Resolution of Nonsingularities, Point-theoreticity, and Metric-admissibility for p-adic Hyperbolic Curves

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Abstract

In this paper, we prove that arbitrary hyperbolic curves over p-adic local fields admit *resolution of nonsingularities* ["RNS"]. This result may be regarded as a generalization of results concerning resolution of nonsingularities obtained by A. Tamagawa and E. Lepage. Moreover, by combining our RNS result with techniques from combinatorial anabelian geometry, we prove that an *absolute version of the geometrically pro-Σ Grothendieck Conjecture for arbitrary hyperbolic curves over p-adic local fields*, where Σ denotes a set of prime numbers of cardinality ≥ 2 that contains p, holds. This settles one of the major open questions in anabelian geometry. Furthermore, we prove — again by applying RNS and combinatorial anabelian geometry — that the various *p-adic versions of the Grothendieck-Teichmüller group* that appear in the literature in fact *coincide*. As a corollary, we conclude that the metrized Grothendieck-Teichmüller group is *commensurably terminal* in the Grothendieck-Teichmüller group. This settles a longstanding open question in combinatorial anabelian geometry.

Contents

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Introduction

Let p be a prime number; Σ a nonempty subset of the set $\mathfrak{Primes}$ of prime numbers. For a connected noetherian scheme S, we shall write Π_S for the *étale fundamental group* of S, relative to a suitable choice of basepoint. For any *field* F of characteristic 0, any *field extension* $F \subseteq E$, and any *algebraic variety* [i.e., a separated, geometrically integral scheme of finite type] Z over F, we shall write $Z_E \stackrel{\text{def}}{=} Z \times_F E$ and denote by $\overline{F}$ an *algebraic closure* [well-defined up to isomorphism] of F and by $G_F \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{F}/F)$ the *absolute Galois group* of F. For any field F of characteristic 0 and any algebraic variety Z over F, we shall write

$$\Pi_Z^{(\Sigma)} \stackrel{\text{def}}{=} \Pi_Z/\mathrm{Ker}(\Pi_{Z_{\overline{F}}} \twoheadrightarrow \Pi_{Z_{\overline{F}}}^{\Sigma}),$$

where $\Pi_{Z_{\overline{F}}} \twoheadrightarrow \Pi_{Z_{\overline{F}}}^{\Sigma}$ denotes the maximal pro-Σ quotient. Here, we recall that $\Pi_Z^{(\Sigma)}$ is often referred to as the *geometrically pro-Σ fundamental group of Z*. We shall write $\mathbb{Q}_p$ for the field of p-adic numbers; $\mathbb{C}_p$ for the p-adic completion of $\overline{\mathbb{Q}}_p$. We shall refer to a finite extension field of $\mathbb{Q}_p$ as a *p-adic local field*. For any hyperbolic curve Z over either the algebraic closure of a mixed characteristic complete discrete valuation field of residue characteristic p or the p-adic completion of such an algebraic closure, we shall write

$$\Pi_Z^{\mathrm{tp}}$$

for the *Σ-tempered fundamental group of Z*, relative to a suitable choice of basepoint [cf. the subsection in Notations and Conventions entitled "Fundamental groups"]. If S_1, S_2 are schemes, then we shall write

$$\mathrm{Isom}(S_1, S_2)$$

for the set of isomorphisms of schemes between S_1 and S_2. If G_1, G_2 are profinite groups, then we shall write

$$\mathrm{OutIsom}(G_1, G_2)$$

for the set of isomorphisms of profinite groups, considered up to composition with an inner automorphism arising from an element $\in G_2$.

In the present paper, we give a complete affirmative answer to the following question:

Does an arbitrary hyperbolic curve over a p-adic local field admit resolution of nonsingularities?

Before continuing, we review the notion of *resolution of nonsingularities*. Let K ($\subseteq \overline{K}$) be a *mixed characteristic complete discrete valuation field of residue characteristic* p; v a valuation on a field F that contains K. Write $\mathcal{O}_K$ for the ring of integers of K; $\mathcal{O}_v$ for the ring of integers determined by v; $\mathfrak{m}_v \subseteq \mathcal{O}_v$ for the maximal ideal of $\mathcal{O}_v$. Then we shall say that v is a *p-valuation [over K]* [cf. Definition 2.2, (i)] if $\mathcal{O}_K = \mathcal{O}_v \cap K$ [which implies that $p \in \mathfrak{m}_v$]. Here, the phrase *"over K"* will be omitted in situations where the base field K is fixed throughout the discussion. We shall say that v is *residue-transcendental* [cf. Definition 2.2, (i)] if it is a p-valuation whose residue field $k_v \stackrel{\text{def}}{=} \mathcal{O}_v/\mathfrak{m}_v$ is a transcendental extension of the residue field of K. Let Z be a *hyperbolic curve* over K. Then we shall say that Z satisfies *Σ-RNS* [i.e., "*Σ-resolution of nonsingularities*" — cf. Definition 2.2, (vii); [Lpg1], Definition 2.1] if the following condition holds:

> Let v be a discrete residue-transcendental p-valuation on the function field $K(Z)$ of Z. Then there exists a connected geometrically pro-Σ finite étale Galois covering $Y \to Z$ such that Y has stable reduction [over *its* base field], and v coincides with the restriction [to $K(Z)$] of a discrete valuation on the function field $K(Y)$ of Y that arises from an irreducible component of the special fiber of the stable model [cf. Definition 2.1, (vi)] of Y.

If $\mathcal{Z}$ is an $\mathcal{O}_K$-scheme, then we shall write $\mathcal{Z}_s$ for the *special fiber* of $\mathcal{Z}$ [i.e., the fiber of $\mathcal{Z}$ over the closed point of Spec $\mathcal{O}_K$]. Let $\mathcal{Z}$ be an $\mathcal{O}_K$-scheme. Then [cf. Definition 2.1, (i), (ii), (iv)]:

(i) We shall say that $\mathcal{Z}$ is a *compactified model* of Z over $\mathcal{O}_K$ if $\mathcal{Z}$ is a proper, flat, normal scheme over $\mathcal{O}_K$ whose generic fiber is the [uniquely determined, up to unique isomorphism] smooth compactification of Z over K.

(ii) Suppose that the cusps of Z are K-rational. Then we shall say that $\mathcal{Z}$ is a *compactified stable model* of Z over $\mathcal{O}_K$ if $\mathcal{Z}$ is a compactified model of Z over $\mathcal{O}_K$ such that the following conditions hold:

- the geometric special fiber of $\mathcal{Z}$ is a *semistable curve* [i.e., a reduced, connected curve each of whose nonsmooth points is an ordinary double point];
- the images of the sections Spec $\mathcal{O}_K \to \mathcal{Z}$ determined by the cusps of Z [which we shall refer to as *cusps* of $\mathcal{Z}$] lie in the smooth locus of $\mathcal{Z}$ and do not intersect each other.
- $\mathcal{Z}$, together with the cusps of $\mathcal{Z}$, determines a pointed stable curve.

Then one verifies immediately, by considering blow-ups of compactified models at specified closed points [cf. also Remark 2.2.2; Proposition 2.4, (iii), (iv)], that, if Σ is a set of cardinality ≥ 2 that contains p [cf. the situation discussed

in Theorem A below], then the *condition of satisfying* Σ*-RNS* discussed above is in fact *equivalent* to the following property, which clarifies the meaning of *"resolution of nonsingularities"*:

> Let $\mathcal{Z}$ be a compactified model of Z over $\mathcal{O}_K$; $z \in \mathcal{Z}_s$ a closed point. Then, after possibly replacing K by a suitable finite extension field of K, there exist
>
> - a connected geometrically pro-Σ finite étale Galois covering $Y \to Z$ of hyperbolic curves over K,
> - a compactified stable model $\mathcal{Y}$ of Y over $\mathcal{O}_K$,
> - a morphism $\mathcal{Y} \to \mathcal{Z}$ of compactified models over $\mathcal{O}_K$ that restricts to the finite étale Galois covering $Y \to Z$,
> - an irreducible component D of $\mathcal{Y}_s$ whose normalization is of genus ≥ 1, and whose image in $\mathcal{Z}_s$ is $z \in \mathcal{Z}_s$.

