mathrm{Hom}_{\mathcal{T}}(E_i, T^p E_j) = 0$ for $\forall p \in \mathbb{Z}$ and $\mu \geq i > j \geq 0$.

iii) An exceptional collection $(E_1, \cdots, E_\mu)$ is called *strongly exceptional* if $\mathrm{Hom}_{\mathcal{T}}(E_i, T^p E_j) = 0$ for all $1 \leq i, j \leq \mu$ and $p \neq 0$.

Theorem. ([Bon],[B-K2]) *Let $\mathcal{T}$ be an enhanced triangulated category of finite type, and let $\mathcal{T}'$ be the full triangulated subcategory of $\mathcal{T}$ generated by a strongly exceptional collection $\mathcal{E} = (E_1, \cdots, E_\mu)$. Then, $\mathcal{T}'$ is right admissible and is equivalent to*

$$D^b(\text{mod-} \oplus_{0 \leq i,j \leq \mu} \mathrm{Hom}_{\mathcal{T}}(E_i, E_j)). \tag{26}$$

We shall sometimes denote the algebra $\oplus_{0 \leq i,j \leq \mu} \mathrm{Hom}_{\mathcal{T}}(E_i, E_j)$ by $\mathrm{Hom}(\mathcal{E}, \mathcal{E})$. and call it the hom-subalgebra.

5. K-group and Auslander-Reiten transformation of $\mathcal{T}_W$.

In this paragraph, we show that the Auslander-Reiten transformation induces an automorphism of the K-group of the category, which is expressed as the product of reflections. Actually, this fact was studied already in the categorical level by Bondal-Polschchuk [**?**].

For a triangulated category $\mathcal{T}$, let us denote by $\mathrm{K}_0(\mathcal{T})$ its Grothendieck group (or K-group), i.e. the quotient group of the free abelian group generated by the equivalence classes $[X]$ of objects X of $\mathcal{T}$ devided by the submodule generated by $[X] + [Z] - [Y]$ for all triangles $X \to Y \to Z \to TX$. We shall denote again by $[X]$ the image element in K_0 of X. Note the relation: $[X] + [TX] = 0$ for any object X. That is, the T induces an endomorphism $[T] = -\mathrm{id}_{\mathrm{K}_0(\mathcal{T})}$. If a set $E_1, \cdots, E_\mu$ of objects generates the triangulated category, then their images $[E_1], \cdots, [E_\mu]$ generateS the K-group over $\mathbb{Z}$.

The Auslander-Reiten translation τ_{AR} is an auto-equivalence of the triangulated category, so that it induces an automorphims of the group $\mathrm{K}_0(\mathcal{T})$, denoted by $[\tau_{AR}]$. In the category $\mathcal{T}_W$ associated to a weight system W, using the expression (22) and the fact (20), we see that it is of finite order h:

$$[\tau_{AR}]^h = [T^{-2\varepsilon_W}] = (-id_{\mathrm{K}_0(\mathcal{T}_W)})^{-2\varepsilon_W} = id_{\mathrm{K}_0(\mathcal{T}_W)}.$$

If $\mathcal{T}$ is of finite type over $\mathbb{C}$, we introduce the Euler pairing by

$$\chi(X, Y) \ := \ \textstyle\sum_{n \in \mathbb{Z}} (-1)^n \mathrm{hom}_{\mathcal{T}}(X, T^n Y)$$

for any two objects X and Y of $\mathcal{T}$. Because of the (co-)homological property of $\hom_{\mathcal{T}}$, it induces a bilinear form on the K-group, which we denote agagin by χ. We equip the K-group with the symmetric bilinear form [47]:

$$I(e,f) \ := \ \chi(e,f) \ + \ \chi(f,e). \tag{27}$$

for $e, f \in \mathrm{K}_0(\mathcal{T})$. The transformation $[\tau_{AR}]$ preserves the bilinear form χ, and, hence, I. Let us express now $[\tau_{AR}]$ as a product of reflections on $\mathrm{K}_0(\mathcal{T})$.

