Categorical Approach to Logic

Lecture 1

Shin-ya Katsumata

Research Institute for Mathematical Sciences, Kyoto University

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Category

A category $\mathbb{C}$ consists of:

- a class or set $\mathbb{C}_0$ of objects,
- a class or set $\mathbb{C}_1$ of morphisms,
- mappings $\mathrm{dom}, \mathrm{cod} \colon \mathbb{C}_1 \to \mathbb{C}_0$ giving a domain and codomain to each morphism,
- $\mathrm{id} \colon \mathbb{C}_0 \to \mathbb{C}_1$ giving the identity to each object,
- $- \circ - \colon \{(g, f) \mid \mathrm{dom}(g) = \mathrm{cod}(f)\} \to \mathbb{C}_1$ called composition.

Category

We stop to introduce some notations:

$$\begin{array}{rcl} I \in \mathbb{C} & \ldots & I \in \mathbb{C}_1 \\ f\colon I \to J & \ldots & \mathrm{dom}(f) = I \wedge \mathrm{cod}(f) = J \\ \mathbb{C}(I, J) & \ldots & \{f \in \mathbb{C}_1 \mid f\colon I \to J\} \end{array}$$

We proceed the definition of category: the above data satisfy

$$\begin{array}{l} \mathrm{id}_I\colon I \to I \\ f\colon I \to J \wedge g\colon J \to K \Rightarrow g \circ f\colon I \to K \\ f \circ \mathrm{id}_I = \mathrm{id}_J \circ f = f \quad (f\colon I \to J) \\ (h \circ g) \circ f = h \circ (g \circ f) \quad (f\colon I \to J, g\colon J \to K, h\colon K \to L) \end{array}$$

Examples of Categories

$\mathbb{C}$	$I \in \mathbb{C}$	$\mathbb{C}(I, J)$
Set	set	functions
Rel	set	binary relations
Mon	monoid	monoid homomorphisms
Vect	vector space	linear functions
Pre	preorder	monotone functions
Top	topological space	continuous functions
Cat	small categorie	functors
$\mathbb{C}^{op}$	I	$\mathbb{C}(J, I)$
$\mathbb{C}_1 \times \mathbb{C}_2$	(I_1, I_2) with $I_i \in \mathbb{C}_i$	$\mathbb{C}_1(I_1, J_1) \times \mathbb{C}_2(I_2, J_2)$

Examples of Categories

A category generated from a single mathematical object:

- For a monoid $(M, 1, \cdot)$, define $\mathbb{C}_0 = \{*\}$ and $\mathbb{C}_1 = M$ with

$$\mathrm{id}_* = 1, \quad g \circ f = g \cdot f.$$

  We obtain a category $C(M, 1, \cdot)$.

- For a preorder $(P, \leqslant)$, define $\mathbb{C}_0 = P$ and $\mathbb{C}_1 = \leqslant$ with

$$\mathrm{id}_I = (I, I), \quad (J, K) \circ (I, J) = (I, K).$$

  We obtain a category $C(P, \leqslant)$.

Below we treat preordered sets / posets as categories.

Size of Categories

We say that a category $\mathbb{C}$ is:

- small if $\mathbb{C}_1$ is a set (hence $\mathbb{C}_0$ is also a set).
- locally small if every $\mathbb{C}(\mathrm{I}, \mathrm{J})$ is a set.
- thin if every $\mathbb{C}(\mathrm{I}, \mathrm{J})$ has cardinality at most 1.

Proposition 1. *Thin small categories bijectively correspond to preorders.*

$\mathrm{Set}, \mathrm{Rel}, \mathrm{Mon}, \mathrm{Vect}, \mathrm{Pre}, \mathrm{Top}$ are all locally small.

Isomorphism

A morphism $g \colon J \to I$ is an inverse of $f \colon I \to J$ if

$$g \circ f = \mathrm{id}_I, \quad f \circ g = \mathrm{id}_J.$$

Proposition 2. *If* $f \colon I \to J$ *has an inverse, it is unique.*

A morphism $f \colon I \to J$ is an isomorphism if f has an inverse.

Two objects $I, J \in \mathbb{C}$ are isomorphic ($I \cong J$) if there exists an isomorphism $f \colon I \to J$.

Slogan in category theory: isomorphic objects are the same.

Terminal Object in a Category

A terminal object in $\mathbb{C}$ consists of:

1. $\mathsf{T} \in \mathbb{C}$.

2. $!_{\mathrm{I}} \colon \mathrm{I} \to \mathsf{T}$ for every $\mathrm{I} \in \mathbb{C}$.

They satisfy: $\mathbb{C}(\mathrm{I}, \mathsf{T}) = \{!_{\mathrm{I}}\}$.

Lemma 3. $\mathrm{id}_{\mathsf{T}} = !_{\mathsf{T}}$.

Proposition 4. *If* $(\mathsf{T}, !)$ *and* $(\mathsf{T}', !')$ *are terminal objects in* $\mathbb{C}$*, then* $\mathsf{T} \cong \mathsf{T}'$.

- Any one-point set with the trivial structure on it is a terminal object in $\mathrm{Set}, \mathrm{Mon}, \mathrm{Vect}, \mathrm{Pre}, \mathrm{Top}$. In Rel, $\emptyset$ is.

Binary Product on a Category

A binary product on $\mathbb{C}$ consists of ($\mathrm{I}_1, \mathrm{I}_2 \in \mathbb{C}$):

1. (Binary) Product object $\mathrm{I}_1 \times \mathrm{I}_2 \in \mathbb{C}$,

2. Tuple $\langle -, - \rangle_{\mathrm{J}}^{\mathrm{I}_1, \mathrm{I}_2} \colon \mathbb{C}(\mathrm{J}, \mathrm{I}_1) \times \mathbb{C}(\mathrm{J}, \mathrm{I}_2) \to \mathbb{C}(\mathrm{J}, \mathrm{I}_1 \times \mathrm{I}_2)$,

3. Projections $\pi_1^{\mathrm{I}_1, \mathrm{I}_2} \colon \mathrm{I}_1 \times \mathrm{I}_2 \to \mathrm{I}_1$ and $\pi_2^{\mathrm{I}_1, \mathrm{I}_2} \colon \mathrm{I}_1 \times \mathrm{I}_2 \to \mathrm{I}_2$.

They satisfy (we omit object annotations):

$$
\begin{aligned}
\langle f_1, f_2 \rangle \circ h &= \langle f_1 \circ h, f_2 \circ h \rangle \\
\pi_i \circ \langle f_1, f_2 \rangle &= f_i \\
\langle \pi_1, \pi_2 \rangle &= \mathrm{id}.
\end{aligned}
$$

Binary Product on a Category

The tuple is actually a bijection:

$$\langle -, - \rangle_{\mathrm{J}}^{\mathrm{I}_1, \mathrm{I}_2} \colon \mathbb{C}(\mathrm{J}, \mathrm{I}_1) \times \mathbb{C}(\mathrm{J}, \mathrm{I}_2) \cong \mathbb{C}(\mathrm{J}, \mathrm{I}_1 \times \mathrm{I}_2).$$

Two binary products on $\mathbb{C}$ give isomorphic product objects:

Proposition 5. *If* $(-\times-, \langle -, - \rangle, \pi_1, \pi_2)$ *and* $(-\times'-, \langle -, - \rangle', \pi_1',$ $\pi_2')$ *are binary products on* $\mathbb{C}$*, then* $\mathrm{I}_1 \times \mathrm{I}_2 \cong \mathrm{I}_1 \times' \mathrm{I}_2$.

Proof. We obtain $\mathbb{C}(\mathrm{J}, \mathrm{I}_1 \times \mathrm{I}_2) \cong \mathbb{C}(\mathrm{J}, \mathrm{I}_1 \times' \mathrm{I}_2)$. Then let $\mathrm{J} = \mathrm{I}_1 \times \mathrm{I}_2$ and $\mathrm{J} = \mathrm{I}_1 \times' \mathrm{I}_2$ and trace $\mathrm{id}_{\mathrm{I}_1 \times \mathrm{I}_2}$ and $\mathrm{id}_{\mathrm{I}_1 \times' \mathrm{I}_2}$. □

Finite / Small Products on a Category

We extend the definition of binary products to n-ary / set-indexed products (What is 0-ary product?).