This *alternative formulation* of the *condition of satisfying* Σ*-RNS* is useful to keep in mind when considering the relationship between Theorem A below and the following result due to A. Tamagawa [cf. [Tama2]], which played an important role in motivating the following result due to E. Lepage [cf. [Lpg1]]:

- Suppose that $\Sigma = \mathfrak{Primes}$, that the residue field of K is *algebraic* over the finite field of cardinality p, and that $\mathcal{Z}$ is the compactified *stable* model of Z over $\mathcal{O}_K$. Then there exist a connected finite étale Galois covering $Y \to Z$ and an irreducible component D of $\mathcal{Y}_s$ as in the above *alternative formulation* [cf. [Tama2], Theorem 0.2, (v)].
- Suppose that $\Sigma = \mathfrak{Primes}$, that K is a *p-adic local field*, and that Z is a hyperbolic Mumford curve over K. Then Z satisfies Σ-RNS [cf. [Lpg1], Theorem 2.7].

Our first main result may be regarded as a generalization of these results [cf. Theorem 2.17]:

Theorem A (**Resolution of nonsingularities for arbitrary hyperbolic curves over p-adic local fields).** *Suppose that $\Sigma \subseteq \mathfrak{Primes}$ is a subset of cardinality ≥ 2 that contains p, and that K is a p-adic local field. Let X be a hyperbolic curve over K; L a mixed characteristic complete discrete valuation field of residue characteristic p that contains K as a topological subfield. Then X_L satisfies Σ-RNS if and only if the residue field of L is algebraic over the finite field of cardinality p.*

In the remainder of the present Introduction, we discuss various anabelian applications/consequences of Theorem A.

First, by applying a certain sophisticated version of the argument applied in the proof of [Tsjm], Theorem 2.2, we obtain the following consequence of Theorem A concerning the *determination of closed points on arbitrary hyperbolic curves via geometric tempered fundamental groups*, which generalizes [Tsjm], Theorem 2.2 [cf. Corollary 2.5, (ii), which in fact applies to hyperbolic curves over *more general p-adic fields*; Remark 2.5.1]:

Theorem B (Determination of closed points on arbitrary p-adic hyperbolic curves by geometric tempered fundamental groups). *Suppose that $\Sigma \subseteq \mathfrak{Primes}$ is a subset of cardinality ≥ 2 that contains p. Let $X^\dagger$, $X^\ddagger$ be hyperbolic curves over $\overline{\mathbb{Q}}_p$. Write $\widetilde{X}^\dagger \to X^\dagger$, $\widetilde{X}^\ddagger \to X^\ddagger$ for the universal geometrically pro-Σ coverings corresponding to $\Pi^{\mathrm{tp}}_{X^\dagger}$, $\Pi^{\mathrm{tp}}_{X^\ddagger}$, respectively. Let $x^\dagger \in X^\dagger(\mathbb{C}_p)$, $x^\ddagger \in X^\ddagger(\mathbb{C}_p)$. Write $X^\dagger_{x^\dagger}$ (respectively, $X^\ddagger_{x^\ddagger}$) for the hyperbolic curve $X^\dagger_{\mathbb{C}_p}\backslash\{x^\dagger\}$ (respectively, $X^\ddagger_{\mathbb{C}_p}\backslash\{x^\ddagger\}$) over $\mathbb{C}_p$. Let*

$$\widetilde{\sigma} : \Pi^{\mathrm{tp}}_{X^\dagger_{x^\dagger}} \stackrel{\sim}{\to} \Pi^{\mathrm{tp}}_{X^\ddagger_{x^\ddagger}}$$

be an isomorphism of topological groups that fits into a commutative diagram

$$\begin{array}{ccc} \Pi^{\mathrm{tp}}_{X^\dagger_{x^\dagger}} & \xrightarrow[\widetilde{\sigma}]{\sim} & \Pi^{\mathrm{tp}}_{X^\ddagger_{x^\ddagger}} \\ \downarrow & & \downarrow \\ \Pi^{\mathrm{tp}}_{X^\dagger} & \xrightarrow[\sigma]{\sim} & \Pi^{\mathrm{tp}}_{X^\ddagger}, \end{array}$$

where the vertical arrows are the natural surjections [determined up to composition with an inner automorphism] induced by the natural open immersions $X^\dagger_{x^\dagger} \hookrightarrow X^\dagger_{\mathbb{C}_p}$, $X^\ddagger_{x^\ddagger} \hookrightarrow X^\ddagger_{\mathbb{C}_p}$ of hyperbolic curves; the lower horizontal arrow σ is the isomorphism of topological groups [determined up to composition with an inner automorphism] induced by a(n) [uniquely determined] isomorphism $\sigma_X : X^\dagger \stackrel{\sim}{\to} X^\ddagger$ of schemes over $\overline{\mathbb{Q}}_p$. Then

$$x^\ddagger \;=\; \sigma_X(x^\dagger).$$

Next, we consider applications of Theorem A to *Grothendieck Conjecture-type* results in anabelian geometry. We begin by recalling the following *question*, which may be considered as an *absolute version of the Grothendieck Conjecture for hyperbolic curves over p-adic local fields*:

> Let $X^\dagger$, $X^\ddagger$ be hyperbolic curves over p-adic local fields. Then is the natural map
>
> $$\mathrm{Isom}(X^\dagger, X^\ddagger) \longrightarrow \mathrm{OutIsom}(\Pi_{X^\dagger}, \Pi_{X^\ddagger})$$
>
> bijective?

This question may be regarded as one of the *major open questions* in anabelian geometry. In this context, we recall that, in the case of the *relative version of the Grothendieck Conjecture for arbitrary hyperbolic curves*, many satisfactory results have been obtained [cf. [PrfGC], Theorem A; [Tama1], Theorem 0.4; [LocAn], Theorem A]. In particular, the first author of the present paper gave a *complete affirmative answer* to the original question posed by A. Grothendieck [i.e., the original "Grothendieck Conjecture"] in quite substantial generality [cf. [LocAn], Theorem A; [AnabTop], Theorem 4.12]. On the other hand, in the case

of the *absolute version of the Grothendieck Conjecture for arbitrary hyperbolic curves over p-adic local fields* [i.e., the *question* of the above display], analogous results had not been obtained previously, due to the existence of *outer isomorphisms of the absolute Galois groups of p-adic local fields that do not arise from isomorphisms of fields* [cf., e.g., [NSW], the Closing Remark preceding Theorem 12.2.7]. In this direction, in some sense the *strongest known result*, prior to the present paper, was the following result [cf. [AbsTopII], Corollary 2.9]:

> *Suppose that* $\Sigma \subseteq \mathfrak{Primes}$ *is a subset of cardinality* ≥ 2 *that contains* p*. Let* $X^\dagger$*,* $X^\ddagger$ *be hyperbolic curves over p-adic local fields. Write*
>
> $$\mathrm{OutIsom}^D(\Pi_{X^\dagger}^{(\Sigma)}, \Pi_{X^\ddagger}^{(\Sigma)}) \subseteq \mathrm{OutIsom}(\Pi_{X^\dagger}^{(\Sigma)}, \Pi_{X^\ddagger}^{(\Sigma)})$$
>
> *for the subset determined by the isomorphisms that induce bijections between the respective sets of decomposition subgroups associated to the closed points of* $X^\dagger$ *and* $X^\ddagger$*. Then the natural map*
>
> $$\mathrm{Isom}(X^\dagger, X^\ddagger) \longrightarrow \mathrm{OutIsom}^D(\Pi_{X^\dagger}^{(\Sigma)}, \Pi_{X^\ddagger}^{(\Sigma)})$$
>
> *is bijective.*

Moreover, in [AbsTopII], [AbsTopIII], the first author developed a technique, called *"Belyi cuspidalization"*, that allows one to reconstruct the decomposition subgroups associated to the closed points of strictly Belyi-type hyperbolic curves and proved that an absolute version of the Grothendieck Conjecture holds for such curves [cf. [AbsTopIII], Theorem 1.9].