Let $\mathcal{E} := (E_1, \cdots, E_\mu)$ be a strongly exceptional collection of $\mathcal{T}$. Assume that $E_1, \cdots, E_\mu$ generate $\mathcal{T}$ and, hence $[E_1], \cdots, [E_\mu]$ generate $\mathrm{K}_0(\mathcal{T})$. Associated to $[E_1], \cdots, [E_\mu]$, we consider two generator systems; $f_1, \cdots, f_\mu$ and $g_1, \cdots, g_\mu$ of $\mathrm{K}_0(\mathcal{T})$ defined by the following relations:

$$[E_i] \ = \ \textstyle\sum_{j=1}^{\mu} \chi(E_i, E_j) f_j \ = \ \sum_{j=1}^{\mu} g_j \chi(E_j, E_i) \tag{28}$$

Here, remark that the matrix $\chi_{\mathcal{E}} := (\chi(E_i, E_j))_{ij=1,\cdots,\mu}$ is an upper triangular matrix with 1 at each diagonal entry so that χ is invertible. Let us denote by $C_{\mathcal{E}} = (C_{\mathcal{E},ij})_{ij=1,\cdots,\mu}$ the inverse matrix $\chi_{\mathcal{E}}^{-1}$ (which is also an upper triangular integral matrix). In fact, using the mapping cone constructions, one can find objects F_i and G_i such that $f_i = [F_i]$ and $g_i = [G_i]$ for $i = 1, \cdots, \mu$.[48] The intersection matrix of them are given by $C_{\mathcal{E}}$ as follows:

$$\chi([F_i],[F_j]) \ = \ \chi([G_i],[G_j]) \ = \ C_{\mathcal{E},ji} \qquad \text{for } i,j = 1, \cdots, \mu. \tag{29}$$

In particular, since $I([F_i],[F_i]) = I([G_i],[G_i]) = 2$ for $1 \leq i \leq \mu$, we define a reflection action $w_e \in O(\mathrm{K}_0(\mathcal{T}), I)$ for each base $e \in \{[F_1], \cdots, [F_\mu], [G_1], \cdots, [G_\mu]\}$ likewise the Picard-Lefschetz formula (c.f. §5): $w_e(u) := u - I(u,e)e$.

Fact 11. *Let $(E_1, \cdots, E_\mu)$ be a strongly exceptional collection, then the transformation $[\tau_{AR}]$ is expressed as the product of reflections associated to the basis:*

$$[\tau_{AR}] \ = \ w_{[F_1]} \cdots w_{[F_\mu]} \ = \ w_{[G_1]} \cdots w_{[G_\mu]} \ .$$

6. Quiver and path algebra associated with $\mathcal{E}$.

In this paragraph, associated with a strongly exceptional collection, we consider a slight generalization of a quiver, and, then, associated to the quiver, we introduce a path algebra with relations, which we shall use in §17 and 18.

Let $\mathcal{E} = (E_1, \cdots, E_\mu)$ be a strongly exceptional collection of a triangulated category $\mathcal{T}$. Then, the associated quiver $\Delta_{\mathcal{E}}$ is a pair

$$\Delta_{\mathcal{E}} \ = \ (\Delta_0, \Delta_1), \tag{30}$$

where $\Delta_0 = \{v_1, \cdots, v_\mu\}$ is a set of μ elements, called the vertices, and Δ_1, called the set of allows, is a multi-set of triplet $(v_i, v_j, \epsilon) \in \Delta_0 \times \Delta_0 \times \{\pm\}$ where

[47] For the purpose of the period map for odd dimensional Milnor fiber, it is important to study the skew symmetric bilnear form: $I_{odd}(e,f) := \chi(e,f) - \chi(f,e)$ (see [Sa19]§6 (6.2.2)), [Sa20] latter half of §4). However, we shall not treat them in the present paper.

[48] Actually, these objects F_i and G_i are constructed by use of mutations and are shown to be exceptional objects ([B-P]).

