A Semantic Formulation of $\top\top$-lifting and Logical Predicates for Computational Metalanguage

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Abstract. A semantic formulation of Lindley and Stark's $\top\top$-lifting is given. We first illustrate our semantic formulation of the $\top\top$-lifting in $\mathbf{Set}$ with several examples, and apply it to the logical predicates for Moggi's computational metalanguage. We then abstract the semantic $\top\top$-lifting as the lifting of strong monads across bifibrations with lifted symmetric monoidal closed structures.

1 Introduction

Logical predicates are a method for extracting submodels of the pure simply typed lambda calculus ($\lambda^{\Rightarrow}$ for short) by induction on type. Logical predicates are widely applied to the reasoning of the properties of $\lambda^{\Rightarrow}$ [23, 9, 24, 16].

We are interested in extending logical predicates to *Moggi's computational metalanguage* (λ_{ml} for short) [18], which has additional types $T\tau$ called *monadic type*. To do so, we need to consider a scheme to calculate a predicate at type $T\tau$ from a predicate at type τ. Recently, Lindley and Stark develop the *leapfrog method* and show the strong normalisation of λ_{ml} in the style of Tait-Girard reducibility [12, 11]. The novelty of the leapfrog method is the operation called $\top\top$-*lifting*, which calculates a reducibility predicate at type $T\tau$ from a reducibility predicate at type τ.

However, Lindley and Stark's $\top\top$-lifting is defined with respect to the syntactic structure of λ_{ml}, and is designed for the proof of the strong normalisation. This paper attempts to provide a semantic aspect of their $\top\top$-lifting. The main contribution of this paper is twofolds:

1. We provide a semantic formulation of Lindley and Stark's $\top\top$-lifting in set theory (section 3). This formulation is carried out by finding a semantic counterpart for each of the building block in $\top\top$-lifting. We instanciate $\top\top$-liftings with well-known strong monads over $\mathbf{Set}$, and show that the logical predicates using the semantic $\top\top$-lifting implies the *basic lemma* of logical predicates.
2. We re-formulate the above semantic $\top\top$-lifting as a construction of *liftings* of strong monads, and give a categorical account of this construction within the framework of fibred category theory (section 4). We then show that the above semantic $\top\top$-lifting and Abadi's $\top\top$-closure operation are instances of $\top\top$-lifting.

2 Preliminaries

Moggi's Computational Metalanguage

We begin with the syntax of λ_{ml}. We define the set of types $\mathbf{Typ}_{ml}$ by the following BNF (we consider a single base type b for simplicity):

$$\mathbf{Typ}_{ml} \ni \tau ::= b \mid \tau \Rightarrow \tau \mid T\tau.$$

Monadic types $T\tau$ are for the programs yielding values of type τ with some computational effect. A *typing context* (ranged over by Γ) is simply a finite sequence of variable-type pairs without any duplication of variables.

The calculus λ_{ml} has two new term constructions related to monadic types: $[-]$ and "let x^τ be M in N". Their typing rules are the following:

$$\frac{\Gamma \vdash M : \tau}{\Gamma \vdash [M] : T\tau} \qquad \frac{\Gamma \vdash M : T\tau \quad \Gamma, x : \tau \vdash N : T\tau'}{\Gamma \vdash \mathsf{let}\ x^\tau\ \mathsf{be}\ M\ \mathsf{in}\ N : T\tau'}$$

The term $[M]$ expresses the value of M involving the trivial computational effect. The term "let x^τ be M in N" expresses a sequential computation of M and N; the term M is first computed, its value is then bound to x^τ and next the term N is computed.

Equational theory of λ_{ml} extends $\beta\eta$ axioms of $\lambda^{\Rightarrow}$ with the following axioms:

$$\mathsf{let}\ x^\tau\ \mathsf{be}\ [M]\ \mathsf{in}\ N = N[M/x] \qquad (T.\beta)$$

$$\mathsf{let}\ x^\tau\ \mathsf{be}\ M\ \mathsf{in}\ [x^\tau] = M \qquad (T.\eta)$$

$$\mathsf{let}\ x^\tau\ \mathsf{be}\ (\mathsf{let}\ y^{\tau'}\ \mathsf{be}\ L\ \mathsf{in}\ M)\ \mathsf{in}\ N = \mathsf{let}\ y^{\tau'}\ \mathsf{be}\ L\ \mathsf{in}\ \mathsf{let}\ x^\tau\ \mathsf{be}\ M\ \mathsf{in}\ N \qquad (T.assoc)$$

Categorical Semantics of $\boldsymbol{\lambda_{ml}}$

A categorical semantics of λ_{ml} is given in a Cartesian closed category $\mathbb{C}$ equipped with a strong monad $\mathcal{T} = (T, \eta, \mu, \theta)$. We omit the formal definition of strong monads; see e.g. [18]. For a morphism $f : A \to TB$ in $\mathbb{C}$, we write $f^\#$ for the morphism $\mu_B \circ Tf : TA \to TB$.

Let B be an object in $\mathbb{C}$. We first assign to each type τ an object $[\![\tau]\!]$ in $\mathbb{C}$ by induction on type:

$$[\![b]\!] = B, \quad [\![\tau \Rightarrow \tau']\!] = [\![\tau]\!] \Rightarrow [\![\tau']\!], \quad [\![T\tau]\!] = T[\![\tau]\!].$$

We extend this assignment to typing contexts by

$$[\![x_1 : \tau_1, \cdots, x_n : \tau_n]\!] = [\![\tau_1]\!] \times \cdots \times [\![\tau_n]\!].$$

The semantics of λ_{ml} in $\mathbb{C}$ is an extension of the standard categorical semantics of $\lambda^{\Rightarrow}$ with the following rules:

- For a well-formed term $\Gamma \vdash [M] : T\tau$, we define

$$[\![[M]]\!] = \eta_{[\![\tau]\!]} \circ [\![M]\!].$$

- For a well-formed term $\Gamma \vdash \mathsf{let}\ x^\tau\ \mathsf{be}\ M\ \mathsf{in}\ N : T\tau'$, we define

$$[\![\mathsf{let}\ x^\tau\ \mathsf{be}\ M\ \mathsf{in}\ N]\!] = [\![N]\!]^\# \circ \theta_{[\![\Gamma]\!],[\![\tau]\!]} \circ \langle \mathrm{id}_{[\![\Gamma]\!]}, [\![M]\!] \rangle$$

3 A Semantic Formulation of $\top\top$-lifting

In [12], Lindley and Stark prove the strong normalisation of λ_{ml} by extending the reducibility predicate technique. The novelty of their method is the operation called $\top\top$-lifting, which calculates a reducibility predicate at a monadic type from that at an ordinary type.

Definition 3.1 ([12], section 3.1).