In the present paper, we apply Theorem A, together with some *combinatorial anabelian geometry*, to reconstruct the *set of* $\mathbb{C}_p$*-valued points* of a hyperbolic curve over $\overline{\mathbb{Q}}_p$ from its geometric tempered fundamental group [cf. Corollary 3.10, which in fact applies to hyperbolic curves over *more general p-adic fields*]:

Theorem C (Reconstruction of $\mathbb{C}_p$-valued points via geometric tempered fundamental groups). *Suppose that* $\Sigma \subseteq \mathfrak{Primes}$ *is a subset of cardinality* ≥ 2 *that contains* p*. Let* X *be a hyperbolic curve over* $\overline{\mathbb{Q}}_p$*. Write* $\widetilde{X} \to X$ *for the universal pro-*Σ *covering corresponding to* Π_X^{tp} *[so* $\mathrm{Gal}(\widetilde{X}/X)$ *may be identified with the pro-*Σ *completion of* Π_X^{tp}*]. Then the set* $\widetilde{X}(\mathbb{C}_p)$ *equipped with its natural action by* $\mathrm{Gal}(\widetilde{X}/X)$ *— hence also, by passing to the set of* $\mathrm{Gal}(\widetilde{X}/X)$*-orbits, the quotient set* $\widetilde{X}(\mathbb{C}_p) \twoheadrightarrow X(\mathbb{C}_p)$ *— may be reconstructed, in a purely combinatorial/group-theoretic way and functorially with respect to isomorphisms of topological groups, from the underlying topological group of* Π_X^{tp}*.*

Note that it follows immediately from Theorem C that, under the assumption that $\Sigma \subseteq \mathfrak{Primes}$ is a subset of cardinality ≥ 2 that contains p, one may reconstruct the *set of closed points*, hence also the *set of associated decomposition subgroups*, of a hyperbolic curve over a p-adic local field, in a purely combinatorial/group-theoretic way and functorially with respect to isomorphisms of topological groups, from its geometrically pro-Σ étale fundamental group [cf. the proof of Theorem 3.11 for more details]. Here, it is also interesting to observe that

this reconstruction of the **set of decomposition subgroups associated to closed points** *from the geometrically pro-Σ étale fundamental group of a hyperbolic curve over a p-adic local field may be thought of as a sort of a* **weak version** *of the* **Section Conjecture**.

In particular, by applying [AbsTopII], Corollary 2.9, together with some *combinatorial anabelian geometry*, we obtain a *complete affirmative answer* to the *question* considered above, i.e., an *absolute version of the Grothendieck Conjecture for arbitrary hyperbolic curves*, as well as for the *configuration spaces* associated to such hyperbolic curves, *over p-adic local fields* [cf. Theorems 3.12; 3.13]:

Theorem D (**Absolute version of the Grothendieck Conjecture for arbitrary hyperbolic curves over *p*-adic local fields**). *Suppose that $\Sigma \subseteq \mathfrak{Primes}$ is a subset of cardinality ≥ 2 that contains p. Let $X^{\dagger}$, $X^{\ddagger}$ be hyperbolic curves over p-adic local fields. Then the natural map*

$$\mathrm{Isom}(X^{\dagger}, X^{\ddagger}) \longrightarrow \mathrm{OutIsom}(\Pi_{X^{\dagger}}^{(\Sigma)}, \Pi_{X^{\ddagger}}^{(\Sigma)})$$

is bijective.

Theorem E (**Absolute version of the Grothendieck Conjecture for configuration spaces associated to arbitrary hyperbolic curves over *p*-adic local fields**). *Let $X^{\dagger}$, $X^{\ddagger}$ be hyperbolic curves over p-adic local fields; $n^{\dagger}$, $n^{\ddagger}$ positive integers. Write $X^{\dagger}_{n^{\dagger}}$ (respectively, $X^{\ddagger}_{n^{\ddagger}}$) for the $n^{\dagger}$-th (respectively, $n^{\ddagger}$-th) configuration space associated to $X^{\dagger}$ (respectively, $X^{\ddagger}$). Then the natural map*

$$\mathrm{Isom}(X^{\dagger}_{n^{\dagger}}, X^{\ddagger}_{n^{\ddagger}}) \longrightarrow \mathrm{OutIsom}(\Pi_{X^{\dagger}_{n^{\dagger}}}, \Pi_{X^{\ddagger}_{n^{\ddagger}}})$$

is bijective.

In the context of the relationship between the theory developed in the present paper and the *Section Conjecture*, it is of interest to note the following consequences of the theory underlying Theorem C [cf. Proposition 2.4, (vii); Proposition 3.5, (iii); Proposition 3.9, (iv) — all of which in fact apply to hyperbolic curves over *more general p-adic fields*]:

Theorem F (**Consequences related to the Section Conjecture over *p*-adic fields**). *Suppose that $\Sigma \subseteq \mathfrak{Primes}$ is a subset that contains p, and that the residue field of K is an algebraic extension of the finite field of cardinality p. Let $l \in \Sigma \setminus \{p\}$; $H \subseteq G_K$ a closed subgroup such that the intersection $H \cap I_K$ of H with the inertia subgroup I_K of G_K admits a surjection to [the profinite group] $\mathbb{Z}_l$. Write Ω for the p-adic completion of $\overline{K}$; $\Omega^H \subseteq \Omega$ for the subfield of Ω fixed by H. Then the following properties hold:*

(i) Let X be a proper hyperbolic curve over K that is in fact defined over a p-adic local field contained [in a fashion compatible with the respective topologies] in K, $\widetilde{X} \to X$ a universal geometrically pro-Σ covering, s_X :

$H \to \Pi_X^{(\Sigma)} \stackrel{\text{def}}{=} \mathrm{Gal}(\widetilde{X}/X)$ *a section of the restriction to* H *of the natural surjection* $\Pi_X^{(\Sigma)} \twoheadrightarrow G_K$. *Write* $\widetilde{X}^{\mathrm{an}}$ *for the topological pro-Berkovich space associated to [i.e., the inverse limit of the underlying topological spaces of the Berkovich spaces associated to the finite subcoverings of]* $\widetilde{X}$. *Then there exists at most one point* $\in \widetilde{X}^{\mathrm{an}}$ *that is fixed by the restriction, via* s_X, *to* H *of the natural action of* $\Pi_X^{(\Sigma)}$ *on the topological pro-Berkovich space* $\widetilde{X}^{\mathrm{an}}$; *if, moreover, the restriction to* H *of the* ***l-adic cyclotomic character*** *of* K *has open image, then there exists a unique such point* $\in \widetilde{X}^{\mathrm{an}}$. *Finally,* s_X *arises from an* Ω^H*-rational point* $\in X(\Omega^H)$ *if and only if its image is a maximal stabilizer* $\subseteq \Pi_X^{(\Sigma)} \times_{G_K} H$ *of a point* $\in \widetilde{X}^{\mathrm{an}}$.

(ii) Suppose that the restriction to H *of the* ***l-adic cyclotomic character*** *of* K *has open image. Let* Y, Z *be [not necessarily proper!] hyperbolic curves over* K *that are in fact defined over a p-adic local field contained [in a fashion compatible with the respective topologies] in* K; $\widetilde{Y} \to Y$, $\widetilde{Z} \to Z$ *universal geometrically pro-*Σ *coverings of* Y, Z, *respectively;* $f : Y \to Z$ *a dominant morphism over* K; $s_Y : H \to \Pi_Y^{(\Sigma)} \stackrel{\text{def}}{=} \mathrm{Gal}(\widetilde{Y}/Y)$, $s_Z : H \to \Pi_Z^{(\Sigma)} \stackrel{\text{def}}{=} \mathrm{Gal}(\widetilde{X}/X)$ *sections of the restrictions to* H *of the respective natural surjections* $\Pi_Y^{(\Sigma)} \twoheadrightarrow G_K$, $\Pi_Z^{(\Sigma)} \twoheadrightarrow G_K$ *such that* s_Y *is mapped, up to* $\Pi_Z^{(\Sigma)}$*-conjugation, by* f *to* s_Z *via the map induced by* f *on geometrically pro-*Σ *fundamental groups. Then* s_Y *arises from an* Ω^H*-rational point* $\in Y(\Omega^H)$ *if and only if* s_Z *arises from an* Ω^H*-rational point* $\in Z(\Omega^H)$.

Here, we observe in passing that it follows immediately from Theorem F, (ii) [i.e., as stated above], together with [BSC], Theorem 5.33, that a similar result to the result stated in Theorem F, (ii), holds if the field K is replaced by a *number field*, but [since global issues over number fields lie beyond the scope of the present paper] we shall not discuss this in detail in the present paper.