$(v_i, v_j, +)$ appears in Δ_1 only when $i \neq j$, $C_{\mathcal{E},ij} < 0$ and $-C_{\mathcal{E},ij}$-times, and $(v_i, v_j, -)$ appears in Δ_1 only when $i \neq j$, $C_{\mathcal{E},ij} > 0$ and $C_{\mathcal{E},ij}$-times. We regard $(v_i, v_j, +) \in \Delta_1$ as an arrow (with positive sign) from the vertex v_i to the vertex v_j, and similarly $(v_i, v_j, -) \in \Delta_1$ as a dotted arrow from v_i to v_j.

Remark 16.1. If one forget from the quiver $\Delta_{\mathcal{E}}$ the directions of the allows but leave only lines or dotted lines together with the vertices, then one obtains automatically the intersection diagram Γ of the symmetric bilinear form I with respect to the system of generators $[F_1], \cdots, [F_\mu]$ or $[G_1], \cdots, [G_\mu]$, i.e. Γ is the diagram for the symmetrization of the matrix (29)) (for instance, [Sa15]I (8.2)).

Associated with the above given quiver $\Delta_{\mathcal{E}}$ (30), the path algebra

$$\mathbb{C}(\Delta_{\mathcal{E}}, R) \tag{31}$$

with relations R is defined as follows. Let $\Delta_1 = \Delta_1^+ \amalg \Delta_1^-$ be the decomposition of the set of arrows into those of positive and negative signs. We regard $\Delta_{\mathcal{E}}^+ := (\Delta_0, \Delta_1^+)$ as the quiver in the classical sense (e.g. [Ri1] 2.1), then by concatenating arrows, one define paths and the path algebra $\mathbb{C}\Delta^+$ as usual ([ibid]). Then,

$$R \ : \ \Delta_1^- \ \longrightarrow \ \mathbb{C}\Delta_{\mathcal{E}}^+$$

is a map such that the image of an arrow $(v_i, v_j, -)$ should belong in the subspace $(v_j \mid v_j) \cdot \mathbb{C}\Delta_{\mathcal{E}}^+ \cdot (v_i \mid v_i)$ spanned by all paths from v_i to v_j (here, we denote by $(v \mid v)$ the path of length 0 at the vertex v). Then, we put

$$\mathbb{C}(\Delta_{\mathcal{E}}, R) \ := \ \mathbb{C}\Delta_{\mathcal{E}}^+ \ / \ (\mathbb{C}\Delta_{\mathcal{E}}^+ \cdot R(\Delta_1^-) \cdot \mathbb{C}\Delta_{\mathcal{E}}^+),$$

where $\mathbb{C}\Delta_{\mathcal{E}}^+ \cdot R(\Delta_1^-) \cdot \mathbb{C}\Delta_{\mathcal{E}}^+$ is the both side ideal of the algebra $\mathbb{C}\Delta_{\mathcal{E}}^+$ generated by the set $R(\Delta_1^-)$, and call it the path algebra with relations R.

Remark 16.2. Assigning to each arrow $(v_i, v_j; +) \in \Delta_1^+$ a morphism $f_{ij} \in \mathrm{Hom}_{\mathcal{T}}(E_i, E_j)$, we can define a ring homomorphism:

$$\mathbb{C}(\Delta_{\mathcal{E}}, R) \ \longrightarrow \ \mathrm{Hom}(\mathcal{E}, \mathcal{E}) := \oplus_{0 \leq i,j \leq \mu} \mathrm{Hom}_{\mathcal{T}}(E_i, E_j)$$

for a suitable choice of relations R. However, in general, the homomorohism may neither be isomorphic nor derived equivalent.