1. *We define the set of* raw continuations *by the following BNF:*

$$K ::= \mathrm{Id} \mid K \circ (x^\tau.N)$$

where the notation $(x^\tau.N)$ *indicates that* N *is a term with a distinguished free variable* x^τ.
A judgement for a raw continuation is a triple $T\tau \vdash_C K : T\tau'$. *Raw continuations are typed by the following rules:*

$$\frac{}{T\tau \vdash_C \mathrm{Id} : T\tau} \qquad \frac{x : \tau \vdash N : T\tau' \quad T\tau' \vdash_C K : T\tau''}{T\tau \vdash_C K \circ (x^\tau.N) : T\tau''}$$

We write $T\tau \vdash_C K$ *to mean that there exists a (unique) type* $T\tau'$ *such that* $T\tau \vdash_C K : T\tau'$ *is derived from the above rules.*

2. *We define an* application $K@M$ *of a term* M *to a continuation* K *by*

$$\mathrm{Id}@M = M, \qquad (K \circ (x^\tau.N))@M = K@(\textsf{let } x^\tau \textsf{ be } M \textsf{ in } N).$$

3. *Given a set* P *of terms of type* τ, *we define a set* $P^{\top\top}$ *of terms of type* $T\tau$ *by*

$$P^\top = \{T\tau \vdash_C K \mid \forall M \in P \,.\, K@[M] \in SN\}$$
$$P^{\top\top} = \{M : T\tau \mid \forall K \in P^\top \,.\, K@M \in SN\}$$

where SN *is the set of strongly normalising terms.*

From this point, we let $\mathcal{T} = (T, \eta, \mu, \theta)$ be a strong monad over $\mathbf{Set}$, and fix a categorical semantics of λ_{ml} with respect to the strong monad $\mathcal{T}$ and the evident CCC structure in $\mathbf{Set}$. We give a semantic formulation of the syntactic $\top\top$-lifting by finding semantic counterparts of continuations, applications and the set SN. This formulation is carried out with respect to the strong monad $\mathcal{T}$. We introduce the following notation: for subsets $X \subseteq I$ and $Y \subseteq J$, by $X \Rrightarrow Y$ we mean the subset $\{f \mid \forall x \in X \,.\, f(x) \in Y\}$ of $I \Rightarrow J$.

To simplify the situation, we assume that all continuations in definition 3.1 have the same type $T\rho$ (this restriction will be relaxed in section 5). We let $R = [\![\rho]\!]$.

Continuation We formulate a continuation as a function

$$f \in [\![\tau]\!] \Rightarrow TR.$$

We explain the idea of this formulation below. We notice that a continuation $T\tau \vdash_C \mathrm{Id} \circ (x^\tau.M) : T\rho$ is equivalent to a context $\mathsf{let}\ x^\tau\ \mathsf{be}\ -\ \mathsf{in}\ M$, and an application of a term to the continuation is equivalent to plugging the term in the hole of the context. The essential information of the context is the body M, and it has the following typing:

$$x : \tau \vdash M : T\rho.$$

Our formulation represents this information as a function $f \in [\![\tau]\!] \Rightarrow TR$.

Application We define an application of an element $x \in [\![T\tau]\!]$ to a continuation $f \in [\![\tau]\!] \Rightarrow TR$ to be $f^\# x$.

The Set SN The set SN is hard-coded in the definition of $P^\top$ and $P^{\top\top}$ since the syntactic $\top\top$-lifting is designed for the proof of the strong normalisation of λ_{ml}. We replace SN with some subset $S \subseteq TR$, and call S a *result predicate*.

We also relax the condition that the set R is given by $[\![\rho]\!]$ with some type ρ; we simply allow R to be any set and call R a *result type*.

Once continuations, applications and the set SN are semantically formulated, it is straightforward to define $P^\top$ and $P^{\top\top}$. We summarise the above discussion as follows:

Definition 3.2. *Let R be a set (called* result type*) and $S \subseteq TR$ be a subset (called* result predicate*).*

1. *A* continuation *is a function $f \in [\![\tau]\!] \Rightarrow TR$.*
2. *We define an* application *of $x \in [\![T\tau]\!]$ to a continuation $f \in [\![\tau]\!] \Rightarrow TR$ to be $f^\# x$.*
3. *Let $P \subseteq [\![\tau]\!]$ be a subset. We define a subset $P^{\top\top} \subseteq [\![T\tau]\!]$ by*

$$P^\top = \{f \in [\![\tau]\!] \Rightarrow TR \mid \forall x \in P \,.\, f(x) \in S\} = P \dot{\Rightarrow} S$$
$$P^{\top\top} = \{x \in [\![T\tau]\!] \mid \forall f \in P^\top \,.\, f^\#(x) \in S\},$$

which is equivalent to

$$P^{\top\top} = \{x \in [\![T\tau]\!] \mid \forall f \in P \dot{\Rightarrow} S \,.\, f^\#(x) \in S\}.$$

We call the operation $(-)^{\top\top}$ the $\top\top$-lifting *of $\mathcal{T}$ with R and $S \subseteq TR$.*

We can also consider the semantic $\top\top$-lifting for binary relations (*binary $\top\top$-lifting* for short) over the semantics of λ_{ml}. Let R be a set and $S \subseteq (TR)^2$ be a subset. A continuation is a pair (f, g) of functions from $[\![\tau]\!]$ to TR. An application of $(x, y) \in [\![T\tau]\!]^2$ to a continuation (f, g) is defined to be $(f^\# x, g^\# y)$. For a binary relation $P \subseteq [\![\tau]\!]^2$, we define $P^{\top\top}$ as follows:

$$P^\top = \{(f, g) \in ([\![\tau]\!] \Rightarrow TR)^2 \mid \forall (x, y) \in P \,.\, (fx, gy) \in S\}$$
$$P^{\top\top} = \{(x, y) \in [\![T\tau]\!]^2 \mid \forall (f, g) \in P^\top \,.\, (f^\# x, g^\# y) \in S\}.$$

Examples of Semantic $\top\top$-liftings

An interesting point is that we can obtain $\top\top$-liftings for various strong monads and result type/predicate pairs. We see some concrete examples of the semantic $\top\top$-lifting below.

Example 3.3. We consider the *lifting monad* $\mathcal{T}_\bot$, which simply adjoins an extra element $\bot$ to a given set. We calculate a $\top\top$-lifting of $\mathcal{T}_\bot$ with the following data:

- The result type R is $\{*\}$ (thus $T_\bot R = \{*, \bot\}$).
- The result predicate S is $\{*\}$.

For a subset $P \subseteq [\![\tau]\!]$, we have $P^{\top\top} = P$.

Example 3.4. We consider the *state monad* $\mathcal{T}_s$ whose functor part is given by $T_s I = M \Rightarrow I \times M$ for some set M. We let $M_0 \subseteq M$ be a subset and calculate a $\top\top$-lifting of $\mathcal{T}_s$ with the following data:

- The result type R is some set.
- The result predicate S is $M_0 \dot{\Rightarrow} R \times M_0$, the set of functions $f \in T_s R$ such that $\forall x \in M_0 \;.\; f(x) \in M_0 \times R$.