Finite products on $\mathbb{C}$ consists of n-ary products for all $n \in \mathbf{N}$. (We assume that 1-ary product satisfy $\prod(I) = I$).

From a terminal object and a binary product on $\mathbb{C}$, we can construct finite products on $\mathbb{C}$:

$$\prod (I_1, ..., I_n) \ = \ (...(I_1 \times I_2) \times I_3 ...) \times I_n.$$

Small products on $\mathbb{C}$ consists of I-ary products for all $I \in \mathrm{Set}$.

- $\mathrm{Set}, \mathrm{Rel}, \mathrm{Mon}, \mathrm{Vect}, \mathrm{Pre}, \mathrm{Top}, \mathrm{Cat}$ have small products.

Finite / Small Products on a Category

In a category $\mathbb{C}$ with finite products, we can consider algebraic structures determined by finite-arity operations and equational axioms, such as monoids, groups, rings, vector spaces, etc.

- A monoid in $\mathbb{C}$ with finite products consists of $M \in \mathbb{C}$, $e\colon 1 \to M$, $m\colon M^2 \to M$ such that

$$
\begin{aligned}
m \circ \langle \mathrm{id}_M, e \circ !_M \rangle &= \mathrm{id}_M \\
m \circ \langle e \circ !_M, \mathrm{id}_M \rangle &= \mathrm{id}_M \\
m \circ \left\langle m \circ \left\langle \pi_1^{M^3}, \pi_2^{M^3} \right\rangle, \pi_3^{M^3} \right\rangle &= m \circ \left\langle \pi_1^{M^3}, m \circ \left\langle \pi_2^{M^3}, \pi_3^{M^3} \right\rangle \right\rangle.
\end{aligned}
$$

Binary Coproduct on a Category

... is a product on $\mathbb{C}^{op}$. It consists of ($I_1, I_2 \in \mathbb{C}$):

1. (Binary) Coproduct object $I_1 + I_2 \in \mathbb{C}$,
2. Cotuple $[-,-]_J^{I_1,I_2} \colon \mathbb{C}(I_1, J) \times \mathbb{C}(I_2, J) \to \mathbb{C}(I_1 + I_2, J)$,
3. Injections $\iota_1^{I_1,I_2} \colon I_1 \to I_1 + I_2$ and $\iota_2^{I_1,I_2} \colon I_2 \to I_1 + I_2$.

They satisfy (we omit object annotations):

$$\begin{aligned} h \circ [f_1, f_2] &= [h \circ f_1, h \circ f_2] \\ [f_1, f_2] \circ \iota_i &= f_i \\ [\iota_1, \iota_2] &= \mathrm{id}. \end{aligned}$$

Finite / Small Coproducts on a Category

Cotuple is also a bijection:

$$[-,-]_{J}^{I_1,I_2} \colon \mathbb{C}(I_1, J) \times \mathbb{C}(I_2, J) \cong \mathbb{C}(I_1 + I_2, J)$$

n-ary / small coproducts are defined similarly.

- $\mathrm{Rel}, \mathrm{Mon}, \mathrm{Vect}, \mathrm{Pre}, \mathrm{Top}, \mathrm{Cat}$ have small coproducts.
- On Set, disjoint union gives small coproducts:

$$\coprod \{I_\lambda\}_{\lambda \in \Lambda} = \{(\lambda, x) \mid \lambda \in \Lambda, x \in I_\lambda\}$$

- On Grp (the category of groups and group homs) the free product of groups give small coproducts.

Exponential on a Category with a Binary Product

An exponential on a category $\mathbb{C}$ with a binary product consists of $(I_1, I_2 \in \mathbb{C})$:

1. Exponential object $I_1 \Rightarrow I_2 \in \mathbb{C}$,
2. Currying $\lambda_J^{I_1,I_2}\colon \mathbb{C}(J \times I_1, I_2) \to \mathbb{C}(J, I_1 \Rightarrow I_2)$,
3. Evaluation $ev^{I_1,I_2}\colon I_1 \Rightarrow I_2 \times I_1 \to I_2$.

They satisfy (below $f \times I_1 = \langle f \circ \pi_1, \pi_2 \rangle$):

$$\begin{aligned} \lambda(f) \circ g &= \lambda(f \circ g \times I_1) \\ ev \circ \lambda(f) \times I_1 &= f \\ \lambda(ev) &= id. \end{aligned}$$

Exponential on a Category with a Binary Product

Currying is a bijection:

$$\lambda_J^{I_1,I_2}\colon \mathbb{C}(J \times I_1, I_2) \cong \mathbb{C}(J, I_1 \Rightarrow I_2).$$

Two exponentials on $\mathbb{C}$ give isomorphic exponential objects.

- On Set, $\mathrm{Set}(I_1, I_2)$ gives an exponential.
- On Pre, $(\mathrm{Pre}(I_1, I_2), \leqslant')$ gives an exponential, where $\leqslant'$ is the pointwise order of monotone functions.
- On Cat, functor category gives an exponential.
- $\mathrm{Rel}, \mathrm{Mon}, \mathrm{Vect}, \mathrm{Top}$ do not have exponentials.

(Bi-) Cartesian Closed Category

A category $\mathbb{C}$ with finite products and an exponential is called a cartesian closed category (CCC).

A category $\mathbb{C}$ with finite products, finite coproducts and an exponential is called a bi-cartesian closed category (bi-CCC).

A poset $\mathbb{C}$ equipped with finite products, finite coproducts and an exponential is called a Heyting algebra.

$$\frac{J \leqslant I_1 \ldots J \leqslant I_n}{J \leqslant \bigwedge (I_1, \ldots, I_n)} \qquad \frac{I_1 \leqslant J \ldots I_n \leqslant J}{\bigvee (I_1, \ldots, I_n) \leqslant J} \qquad \frac{I \wedge J \leqslant K}{I \leqslant J \Rightarrow K}$$

Every bi-CCC is Distributive

Proposition 6. *In any bi-CCC, the following is an isomorphism.*

$$[\iota_1 \times J, \ldots, \iota_n \times J] \colon \coprod (I_1, \ldots, I_n) \times J \to \coprod (I_1 \times J, \ldots, I_n \times J)$$

The category with finite products and finite coproducts such that the above morphism is an isomorphism is called distributive.

Corollary 7. *Every bi-CCC is distributive.*

Corollary 8. *Every Heyting algebra is a distributive lattice.*

Categorical Approach to Logic

Lecture 2

Shin-ya Katsumata

Research Institute for Mathematical Sciences, Kyoto University

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Simply Typed Lambda Calculus

We introduce the simply typed lambda calculus (STLC) and the $\beta\eta$-equational theory on it, and relate it to bi-CCCs.

STLC extends the propositional fragment of NJ with a syntactic representation of proofs (called λ-term):

$$\frac{\vdots}{\Gamma \vdash \tau} \quad \Rightarrow \quad \frac{\vdots}{\Gamma \vdash M : \tau}.$$

The $\beta\eta$-equational theory extends the proof reduction $\triangleright$ of NJ to reflect the equality of computational contents of λ-terms / proofs.

Simply Typed Lambda Calculus

We assume:

$$\begin{array}{rcl} \text{variables } x, y, z, \ldots & \in & \mathrm{Var}\ (\text{countably infinite}) \\ \text{base types } b, \ldots & \in & B \end{array}$$

We define the set $\mathrm{Typ}(B)$ of types by

$$\tau ::= b \mid \bigwedge(\tau, \ldots, \tau) \mid \bigvee(\tau, \ldots, \tau) \mid \tau \Rightarrow \tau.$$

We define a context to be a finite sequence

$$x_1 \colon \tau_1, \ldots, x_n \colon \tau_n$$

of variable-type pairs such that variables are all different.

Simply Typed Lambda Calculus

We define λ-terms by

$$
\begin{array}{rcl}
M & ::= & x \\
(\wedge I, \wedge E) & | & (M, \ldots, M) \mid \pi_i^{\tau, \ldots, \tau}(M) \\
(\vee I, \vee E) & | & \iota_i(M) \mid \delta(M, x^\tau.M, \ldots, x^\tau.M) \\
(\Rightarrow I, \Rightarrow E) & | & (\lambda x.M) \mid (M\,M)^\tau
\end{array}
$$

This way of annotating types is non-standard!