Example. Let us consider a strongly exceptional collection $\mathcal{E} = (E_1, E_2, E_3)$ such that $\chi_{\mathcal{E}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $C_{\mathcal{E}} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$. Then the associated quiver is a Dynkin quiver $\Delta_{\mathcal{E}} = \circ \to \circ \to \circ$ of type A_2 and $\mathbb{C}\Delta_{\mathcal{E}}$ is the path algebra of type A_2. On the other hand, there are two cases of the structure of the hom-subalgebra $\mathrm{Hom}(\mathcal{E}, \mathcal{E}) := \oplus_{1 \leq i \leq j \leq 3} \mathrm{Hom}(E_i, E_j)$ according as the product $\mathrm{Hom}(E_1, E_2) \times \mathrm{Hom}(E_2, E_3) \to \mathrm{Hom}(E_1, E_3)$ is a) non-zero or b) zero. Then the homomorphism $\mathbb{C}\Delta_{\mathcal{E}} \to \mathrm{Hom}(\mathcal{E}, \mathcal{E})$ assigning the two arrows in $\Delta_{\mathcal{E}}$ to the base of $\mathrm{Hom}(E_1, E_2)$ and $\mathrm{Hom}(E_2, E_3)$, respectively, is isomorphic in the case a) but neither isomorphic nor derived equivalent in the case b).

17 The category of matrix factorizations: Case of $\varepsilon_W = 1$.

In this section, we study the category $\mathcal{T}_W$ associated with a weight system W with positive smallest exponent $\varepsilon_W > 0$. Recall that, in this case, the weight systems are classified into types A_l ($l \geq 1$), D_l ($l \geq 4$), E_6, E_7 and E_8 (see Table 8), and that the associated polynomials f_W of type W are the simple polynomials (see Table 2).

Then the following theorem is proven in [K-S-T 1] (c.f. also [Ue]).

Theorem. *Let W be a regular system of weights of type ADE. For any Dynkin quiver $\vec{\Delta}$ of type W (see Note below), there exists a unique, up to shifts by a power of τ, strongly exceptional collection $\mathcal{E}$ of the category $\mathcal{T}_W$ such that i) associated quiver $\Delta_{\mathcal{E}}$ is equal to $\vec{\Delta}$ and ii) the path algebra $\mathbb{C}\Delta_{\mathcal{E}}$ is isomorphic to the hom-subalgebra $\mathrm{Hom}_{\mathcal{T}_W}(\mathcal{E}, \mathcal{E})$. The category is generated by the collection $\mathcal{E}$ so that one has the equivalent $D^b(\mathrm{mod}\text{-}\mathbb{C}\vec{\Delta}) \simeq \mathcal{T}_W$ as a triangulated category In particular, the set of indecomposable objects in the K-group of $\mathcal{T}_W$ form the root system of the type of W.*

Note. Here, by a Dynkin quiver, we mean a simply-laced Dynkin diagram of type ADE, whose edges are labeled by an orientation.

A sketch of proof. According to the works by the authors [Ei], [A-R1] and [Au], the Auslander-Reiten quiver for the triangulated category $\mathrm{HMF}_{\hat{\mathcal{O}}}(f_W)$ of non-graded matrix factorizations over the local rings $\mathcal{O}$ and $\hat{\mathcal{O}}$ are well know to be isomorphic to the both side oriented Dynkin quiver $\overleftrightarrow{\Delta}$ of type W. There is a natural forgettful functor $\mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W) \to \mathrm{HMF}_{\hat{\mathcal{O}}}(f_W)$ by forgetting the graded structure. In fact, this is nothing but the τ-orbifold quotient functor. Then, by "lifting" the results to the graded category, in [K-S-T 1], we list up explicitly a complete list of indecomposable objects and irreducible morphisms in $\mathrm{HMF}^{\mathrm{gr}}_{A_W}$. From these data, we can show an existence of a strongly exceptional collection of indecomposable objects in $\mathrm{HMF}^{\mathrm{gr}}_{A_W}$ corresponding to any given Dynkin quiver $\vec{\Delta}$ of type W

Then, applying a theorem by Bondal-Kapranov, we see that $D^b(\mathrm{mod}\text{-}\mathbb{C}\vec{\Delta})$ is a full triangulated admissible subcategory of $\mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W)$.