For a subset $P \subseteq [\![\tau]\!]$, we expand the definition of $P^{\top\top}$ and obtain

$$P^{\top\top} = \{f \in T_s[\![\tau]\!] \mid \forall g \in P \times M_0 \dot{\Rightarrow} R \times M_0 \;.\; g \circ f \in M_0 \dot{\Rightarrow} R \times M_0\}.$$

In fact, $P^{\top\top}$ can be characterised as follows:

$$P^{\top\top} = \begin{cases} M_0 \dot{\Rightarrow} P \times M_0 & (\emptyset \subsetneq R \times M_0 \subsetneq R \times M) \\ T_s[\![\tau]\!] & (\text{otherwise}) \end{cases}$$

Below we prove the first case of this characterisation; the second case is trivial. We first prove

$$P \times M_0 = \{i \in [\![\tau]\!] \times M \mid \forall g \in P \times M_0 \dot{\Rightarrow} R \times M_0 \;.\; g(i) \in R \times M_0\}.$$

($\subseteq$) Easy. ($\supseteq$) Let $x \notin P \times M_0$. From the assumption on $R \times M_0$, we can take two elements $s \in R \times M_0$ and $s' \in (R \times M) \backslash (R \times M_0)$. We then define the following function $g \in [\![\tau]\!] \times M \Rightarrow R \times M$:

$$g(x) = \begin{cases} s & (x \in P \times M_0) \\ s' & (x \notin P \times M_0) \end{cases}$$

which is clearly included in $P \times M_0 \dot{\Rightarrow} R \times M_0$. However $g(x) \notin R \times M_0$, so we conclude that $x \notin (r.h.s.)$. Therefore

$$\begin{aligned} & f \in M_0 \dot{\Rightarrow} P \times M_0 \\ \iff & \forall x \in M_0 \;.\; \forall g \in P \times M_0 \dot{\Rightarrow} R \times M_0 \;.\; g(f(x)) \in R \times M_0 \\ \iff & f \in P^{\top\top}. \end{aligned}$$

Example 3.5. We calculate a binary $\top\top$-lifting of the lifting monad $T_\bot$ with the following data:

- The result type R is a one-point set $\{*\}$. We have $T_\bot R = \{\bot, *\}$.
- The result predicate $S \subseteq (T_\bot R)^2$ is $\{(x, y) \in (T_\bot R)^2 \mid (x = * \implies y = *)\}$.

For a subset $P \subseteq [\![\tau]\!]$, we obtain $P^{\top\top} = P \cup \{(\bot, \bot)\}$.

Example 3.6. We consider the *finite powerset monad* $\mathcal{T}_p$, whose functor part is given by $T_p(X) = \{x \subseteq X \mid x \text{ is a finite set}\}$. We calculate a binary $\top\top$-lifting wf $\mathcal{T}_p$ with the following data:

– The result type R is a one-point set $\{*\}$. We have $T_pR = \{\emptyset, R\}$.
– The result predicate $S \subseteq (T_pR)^2$ is $\{(x, y) \in (T_pR)^2 \mid x = R \implies y = R\}$.

We identify a function $f \in [\![\tau]\!] \Rightarrow T_pR$ and a subset (written with the capital letter of the function) $F = \{x \in [\![\tau]\!] \mid f(x) = R\} \subseteq [\![\tau]\!]$. Under this identification, for each $x \in T_p[\![\tau]\!]$, we have

$$f^{\#}x = R \iff \bigcup_{e \in x} fe = R \iff \exists e \in x \,.\, e \in F.$$

For a subset $P \subseteq [\![\tau]\!]$, we expand the definition of $P^{\top\top}$ and obtain

$$\begin{aligned} P^{\top\top} = \{(p, q) \in (T_p[\![\tau]\!])^2 \mid \forall F, G \subseteq [\![\tau]\!] \,.\, (\forall (x, y) \in P \,.\, x \in F \implies y \in G) \implies \\ \forall e \in p \,.\, e \in F \implies \exists e' \in q \,.\, e' \in G\}. \end{aligned}$$

This is not intuitive, but interestingly we have the following characterisation of $P^{\top\top}$:

$$P^{\top\top} = \{(p, q) \mid \forall a \in p \,.\, \exists b \in q \,.\, (a, b) \in P\}. \tag{1}$$

This appears in the pattern of defining *pre-bisimulation relation* in concurrency.

The rest of this example is the proof of equation 1. ($\subseteq$) Let $(p, q) \in P^{\top\top}$ and $a \in p$. We show $\exists b \in q \,.\, (a, b) \in P$. We supply $\{a\}$ and $\{b \mid (a, b) \in P\}$ to F and G in the definition of $(p, q) \in P^{\top\top}$. We obtain

$$\begin{aligned} &(\forall (x, y) \in P \,.\, x = a \implies (a, y) \in P\}) \\ \implies &(\forall e \in p \,.\, e = a \implies \exists e' \in q \,.\, (a, e') \in P\}) \end{aligned}$$

whose premise part is trivially true. By letting e be a in the conclusion part of the above formula, we obtain $\exists e' \in q \,.\, (a, e') \in P$. ($\supseteq$) We take $p, q \in T_p[\![\tau]\!]$ such that $\forall a \in p \,.\, \exists b \in q \,.\, (a, b) \in P$. Let $F, G \subseteq [\![\tau]\!]$, $e \in p$ and assume $\forall (x, y) \in P \,.\, x \in F \implies y \in G$ (we call this assumption (*)) and $e \in F$. We show $\exists e' \in q \,.\, e' \in G$. Since $e \in p$, there exists $e' \in q$ such that $(e, e') \in P$. From (*), we have $e \in F \implies e' \in G$. Thus e' gives a witness of $\exists e' \in q \,.\, e' \in G$.

Logical Predicates for λ_{ml} Using $\top\top$-lifting

The semantic $\top\top$-lifting constructs a subset of $[\![T\tau]\!]$ from a subset of $[\![\tau]\!]$. This construction is suitable for extending the concept of *logical predicates* to λ_{ml}. We show that a logical predicate using the semantic $\top\top$-lifting extract a submodel of λ_{ml}. We fix a result type R and a result predicate $S \subseteq TR$, and consider the $\top\top$-lifting determined by R and S.

Definition 3.7. *A* $\top\top$-logical predicate *is a type-indexed family* $\{P^\tau \subseteq [\![\tau]\!]\}_{\tau \in \mathbf{Typ}_{ml}}$ *of subsets satisfying*

$$P^{T\tau} = (P^\tau)^{\top\top}, \quad P^{\tau \Rightarrow \tau'} = P^\tau \dot{\Rightarrow} P^{\tau'}.$$

For a typing context $\Gamma = x_1 : \tau_1, \cdots, x_n : \tau_n$, by P^Γ we mean the product $P_1^\tau \times \cdots \times P_n^\tau$, which is a subset of $[\![\Gamma]\!]$.