Pros: we can easily recover the types lost by elimination rules.

Bound / Free Variables and Substitution

The λ-terms of the form:

$$(\lambda x.M) \quad \text{or} \quad \delta(\ldots, x^{\tau}.M, \ldots)$$

binds any free occurrence of x in M.

λ-terms are identified up to the renaming of bound variables.

$M[N/x]$ is the λ-term obtained by substituting every free occurence of x in M with N.

Before the substitution, we rename bound variables of M as necessary to avoid the capture of free variables in N.

Typing Rules

We inductively define a ternaly relation $\Gamma \vdash M : \tau$ by

$$\frac{\Gamma_i = x : \tau}{\Gamma \vdash x : \tau} \qquad \frac{\Gamma \vdash M_1 : \tau_1 \quad \cdots \quad \Gamma \vdash M_n : \tau_n}{\Gamma \vdash (M_1, \ldots, M_n) : \bigwedge(\tau_1, \ldots, \tau_n)} \qquad \frac{\Gamma \vdash M : \bigwedge(\tau_1, \ldots, \tau_n)}{\Gamma \vdash \pi_i^{\tau_1, \ldots, \tau_n}(M) : \tau_i}$$

$$\frac{\Gamma \vdash M : \tau_i}{\Gamma \vdash \iota_i(M) : \bigvee(\tau_1, \ldots, \tau_n)}$$

$$\frac{\Gamma \vdash M : \bigvee(\tau_1, \ldots, \tau_n) \quad \Gamma, x_1 : \tau_1 \vdash M_1 : \sigma \quad \cdots \quad \Gamma, x_n : \tau_n \vdash M_n : \sigma}{\Gamma \vdash \delta(M, x_1^{\tau_1}.M_1, \ldots, x_n^{\tau_n}.M_n) : \sigma}$$

$$\frac{\Gamma, x : \tau \vdash M : \sigma}{\Gamma \vdash (\lambda x.M) : \tau \Rightarrow \sigma} \qquad \frac{\Gamma \vdash M : \tau \Rightarrow \sigma \quad \Gamma \vdash N : \tau}{\Gamma \vdash (M\,N)^{\tau} : \sigma}$$

Typing Rules

Typing rules correspond to NJ:

$$\frac{\Gamma_i = \tau}{\Gamma \vdash \tau} \qquad \frac{\Gamma \vdash \tau_1 \quad \Gamma \vdash \tau_n}{\Gamma \vdash \bigwedge(\tau_1, \ldots, \tau_n)} \qquad \frac{\Gamma \vdash \bigwedge(\tau_1, \ldots, \tau_n)}{\Gamma \vdash \tau_i}$$

$$\frac{\Gamma \vdash \tau_i}{\Gamma \vdash \bigvee(\tau_1, \ldots, \tau_n)} \qquad \frac{\Gamma \vdash \bigvee(\tau_1, \ldots, \tau_n) \quad \Gamma, \tau_1 \vdash \sigma \cdots \Gamma, \tau_n \vdash \sigma}{\Gamma \vdash \sigma}$$

$$\frac{\Gamma, \tau \vdash \sigma}{\Gamma \vdash \tau \Rightarrow \sigma} \qquad \frac{\Gamma \vdash \tau \Rightarrow \sigma \quad \Gamma \vdash \tau}{\Gamma \vdash \sigma}$$

Typing Rules

Lemma 1. $\Gamma, x : \tau \vdash M : \sigma$ *and* $\Gamma \vdash N : \tau$ *implies* $\Gamma \vdash M[N/x] : \sigma$*.*

Lemma 2. *Each triple* $\Gamma \vdash M : \tau$ *is derived in a unique way.*

βη-Equational Theory on STLC

The βη-equational theory equates the λ-terms that have equal computational contents.

The theory inductively defines a relation:

$$\Gamma \vdash M = N : \tau$$

It subsumes the proof reduction relation of NJ:

$$\Pi \, \frac{\vdots}{\Gamma \vdash \varphi} \quad \rhd \quad \Pi' \, \frac{\vdots}{\Gamma \vdash \varphi}$$

Every $\{(M, N) \mid \Gamma \vdash M = N : \tau\}$ is designed to be an equivalence relation on $\{M \mid \Gamma \vdash M : \tau\}$.

$\beta\eta$-Equational Theory on STLC

1. Equivalence relation

$$\frac{\Gamma \vdash N = M : \tau}{\Gamma \vdash M = N : \tau} \qquad \frac{}{\Gamma \vdash M = M : \tau} \qquad \frac{\Gamma \vdash M = N : \tau \quad \Gamma \vdash N = L : \tau}{\Gamma \vdash M = L : \tau}$$

2. Congruence

$$\frac{\Gamma \vdash M_1 = N_1 : \tau_1 \quad \Gamma \vdash M_n = N_n : \tau_n}{\Gamma \vdash (M_1, \ldots, M_n) = (N_1, \ldots, N_n) : \bigwedge (t_1, \ldots, \tau_n)}$$

$$\frac{\Gamma \vdash M = N : \bigwedge (\tau_1, \ldots, \tau_n)}{\Gamma \vdash \pi_i(M) = \pi_i(N) : \tau_i}$$

etc.

$\beta\eta$-Equational Theory on STLC $(\wedge, \Rightarrow)$

3. $\beta\eta$-equational axioms.

$$\frac{\Gamma \vdash M_1 : \tau_1, \ldots, \Gamma \vdash M_n : \tau_n}{\Gamma \vdash \pi_i^{\tau_1,\ldots,\tau_n}(M_1, \ldots, M_n) = M_i : \tau_i}$$

$$\frac{\Gamma \vdash M : \bigwedge(\tau_1, \ldots, \tau_n)}{\Gamma \vdash (\pi_1^{\tau_1,\ldots,\tau_n}(M), \ldots, \pi_n^{\tau_1,\ldots,\tau_n}(M)) = M : \bigwedge(\tau_1, \ldots, \tau_n)}$$

$$\frac{\Gamma, x : \tau \vdash M : \sigma \quad \Gamma \vdash N : \tau}{\Gamma \vdash ((\lambda x.M)N)^{\tau} = M[N/x] : \sigma} \qquad \frac{\Gamma \vdash M : \tau \Rightarrow \sigma}{\Gamma \vdash (\lambda x.(Mx)^{\tau}) = M : \tau \Rightarrow \sigma}$$

$\beta\eta$-Equational Theory on STLC ($\vee$)

$$\frac{\Gamma \vdash N : \tau_i \quad \Gamma, x_1 : \tau_1 \vdash M_1 : \sigma \quad \ldots \quad \Gamma, x_n : \tau_n \vdash M_n : \sigma}{\Gamma \vdash \delta(\iota_i(N), \ldots, x_i^{\tau_i}.M_i, \ldots) = M_i[N/x] : \sigma}$$

$$\frac{\Gamma \vdash M : \bigvee(\tau_1, \ldots, \tau_n)}{\Gamma \vdash \delta(M, \ldots, x_i^{\tau_i}.\iota_i(x_i), \ldots) = M : \bigvee(\tau_1, \ldots, \tau_n)}$$

4. Commuting Conversion

$$\frac{\Gamma, x : \sigma \vdash N : \rho, \quad \Gamma \vdash M : \bigvee(\tau_1, \ldots, \tau_n), \quad \Gamma, x_i : \tau_i \vdash M_i : \sigma}{\Gamma \vdash N[\delta(M, \ldots, x_i^{\tau_i}.M_i, \ldots)/x] = \delta(M, \ldots, x_i^{\tau_i}.N[M_i/x], \ldots) : \rho}$$

Interpreting STLC in a bi-CCC

Let $\mathbb{C}$ be a bi-CCC.

We interpret types and terms of STLC by means of mathematical (categorical) structures.