On the other hand, we observe the equalities:

$\#\{$ indecomposable objects of $\mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W)$ up to the shift functor T $\}$
$= \#\{$ the positive roots for the root system of type ADE $\}$ $(= \frac{l \cdot h}{2})$
$= \#\{$ indecomposable objects of $D^b(\mathrm{mod}\text{-}\mathbb{C}\vec{\Delta})$ up to the shift functor $T\}$
(here we apply Gabriel's theorem [Ga] to show the second equality).

These give a proof of Theorem: $D^b(\mathrm{mod}\text{-}\mathbb{C}\vec{\Delta}) \simeq \mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W)$.

Remark. As in the A_l case [Ta2], a stability condition (Bridgeland [Bri 1]) can be naturally given by the grading of matrix factorizations. The abelian category associated to the stability condition (as a full subcategory of $\mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W)$) is equivalent to the category $\mathrm{mod}\text{-}\mathbb{C}\vec{\Delta}$ of finite modules over the path algebra of a Dynkin quiver $\vec{\Delta}$, whose orientation is the principal orientation introduced in [Sa22].

18 The category of matrix factorizations: Case of $\varepsilon_W = -1$

In this section, we describe the category $\mathcal{T}_W$ associated to a regular system of weights W with $\varepsilon_W = -1$ and $a_0 = 0$. Recall Table 10 that there are 14+8 such weight systems. We show an existence of strongly exceptional collection $\mathcal{E}$ of the category, whose associated quiver $\Delta_{\mathcal{E}}$ has the shape Table 14. such that the path algebra $\mathbb{C}\Delta_{\mathcal{E}}$ is isomorphic to the hom-subalgebra $\mathrm{Hom}(\mathcal{E}, \mathcal{E})$. As a part of Theorem, we observe further that the set $A := \{\alpha_i\}$ of length's α_i $(i = 1, \cdots, r)$ of the branches of the quiver is equal to the signature set $A(W)$ (13) of the weight system W. Recall that the signatures in $A(W)$ for the 14 weight systems are called Dolgachev numbers, and that it is equal to the set of Gabrielov numbers (recall Table 12) for the $*$-dual weight system W^* (§14, Fact 7.) This implies the isomorphism (as abstract lattices with a symmetric bilinear forms)

$$\mathrm{K}_0(\mathcal{T}_W) \ \simeq \ \mathrm{H}_2(X_{W^*,1}, \mathbb{Z})$$

of the K-group of the category for the weight system W (see (27)) and the middle homology group of the Milnor fiber (see Footnote 33) of the dual weight system W^*. In this sence, the Mirror symmetry in the homology group level is confirmed. However, the characterization of the subset in LHS corresponding to R_{W^*} we should remind that the part iv), v) and vi) in §12 are still unsolved (which are necessary to construct a flat structure), and, in that sense, the question which root system, $\mathrm{K}_0(\mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_W))$ or $\mathrm{K}_0(\mathrm{HMF}^{\mathrm{gr}}_{A_W}(f_{W^*}))$ to choose as for the input data iii) in order to answer to the question in §14, is still open and unsolved.

Definition. We introduce a quiver associated to a signature set A, where

$$A = \{\alpha_1, \cdots, \alpha_r\}$$

is a multiset of r number of positive integers for some $r \in \mathbb{Z}_{\geq 0}$. Then the quiver Δ_A of type A is defined by the following figure.