Theorem 3.8 (Basic Lemma). *Let P be a $\top\top$-logical predicate. For any well-formed term $\Gamma \vdash M : \tau$, we have $[\![M]\!] \in P^\Gamma \dot{\Rightarrow} P^\tau$.*

Proof. We show the following properties on the $\top\top$-lifting. Let $X \subseteq I$ and $Y \subseteq J$ be subsets.

1. $\eta_I \in X \dot{\Rightarrow} X^{\top\top}$. Let $x \in X$. Then for any $f \in X \dot{\Rightarrow} S$, we have $f^\#(\eta_I(x)) = f(x) \in S$. Therefore $\eta_I(x) \in X^{\top\top}$.
2. $\mu_I \in (X^{\top\top})^{\top\top} \dot{\Rightarrow} X^{\top\top}$. Let $x \in (X^{\top\top})^{\top\top}$ and $f \in X \dot{\Rightarrow} S$. We show $f^\#(\mu_I(x)) \in S$. It is easy to show that $f \in X \dot{\Rightarrow} S$ implies $f^\# \in X^{\top\top} \dot{\Rightarrow} S$, hence $(f^\#)^\# \in (X^{\top\top})^{\top\top} \dot{\Rightarrow} S$. Notice that $f^\#(\mu_I(x)) = (f^\#)^\#(x)$. Therefore $f^\#(\mu_I(x)) \in S$.
3. $\theta_{I,J} \in X \times Y^{\top\top} \dot{\Rightarrow} (X \times Y)^{\top\top}$. Let $a \in X, b \in Y^{\top\top}$ and $f \in X \times Y \dot{\Rightarrow} S$. We show $f^\# \circ \theta_{I,J}(a, b) \in S$. We note that the strength $\theta_{I,J}$ is given by $\theta_{I,J}(a, b) = T(\lambda b \in B \,.\, (a, b))(b)$ as $\mathbf{Set}$ is a well-pointed category (see e.g. [18]). Thus $f^\# \circ \theta_{I,J}(a, b) = (\lambda b \in B \,.\, f(a, b))^\#(b)$. Since $\lambda b \in B \,.\, f(a, b) \in Y \dot{\Rightarrow} S$, for each $b \in Y^{\top\top}$ we have $(\lambda b \in B \,.\, f(a, b))^\#(b) \in S$. Therefore $f^\# \circ \theta_{I,J}(a, b) \in S$
4. $f \in X \dot{\Rightarrow} Y$ implies $Tf \in X^{\top\top} \dot{\Rightarrow} Y^{\top\top}$. Let $x \in X^{\top\top}$ and $g \in Y \dot{\Rightarrow} S$. We show $g^\#(Tf(x)) = (g \circ f)^\#(x) \in S$. This holds from $g \circ f \in X \dot{\Rightarrow} S$ and the definition of $x \in X^{\top\top}$.
5. From 2 and 4, $f \in X \dot{\Rightarrow} Y^{\top\top}$ implies $f^\# \in X^{\top\top} \dot{\Rightarrow} Y^{\top\top}$.

We prove the theorem by induction on derivation of a well-formed term $\Gamma \vdash M : \tau$. We omit the cases for the syntax constructions inherited from $\lambda^{\Rightarrow}$; see e.g. [2]. The cases new to λ_{ml} is the following.

- Case $\Gamma \vdash [M] : T\tau$. From IH, we have $[\![M]\!] : P^\Gamma \dot{\Rightarrow} P^\tau$. From 1, we have $[\![[M]]\!] = \eta_{[\![\tau]\!]} \circ [\![M]\!] : P^\Gamma \dot{\Rightarrow} P^{T\tau}$.
- Case $\Gamma \vdash \mathsf{let}\ x^\tau\ \mathsf{be}\ M\ \mathsf{in}\ N : T\tau'$ with well-formed terms $\Gamma \vdash M : T\tau$ and $\Gamma, x : \tau \vdash N : T\tau'$. From IH, $[\![M]\!] : P^\Gamma \dot{\Rightarrow} P^{T\tau}$ and $[\![N]\!] : P^\Gamma \times P^\tau \dot{\Rightarrow} P^{T\tau'}$. From 3 and 5, we have $[\![N]\!]^\# \circ \theta_{[\![\Gamma]\!],[\![\tau]\!]} : P^\Gamma \times P^{T\tau} \dot{\Rightarrow} P^{T\tau'}$. Therefore $[\![\mathsf{let}\ x^\tau\ \mathsf{be}\ M\ \mathsf{in}\ N]\!] = [\![N]\!]^\# \circ \theta_{[\![\Gamma]\!],[\![\tau]\!]} \circ \langle \mathrm{id}_{[\![\Gamma]\!]}, [\![M]\!]\rangle : P^\Gamma \dot{\Rightarrow} P^{T\tau'}$.

□

4 A Categorical Generalisation of $\top\top$-lifting

In the proof of theorem 3.8, we notice that the operation $(-)^{\top\top}$ resembles an endofunctor (claim 4) equipped with morphisms constituting a strong monad (claim 1,2,3). It is indeed a strong monad over the category $\mathbf{Sub}(\mathbf{Set})$ of subsets and functions respecting subsets (example 4.3). Furthermore, the strong monad $(-)^{\top\top}$ makes the following diagram commute:

$$\begin{array}{ccc} \mathbf{Sub}(\mathbf{Set}) & \xrightarrow{(-)^{\top\top}} & \mathbf{Sub}(\mathbf{Set}) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbf{Set} & \xrightarrow[T]{} & \mathbf{Set} \end{array}$$

where $\pi : \mathbf{Sub}(\mathbf{Set}) \to \mathbf{Set}$ is the evident forgetful functor. This suggests that we can understand the semantic $\top\top$-lifting as a *construction* of such a strong monad from $\mathcal{T}$.

We give a categorical generalisation of this construction using fibrations and symmetric monoidal closed structures. We replace π with a bifibration $p : \mathbb{E} \to \mathbb{B}$ equipped with a lifted symmetric monoidal closed structure (definition 4.2). We then capture the semantic $\top\top$-lifting as a construction of a strong monad over $\mathbb{E}$ from that over $\mathbb{B}$.

We borrow some notations from 2-category theory. We use $\bullet$ and $*$ for the vertical and horizontal compositions of natural transformations, respectively. We overload $\circ$ with the notation for the composition of functors, as well as for the composition of a functor and a natural transformation.