- A type τ is interpreted as an object $[\![\tau]\!] \in \mathbb{C}$.
- A judgement $x_1 \colon \tau_1, \ldots, x_n \colon \tau_n \vdash M \colon \tau$ is interpreted as a morphism

$$\prod([\![\tau_1]\!], \ldots, [\![\tau_n]\!]) \to [\![\tau]\!].$$

Interpreting Types

Fix an assignment of $[\![b]\!]_0 \in \mathbb{C}$ to each $b \in B$.

We extend it to an assignment of $[\![\tau]\!] \in \mathbb{C}$ to each $\tau \in \mathrm{Typ}(B)$:

$$
\begin{aligned}
[\![b]\!] &= [\![b]\!]_0 \\
[\![\bigwedge (\tau_1, \ldots, \tau_n)]\!] &= \prod [\![(\tau_1, \ldots, \tau_n)]\!] \\
[\![\bigvee (\tau_1, \ldots, \tau_n)]\!] &= \coprod [\![(\tau_1, \ldots, \tau_n)]\!] \\
[\![\tau \Rightarrow \sigma]\!] &= [\![\tau]\!] \Rightarrow [\![\sigma]\!].
\end{aligned}
$$

We further extend it to contexts:

$$[\![x_1 : \tau_1, \ldots, x_n : \tau_n]\!] = \prod ([\![\tau_1]\!], \ldots, [\![\tau_n]\!]).$$

Interpreting Judgements $(\wedge, \Rightarrow)$

Following the unique derivation of $\Gamma \vdash M : \tau$, we construct a morphism $[\![M]\!]^{\Gamma}_{\tau} : [\![\Gamma]\!] \to [\![\tau]\!]$ as follows:

$$\frac{\Gamma_i = (x, \tau)}{\pi_i : [\![\Gamma]\!] \to [\![\tau_i]\!]}$$

$$\frac{f_1 : [\![\Gamma]\!] \to [\![\tau_1]\!] \quad f_n : [\![\Gamma]\!] \to [\![\tau_n]\!]}{\langle f_1, \ldots, f_n \rangle : [\![\Gamma]\!] \to [\![\bigwedge(\tau_1, \ldots, \tau_n)]\!]}, \qquad \frac{f : [\![\Gamma]\!] \to [\![\bigwedge(\tau_1, \ldots, \tau_n)]\!]}{\pi_i \circ f : [\![\Gamma]\!] \to [\![\tau_i]\!]}$$

$$\frac{f : [\![\Gamma, x : \tau]\!] \to [\![\sigma]\!]}{\lambda(f \circ a) : [\![\Gamma]\!] \to [\![\tau \Rightarrow s]\!]} \qquad \frac{f : [\![\Gamma]\!] \to [\![\tau \Rightarrow \sigma]\!] \quad g : [\![\Gamma]\!] \to [\![\tau]\!]}{ev \circ \langle f, g \rangle : [\![\Gamma]\!] \to [\![\sigma]\!]}$$

Here $a : [\![\Gamma]\!] \times [\![\tau]\!] \to [\![\Gamma, x : \tau]\!]$ is an iso.

Interpreting Judgements ($\vee$)

$$\frac{f\colon [\![\Gamma]\!] \to [\![\tau_i]\!]}{\iota_i \circ f\colon [\![\Gamma]\!] \to [\![\bigvee(\tau_1, \ldots, \tau_n)]\!]}$$

$$\frac{f\colon [\![\Gamma]\!] \to [\![\bigvee(\tau_1, \ldots, \tau_n)]\!] \qquad [\![g_i]\!]\colon [\![\Gamma, x_i\colon \tau_i]\!] \to [\![\sigma]\!]}{[g_1, \ldots, g_n] \circ d \circ \langle \mathrm{id}_{[\![\Gamma]\!]}, f\rangle\colon [\![\Gamma]\!] \to [\![\sigma]\!]}$$

The composite is the following diagram:

$$[\![\Gamma]\!] \to [\![\Gamma]\!] \times [\![\bigvee(\tau_1, \ldots, \tau_n)]\!] \to \coprod [\![\Gamma, x_i\colon \tau_i]\!] \to [\![\sigma]\!].$$

Soundnes and Completeness

Theorem 3. *Suppose* $\Gamma \vdash M : \tau$ *and* $\Gamma \vdash N : \tau$. *TFAE:*

1. $\Gamma \vdash M = N : \tau$.

2. *For any bi-CCC* $\mathbb{C}$ *and an assignment* $[\![-]\!]_0$ *of an* $\mathbb{C}$*-object to each* $b \in B$, *the induced* $[\![-]\!]$ *satisfies*

$$[\![M]\!]^{\Gamma}_{\tau} = [\![N]\!]^{\Gamma}_{\tau} : [\![\Gamma]\!] \to [\![\tau]\!].$$

Souneness (1⇒2) is easy. The converse is in the next slide.

Completeness

Fix a distinguished $\Box \in \mathrm{Var}$. We define the category ΛB by

$$\Lambda B_0 = \mathrm{Typ}(B), \quad \Lambda B(\tau, \sigma) = \{M^= \mid \Box : \tau \vdash M : \sigma\}.$$

Here, $M^=$ is the equivalence class of M by $\{(M, N) \mid \Box : \tau \vdash M = N : \sigma\}$. The composition is substitution: $N^= \circ M^= = (N[M/\Box]^=)$.

Proposition 4. ΛB *is a bi-CCC.*

(Sketch of 2⇒1) Let $\Gamma = x_1 : \tau_1, \ldots, x_n : \tau_n$. Show that the interpretation in ΛB with $[\![b]\!]_0 = b$ satisfies $[\![M]\!]^\Gamma_\sigma = (M[\pi_i \Box / x_i])^=$. Then

$$\begin{aligned} [\![M]\!]^\Gamma_\sigma = [\![N]\!]^\Gamma_\sigma \;&\Rightarrow\; \Box : \bigwedge (\tau_1, \ldots, \tau_n) \vdash M[\pi_i \Box / x_i] = N[\pi_i \Box / x_i] : \sigma \\ &\Rightarrow\; \Gamma \vdash M = N : \sigma. \end{aligned}$$

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Shin-ya Katsumata

Research Institute for Mathematical Sciences, Kyoto University

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Functor

A functor F from $\mathbb{C}$ to $\mathbb{D}$ ($F\colon \mathbb{C} \to \mathbb{D}$) consists of:

1. A mapping $F_0\colon \mathbb{C}_0 \to \mathbb{D}_0$
2. A mapping $F_1\colon \mathbb{C}_1 \to \mathbb{D}_1$.

Both are written F for simplicity. They satisfy:

$$f\colon I \to J \Rightarrow Ff\colon FI \to FJ, \quad F(\mathrm{id}_I) = \mathrm{id}_{FI}, \quad F(g \circ f) = Fg \circ Ff.$$

Thus F_1 can be restricted to

$$F_{I,J}\colon \mathbb{C}(I,J) \to \mathbb{D}(FI,FJ).$$

Examples of Functors

Forgetful functor $\mathsf{U}\colon \mathrm{Mon}, \mathrm{Vect}, \mathrm{Top}, \mathrm{Pre} \to \mathrm{Set}$

Free construction $\mathsf{F}\colon \mathrm{Set} \to \mathrm{Mon}, \mathrm{Vect}$

Functors $\mathrm{Eq}, \mathrm{Ch}\colon \mathrm{Set} \to \mathrm{Pre}$:

$$\mathrm{Eq}\,\mathrm{I} = (\mathrm{I}, \{(i, i) \mid i \in \mathrm{I}\}), \qquad \mathrm{Ch}\,\mathrm{I} = (\mathrm{I}, \mathrm{I} \times \mathrm{I}).$$

Functors $\mathrm{D}, \mathrm{iD}\colon \mathrm{Set} \to \mathrm{Top}$:

$$\mathrm{D}\,\mathrm{I} = (\mathrm{I}, 2^{\mathrm{I}}), \qquad \mathrm{iD} = (\mathrm{I}, \{\emptyset, \mathrm{I}\}).$$

The Category of Small Categories

For $\mathsf{F}\colon \mathbb{C} \to \mathbb{D}$ and $\mathsf{G}\colon \mathbb{D} \to \mathbb{E}$, we define $\mathsf{G} \circ \mathsf{F}\colon \mathbb{C} \to \mathbb{E}$ by

$$(\mathsf{G} \circ \mathsf{F})_0 \mathrm{I} = \mathsf{G}_0(\mathsf{F}_0 \mathrm{I}), \quad (\mathsf{G} \circ \mathsf{F})_1 \mathsf{f} = \mathsf{G}_1(\mathsf{F}_1 \mathsf{f}).$$

This is indeed a functor. We define the category Cat by

$$\mathrm{Cat}_0 = \{\text{small cats.}\} \quad \mathrm{Cat}(\mathbb{C}, \mathbb{D}) = \{\mathsf{F}\colon \mathbb{C} \to \mathbb{D} \text{ functor}\}.$$

Some related functors:

- $\mathrm{C}\colon \mathrm{Pre}, \mathrm{Mon} \to \mathrm{Cat}$.
- $\mathrm{Obj}\colon \mathrm{Cat} \to \mathrm{Set}$ mapping $\mathbb{C}$ to $\mathbb{C}_0$.