Next, we recall that one of the key ingredients in the theory of *p-adic arithmetic cuspidalizations* developed in [Tsjm], §2, is Lepage's *resolution of nonsingularities* [i.e., [Lpg1], Theorem 2.7], which may be regarded as a special case of Theorem A. Our next result is obtained by applying this theory of p-adic arithmetic cuspidalizations [i.e., [Tsjm], §2], together with some elementary observations concerning the *lengths of nodes* of stable models of hyperbolic curves [cf. Proposition 3.15], to prove that the various *p-adic versions of the Grothendieck-Teichmüller group* that appear in the literature [cf. [Tsjm], Remark 2.1.2] in fact *coincide* and are *commensurably terminal in the Grothendieck-Teichmüller group* [cf. Theorem 3.16; Corollary 3.17]:

Theorem G (**Equality and commensurable terminality of various *p*-adic versions of the Grothendieck-Teichmüller group**). *Suppose that* $\Sigma = \mathfrak{Primes}$. *Write* $X \stackrel{\text{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0, 1, \infty\}$;

$$\mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$$

for the Grothendieck-Teichmüller group [cf. [CmbCsp], Remark 1.11.1];

$$\mathrm{GT}^{\mathrm{M}} \subseteq \mathrm{GT} \ (\subseteq \mathrm{Out}(\Pi_X))$$

for the metrized Grothendieck-Teichmüller group [cf. [CbTpIII], Remark 3.19.2];

$$\mathrm{GT}_p^{\mathrm{tp}} \stackrel{\mathrm{def}}{=} \mathrm{GT} \cap \mathrm{Out}(\Pi_X^{\mathrm{tp}}) \subseteq \mathrm{Out}(\Pi_X)$$

[cf. the subsection in Notations and Conventions entitled "Fundamental groups"; [Tsjm], Definition 2.1]. Then the natural inclusion

$$\mathrm{GT}^{\mathrm{M}} \subseteq \mathrm{GT}_p^{\mathrm{tp}}$$

of subgroups of GT *is an* **equality**. *In particular, it holds that*

$$\mathrm{GT}^{\mathrm{M}} = \mathrm{GT}_p = \mathrm{GT}^{\mathrm{G}} = \mathrm{GT}_p^{\mathrm{tp}}$$

[cf. [Tsjm], Remark 2.1.2]. Moreover, $\mathrm{GT}^{\mathrm{M}} = \mathrm{GT}_p = \mathrm{GT}^{\mathrm{G}} = \mathrm{GT}_p^{\mathrm{tp}}$ *is commensurably terminal in* GT, *i.e., the commensurator* $C_{\mathrm{GT}}(\mathrm{GT}^{\mathrm{M}})$ *of* GT^{M} *in* GT *is equal to* GT^{M}.

Note that the *commensurable terminality* in the final portion of Theorem G may be regarded as an *affirmative answer* to the question posed in the discussion immediately preceding Theorem E in [CbTpIII], Introduction.

Finally, we conclude with an interesting complement to the theory of p-adic arithmetic cuspidalizations by applying Theorem C, together with the theory of *metric-admissibility* developed in [CbTpIII], §3, to construct certain *p-adic arithmetic cuspidalizations* of the geometric tempered fundamental group of a hyperbolic curve over $\overline{\mathbb{Q}}_p$ equipped with certain relatively mild auxiliary data [cf. Definition 3.19; Theorem 3.20; Remarks 3.20.1, 3.20.2].

The contents of the present paper may be summarized as follows:

In §1, we discuss in detail certain *local computations* — motivated by [Lpg1], Proposition 2.4, but formulated entirely in the language of *schemes* and *formal schemes*, i.e., without resorting to the use of notions from the theory of Berkovich spaces — concerning *iterates of the p-th power morphism of the multiplicative group scheme* $\mathbb{G}_{\mathrm{m}}$ over the ring of integers of a mixed characteristic discrete valuation field of residue characteristic p. By restricting such morphisms to suitable formal neighborhoods, we conclude that *smooth curves of genus* ≥ 1 appear in the special fibers of suitable models of the domain curves of such morphisms, i.e., as certain *Artin-Schreier coverings* of curves of genus 0 [cf. Proposition 1.6; Remark 1.6.2].

In §2, we begin by discussing various generalities concerning models of a hyperbolic curve over a mixed characteristic complete discrete valuation field [cf. Definition 2.1; Proposition 2.3]. In particular, we discuss the definition of the notion of Σ-RNS [cf. Definition 2.2, (vii)], together with closely related basic properties of this notion [cf. Propositions 2.4; Corollary 2.5]. Another important

notion in this context is the *purely combinatorial* notion of *VE-chains* associated to a hyperbolic curve over a mixed characteristic complete discrete valuation field [cf. Definition 2.2, (iii)]. This notion is closely related to the *topological Berkovich space* associated to the hyperbolic curve [cf. Proposition 2.3, (vii), (viii)]. Finally, we apply certain constructions involving *p-divisible groups* to extend the Artin-Schreier coverings constructed locally in §1 to coverings of an arbitrary hyperbolic curve over a p-adic local field [cf. Propositions 2.12, 2.13; Theorem 2.16]. This leads naturally to a proof of Theorem A [cf. Theorem 2.17], hence also, by combining Theorem A with the theory of VE-chains developed in the earlier portion of §2, together with some combinatorial anabelian geometry, of Theorem B.

In §3, we begin by recalling the well-known *classification* of the *points* of the *topological Berkovich space* associated to a proper hyperbolic curve over a mixed characteristic complete discrete valuation field via the notion of *type i points*, where $i \in \{1, 2, 3, 4\}$. Next, we introduce a certain *combinatorial classification* of the *VE-chains* considered in §2 and observe that this classification of VE-chains leads naturally to a *purely combinatorial characterization* of the well-known classification via type i points mentioned above. We then apply the theory of §2 to give a *group-theoretic characterization*, motivated by [but by no means identical to] the characterization of [Lpg2], §4, of the type i points in terms of the *geometric Σ-tempered fundamental group* of the hyperbolic curve. The theory surrounding this group-theoretic characterization leads naturally to proofs of Theorems C and F. Moreover, by combining this group-theoretic characterization with [AbsTopII], Corollary 2.9; [HMM], Theorem A, we obtain proofs of Theorems D and E [cf. Theorems 3.12; 3.13]. We then switch gears to discuss *metric-admissibility* for p-adic hyperbolic curves. This discussion of metric-admissibility yields, in particular, a proof of Theorem G and leads naturally to the discussion of *p-adic arithmetic cuspidalizations* associated to geometric tempered fundamental groups equipped with certain relatively mild auxiliary data [cf. Definition 3.19; Theorem 3.20; Remarks 3.20.1, 3.20.2] mentioned above.

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Notations and Conventions

Numbers:

The notation $\mathfrak{Primes}$ will be used to denote the set of prime numbers. The notation $\mathbb{N}$ will be used to denote the set of nonnegative integers. The notation $\mathbb{Q}_{\geq 0}$ will be used to denote the additive monoid of nonnegative rational numbers. The notation $\mathbb{Z}$ will be used to denote the additive group or ring of integers. The notation $\mathbb{Q}$ will be used to denote the field of rational numbers. The notation $\mathbb{R}$ will be used to denote the field of real numbers. For each $x \in \mathbb{R}$, the notation $\lfloor x \rfloor$ will be used to denote the greatest integer $\leq x$. If p is a prime number, then the notation $\mathbb{Q}_p$ will be used to denote the field of p-adic numbers; the notation $\mathbb{Z}_p$ will be used to denote the additive group or ring of p-adic integers; the notation $\overline{\mathbb{Q}}_p$ will be used to denote an algebraic closure of $\mathbb{Q}_p$; the notation $\mathbb{C}_p$ will be used to denote the p-adic completion of $\overline{\mathbb{Q}}_p$. We shall refer to a finite extension field of $\mathbb{Q}_p$ as a *p-adic local field*.

Monoids:

Let M be a commutative monoid. In this subsection, we regard the set of positive integers $\mathbb{N}_{\geq 1}$ as a directed set via its *multiplicative* structure, i.e., $i \leq j \Leftrightarrow i \mid j$. For $i \in \mathbb{N}_{\geq 1}$, write $M_i \stackrel{\text{def}}{=} M$. For $i, j \in \mathbb{N}_{\geq 1}$ such that $i \leq j$, write $\phi_{i,j} : M_i \to M_j$ for the homomorphism of monoids obtained by multiplication by $\frac{j}{i}$. Then we shall refer to the inductive limit of the inductive system $\{M_i, \phi_{i,j}\}$ on the directed set $\mathbb{N}_{\geq 1}$ as the *perfection* of M.

Rings and fields:

Let R be a ring. Then we shall write $R^\times$ for the multiplicative group of units of the ring.

Let F be a perfect field, p a prime number. Then the notation $\overline{F}$ will be used to denote an algebraic closure of F; $G_F \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{F}/F)$.