4.1 Preliminaries

Symmetric Monoidal Close Category We assume that the reader is familiar with *symmetric monoidal closed categories*. We reserve symbols $\mathbf{I}, \otimes, \multimap$ for unit objects, tensor products and exponentials. A *symmetric monoidal functor* is a functor $F : \mathbb{C} \to \mathbb{D}$ between symmetric monoidal categories $\mathbb{C}, \mathbb{D}$ together with morphisms $m_{\mathbf{I}} : \mathbf{I}_{\mathbb{D}} \to F\mathbf{I}_{\mathbb{C}}$ and $m_{X,Y} : FX \otimes_{\mathbb{D}} FY \to F(X \otimes_{\mathbb{C}} Y)$ satisfying certain coherence laws (see e.g. [14]).

Example 4.1. 1. The category $\mathbf{Set}$ has a symmetric monoidal closed structure given by a chosen CCC structure.

2. The category $\omega\mathbf{CPPO}$ of pointed ω-CPOs and strict ω-continuous functions has a symmetric monoidal closed structure given by Sierpinski space $\mathbf{O} = \{\bot \sqsubseteq \top\}$, smash products and strict ω-continuous function spaces.
3. The functor $\times : (\omega\mathbf{CPPO})^2 \to \mathbf{Set}$ sending a pair (X, Y) of pointed ω-CPOs to the binary product $X \times Y$ of carrier sets is a symmetric monoidal functor.

Strong Monad A *strong monad* $\mathcal{T}$ over a symmetric monoidal category $\mathbb{B}$ is a tuple (T, η, μ, θ) such that (T, η, μ) is an ordinary monad over $\mathbb{B}$ and $\theta_{X,Y} : X \otimes TY \to T(X \otimes Y)$ is a natural transformation called *tensorial strength* satisfying certain coherence laws (see e.g. [10]). A *strong monad morphism* from $\mathcal{T} = (T, \eta, \mu, \theta)$ to $\mathcal{T}' = (T', \eta', \mu', \theta')$ is a natural transformation $\sigma : T \to T'$ satisfying

$$\mu' \bullet (\sigma * \sigma) = \sigma \bullet \mu, \quad \eta' = \sigma \bullet \eta, \quad \theta'_{X,Y} \circ (X \otimes \sigma_Y) = \sigma_{X \otimes Y} \circ \theta_{X,Y}.$$

Fibration We assume that the reader is familiar with preliminaries on fibration. A good reference is [7].

Definition 4.2. *A functor $p : \mathbb{E} \to \mathbb{B}$ is a* bifibration with a lifted symmetric monoidal closed structure *if p is a preordered bifibration, $\mathbb{E}$ and $\mathbb{B}$ are symmetric monoidal closed categories and p strictly preserves the symmetric monoidal closed structure in $\mathbb{E}$. We use dot notation $\dot{\mathbf{I}}$, $\dot{\otimes}$, $\dot{\multimap}$ to denote the symmetric monoidal closed structure in $\mathbb{E}$ which are sent to the symmetric monoidal closed structure $\mathbf{I}, \otimes, \multimap$ in $\mathbb{B}$ by p.*

Example 4.3. We define a category $\mathbf{Sub}(\mathbf{Set})$ by the following data: an object is a pair (X, I) where X is a subset of I, and a morphisms from (X, I) to (Y, J) is a function in $X \dot{\Rightarrow} Y$. The category $\mathbf{Sub}(\mathbf{Set})$ has the following CCC structure:

$$\begin{aligned} \dot{\mathbf{1}} &= (\{*\}, \{*\}) \\ (X, I) \mathbin{\dot{\times}} (Y, J) &= (\{(i, j) \mid i \in X \wedge j \in Y\}, I \times J) \\ (X, I) \mathbin{\dot{\Rightarrow}} (Y, J) &= (X \mathbin{\dot{\Rightarrow}} Y, I \Rightarrow J). \end{aligned}$$

(here the reader should not worry about the confusion caused by a clash of the notation $\dot{\Rightarrow}$). This structure is strictly preserved by the evident forgetful functor π : $\mathbf{Sub}(\mathbf{Set}) \to \mathbf{Set}$, which is actually a partial-order bifibration. Therefore π is a bifibration with a lifted symmetric monoidal closed structure.

One good property of the class of bifibrations with lifted symmetric monoidal closed structures is the closure under change-of-base along symmetric monoidal functors.

Proposition 4.4 (e.g. [5]). *Let $p : \mathbb{E} \to \mathbb{B}$ be a bifibration with a lifted symmetric monoidal closed structure and $F : \mathbb{C} \to \mathbb{B}$ be a symmetric monoidal functor. Then the change-of-base of p along F is again a bifibration with a lifted symmetric monoidal closed structure.*

Example 4.5. We consider the following change-of-base of π along $\times$:

$$\begin{array}{ccc} \mathbf{Rel}(\omega\mathbf{CPPO}) & \longrightarrow & \mathbf{Sub}(\mathbf{Set}) \\ {\scriptstyle \pi_2}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \pi} \\ (\omega\mathbf{CPPO})^2 & \underset{\times}{\longrightarrow} & \mathbf{Set} \end{array}$$

From proposition 4.4, π_2 is again a bifibration with a lifted symmetric monoidal closed structure. An object in $\mathbf{Rel}(\omega\mathbf{CPPO})$ is a triple (X, I, J) where I, J are pointed ω-CPOs and X is an arbitrary subset of $I \times J$, that is, a binary relation between I and J. A morphism in $\mathbf{Rel}(\omega\mathbf{CPPO})$ from (X, I, J) to (X', I', J') is a pair $(f : I \to I', g : J \to J')$ of strict ω-continuous functions such that $f \times g \in X \dot{\Rightarrow} X'$. We can similarly derive the category of n-ary relations between ω-CPOs by change-of-base.

4.2 $\top\top$-lifting as a Construction of Liftings of Strong Monads

We fix a bifibration $p : \mathbb{E} \to \mathbb{B}$ with a lifted symmetric monoidal closed structure. We define a fibration of *lifted strong monads* which is suitable for characterising the $\top\top$-lifting.

Definition 4.6. *1. We say that a strong monad $\dot{\mathcal{T}} = (\dot{T}, \dot{\eta}, \dot{\mu}, \dot{\theta})$ over $\mathbb{E}$ is a* lifting *of a strong monad $\mathcal{T} = (T, \eta, \mu, \theta)$ over $\mathbb{B}$ if the following holds:*

$$p \circ \dot{T} = T \circ p, \quad p \circ \dot{\eta} = \eta \circ p, \quad p \circ \dot{\mu} = \mu \circ p, \quad p(\dot{\theta}_{X,Y}) = \theta_{pX,pY}.$$

2. We write $\mathbf{Mon}(\mathbb{B})$ for the category of strong monads over $\mathbb{B}$ and strong monad morphisms between them.