Adjunction

An adjunction consists of the following data:

1. Functors $L\colon \mathbb{C} \to \mathbb{D}$ and $R\colon \mathbb{D} \to \mathbb{C}$.

2. A bijection for each $I \in \mathbb{C}$, $J \in \mathbb{D}$:

$$\varphi_{I,J}\colon \mathbb{D}(LI, J) \cong \mathbb{C}(I, RJ)$$

They satisfy:

$$\varphi(h \circ g \circ Lf) = Rh \circ \varphi(g) \circ f.$$

The notation for an adjunction is $(L, R, \varphi)\colon \mathbb{C} \to \mathbb{D}$.

Adjunction

$L \dashv R$ ... $\exists \varphi.(L, R, \varphi)\colon \mathbb{C} \to \mathbb{D}$ is an adjunction

$L\colon \mathbb{C} \to \mathbb{D}$ is a left adjoint ... $\exists R\colon \mathbb{D} \to \mathbb{C}. L \dashv R$

$R\colon \mathbb{D} \to \mathbb{C}$ is a right adjoint ... $\exists L\colon \mathbb{C} \to \mathbb{D}. L \dashv R$

$\eta_I = \varphi(id_{LI})\colon I \to RLI$ ... the unit of $(L, R, \varphi)\colon \mathbb{C} \to \mathbb{D}$

$\varepsilon_I = \varphi^{-1}(id_{RI})\colon LRI \to I$ ... the counit of $(L, R, \varphi)\colon \mathbb{C} \to \mathbb{D}$

Lemma 1. *If* (L, R, φ) *and* (L, R', φ') *are adjunctions then for every* $I \in \mathbb{D}$, $RI \cong R'I$ *and this isomorphism is natural on* I.

Representation of $\mathbb{D}(\mathrm{L}\,-,\mathrm{I})$

Let $\mathrm{I} \in \mathbb{D}$ and $\mathrm{L}\colon \mathbb{C} \to \mathbb{D}$.

A representation of $\mathbb{D}(\mathrm{L}\,-,\mathrm{I})$ consists of:

1. Representing object $\mathrm{R} \in \mathbb{C}$,
2. Counit $\varepsilon\colon \mathrm{L}\,\mathrm{R} \to \mathrm{I}$.

They satisfy: the following is a bijection for all $\mathrm{J} \in \mathbb{C}$.

$$\varepsilon \circ \mathrm{L}\,-\colon \mathbb{C}(\mathrm{J},\mathrm{R}) \to \mathbb{D}(\mathrm{L}\,\mathrm{J},\mathrm{I})$$

Representation of $\mathbb{D}(L-, I)$ with Inverse

Let $I \in \mathbb{D}$ and $L \colon \mathbb{C} \to \mathbb{D}$.

A representation of $\mathbb{D}(L-, I)$ with inverse consists of:

1. Representing object $R \in \mathbb{C}$,
2. A mapping $\varphi_J \colon \mathbb{D}(LJ, I) \to \mathbb{C}(J, R)$,
3. Counit $\varepsilon \colon LR \to I$.

They satisfy:

$$\varphi_J(f \circ Lg) = \varphi_K(f) \circ g, \quad \varepsilon \circ L(\varphi_J(f)) = f, \quad \varphi_R(\varepsilon) = \mathrm{id}_R.$$

(this is a local terminology in this document)

Universal Arrow

Let $I \in \mathbb{D}$ and $L \colon \mathbb{C} \to \mathbb{D}$.

A universal arrow from L to I consists of:

1. $R \in \mathbb{C}$,

2. $\varepsilon \colon L R \to I$.

They satisfy: for all $J \in \mathbb{C}$ and $f \colon L J \to I$, there exists a unique $m \colon J \to R$ such that $\varepsilon \circ L m = f$.

Representation of $\mathbb{D}(\mathrm{L}-,\mathrm{I}) \cong$ Universal Arrow

Theorem 2. *Let* $\mathrm{I} \in \mathbb{D}$ *and* $\mathrm{L}\colon \mathbb{C} \to \mathbb{D}$*. There is a bijective correspondence between:*

1. *A representation* $(\mathrm{R}, \varepsilon)$ *of* $\mathbb{D}(\mathrm{L}-,\mathrm{I})$*.*
2. *A representation* $(\mathrm{R}, \varphi, \varepsilon)$ *of* $\mathbb{D}(\mathrm{L}-,\mathrm{I})$ *with inv.*
3. *A universal arrow* $(\mathrm{R}, \varepsilon)$ *from* L *to* I*.*

Representation of $\mathbb{C}(I, R-) \cong$ Universal Arrow

Let $I \in \mathbb{C}$ and $R\colon \mathbb{D} \to \mathbb{C}$. A pair $(L \in \mathbb{D}, \eta\colon I \to RL)$ is:

1. a representation of $\mathbb{C}(I, R-)$ if $R - \circ\eta\colon \mathbb{D}(L, J) \to \mathbb{C}(I, RJ)$ is a bijection for every $J \in \mathbb{D}$.

2. a universal arrow from I to R if for every $J \in \mathbb{D}$ and $f\colon I \to RJ$, there exists a unique $m\colon L \to J$ such that $Rm \circ \eta = f$.

Theorem 3. *Representations of $\mathbb{C}(I, R-)$ bijectively correspond to universal arrows from I to R.*

Adjunction = Representations of all $\mathbb{D}(\mathrm{L}-, \mathrm{I})$

Theorem 4. *Let* $\mathrm{L}\colon \mathbb{C} \to \mathbb{D}$*. There is a bijective correspondence between:*

1. *An assignment of a representation* $(\mathrm{R}\,\mathrm{I}, \varphi, \varepsilon^{\mathrm{I}})$ *of* $\mathbb{D}(\mathrm{L}-, \mathrm{I})$ *with inv. to each* $\mathrm{I} \in \mathbb{D}$*.*

2. *An adjunction* $(\mathrm{L}, \mathrm{R}, \varphi)\colon \mathbb{C} \to \mathbb{D}$*.*

A Binary Product on $\mathbb{C}$ = Reps. of all $\mathbb{C}^2(\Delta -, \mathrm{I})$

Define $\Delta\colon \mathbb{C} \to \mathbb{C}^2$ by $\Delta \mathrm{I} = (\mathrm{I}, \mathrm{I})$ and $\Delta \mathrm{f} = (\mathrm{f}, \mathrm{f})$.

Suppose that $\mathbb{C}$ has a binary product. Then every

$$(\mathrm{I}_1 \times \mathrm{I}_2, \langle -, - \rangle^{\mathrm{I}_1, \mathrm{I}_2}, (\pi_1^{\mathrm{I}_1, \mathrm{I}_2}, \pi_2^{\mathrm{I}_1, \mathrm{I}_2}))$$

is a representation of $\mathbb{C}^2(\Delta -, (\mathrm{I}_1, \mathrm{I}_2))$ with inv.