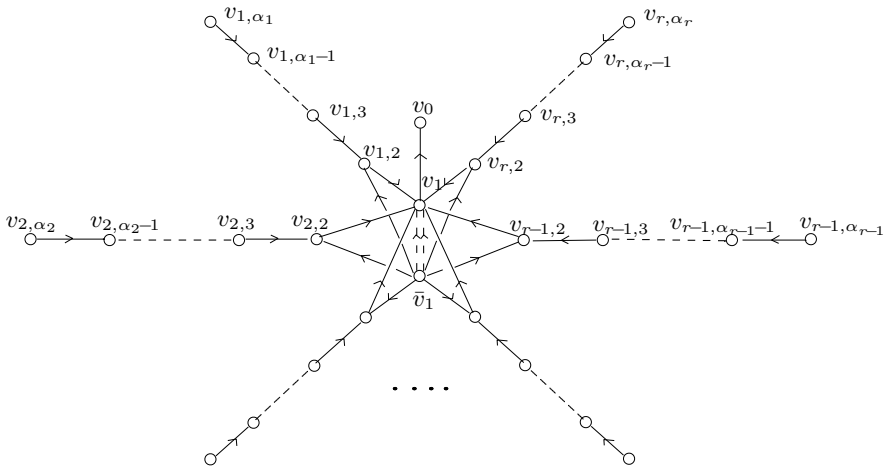


Table 14. $\Delta_A = (\Pi_A, E_A)$

where

$$\begin{aligned}\Pi_A &:= \{v_0, v_1, \overline{v}_1\} \amalg \amalg_{i=1}^{r}\{v_{i,2}, \cdots, v_{i,\alpha_i}\},\\ E_A &:= \{(v_1v_0;+), (\overline{v}_1, v_1;-)_1, (\overline{v}_1, v_1;-)_2\}\\ &\quad \amalg \amalg_{i=1}^{r}\{(v_{i,2}, v_{i,1};+), \cdots, (v_{i,\alpha_i}, v_{i,\alpha_i-1};+), (\overline{v}_1, v_{i,2};+)\}.\end{aligned}$$

Here, we distinguish the two arrows of negative sign by the subscripts 1 and 2.

Theorem. *Let W be a regular system of weights with $\varepsilon_W = -1$ and $a_0 = 0$. We choose and fix a polynomial f_W of type W. Let $\mathcal{T}_W$ (24) be triangulated category associated to f_W. Then, there exists a strongly exceptional collection $\mathcal{E} = (E_1, \cdots, E_{\nu_W})$ of the category $\mathcal{T}_W$ satisfying the following properties.*

1) *The objects $E_1, \cdots E_{\nu_W}$ generate the category $\mathcal{T}_W$.*

2) *The quiver $\Delta_{\mathcal{E}}$ associated to the collection $\mathcal{E}$ is equal to Δ_A (*Table 14*), where A is equal to the signature set $A(W)$ of W.*

3) *If the $(v_i, v_j;+)$ is a real arrow of Δ_A, then $\mathrm{Hom}_{\mathcal{T}_W}(E_i, E_j)$ is a vector space of rank $-C_{\mathcal{E},ij}(= 1)$. If, further, the arrow $(v_i, v_j;+)$ is on the branches of Δ_A, then $\mathrm{Hom}_{\mathcal{T}_W}(E_i, E_j)$ is spanned by irreducible homomorphisms.*

4) *The assignments*

$$(v_i, v_j, +) \longmapsto f_{ij}$$

of a base f_{ij} of $\mathrm{Hom}_{\mathcal{T}_W}(E_i, E_j)$ to each arrow $(v_i, v_j;+)$ of $\Delta_{\mathcal{E}}^{+}$ together with suitable choices, depending on f_W, of the relations

$$\begin{aligned}R((\overline{v}_1, v_1;-)_1) &= \textstyle\sum_{i=1}^{r} \lambda_{1,i}\ (\overline{v}_1, v_{i,2};+) \circ (v_{i,2}, v_1;+),\\ R((\overline{v}_1, v_1;-)_2) &= \textstyle\sum_{i=1}^{r} \lambda_{2,i}\ (\overline{v}_1, v_{i,2};+) \circ (v_{i,2}, v_1;+),\end{aligned}$$

induce an isomorphism:

$$\mathbb{C}(\Delta_{A(W)}, R) \ \simeq\ \mathrm{Hom}_{\mathcal{T}_W}(\mathcal{E}, \mathcal{E}) := \oplus_{i,j\in\Pi_A(W)}\mathrm{Hom}_{\mathcal{T}_W}(E_i, E_j)$$

between the path algebra and the hom-subalgebra.