Suppose that F is a *valuation field*, i.e., a field equipped with a *valuation map* [cf., e.g., the axioms of [EP], Eq. (2.1.2)]. Then we shall write $\mathcal{O}_F$ for the ring of integers of F; $\mathfrak{m}_F$ for the maximal ideal of $\mathcal{O}_F$. Thus, the valuation map on F induces an *isomorphism of ordered abelian groups* between $F^\times/\mathcal{O}_F^\times$ and the value group of the valuation field F. In particular, any valuation map on the field F is determined, up to unique isomorphism [in the evident sense], by the ring of integers $\subseteq F$ associated to the valuation map. In the present paper,

> we shall use the term *valuation* to refer to an isomorphism class of valuation maps on a field F, i.e., the collection of valuation maps that give rise to the same ring of integers $\subseteq F$.

We shall refer to a specific valuation map within a given isomorphism class of valuation maps on a field as a *normalized valuation*, i.e., an isomorphism class of valuation maps on the field, together with a specific valuation map [belonging to this class], which we shall refer to as a *normalization*. If the value group of the

valuation field F is isomorphic, as an ordered abelian group, to a subgroup of the underlying additive group of $\mathbb{R}$, then we shall say that F is a *real valuation field*. If F is a mixed characteristic real valuation field of residue characteristic p, then the notation v_p will be used to denote the normalized valuation on F whose normalization is determined by the condition that $v_p(p) = 1 \in \mathbb{R}$. If F is a henselian valuation field of characteristic 0, then we shall write $I_F \subseteq G_F$ for the inertia subgroup; $F \subseteq F^{\mathrm{ur}}$ ($\subseteq \overline{F}$) for the maximal unramified extension. If F is a real henselian valuation field of characteristic 0, then we shall write $\widehat{F}^{\mathrm{ur}}$ for the completion of F^{ur} [cf. Remark 2.2.4 for more details].

Topological groups:

Let G be a topological group; $H \subseteq G$ a closed subgroup of G. Then we shall write G^{ab} for the abelianization of G; $C_G(H)$ for the *commensurator* of $H \subseteq G$, i.e.,

$$C_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid H \cap g \cdot H \cdot g^{-1} \text{ is of finite index in } H \text{ and } g \cdot H \cdot g^{-1}\}.$$

We shall say that the closed subgroup H is *commensurably terminal* in G if $H = C_G(H)$. Let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty subset. Then we shall write G^{Σ} for the pro-Σ completion of G.

We shall write $\mathrm{Aut}(G)$ for the group of continuous automorphisms of the topological group G, $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ for the subgroup of inner automorphisms of G, and $\mathrm{Out}(G) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)$. Suppose that G is *center-free*. Then we have a natural exact sequence of groups

$$1 \longrightarrow G \ \ (\stackrel{\sim}{\rightarrow} \mathrm{Inn}(G)) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1.$$

If J is a group, and $\rho : J \to \mathrm{Out}(G)$ is a homomorphism, then we shall denote by

$$G \stackrel{\mathrm{out}}{\rtimes} J$$

the group obtained by pulling back the above exact sequence of groups via ρ. Thus, we have a *natural exact sequence* of groups

$$1 \longrightarrow G \longrightarrow G \stackrel{\mathrm{out}}{\rtimes} J \longrightarrow J \longrightarrow 1.$$

Suppose further that the topology of G admits a *countable basis* consisting of *characteristic open subgroups* of G. Then one verifies immediately that the topology of G induces a *natural topology* on the group $\mathrm{Aut}(G)$, hence on the group $\mathrm{Out}(G)$. In particular, one verifies easily that if J is a *topological group*, and $\rho : J \to \mathrm{Out}(G)$ is *continuous*, then $G \stackrel{\mathrm{out}}{\rtimes} J$ admits a natural *topological group* structure.

Let G_1, G_2 be profinite groups. Then we shall write

$$\mathrm{OutIsom}(G_1, G_2)$$

for the set of isomorphisms of profinite groups, considered up to composition with an inner automorphism arising from an element $\in G_2$.

Semi-graphs:

Let Γ be a connected semi-graph [cf. [SemiAn], §1]. Then we shall refer to the dimension over $\mathbb{Q}$ of the first homology module of Γ [with coefficients in $\mathbb{Q}$] $H_1(\Gamma, \mathbb{Q})$ as the *loop-rank* of Γ. We shall say that Γ is *untangled* if every closed edge of Γ abuts to two distinct vertices.

Schemes:

Let K be a field; $K \subseteq L$ a field extension; X an algebraic variety [i.e., a separated, geometrically integral scheme of finite type] over K. Then we shall write $X_L \stackrel{\text{def}}{=} X \times_K L$. Suppose that X is a smooth proper curve [i.e., a smooth, proper algebraic variety of dimension 1] over K. Then we shall write $J(X)$ for the Jacobian of X. Let p be a prime number; A a semi-abelian variety or a p-divisible group over K. Then we shall write T_pA for the p-adic Tate module associated to [the p-power torsion points valued in some fixed algebraic closure of K of] A. Suppose that K is a valuation field. Let $\mathcal{X}$ be an $\mathcal{O}_K$-scheme. Then we shall write $\mathcal{X}_s$ for the special fiber of $\mathcal{X}$ [i.e., the fiber of $\mathcal{X}$ over the closed point of Spec $\mathcal{O}_K$].

Let S_1, S_2 be schemes. Then we shall write

$$\mathrm{Isom}(S_1, S_2)$$

for the set of isomorphisms of schemes between S_1 and S_2.

Curves:

We shall use the term *"hyperbolic curve"* [i.e., a *family of hyperbolic curves* over the spectrum of a field] as defined in [MT], §0. We shall use the term *"n-th configuration space"* as defined in [MT], Definition 2.1, (i).

Log schemes:

If $X^{\log}$ is a fine log scheme, then we shall write

- X for the underlying scheme of $X^{\log}$;
- $\mathcal{M}_X$ for the étale sheaf of monoids on X that defines the log structure of $X^{\log}$;
- $M_X \stackrel{\text{def}}{=} \mathcal{M}_X/\mathcal{O}_X^\times$, which we shall refer to as the *characteristic* of $X^{\log}$ [cf. [MT], Definition 5.1, (i)].

Let K be a *complete discrete valuation field*; Y a *hyperbolic curve* over K; $\mathcal{Y}$ a *compactified semistable model* of Y over $\mathcal{O}_K$ [cf. Definition 2.1, (ii)]. Write $S \stackrel{\text{def}}{=}$ Spec $\mathcal{O}_K$; $S^{\log}$ for the log scheme determined by the log structure associated to the closed point of S. Then it follows immediately from [Hur], §3.7, §3.8,

that the multiplicative monoid of sections of $\mathcal{O}_{\mathcal{Y}}$ that are *invertible* on [the open subscheme of $\mathcal{Y}$ determined by] Y determines a *natural log structure* on $\mathcal{Y}$. Denote the resulting *log scheme* by $\mathcal{Y}^{\log}$. Then one verifies immediately that the natural morphism of schemes $\mathcal{Y} \to S$ extends to a a *proper, log smooth morphism* $\mathcal{Y}^{\log} \to S^{\log}$ of fine log schemes. Let y be a geometric point of $\mathcal{Y}_s$. Write M_y^{pf} for the *perfection* of the stalk of the *characteristic* $M_{\mathcal{Y}}$ at y. Then one verifies immediately that, if the image of y in $\mathcal{Y}_s$ is a *smooth point that is not a cusp* (respectively, a *cusp*; a *node*), then

$$M_y^{\mathrm{pf}} \xrightarrow{\sim} \mathbb{Q}_{\geq 0} \text{ (respectively, } M_y^{\mathrm{pf}} \xrightarrow{\sim} \mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0};\ M_y^{\mathrm{pf}} \xrightarrow{\sim} \mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0}).$$

Fundamental groups:

For a connected noetherian scheme S, we shall write Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint. Let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty subset; K a perfect field; X an algebraic variety over K. Then we shall write

$$\Delta_X \stackrel{\mathrm{def}}{=} \Pi_{X_{\overline{K}}}; \quad \Pi_X^{(\Sigma)} \stackrel{\mathrm{def}}{=} \Pi_X / \mathrm{Ker}(\Delta_X \twoheadrightarrow \Delta_X^{\Sigma}),$$

where $\Delta_X \twoheadrightarrow \Delta_X^{\Sigma}$ denotes the natural surjection. We shall refer to Δ_X^{Σ} (respectively, $\Pi_X^{(\Sigma)}$) as the *geometric pro-Σ fundamental group* (respectively, *geometrically pro-Σ fundamental group*) of X.