Theorem 5. *A binary product on $\mathbb{C}$ is exactly an assignment of a representation of $\mathbb{C}^2(\Delta -, \mathrm{I})$ (with inv.) to each $\mathrm{I} \in \mathbb{C}^2$.*

Theorem 6. *A binary product on $\mathbb{C}$ bijectively corresponds to an adjunction $(\Delta, \times, \varphi)\colon \mathbb{C} \to \mathbb{D}$.*

The Adjunction $- \times \mathrm{I} \dashv \mathrm{I} \Rightarrow -$

Suppose that $\mathbb{C}$ has a binary product $\times \colon \mathbb{C}^2 \to \mathbb{C}$ and an exponential. Every $\mathrm{I} \in \mathbb{C}$ determines $- \times \mathrm{I} \colon \mathbb{C} \to \mathbb{C}$. Then

$$(\mathrm{I}_1 \Rightarrow \mathrm{I}_2, \lambda^{\mathrm{I}_1,\mathrm{I}_2}, \mathrm{ev}^{\mathrm{I}_1,\mathrm{I}_2})$$

is exactly a representation of $\mathbb{D}(- \times \mathrm{I}_1, \mathrm{I}_2)$ with inv.

Theorem 7. *Let $\mathbb{C}$ be a category with a binary product. An exponential on $\mathbb{C}$ is exactly an assignment of a representation of $\mathbb{D}(- \times \mathrm{I}_1, \mathrm{I}_2)$ (with inv.) to each $\mathrm{I}_1, \mathrm{I}_2 \in \mathbb{C}$.*

Theorem 8. *An exponential on $\mathbb{C}$ consists of an adjunction $(- \times \mathrm{I}, \mathrm{I} \Rightarrow -, \lambda^{\mathrm{I}}) \colon \mathbb{C} \to \mathbb{C}$ for every $\mathrm{I} \in \mathbb{C}$.*

Strict Map of Adjunction

Let $(L, R, \varphi)\colon \mathbb{C} \to \mathbb{D}$ and $(L', R', \varphi')\colon \mathbb{C}' \to \mathbb{D}'$ be adjunctions. A strict map of adjunctions consists of

1. $F\colon \mathbb{C} \to \mathbb{C}'$ and $G\colon \mathbb{D} \to \mathbb{D}'$

such that

$$G \circ L = L' \circ F, \quad F \circ R = R' \circ G, \quad F\varphi(f) = \varphi'(G f).$$

Strict Preservation of bi-CC structure

Let $\mathbb{C}$, $\mathbb{D}$ be bi-CCCs. We say that $F\colon \mathbb{C} \to \mathbb{D}$ strictly preserves the bi-CC structure if the following are strict map of adjunctions.

$$(F, F^n)\colon (\Delta^{\mathbb{C}}, \prod{}^{\mathbb{C}}, \varphi^{\mathbb{C}}) \to (\Delta^{\mathbb{D}}, \prod{}^{\mathbb{D}}, \varphi^{\mathbb{D}})$$

$$(F^n, F)\colon (\coprod{}^{\mathbb{C}}, \Delta^{\mathbb{C}}, \varphi^{\mathbb{C}}) \to (\coprod{}^{\mathbb{D}}, \Delta^{\mathbb{D}}, \varphi^{\mathbb{D}})$$

$$(F, F)\colon (- \times I, I \Rightarrow -, \lambda^{I}) \to (- \times FI, FI \Rightarrow -, \lambda^{FI}) \quad (I \in \mathbb{C})$$

Another way to put this: F commutes with $\prod, \coprod, \Rightarrow$.

The Category of bi-CCCs

We define $\mathrm{biCCCat}$ by

$$(\mathrm{biCCCat})_0 = \{\mathbb{C} \text{ small bi-CCC}\}$$
$$\mathrm{biCCCat}(\mathbb{C}, \mathbb{D}) = \{\mathsf{F}\colon \mathbb{C} \to \mathbb{D} \text{ strictly preserve bi-CC struct.}\}$$

We write $\mathrm{Obj}\colon \mathrm{biCCCat} \to \mathrm{Set}$ extracting the object part.

For instance, $\Lambda\mathrm{B} \in \mathrm{biCCCat}$, and $\mathrm{Obj}(\Lambda\mathrm{B}) = \mathrm{Typ}(\mathrm{B})$.

Universal Property of $\Lambda\mathrm{B}$

Let $\mathbb{C} \in \mathrm{biCCCat}$ and $[\![-]\!]_0 \colon \mathrm{B} \to \mathrm{Obj}(\mathbb{C})$. We write $[\![-]\!]$ for the interpretation of the STLC over B obtained from $[\![-]\!]_0$.

1. $[\![-]\!]$ extends to a morphism in $\mathrm{biCCCat}(\Lambda\mathrm{B}, \mathbb{C})$.

2. $\mathrm{Obj}([\![-]\!]) \circ \eta_{\mathrm{B}} \colon \mathrm{B} \hookrightarrow \mathrm{Typ}\,\mathrm{B} \to \mathrm{Obj}(\mathbb{C}) = [\![-]\!]_0$.

3. If $\mathsf{F} \colon \Lambda\mathrm{B} \to \mathbb{C}$ satisfies $\mathrm{Obj}(\mathsf{F}) \circ \eta_{\mathrm{B}} = [\![-]\!]_0$, then $\mathsf{F} = [\![-]\!]$.

Thus $(\Lambda\mathrm{B}, \eta_{\mathrm{B}} \colon \mathrm{B} \hookrightarrow \mathrm{Typ}(\mathrm{B}))$ is a universal arrow from B to Obj.

Theorem 9. *We have* $\Lambda \dashv \mathrm{Obj}$*.*

STLC over B / $\beta\eta$-equality = free bi-CCC over B

Universal Property of $\Lambda \mathrm{B}$

Though we characterised Λ as a left adjoint to $\mathrm{Obj}\colon \mathrm{biCCCat} \to \mathrm{Set}$, $(\Lambda \mathrm{B}, \eta_{\mathrm{B}}\colon \mathrm{B} \hookrightarrow \mathrm{Typ}(\mathrm{B}))$ satisfies the universal property for large bi-CCCs as well.

Theorem 10. *For any bi-CCC* $\mathbb{C}$ *and an assignment of* $[\![\mathrm{b}]\!]_0 \in \mathbb{C}$ *to each* $\mathrm{b} \in \mathrm{B}$*,* $[\![-]\!]\colon \Lambda \mathrm{B} \to \mathbb{C}$ *is the unique functor that strictly preserves the bi-CC structure and that satisfies* $[\![\mathrm{b}]\!] = [\![\mathrm{b}]\!]_0$*.*

Categorical Approach to Logic

Lecture 4

Shin-ya Katsumata

Research Institute for Mathematical Sciences, Kyoto University

Kisoron Summer School, August 2013

Intuitionistic First-Order Predicate Logic NJ

Each NJ is built on a language $\Sigma, \Pi \colon \mathbb{N} \to \mathrm{Set}$:

- Σn ... the set of n-ary function symbols
- Πn ... the set of n-ary predicate symbols

For instance, the NJ built on the language

$$\Sigma 0 = \{z\}, \quad \Sigma 1 = \{s\}, \quad \Pi 2 = \{\leqslant\}, \quad \text{otherwise } \Sigma n = \Pi n = \emptyset$$

can be used to talk about natural numbers and their inequality.

Intuitionistic First-Order Predicate Logic NJ

The NJ over a language $L = (\Sigma, \Pi)$ specifies three concepts:

1. (L-)Terms $x_1, \ldots, x_l \vdash t$.

2. (L-)Formulas $x_1, \ldots, x_l \vdash \varphi$.

3. Sequents $x_1, \ldots, x_l \mid \varphi_1, \ldots, \varphi_m \Rightarrow \varphi$.

The leading part $x_1, \ldots, x_l$ (context) is a finite sequence of distinct variables.

Contexts are ranged over by Δ.

$\mathrm{NJ}^{\Sigma,\Pi}$

Terms

$$\frac{\Delta_i = x}{\Delta \vdash x} \qquad \frac{\Delta \vdash t_1 \quad \cdots \quad \Delta \vdash t_n \quad f \in \Sigma n}{\Delta \vdash f(t_1, \ldots, t_n)}$$

Formulas

$$\frac{\Delta \vdash t_1 \quad \cdots \quad \Delta \vdash t_n \quad p \in \Pi n}{\Delta \vdash p(t_1, \ldots, t_n)} \qquad \frac{\Delta, x \vdash \varphi}{\Delta \vdash \forall x.\varphi} \qquad \frac{\Delta, x \vdash \varphi}{\Delta \vdash \exists x.\varphi}$$

and the rules for $\bigwedge, \bigvee, \Rightarrow$.