Remark. For the 14 exceptional weight systems W, the period domain for the period map for the primitive form of type W is introduced [Sa23] (c.f. [Ao]) as

$$\mathcal{B}_V \ :=\ \{\ \varphi \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid \ker(\varphi) < 0\}$$

where $V := (Q_W \otimes \mathbb{R}, I)$ is the real vector space equipped with a quadratic form I of the signature $(l + 2, 0, 2)$, and $\ker(\varphi) < 0$ means that the restriction of I to the subspace $\ker(\varphi)$ is negative definite. It is interesting to know whether the domain is, through the identification $\mathrm{K}_0(\mathcal{T}_{W^*}) \simeq (Q_W, -I)$ (due to the above Theorem), equal to the image of the space of the stability conditions (Bridgeland [Bri 1]) for the category $\mathcal{T}_{W^*}$ for the $*$-dual weight system or not. The ring of "automorphic forms" (in suitable sense) on $\mathcal{B}_V$ with respect to the group W_W is expected to carry the flat structure. For some recent developments on the geometry of the modular varieties for the orthogonal groups $O(2, n)$, we refer to [Bo1, Bo2, Bo3] and [G-H-S 1, G-H-S 2, G-H-S 3].

19 Appendix. McKay correspondence and its Inverse.

1. McKay correspondence (1979) [Mc].

We recall McKay correspondence in its original form [Mc]. For its further understanding from a categorical view point, see [B-K-R].

Let ρ be the faithful representation of the Kleinean group $\tilde{G}$ into $SU(2)$. Let $\{\rho_0 = 1, \rho_2, \ldots, \rho_n\}$ be the set of isomorphism classes of all irreducible representations of $\tilde{G}$. Consider the decomposition

$$\rho \otimes \rho_j \quad = \quad \sum_{i=\phi}^{n} n_{ij}\rho_i \qquad (j = 0, \ldots, n)$$

for some nonnegative integers $n_{ij} \in \mathbb{Z}_{\geq 0}$. Then, one has:

0) $n_{ij} \in \{0, 1\}$

i) $n_{ii} = 0 \quad (i = 0, \ldots, n)$

ii) $n_{ij} = n_{ji}$

iii) $\tilde{C} := 2I_{n+1} - (n_{ij})_{i,j=0}^{n}$ is negative semi-definite with 1-dimensional kernel.

Actually, from these properties, it is not hard see that $\tilde{C}$ is an affine Cartan matrix of one of types $\tilde{A}_l, \tilde{D}_l$ or $\tilde{E}_6, \tilde{E}_7$ or $\tilde{E}_8$, and that the matrix C obtained by deleting column and low for the trivial representation is a Cartan matrix of one of types A_l, D_l, E_6, E_7 or E_8 (see Table 3.). The correspondence:

(MC): $\qquad \tilde{G} \longmapsto \Gamma :=$ the graph associated to C

induces the bijection, called the McKay correspondence:

$$\{\text{Kleinean groups}\} \quad \xrightarrow{\sim} \quad \{\text{Simply laced Coxeter-Dynkin graphs of finite type}\}$$

McKay wrote [Mc] "Would not the Greeks appreciate the result that the simple Lie algebras may be derived from Platonic solids?".

2. Gonzalez-Verdier interpretation of McKay correspondence

The work by Gonzalez-Verdier [G-V] says that the representations ρ_i are interpreted as vector bundles $\tilde{V}_i$ on the resolution $\tilde{X}_0$ of the singularity X_0. Then, the 1-st Chern classes $c_1(\tilde{V}_i)$ of the vector bundles form the dual basis to the homology classes of the exceptional curves $[E_i]$ in $\tilde{X}_0$. That is: $c_1(\tilde{V}_i)$ form the fundamental weight for the simple root system.