Let p be a prime number, $\Sigma \subseteq \mathfrak{Primes}$ a nonempty subset. Suppose that K is a *mixed characteristic complete discrete valuation field of residue characteristic p*. Write Ω for the p-adic completion of $\overline{K}$. Let F be one of the [topological] fields K, $\overline{K}$, and Ω; X a hyperbolic curve over F. If $F = \Omega$, then we shall write

$$\Pi_X^{\mathrm{tp}}$$

for the *Σ-tempered fundamental group* of X, i.e., in the terminology of [CbTpIII], Definition 3.1, (ii), the "Σ-tempered fundamental group" of the smooth log curve over F determined by X [where we note that one verifies immediately that the field "$^{\square}\overline{K}^{\wedge}$" of *loc. cit.* may be taken to to be the field $F = \Omega$ of the present discussion, and that the discussion of *loc. cit.* may be applied to the situation of the present discussion even if the X of the present discussion *does not descend* to the field "$^{\square}K$" of *loc. cit.*]. Here, we recall that, when $F = \Omega$, it follows immediately from [André], Proposition 4.3.1 [and the surrounding discussion], that the *universal topological coverings* of finite coverings corresponding to characteristic open subgroups of Π_X^{tp} of finite index determine a *countable* collection of *characteristic open subgroups* of Π_X^{tp} that form a *basis* of the topology of Π_X^{tp}, hence determine a *natural topology* on $\mathrm{Out}(\Pi_X^{\mathrm{tp}})$. Thus, if $F = K$, then we obtain a *natural continuous outer action*

$$G_K \quad \longrightarrow \quad \mathrm{Out}(\Pi_{X_\Omega}^{\mathrm{tp}})$$

and hence, since $\Pi^{\mathrm{tp}}_{X_\Omega}$ is *center-free* [cf., e.g., [CbTpIII], Proposition 3.3, (i), (ii); [MT], Proposition 1.4], a *topological group*

$$\Pi^{\mathrm{tp}}_X \stackrel{\mathrm{def}}{=} \Pi^{\mathrm{tp}}_{X_\Omega} \overset{\mathrm{out}}{\rtimes} G_K,$$

which we refer to as the *geometrically* Σ*-tempered fundamental group* of X. Moreover, if $F = K$, then Π^{tp}_X is equipped with a *natural continuous surjection* $\Pi^{\mathrm{tp}}_X \twoheadrightarrow G_K$, whose kernel [which in fact may be naturally identified with $\Pi^{\mathrm{tp}}_{X_\Omega}$] we denote by $\Delta^{\Sigma\text{-tp}}_X \stackrel{\mathrm{def}}{=} \Pi^{\mathrm{tp}}_{X_\Omega}$ and refer to as the *geometric* Σ*-tempered fundamental group* of X. If $F = \overline{K}$, then, after possibly replacing K by a finite extension of K, we may assume that X descends to a hyperbolic curve X_K over K [so that X may be naturally identified with "$(X_K)_{\overline{K}}$"], and we shall refer to

$$\Pi^{\mathrm{tp}}_X \stackrel{\mathrm{def}}{=} \Pi^{\mathrm{tp}}_{X_\Omega}$$

as the Σ*-tempered fundamental group* of X. Finally, we note that, in the case where $F = \overline{K}$, it follows immediately from [CanLift], Proposition 2.3, (ii) [cf. also [CbTpIII], Proposition 3.3, (i)], together with the definition of the Σ-tempered fundamental group, that the *pro-*Σ *completion* of $\Pi^{\mathrm{tp}}_X = \Pi^{\mathrm{tp}}_{X_\Omega}$ may be naturally identified, for suitable choices of basepoints, with the *geometric pro-*Σ *fundamental group* Δ^{Σ}_X of X.

1 Local construction of Artin-Schreier extensions in the special fiber

Let p be a prime number. In the present section, we perform various *local computations concerning iterates of the* p*-th power morphism of the multiplicative group scheme* $\mathbb{G}_{\mathrm{m}}$ over the ring of integers of a mixed characteristic discrete valuation field of residue characteristic p. As a consequence, by restricting such morphisms to suitable formal neighborhoods, we conclude that *smooth curves of genus* ≥ 1 *appear in the special fibers of suitable models of the domain curves of such morphisms* [cf. Proposition 1.6; Remark 1.6.2]. This observation will be applied in §2 to prove that arbitrary hyperbolic curves over p-adic local fields admit resolution of nonsingularities. The contents of this section may be regarded as an alternative and somewhat more detailed discussion of [Lpg1], Proposition 2.4, that is phrased entirely in the language of *schemes* and *formal schemes* and does not resort to the use of Berkovich spaces.

First, we begin with several elementary lemmas concerning the p-adic valuations of the coefficients of certain polynomials and power series [cf. Lemmas 1.1, 1.2, 1.3].

Lemma 1.1. *Let K be a mixed characteristic discrete valuation field of residue characteristic p. Suppose that K contains a primitive p-th root of unity $\zeta_p \in K$. Write $\pi \stackrel{\text{def}}{=} 1 - \zeta_p \in K$;*

$$f(x) \stackrel{\text{def}}{=} \pi^{-p}\big((1+\pi x)^p - 1\big) \in K[x],$$

where x denotes an indeterminate. Then it holds that

$$f(x) - (x^p - x) \in x \cdot \mathfrak{m}_K[x] \subseteq \mathcal{O}_K[x].$$

Proof. First, write $c \stackrel{\text{def}}{=} p + (-\pi)^{p-1}$. Then since $\pi = 1 - \zeta_p \in K$, it holds that

$$c = p + (-\pi)^{p-1} = p + (-\pi)^{p-1} + \pi^{-1}((1-\pi)^p - 1).$$

Observe that these equalities imply that $v_p(c) > 1$. In particular, it holds that $v_p(p + (-\pi)^{p-1}) > 1$, hence that $v_p(p + \pi^{p-1}) > 1$.

Next, observe that, if we write

$$f(x) = \sum_{1 \leq i \leq p} a_i x^i \in K[x],$$

then since $v_p(p) = 1 = v_p(\pi^{p-1})$, and $v_p(p + \pi^{p-1}) > 1$, it holds that

$$a_1 = \frac{p}{\pi^{p-1}}, \quad a_p = 1, \quad v_p(a_1 + 1) > 0, \quad v_p(a_i) > 0 \quad (\forall i \neq 1, p).$$

Thus, we conclude that $f(x) - (x^p - x) \in x \cdot \mathfrak{m}_K[x]$. This completes the proof of Lemma 1.1. □

Lemma 1.2. *Let K be a mixed characteristic discrete valuation field of residue characteristic p; n a positive integer. Write*

$$f(x) = 1 + \sum_{i \geq 1} q_i x^i \in K[[x]]$$

— where x denotes an indeterminate — for the n-th root of $1 + x \in K[[x]]$ whose constant term is equal to 1. Then it holds that

$$v_p(q_1) = -v_p(n), \quad v_p(q_i) \geq -i v_p(n) - v_p(i!) \geq -i\left(v_p(n) + \frac{1}{p-1}\right) \quad (\forall i \geq 2)$$

[cf. the well-known elementary fact that $v_p(i!) \leq \frac{i}{p-1}$]. If, moreover, n is prime to p, then $v_p(q_i) \geq 0$ for each positive integer i.

Proof. First, we observe, by considering the Taylor expansion of $(1+x)^{\frac{1}{n}}$ at 0, that

$$q_i = \frac{1}{i!} \prod_{0 \leq k \leq i-1} \left(\frac{1}{n} - k\right)$$

for each positive integer i. The equality and inequalities of the second display of the statement of Lemma 1.2 follow immediately.