$NJ^{\Sigma,\Pi}$

Inference Rules ($\Gamma = \varphi_1, \ldots, \varphi_n$)

$$\frac{\Delta, x \mid \Gamma \Rightarrow \varphi \quad \Delta \vdash \varphi_1 \quad \ldots \quad \Delta \vdash \varphi_n}{\Delta \mid \Gamma \Rightarrow \forall x.\varphi} \qquad \frac{\Delta \mid \Gamma \Rightarrow \forall x.\varphi \quad \Delta \vdash t}{\Delta \mid \Gamma \Rightarrow \varphi[t/x]}$$

$$\frac{\Delta \mid \Gamma \Rightarrow \varphi[t/x]}{\Delta \mid \Gamma \Rightarrow \exists x.\varphi} \qquad \frac{\Delta \mid \Gamma \Rightarrow \exists x.\varphi \quad \Delta, x \mid \Gamma, \varphi \Rightarrow \varphi' \quad \Delta \vdash \varphi'}{\Delta \mid \Gamma \Rightarrow \varphi'}$$

(red part: eigenvariable condition)

and the rules for $\bigwedge, \bigvee, \Rightarrow$.

Lawvere's Hyperdoctrine

$$|-| \circ \mathrm{p} \colon \mathbb{B}^{\mathrm{op}} \to \mathrm{Mod}_{\mathrm{L}} \to \mathrm{Cat}$$

- $\mathrm{Mod}_{\mathrm{L}}$ is the category of models of propositional L-logic.
- $\mathbb{B}$ is the category interpreting terms / substitutions.
- p assigns to $\mathrm{I} \in \mathbb{B}$ the poset $\mathrm{p}\,\mathrm{I}$ of predicates over I, which has the structure of L-model.
- p captures the situation that term substitutions contravariantly act on predicates.
- $|-| \circ \mathrm{p}$ satisfy certain properties for modelling universal / existensial quantification.

Hyperdoctrine for NJ

We employ $|-|\colon \mathrm{Heyt} \to \mathrm{Cat}$ for hyperdoctrines for NJ.

Definition 1. *A hyperdctorine for NJ consists of:*

1. *a category* $\mathbb{B}$ *with finite products,*
2. *a functor* $\mathrm{p}\colon \mathbb{B}^{\mathrm{op}} \to \mathrm{Heyt}$*.*

They satisfy:

1. *for each projection* $\pi_1^{\mathrm{I,J}}\colon \mathrm{I} \times \mathrm{J} \to \mathrm{I}$ *in* $\mathbb{B}$*,* $|\mathrm{p}\,\pi_1^{\mathrm{I,J}}|\colon |\mathrm{p}\,\mathrm{I}| \to |\mathrm{p}(\mathrm{I} \times \mathrm{J})|$ *has the left adjoint* $\exists^{\mathrm{I,J}}$ *and right adjoint* $\forall^{\mathrm{I,J}}$*,*
2. *these adjoints satisfy Beck-Chevalley condition.*

Example: Powerset Hyperdoctrine

We define $2^{-}\colon \mathrm{Set}^{\mathrm{op}} \rightarrow \mathrm{Heyt}$ by

$$\mathrm{I} \mapsto (2^{\mathrm{I}}, \subseteq), \quad (\mathrm{f}\colon \mathrm{I} \mapsto \mathrm{J}) \mapsto (\mathrm{f}^{-1}\colon 2^{\mathrm{J}} \rightarrow 2^{\mathrm{I}}).$$

Theorem 2. *The above functor is a hyperdctorine for NJ.*

The left / right adjoints $\exists^{\mathrm{I},\mathrm{J}} \dashv |2^{\pi_1^{\mathrm{I},\mathrm{J}}}| \dashv \forall^{\mathrm{I},\mathrm{J}}$ are

$$\forall^{\mathrm{I},\mathrm{J}} X = \{ i \in \mathrm{I} \mid \forall j.(i,j) \in X \}, \quad \exists^{\mathrm{I},\mathrm{J}} X = \{ i \in \mathrm{I} \mid \exists j.(i,j) \in X \}.$$

Example: Heyting-Valued Hyperdoctrine

Let H be a complete Heyting algebra (cHA). The set of functions $\mathrm{I} \Rightarrow \mathrm{H}$ is also a cHA by the pointwise $\bigwedge, \bigvee, \Rightarrow$.

We define $- \Rightarrow \mathrm{H} \colon \mathrm{Set}^{\mathrm{op}} \to \mathrm{Heyt}$ by

$$\mathrm{I} \mapsto \mathrm{I} \Rightarrow \mathrm{H}, \quad (f \colon \mathrm{I} \to \mathrm{J}) \mapsto (- \circ f \colon \mathrm{J} \Rightarrow \mathrm{H} \to \mathrm{I} \Rightarrow \mathrm{H}).$$

Theorem 3. *The above functor is a hyperdctorine for NJ.*

The left / right adjoints $\exists^{\mathrm{I},\mathrm{J}} \dashv |\pi_1^{\mathrm{I},\mathrm{J}} \Rightarrow \mathrm{H}| \dashv \forall^{\mathrm{I},\mathrm{J}}$ are

$$(\forall^{\mathrm{I},\mathrm{J}} X)(i) = \bigwedge_{j \in \mathrm{J}} X(i,j), \quad (\exists^{\mathrm{I},\mathrm{J}} X)(i) = \bigvee_{j \in \mathrm{J}} X(i,j).$$

Base category of Hyperdoctrine of $\mathrm{NJ}^{\Sigma,\Pi}$

We fix an enumeration $\overbrace{v_1, \ldots, v_l}^{\Delta_l}, \ldots$ of variables.

Define $\mathbb{B}_\Sigma$ by:

$$(\mathbb{B}_\Sigma)_0 = \mathrm{N}, \quad \mathbb{B}_\Sigma(l, m) = \{(t_1, \ldots, t_m) \mid \Delta_l \vdash t_i\}$$

$$\mathrm{id}_l = (v_1, \ldots, v_l), \quad (s_1, \ldots, s_n) \circ (t_1, \ldots, t_m) = (s_1[t_i/v_i], \ldots, s_n[t_i/v_i])$$

For $t\colon l \to m$, we write t^* for the substitution

$$-[t_1/v_1, \ldots, t_m/v_m]$$

to terms / formulas.

Proposition 4. $\mathbb{B}_\Sigma$ *has finite products.*

Some Lemmas

For any $f\colon \mathfrak{l} \to \mathfrak{m}$ in $\mathbb{B}_\Sigma$,

1. $\Delta_\mathfrak{m} \vdash \varphi$ implies $\Delta_\mathfrak{l} \vdash t^*\varphi$.

2. $\Delta_\mathfrak{m} \,|\, \varphi \Rightarrow \varphi'$ implies $\Delta_\mathfrak{l} \,|\, t^*\varphi \Rightarrow t^*\varphi'$.

3. t^* commutes with $\bigwedge, \bigvee, \to$, that is,

$$
\begin{aligned}
t^*\Big(\bigwedge(\varphi_1, \ldots, \varphi_\mathfrak{l})\Big) &= \bigwedge(t^*\varphi_1, \ldots, t^*\varphi_\mathfrak{l}) \\
t^*\Big(\bigvee(\varphi_1, \ldots, \varphi_\mathfrak{l})\Big) &= \bigvee(t^*\varphi_1, \ldots, t^*\varphi_\mathfrak{l}) \\
t^*(\varphi_1 \to \varphi_2) &= t^*\varphi_1 \to t^*\varphi_2.
\end{aligned}
$$

Functor of Hyperdoctrine of $\mathrm{NJ}^{\Sigma,\Pi}$

For $l \in \mathbb{B}_\Sigma$, define $q_{\Sigma,\Pi} l$ by

$$q_{\Sigma,\Pi} l = (\{\varphi \mid \Delta_l \vdash \varphi\}, \quad \{(\varphi, \varphi') \mid \Delta_l \mid \varphi \vdash \varphi'\}).$$

We write $\lesssim_l$ for the preorder part of $q_{\Sigma,\Pi} l$.