3. We define a category $\mathbf{Mon}_l(\mathbb{E})$ using the following data:

- *An object in $\mathbf{Mon}_l(\mathbb{E})$ is a pair of a strong monad $\dot{\mathcal{T}}$ over $\mathbb{E}$ and a strong monad $\mathcal{T}$ over $\mathbb{B}$ such that $\dot{\mathcal{T}}$ is a lifting of $\mathcal{T}$. We sometimes represent an object in $\mathbf{Mon}_l(\mathbb{E})$ simply by a strong monad over $\mathbb{E}$ when its underlying strong monad over $\mathbb{B}$ is clear from the context.*
- *A morphism in $\mathbf{Mon}_l(\mathbb{E})$ is a pair of strong monad morphisms $\dot{\alpha} : \dot{\mathcal{T}} \to \dot{\mathcal{T}}'$ and $\alpha : \mathcal{T} \to \mathcal{T}'$ such that $p \circ \dot{\alpha} = \alpha \circ p$.*

4. We write $\mathbf{Mon}(p) : \mathbf{Mon}_l(\mathbb{E}) \to \mathbf{Mon}(\mathbb{B})$ for the following forgetful functor:

$$\mathbf{Mon}(p)(\dot{\mathcal{T}}, \mathcal{T}) = \mathcal{T}, \quad \mathbf{Mon}(p)(\dot{\alpha}, \alpha) = \alpha.$$

Theorem 4.7. *$\mathbf{Mon}(p)$ is a fibration.*

Proof. See appendix A.1 □

We are ready to give a categorical account of the semantic $\top\top$-lifting. We capture the $\top\top$-lifting as a construction of a lifting of a strong monad over $\mathbb{E}$ from that over $\mathbb{B}$. For this construction, *continuation monads* play a crucial role. We observe the following facts.

- For each object I in $\mathbb{B}$, an endofunctor $(- \multimap I) \multimap I$ over $\mathbb{B}$ is a strong monad (called *continuation monad*). Particularly, for a strong monad $\mathcal{T}$ over $\mathbb{B}$ and an object R in $\mathbb{B}$, we have a continuation monad $(- \multimap TR) \multimap TR$ and a strong monad morphism
  $$\sigma : \mathcal{T} \longrightarrow (- \multimap TR) \multimap TR$$
  whose component at an object I in $\mathbb{B}$ is given by the following transposition (object annotations are omitted):
  $$\frac{TI \otimes (I \multimap TR) \xrightarrow{s} (I \multimap TR) \otimes TI \xrightarrow{\theta} T((I \multimap TR) \otimes I) \xrightarrow{@^{\#}} TR}{\sigma_I = \lambda(@^{\#} \circ \theta \circ s) : TI \longrightarrow (I \multimap TR) \multimap TR}$$
  where s and $@$ are a symmetry and an evaluation morphisms in $\mathbb{B}$, respectively.
- Let S be an object in $\mathbb{E}$ above TR and consider a continuation monad $(- \mathrel{\dot{\multimap}} S) \mathrel{\dot{\multimap}} S$ over $\mathbb{E}$. It is a lifting of $(- \multimap TR) \multimap TR$ since p strictly preserves the symmetric monoidal closed structure in $\mathbb{E}$.

The following diagram summarises these facts in $\mathbf{Mon}(p)$:

$$\begin{array}{ccc} & (- \mathrel{\dot{\multimap}} S) \mathrel{\dot{\multimap}} S & \mathbf{Mon}_l(\mathbb{E}) \\ & & \downarrow {\scriptstyle \mathbf{Mon}(p)} \\ \mathcal{T} \xrightarrow[\sigma]{} & (- \multimap TR) \multimap TR & \mathbf{Mon}(\mathbb{B}) \end{array}$$

We now consider a Cartesian lifting of σ.

$$\begin{array}{ccc} \sigma^*((- \mathbin{\dot{\multimap}} S) \mathbin{\dot{\multimap}} S) \overset{\overline{\sigma}}{\cdots\cdots>} (- \mathbin{\dot{\multimap}} S) \mathbin{\dot{\multimap}} S & & \mathbf{Mon}_l(\mathbb{E}) \\ & & \big\downarrow {\scriptstyle \mathbf{Mon}(p)} \\ \mathcal{T} \xrightarrow[\sigma]{\qquad\qquad} (- \multimap TR) \multimap TR & & \mathbf{Mon}(\mathbb{B}) \end{array}$$

We claim that the vertex $\sigma^*((- \mathbin{\dot{\multimap}} S) \mathbin{\dot{\multimap}} S)$, which is by definition a lifting of $\mathcal{T}$, gives the $\top\top$-lifting of $\mathcal{T}$. There are two sets of evidence supporting our claim.

– The set-theoretic $\top\top$-lifting in section 3 is an instance of this generalised $\top\top$-lifting. We work in the fibration $\pi : \mathbf{Sub}(\mathbf{Set}) \to \mathbf{Set}$ from example 4.3. Subsequently, for any strong monad $\mathcal{T}$ and subsets $X \subseteq I$ and $S \subseteq TR$, we have:

$$\begin{aligned} \sigma^*((X \mathbin{\dot{\Rightarrow}} S) \mathbin{\dot{\Rightarrow}} S) &= \{x \in TI \mid \sigma^*(x) \in ((X \mathbin{\dot{\Rightarrow}} S) \mathbin{\dot{\Rightarrow}} S)\} \\ &= \{x \in TI \mid \forall f \in X \mathbin{\dot{\Rightarrow}} S \;.\; \sigma^*(x)(f) \in S\} \\ &= \{x \in TI \mid \forall f \in X \mathbin{\dot{\Rightarrow}} S \;.\; f^{\#}x \in S\} \\ &= X^{\top\top}. \end{aligned}$$

– Let D, E be pointed ω-CPOs and R be an arbitrary subset of $D \times E$. In [1], Abadi considered the following closure operation $(-)^{\top\top}$ as a semantic abstraction of Pitts' syntactic $\top\top$-closure operation [21]:

$$\begin{aligned} R^\top &= \{(f, g) \in [D \to_\perp \mathbf{O}] \times [E \to_\perp \mathbf{O}] \mid \forall (x, y) \in R \;.\; fx = gy\} \\ R^{\top\top} &= \{(x, y) \in D \times E \mid \forall (f, g) \in R^\top \;.\; fx = gy\} \end{aligned}$$

where $[- \to_\perp -]$ denotes strict ω-continuous function spaces.
The above closure operation is an instance of our semantic $\top\top$-lifting. We work in the fibration $\pi_2 : \mathbf{Rel}(\omega\mathbf{CPPO}) \to (\omega\mathbf{CPPO})^2$ from example 4.5. The $\top\top$-lifting of the *identity monad* over $(\omega\mathbf{CPPO})^2$ with the following data coincides with Abadi's $\top\top$-closure operation.
  - The result type R is $(\mathbf{O}, \mathbf{O})$.
  - The result predicate S is $(\{(\perp, \perp), (\top, \top)\}, (\mathbf{O}, \mathbf{O}))$.