Next, suppose that n is prime to p. For each positive integer m, write $Q_m \subseteq K$ for the $\mathcal{O}_K$-subalgebra generated by $\{q_j\}_{1 \leq j \leq m-1}$. Write $Q_0 \stackrel{\text{def}}{=} \mathcal{O}_K$. Then, by comparing the coefficients of x^m in the left- and right-hand sides of the equality

$$\left(1 + \sum_{i \geq 1} q_i x^i\right)^n = 1 + x,$$

we conclude that $nq_m \in Q_{m-1}$. In particular, since n is prime to p, it holds that $q_m \in Q_{m-1}$. Thus, by induction, we conclude that $q_i \in \mathcal{O}_K$ for each positive integer i. This completes the proof of Lemma 1.2. $\square$

Lemma 1.3. *Let K be a mixed characteristic discrete valuation field of residue characteristic p; n a positive integer;*

$$g(s) = 1 + \sum_{i \geq 1} a_i s^i \in \mathcal{O}_K[[s]] \setminus \{1\},$$

where s denotes an indeterminate. Write i_0 for the smallest positive integer i such that $a_i \neq 0$; 0_s for the $\mathcal{O}_K$-valued point of Spec $\mathcal{O}_K[[s]]$ *obtained by mapping $s \mapsto 0$. Then the following hold:*

(i) Suppose that $v_p(a_i) \geq 2v_p(n)$ for each $i \geq 1$. Then $g(s)$ admits an n-th root

$$1 + \sum_{i \geq 1} b_i s^i \in \mathcal{O}_K[[s]].$$

(ii) Suppose that there exists an element $x \in \mathcal{O}_K$ such that $v_p(x) = \frac{1}{3i_0(p-1)}$. Then there exist

- *a positive integer j,*
- *an element $b \in \mathcal{O}_K$ satisfying the inequalities $v_p(b) \geq 1$ and $\frac{2}{3(p-1)} \leq v_p(b) - \lfloor v_p(b) \rfloor < \frac{1}{p-1}$, and*
- *an isomorphism $h : \mathcal{O}_K[[s]] \stackrel{\sim}{\to} \mathcal{O}_K[[s]]$ of topological $\mathcal{O}_K$-algebras such that h maps $0_s \mapsto 0_s$ and $h(g(x^j s)) = 1 + bs^{i_0}$.*

(iii) Suppose that $g(s) = 1 + a_{i_0} s^{i_0}$, where a_{i_0} satisfies the inequalities $v_p(a_{i_0}) \geq 1$ and $\frac{2}{3(p-1)} \leq v_p(a_{i_0}) - \lfloor v_p(a_{i_0}) \rfloor < \frac{1}{p-1}$. Write $\mu \stackrel{\text{def}}{=} \lfloor v_p(a_{i_0}) \rfloor - 1 \geq 0$. Then $g(s)$ admits a p^μ-th root

$$g(s)^{\frac{1}{p^\mu}} = 1 + \sum_{i \geq 1} c_{i_0 i} s^{i_0 i} \in \mathcal{O}_K[[s]],$$

where $1 + \frac{2}{3(p-1)} \leq v_p(c_{i_0}) < \frac{p}{p-1}$, $v_p(c_{2i_0}) \geq 2\big(1 + \frac{2}{3(p-1)}\big) - 1$, and $v_p(c_{i_0 i}) \geq i\big(1 - \frac{1}{3(p-1)}\big)$ for each positive integer $i > 2$.

(iv) In the notation of (iii), suppose that K contains a primitive p-th root of unity $\zeta_p \in K$ and an element $c \in K$ such that

$$c^{i_0} = \frac{\pi^p}{c_{i_0}},$$

where we write $\pi \stackrel{\text{def}}{=} 1 - \zeta_p$. [Note that since $v_p(c_{i_0}) < \frac{p}{p-1}$, it holds that $v_p(c) > 0$.] Write

$$\sum_{i \geq 1} d_{i_0 i} s^{i_0 i} \stackrel{\text{def}}{=} \pi^{-p}(g(cs)^{\frac{1}{p^\mu}} - 1) \in K[[s]].$$

Then it holds that $d_{i_0} = 1$, $v_p(d_{i_0}) = 0 > v_p(c^{i_0}) - \frac{p}{p-1}$, $v_p(d_{2i_0}) > \sup\{v_p(c^{2i_0}) - \frac{p}{p-1}, 0\}$, and

$$v_p(d_{i_0 i}) > i \cdot \sup\left\{\left(1 - \frac{1}{3(p-1)}\right), v_p(c^{i_0})\right\} - \frac{p}{p-1} \geq 0$$

for each positive integer $i > 2$. Moreover,

$$v_p(d_{i_0 i}) > v_p(c^{i_0 i})$$

for each sufficiently large positive integer i.

Proof. First, we verify assertion (i). Note that $g(s)$ admits an n-th root

$$1 + \sum_{i \geq 1} b_i s^i \in K[[s]].$$

Thus, it suffices to verify that $v_p(b_i) \geq 0$ for each positive integer i. Note that if $v_p(n) \geq 1$ ($\geq \frac{1}{p-1}$), then

$$2i v_p(n) - i\left(v_p(n) + \frac{1}{p-1}\right) \geq 0.$$

Thus, by applying Lemma 1.2 [where we take "x" to be the element $\sum_{i \geq 1} a_i s^i$], together with our assumption that $v_p(a_i) \geq 2v_p(n)$ for each positive integer i, we conclude that $v_p(b_i) \geq 0$ for each positive integer i. This completes the proof of assertion (i).

Next, we verify assertion (ii). Fix an element $x \in \mathcal{O}_K$ such that $v_p(x) = \frac{1}{3i_0(p-1)}$. For each pair of positive integers (i, j), write $b_{i,j} \stackrel{\text{def}}{=} x^{ij} a_i$. Note that it follows immediately from our assumption on $v_p(x)$ [i.e., by thinking of the real line $\mathbb{R}$ modulo integral multiples of $\frac{1}{3(p-1)} = i_0 v_p(x)$] that there exists a positive integer j such that

- $v_p(b_{i_0,j}) \geq 1$,
- $\frac{2}{3(p-1)} \leq v_p(b_{i_0,j}) - \lfloor v_p(b_{i_0,j}) \rfloor < \frac{1}{p-1}$, and

- $v_p(b_{i,j}) \geq v_p(b_{i_0,j}) + 2v_p(i_0)$ for each positive integer $i > i_0$.

Fix such a positive integer j and write $b \stackrel{\text{def}}{=} b_{i_0,j}$. Then the existence of an isomorphism $h : \mathcal{O}_K[[s]] \stackrel{\sim}{\to} \mathcal{O}_K[[s]]$ of topological $\mathcal{O}_K$-algebras such that h maps $0_s \mapsto 0_s$ and $h(g(x^j s)) = 1 + bs^{i_0}$ follows immediately from Lemma 1.3, (i), where we take "n" to be i_0 and "$g(s)$" to be

$$g^{\dagger}(s) \stackrel{\text{def}}{=} b^{-1}s^{-i_0}(g(x^j s) - 1)$$

[so $g(x^j s) = 1 + bs^{i_0}g^{\dagger}(s)$]. [That is to say, h is defined by taking $h(s(g^{\dagger}(s))^{\frac{1}{n}}) \stackrel{\text{def}}{=} s$, where "$(g^{\dagger}(s))^{\frac{1}{n}}$" denotes the n-th root of Lemma 1.3, (i).] This completes the proof of assertion (ii).

Assertion (iii) follows immediately from Lemma 1.2 [applied to $x = a_{i_0}s^{i_0}$], together with the elementary fact that $v_p(i!) \leq 1$ for $i = 2$.

Finally, we verify assertion (iv). It follows immediately from the various definitions involved that

$$d_{i_0 i} = c_{i_0 i}c_{i_0}^{-i}\pi^{p(i-1)}.$$

In particular, it holds that $d_{i_0} = 1$. Moreover, since $v_p(\pi) = \frac{1}{p-1}$, and $v_p(c_{i_0}) < \frac{p}{p-1}$ [cf. Lemma 1.3, (iii)], it holds that

$$v_p(d_{i_0 i}) = v_p(c_{i_0 i}) - iv_p(c_{i_0}) + \frac{p(i-1)}{p-1} > v_p(c_{i_0 i}) - \frac{p}{p-1}.$$

Suppose that $i = 2$ (respectively, $i \geq 3$). Then it follows immediately from Lemma 1.3, (iii), that

$$v_p(c_{i_0 i}) - \frac{p}{p-1} \geq 2\left(1 + \frac{2}{3(p-1)}\right) - 1 - \frac{p}{p-1} = \frac{1}{3(p-1)} > 0$$

(respectively,

$$v_p(c_{i_0 i}) - \frac{p}{p-1} \geq i\left(1 - \frac{1}{3(p-1)}\right) - \frac{p}{p-1} \geq 3\left(1 - \frac{1}{3(p-1)}\right) - \frac{p}{p-1} = \frac{2p-4}{p-1} \geq 0).$$