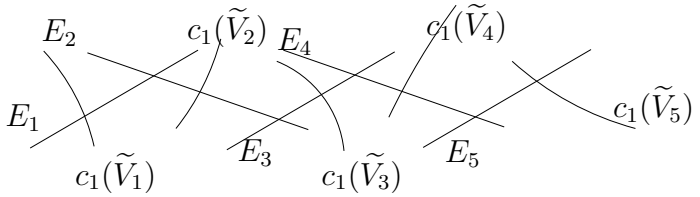
Let $\rho_i : G \to GL(V_i)$ be an irred. repr. of G. So G acts on $\mathbb{C}^2 \times V_i$ diagonally. Then the diagram (not precise)

$$\begin{array}{ccc} (\mathbb{C}^2 \times V_i)/G & \leftarrow & \widetilde{V}_i \\ \downarrow & & \downarrow \\ \mathbb{C}^2/G \simeq X_0 & \leftarrow & \widetilde{X}_0 \end{array}$$

defines an irreducible vector bundle $\widetilde{V}_i$ on $\widetilde{X}_0$

Theorem (Gonzalez-Verdier 1984)**.** *The first Chern class $c_1(\widetilde{V}_i)$ of $\widetilde{V}_i$ defines a divisor (a smooth curve) in $\widetilde{X_0}$, which is transversal to exactly one irreducible component, say E_i, of $E = \pi^{-1}(0)$. That is: $c_1(\widetilde{V}_1), \cdots, c_1(\widetilde{V}_l) \in H^2(\widetilde{X}_0, \mathbb{C})$ forms the dual basis of $[E_1], \ldots, [E_l] \in H_2(\widetilde{X}_0, \mathbb{Z}) \simeq Q := \overset{l}{\underset{i=1}{\oplus}} \mathbb{Z}\rho_i^v$.*

Table 15. The first Chern classes of irreducible vector bundles over $\tilde{X}_{A_4,0}$.



3. The inverse of McKay correspondence: $\Gamma \mapsto W_\Gamma \mapsto \langle A(W_\Gamma)\rangle$.

Let us construct conceptually the inverse of the McKay correspondence (MC) (through regular systems of weights) without using the classification.

Let a simplly-laced dynkin diagram Γ (or, equivalently a Cartan matrix C of finite type) be given. The data determine the Coxeter-Killing transformation c and using its eigenvalues, as we did in §8, we obtain the system of exponents $m_1, \cdots, m_\mu$. Then, as was discussed in §8, the generating function (1) of the exponents decomposes as (2) so that we obtain a simple weight system $W = W_\Gamma$.

How to recover the Kleinean group $\tilde{G}$ from a simple weight system W?

Let W be a simple weight system (i.e. $\varepsilon_W > 0$, see §8 Fact 1). Let f_W be the simple polynomial of the type W (9), and let us consider the associated hypersurface $X_{W,0}$ (11) (the simple singularity). Due to Fact 2 in §8 and Theorem in §1, the fundamental group of $X_{W,0} \setminus \{0\}$ is nothing but the isomorphic to the Kleinean group to define the simple singularity. on the other hand, we can determine the fundamental group purely arithmetically as follows.

Fact 12. *Let W be either a simple weight system or one of the 14 non-degenerate weight systems with $\varepsilon_W = -1$ and $a_0 = 0$. Then, we have the following isomorphism:*

$$\pi_1(X_{W,0} \setminus \{0\}, *) \ \simeq \ \langle A(W)\rangle,$$

where we recall that $A(W)$ is the signature set of W (see §11 a) Fact 5.*) and that $\langle\{p,q,r\}\rangle := \langle p,q,r\rangle$ denotes the group defined in* §1.

Proof. Combining a result of Mumford, which we quote below, and the description of the singularity $X_{W,0}$ in §11 a) Fact 6. (see also its following Example), we get the result. □

Theorem (Mumford 1961)**.** *Let X_0 be a two dimensional normal singularity, and let $\tilde{X}_0 \to X_0$ be a resolution of the singularity such that the exceptional set $E := \pi^{-1}(0)$ is a union of $\mathbb{P}^1$ such that the intersection diagram is a tree. Then, by the use of the data of the tree (details are omitted), one can write down $\pi_1(X_0 \setminus \{0\}, *)$ by suitable generators and relations, explicitly.*

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