We write $\mathcal{T}^{\top\top}$ for $\sigma^*((- \mathbin{\dot{\multimap}} S) \mathbin{\dot{\multimap}} S)$.

5 Multiple Result Types

We relax the restriction we imposed on the result type in section 3. Let $p : \mathbb{E} \to \mathbb{B}$ be a bifibration with a lifted symmetric monoidal closed structure and $\mathcal{T}$ be a strong monad over $\mathbb{B}$.

Theorem 5.1. *If p has fibred (finite/small) products, then so does $\mathbf{Mon}(p)$.*

Proof. See appendix A.2. □

Let $\{(S_k, R_k)\}_{k\in K}$ be a set of pairs of objects in $\mathbb{E}$ and $\mathbb{B}$ such that $pS_k = TR_k$ for all $k \in K$. For each $k \in K$, the pair (S_k, R_k) determines a $\top\top$-lifting $\mathcal{T}^{\top\top_k}$. They are all liftings of $\mathcal{T}$, so we consider the following fibred product in $\mathbf{Mon}_l(\mathbb{E})_{\mathcal{T}}$:

$$\bigwedge_{k\in K} \mathcal{T}^{\top\top_k}$$

which is again a lifting of $\mathcal{T}$.

Example 5.2. We flip the relation S in example 3.6 and obtain the following $\top\top$-lifting:

$$P^{\top\top'} = \{(p, q) \mid \forall b \in q \,.\, \exists a \in p \,.\, (a, b) \in P\}.$$

The intersection

$$P^{\top\top} \wedge P^{\top\top'} = \{(p, q) \mid (\forall b \in q \,.\, \exists a \in p \,.\, (a, b) \in P) \wedge (\forall a \in p \,.\, \exists b \in q \,.\, (a, b) \in P)\}$$

coincides with the pattern of bisimulation.

6 Related Work

This work has been inspired by Lindley and Stark's paper [12] and Lindley's thesis [11]. Lindley and Stark introduce the syntactic $\top\top$-lifting for λ_{ml} and prove the strong normalisation of λ_{ml}. In the latter part of [12], they also discuss an extension of the syntactic $\top\top$-lifting to other types such as sum types. However, this extension has not been covered here.

Operations which are similar to Lindley and Stark's $\top\top$-lifting have previously appeared in several other studies. Some examples of these studies are: the reducibility technique for linear logic by Girard [4], Parigot's work on the second order classical natural deduction [20], Pitts' $\top\top$-closure operation [21] and Melliès and Vouillon's *biorthogonality* [15]. In addition, Abadi gives a semantic formulation of Pitts' $\top\top$-closure operation and discusses the relationship between $\top\top$-closed relations (those which satisfy $R = R^{\top\top}$) and admissibility [1]. The $\top\top$-closed relations are applied to the verification of the correctness of program transformations [8, 19], and to the characterisation of the observational equivalence for a language with local states [22].

Categorical study of logical predicates established in [13, 17] is generalised by Hermida using fibrational category theory [6]. The key observation of his generalisation is that logical predicates with respect to a fibration $p : \mathbb{E} \to \mathbb{B}$ employ a CCC structure in $\mathbb{E}$ which is strictly preserved by p. This observation leads us to consider liftings of strong monads and bifibrations with lifted symmetric monoidal closed structures.

In general, there are many liftings of a strong monad. In [3], Larrecq, Lasota and Nowak propose a construction method of liftings of strong monads using factorisation systems. Their method appears to be fundamentally different from our semantic $\top\top$-lifting. However, some of their examples of liftings of strong monads over $\mathbf{Set}$ can also be calculated with our method. It will be interesting to establish a formal relationship between their lifting of strong monads and the semantic $\top\top$-lifting developed by us.

7 Conclusion

We semantically formulated Lindley and Stark's $\top\top$-lifting and showed that it provides a satisfactory construction method of logical predicates for λ_{ml}. We also examined several examples of the semantic $\top\top$-lifting of strong monads over $\mathbf{Set}$.

We then categorically re-formulated the $\top\top$-lifting as a lifting of a monad along a bifibration with a symmetric monoidal closed structure using continuation monads. This generalisation subsumes the set-theoretic $\top\top$-lifting in section 3 and Abadi's $\top\top$-lifting.

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A Proof

A.1 Proof of theorem 4.7

When $p : \mathbb{E} \to \mathbb{B}$ is a fibration, $p \circ - : [\mathbb{E}, \mathbb{E}] \to [\mathbb{E}, \mathbb{B}]$ is also a fibration. Then an endofunctor F over $\mathbb{E}$ is a lifting of an endofunctor G over $\mathbb{B}$ if and only if F is above $G \circ p$ in the fibration $p \circ -$.

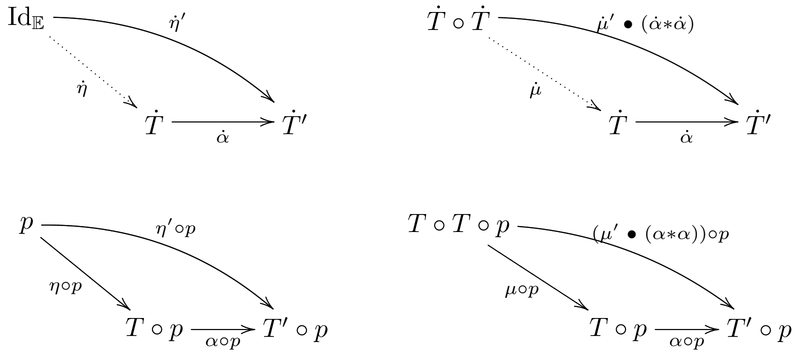
Let $\mathcal{T}, \mathcal{T}'$ be strong monads over $\mathbb{B}$, $\alpha : \mathcal{T} \to \mathcal{T}'$ be a strong monad morphism and $\dot{\mathcal{T}}'$ be a strong monad over $\mathbb{E}$ which is a lifting of $\mathcal{T}'$. We construct a monad $\dot{\mathcal{T}} = (\dot{T}, \dot{\eta}, \dot{\mu}, \dot{\theta})$ together with a strong monad morphism $\dot{\alpha} : \dot{\mathcal{T}} \to \dot{\mathcal{T}}'$ which is Cartesian above α.