Proposition 5. *Each $q_{\Sigma,\Pi} l$ is a bi-cartesian closed preorder.*

For $t \colon l \to m$ in $\mathbb{B}_\Sigma$, define $q_{\Sigma,\Pi} t \colon q_{\Sigma,\Pi} m \to q_{\Sigma,\Pi} l$ by

$$q_{\Sigma,\Pi} t(\varphi) = t^* \varphi.$$

Proposition 6. *Each $q_{\Sigma,\Pi} t$ is a functor that strictly preserves the bi-cartesian closed structure.*

Functor of Hyperdoctrine of $\mathrm{NJ}^{\Sigma,\Pi}$

The quotient of $\mathfrak{q}_{\Sigma,\Pi}\mathfrak{l}$ by $\lesssim_{\mathfrak{l}} \cap \gtrsim_{\mathfrak{l}}$ is a Heyting algebra, say $\mathfrak{p}_{\Sigma,\Pi}\mathfrak{l}$.

Theorem 7. *The assignment* $\mathfrak{l} \mapsto \mathfrak{p}_{\Sigma,\Pi}\mathfrak{l}$ *extends to a functor*

$$\mathfrak{p}_{\Sigma,\Pi} \colon \mathbb{B}_{\Sigma}^{\mathrm{op}} \to \mathrm{Heyt}.$$

Below we identify a formula $\Delta_{\mathfrak{l}} \vdash \varphi$ and its equivalence class by $\lesssim_{\mathfrak{l}} \cap \gtrsim_{\mathfrak{l}}$.

We often write $\mathfrak{p}$ for $\mathfrak{p}_{\Sigma,\Pi}$.

Adjoints of Hyperdoctrine of $\mathrm{NJ}^{\Sigma,\Pi}$

The projection $\pi_1^{l,m}$ in $\mathbb{B}_\Sigma$ is $(\nu_1, \ldots, \nu_l)$.

$$|p\pi_1^{l,m}| \colon |p\,l| \to |p(l+m)|, \quad |p\pi_1^{l,m}|(\varphi) = (\pi_1^{l,m})^* \varphi = \varphi.$$

In $\mathrm{NJ}^{\Sigma,\Pi}$ we have

$$\Delta_l \,|\, \varphi \vdash \forall \nu_{l+1}.\varphi' \iff \Delta_{l+1} \,|\, \varphi = |p\pi_1^{l,1}|(\varphi) \vdash \varphi'$$
$$\Delta_l \,|\, \exists \nu_{l+1}.\varphi \vdash \varphi' \iff \Delta_{l+1} \,|\, \varphi \vdash \varphi' = |p\pi_1^{l,1}|(\varphi')$$

Corollary 8. $(\exists \nu_{l+1}. -) \dashv |p\pi_1^{l,1}| \dashv (\forall \nu_{l+1}. -)$.

Universal / existential quantification $\Leftrightarrow$ R / L adjoint of $p(\pi)$

Beck-Chevalley Condition

The right adjoint $\forall^{I,J}$ of $|p\,\pi_1^{I,J}|\colon |p\,I| \longrightarrow |p(I \times J)|$ satifies Beck-Chevalley condition if for any $f\colon I \to K$,

$$|p\,f| \circ \forall^{K,J} = \forall^{I,J} \circ |p(f \times J)|.$$

Similarly, the left adjoint $\exists^{I,J}$ of $|p\,\pi_1^{I,J}|\colon |p\,I| \longrightarrow |p(I \times J)|$ satifies Beck-Chevalley condition if for any $f\colon I \to K$,

$$|p\,f| \circ \exists^{K,J} = \exists^{I,J} \circ |p(f \times J)|.$$

Beck-Chevalley $\Leftrightarrow$ Quantifier commutes with substitution

(They are a special version of a more general BC condition.)

Beck-Chevalley Condition

For $t = (t_1, \ldots, t_m) \colon l \to m$ in $\mathbb{B}_\Sigma$,

$$t \times 1 = (t_1, \ldots, t_m, v_{l+1}), \qquad |p(t \times 1)|(\varphi) = \varphi[t_i/v_i, v_{l+1}/v_{m+1}].$$

Theorem 9. $\forall v_{m+1}. -$ *and* $\exists v_{m+1}. -$ *satisfy Beck-Chevalley conditions.*

For any $t \colon l \to m$ in $\mathbb{B}_\Sigma$,

$$\begin{aligned}
|p\,t| \circ (\forall v_{m+1}. -)(\varphi) &= (\forall v_{m+1}.\varphi)[t_i/v_i] \\
&= (\forall v_{l+1}.\varphi[v_{l+1}/v_{m+1}])[t_i/v_i] \\
&= (\forall v_{l+1}.\varphi[t_i/v_i, v_{l+1}/v_{m+1}]) \\
&= (\forall v_{l+1}. -) \circ |p(t \times 1)|(\varphi).
\end{aligned}$$

The Hyperdoctrine of $\mathrm{NJ}^{\Sigma,\Pi}$

Theorem 10. *The functor* $\mathrm{p}_{\Sigma,\Pi}\colon \mathbb{B}_{\Sigma}^{\mathrm{op}} \to \mathrm{Heyt}$ *constructed from* $\mathrm{NJ}^{\Sigma,\Pi}$ *is a hyperdctorine for NJ.*

Structure of NJ

Hyperdoctrines for NJ captures the structures that are common among all NJs.

Each $\mathrm{NJ}^{\Sigma,\Pi}$ is interpreted using a Σ, Π-structure.

Definition 11. *Let* $\mathrm{p}\colon \mathbb{B}^{\mathrm{op}} \to \mathrm{Heyt}$ *be a hyperdoctrine for NJ. A* Σ, Π*-structure in* p *consists of*

1. *a universe* $\mathrm{U} \in \mathbb{B}$*,*
2. *a morphism* $[\![\mathrm{f}]\!]\colon \mathrm{U}^{\mathrm{n}} \to \mathrm{U}$ *for every* $\mathrm{f} \in \Sigma\mathrm{n}$*,*
3. *an object* $[\![\mathrm{p}]\!] \in \mathrm{p}(\mathrm{U}^{\mathrm{n}})$ *for every* $\mathrm{p} \in \Pi\mathrm{n}$*.*

Interpretation of $\mathrm{NJ}^{\Sigma,\Pi}$

The following is the principle of the interpretation of $\mathrm{NJ}^{\Sigma,\Pi}$:

$$
\begin{array}{rcl}
x_1, \cdots, x_l \vdash t & \ldots & [\![t]\!] : U^l \to U \\
x_1, \cdots, x_l \vdash \varphi & \ldots & [\![\varphi]\!] \in \mathrm{p}\,U^l \\
x_1, \ldots, x_l \mid \varphi_1, \ldots, \varphi_m \Rightarrow \varphi & \ldots & \bigwedge([\![\varphi_1]\!], \ldots, [\![\varphi_m]\!]) \leqslant [\![\varphi]\!] \quad (\mathrm{p}\,U^l).
\end{array}
$$

Question 12. *Fill out the detail of the interpretation.*

Soundness and Completeness

Theorem 13. *Let* Δ *be a context and* $\Delta \vdash \varphi_1, \ldots, \varphi_n, \varphi$ *be formulas in* $\mathrm{NJ}^{\Sigma,\Pi}$*. TFAE:*

1. $\Delta \,|\, \varphi_1, \ldots, \varphi_n \Rightarrow \varphi$.
2. *For any hyperdoctrine* $\mathrm{p}\colon \mathbb{B}^{\mathrm{op}} \to \mathrm{Heyt}$ *for NJ and* Σ, Π*-structure* $(\mathrm{U}, [\![-]\!])$*, the following holds in* $\mathrm{p}(\mathrm{U}^{|\Delta|})$*:*

$$\bigwedge ([\![\varphi_1]\!], \ldots, [\![\varphi_n]\!]) \leqslant [\![\varphi]\!].$$

Proof. (1⇒2) Easy. (2⇒1) Use the hyperdoctrine $\mathrm{p}_{\Sigma,\Pi}$ and the canonical Σ, Π-structure in this hyperdoctrine. □