– We define the endofunctor $\dot{T} : \mathbb{E} \to \mathbb{E}$ to be the vertex $(\alpha \circ p)^* \dot{T}'$ of the following Cartesian lifting of $\alpha \circ p$ in the fibration $p \circ -$:

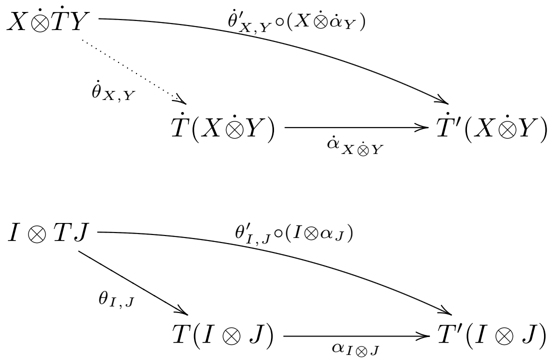
$$\begin{array}{ccc} (\alpha \circ p)^* \dot{T}' & \xrightarrow{\overline{(\alpha \circ p)}(\dot{T}')} & \dot{T}' \\ & & \\ T \circ p & \xrightarrow[\alpha \circ p]{} & T' \circ p \end{array}$$

We define $\dot{\alpha} = \overline{(\alpha \circ p)}(\dot{T}')$.

– We define the unit $\dot{\eta}$ and the multiplication $\dot{\mu}$ by the morphisms obtained from the universal property of the Cartesian morphism $\dot{\alpha}$ in the fibration $p \circ -$:



– For objects X, Y in $\mathbb{E}$ above objects I, J in $\mathbb{B}$ respectively, we define the strength $\dot{\theta}_{X,Y}$ as follows:



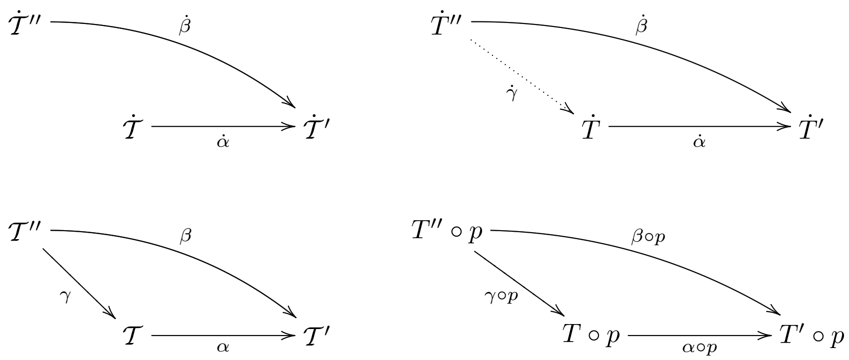
We can easily verify that $\dot{\eta}, \dot{\mu}, \dot{\theta}$ satisfy the law of strong monad using the fact that p is faithful (since p is a preordered fibration). For example, to show $\dot{\mu}_X \circ \dot{T}(\dot{\eta}_X) = \mathrm{id}_X$ for each object X in $\mathbb{E}$, we calculate:

$$p(\dot{\mu}_X \circ \dot{T}(\dot{\eta}_X)) = \mu_{pX} \circ T(\eta_{pX}) = \mathrm{id}_{pX} = p(\mathrm{id}_X).$$

Since p is faithful, we conclude that $\dot{\mu}_X \circ \dot{T}(\dot{\eta}_X) = \mathrm{id}_X$.

The morphism $\dot{\alpha}$ is clearly a monad morphism from the construction of $\dot{\eta}, \dot{\mu}, \dot{\theta}$.

To see that $\dot{\alpha}$ is a Cartesian morphism, we consider a situation in $\mathbf{Mon}(p)$ described in the left diagram:



This situation induces the right diagram in $p \circ -$. From the universal property of $\dot{\alpha}$, we obtain a unique morphism $\dot{\gamma} : \dot{\mathcal{T}}'' \to \dot{\mathcal{T}}$ above $\gamma \circ p$ satisfying $\dot{\alpha} \bullet \dot{\gamma} = \dot{\beta}$. To verify that $\dot{\gamma}$ is a strong monad morphism, we use the universal property of $\dot{\alpha}$. We show $\dot{\gamma} \bullet \dot{\eta}'' = \dot{\eta}$ as an example. First, $\dot{\gamma} \bullet \dot{\eta}''$ and $\dot{\eta}$ are above $\eta \circ p$ in the fibration $p \circ -$. Next, we have

$$\dot{\alpha} \bullet \dot{\gamma} \bullet \dot{\eta}'' = \dot{\beta} \bullet \dot{\eta}'' = \dot{\eta}' = \dot{\alpha} \bullet \dot{\eta}$$

From the universal property of $\dot{\alpha}$, we have $\dot{\gamma} \bullet \dot{\eta}'' = \dot{\eta}$. We can similarly verify the other equations of the law of strong monad morphism. □

A.2 Proof of theorem 5.1

(Sketch) Let $\mathcal{T} = (T, \eta, \mu, \theta)$ be a strong monad over $\mathbb{B}$, K be a (finite) set and suppose that we have a lifting $\dot{\mathcal{T}}_k = (\dot{T}_k, \dot{\eta}_k, \dot{\mu}_k, \dot{\theta}_k)$ of $\mathcal{T}$ for each $k \in K$.

The fibred product $\hat{\dot{\mathcal{T}}} = (\hat{\dot{T}}, \hat{\dot{\eta}}, \hat{\dot{\mu}}, \hat{\dot{\theta}})$ of $\dot{\mathcal{T}}_k$ is given as follows.

- The functor part is defined by $\hat{\dot{T}}X = \bigwedge_{k \in K} \dot{T}_k X$. We write $\pi^k_X : \hat{\dot{T}}X \to \dot{T}_k X$ for the k-th projection.
- We observe that for objects X, Y in $\mathbb{E}$ and a morphism $f : pX \to pY$ in $\mathbb{B}$, we have the following natural isomorphism:

$$\mathbb{E}_f(X, \hat{\dot{T}}Y) \cong \mathbb{E}_{pX}(X, f^*(\hat{\dot{T}}Y)) \cong \mathbb{E}_{pX}\left(X, \bigwedge_{k \in K} f^* \dot{T}_k\right) \cong \prod_{k \in K} \mathbb{E}_f(X, \dot{T}_k Y).$$

  We write ϕ for the right-to-left part of the above isomorphism. The unit, multiplication and strength is then defined by:

$$\begin{aligned} \hat{\dot{\eta}}_X &= \phi\langle(\dot{\eta}_k)_X\rangle_{k \in K} \\ \hat{\dot{\mu}}_X &= \phi\langle(\dot{\mu}_k)_X \circ \dot{T}_k(\pi^k_X) \circ \pi^k_X\rangle_{k \in K} \\ \hat{\dot{\theta}}_{X,Y} &= \phi\langle(\dot{\theta}_k)_{X,Y} \circ (X \dot{\otimes} \pi^k_Y)\rangle_{k \in K} \end{aligned}$$

The reader can verify that $\hat{\dot{\mathcal{T}}}$ is indeed a strong monad, and is a fibred product of $\{\dot{\mathcal{T}}_k\}_{k \in K}$.