Thus, we conclude that $v_p(d_{2i_0}) > 0$, and

$$v_p(d_{i_0 i}) > i\left(1 - \frac{1}{3(p-1)}\right) - \frac{p}{p-1} \geq 0$$

for each positive integer $i > 2$. The remainder of assertion (iv) follows immediately from the inequalities already obtained, together with the inequalities

$$\begin{aligned} \frac{p}{p-1} &> \inf\left\{1 - \frac{1}{3(p-1)}, \frac{1}{2 \cdot 3(p-1)} + \frac{p}{2(p-1)}\right\} \\ &> \frac{1}{3(p-1)} = \frac{p}{p-1} - \left(1 + \frac{2}{3(p-1)}\right) \\ &\geq v_p(c^{i_0}) \end{aligned}$$

[cf. Lemma 1.3, (iii)]. This completes the proof of assertion (iv), hence of Lemma 1.3. □

Definition 1.4. Let K be a mixed characteristic discrete valuation field of residue characteristic p; $c \in \mathfrak{m}_K \setminus \{0\}$. Then we shall write

$$\phi_c : B_c \longrightarrow \text{Spec } \mathcal{O}_K[[t]]$$

— where t denotes an indeterminate — for the blow-up of Spec $\mathcal{O}_K[[t]]$ with center given by the closed subscheme defined by the ideal (c, t); $b_c \in B_c$ for the generic point of the exceptional irreducible component of the special fiber of B_c [i.e., the fiber of B_c over the closed point of Spec $\mathcal{O}_K$];

$$U_c = \text{Spec } \mathcal{O}_K[[t]][s_c]/(cs_c - t) \subseteq B_c$$

— where s_c denotes an indeterminate, but, by a slight abuse of notation, we shall also use the notation "s_c" to denote the element of $\Gamma(U_c, \mathcal{O}_{U_c})$ determined by the indeterminate "s_c"— for the open subscheme obtained by removing the strict transform of the special fiber of Spec $\mathcal{O}_K[[t]]$. Note that it follows immediately from the various definitions involved that ϕ_c induces a morphism

$$\widehat{\phi}_c : \widehat{U}_c \stackrel{\text{def}}{=} \text{Spec } \mathcal{O}_K[[s_c]] \longrightarrow \text{Spec } \mathcal{O}_K[[t]]$$

over $\mathcal{O}_K$ that maps t to cs_c.

Proposition 1.5 (Local construction of Artin-Schreier extensions in the special fiber I). *We maintain the notation of Definition 1.4. Suppose that K contains a primitive p-th root of unity $\zeta_p \in K$. Write k for the residue field of K; $\pi \stackrel{\text{def}}{=} 1 - \zeta_p \in K$;*

$$f : \text{Spec } \mathcal{O}_K[[t]] \longrightarrow \text{Spec } \mathcal{O}_K[[t]]$$

for the [manifestly] finite flat morphism over $\mathcal{O}_K$ corresponding to the homomorphism of topological $\mathcal{O}_K$-algebras that maps $t \mapsto (1+t)^p - 1$. Then f induces a finite morphism

$$\tilde{f} : B_\pi \longrightarrow B_{\pi^p}$$

over $\mathcal{O}_K$ that maps $b_\pi \mapsto b_{\pi^p}$ and induces a finite flat morphism $U_\pi \to U_{\pi^p}$, whose induced morphism on special fibers is the morphism induced on spectra by the injective homomorphism

$$k[s_{\pi^p}] \longrightarrow k[s_\pi]$$

over k that maps $s_{\pi^p} \mapsto s_\pi^p - s_\pi$.

Proof. First, *observe* that we have an inclusion of ideals in $\mathcal{O}_K[[t]]$

$$(\pi^p, (1+t)^p - 1) \subseteq (\pi, t).$$

This *observation*, together with Lemma 1.1, implies, by the definition of the blow-up, that the morphism f functorially induces a [proper and quasi-finite,

hence] finite morphism $\tilde{f} : B_\pi \to B_{\pi^p}$ over $\mathcal{O}_K$, which, in turn, induces a [manifestly] finite flat morphism

$$U_\pi \longrightarrow U_{\pi^p}$$

over $\mathcal{O}_K$ such that

$$t \mapsto (1+t)^p - 1, \quad s_{\pi^p} \mapsto \pi^{-p}\big((1+\pi s_\pi)^p - 1\big)$$

[cf. Lemma 1.1]. Indeed, it follows from Lemma 1.1 that this morphism $U_\pi \to U_{\pi^p}$ determines a dominant morphism

$$\text{Spec } k[s_\pi] \longrightarrow \text{Spec } k[s_{\pi^p}]$$

[between open subschemes of the respective exceptional irreducible components of the special fibers of B_π, B_{π^p}] that corresponds to the injective homomorphism

$$k[s_{\pi^p}] \longrightarrow k[s_\pi]$$

over k that maps $s_{\pi^p} \mapsto s_\pi^p - s_\pi$. This completes the proof of Proposition 1.5. □

Remark 1.5.1. In the notation of Proposition 1.5, write $\mathbb{G}_{\mathrm{m}} = \text{Spec } \mathcal{O}_K[u, \frac{1}{u}]$ for the multiplicative group scheme over $\mathcal{O}_K$, where u denotes an indeterminate;

$$\iota : \text{Spec } \mathcal{O}_K[[t]] \longrightarrow \mathbb{G}_{\mathrm{m}}$$

for the morphism that corresponds to the homomorphism over $\mathcal{O}_K$ that maps $u \mapsto 1 + t$. Then we have a commutative diagram

$$\begin{array}{ccccc} U_\pi & \xrightarrow[\phi_\pi|_{U_\pi}]{} & \text{Spec } \mathcal{O}_K[[t]] & \xrightarrow[\iota]{} & \mathbb{G}_{\mathrm{m}} \\ {\scriptstyle \tilde{f}|_{U_\pi}}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle p}\downarrow \\ U_{\pi^p} & \xrightarrow[\phi_{\pi^p}|_{U_{\pi^p}}]{} & \text{Spec } \mathcal{O}_K[[t]] & \xrightarrow[\iota]{} & \mathbb{G}_{\mathrm{m}}, \end{array}$$

where the right-hand vertical arrow denotes the p-th power morphism; the vertical arrows are *finite flat morphisms of degree* p [cf. Proposition 1.5]; the second square is *cartesian*; the first square is *cartesian*, up to taking the *normalization* of the fiber product that would make the first square "truly cartesian".

Proposition 1.6 (Local construction of Artin-Schreier extensions in the special fiber II). *We maintain the notation of Remark 1.5.1. Let*

$$g(t) = 1 + \sum_{i \geq 1} a_i t^i \in \mathcal{O}_K[[t]] \setminus \{1\}.$$

Write i_0 for the smallest positive integer i such that $a_i \neq 0$;

$$\lambda_g : \text{Spec } \mathcal{O}_K[[t]] \longrightarrow \text{Spec } \mathcal{O}_K[[t]]$$

for the morphism over $\mathcal{O}_K$ corresponding to the homomorphism of topological $\mathcal{O}_K$-algebras that maps $t \mapsto g(t) - 1$. Then the following hold:

(i) *After possibly replacing K by a suitable finite field extension of K, there exist a nonnegative integer μ, elements $c_1, c_2 \in \mathfrak{m}_K \setminus \{0\}$, an isomorphism*

$$\lambda_h : \text{Spec } \mathcal{O}_K[[s_{c_1}]] \tilde{\to} \text{Spec } \mathcal{O}_K[[s_{c_1}]]$$

over $\mathcal{O}_K$, and a morphism

$$\xi_g : \text{Spec } \mathcal{O}_K[[s_{c_1}]] \longrightarrow \mathbb{G}_{\mathrm{m}}$$

over $\mathcal{O}_K$ satisfying the following conditions:

- *Write 0_{c_1} for the $\mathcal{O}_K$-valued point of* Spec $\mathcal{O}_K[[s_{c_1}]]$ *obtained by mapping $s_{c_1} \mapsto 0$. Then λ_h maps $0_{c_1} \mapsto 0_{c_1}$, and ξ_g maps 0_{c_1} to the identity element of $\mathbb{G}_{\mathrm{m}}(\mathcal{O}_K)$.*
- *There exists a commutative diagram*

$$\begin{array}{ccccc} \text{Spec } \mathcal{O}_K[[s_{c_1}]] & \xrightarrow[\widehat{\phi}_{c_1}]{} & \text{Spec } \mathcal{O}_K[[t]] & \xrightarrow[\lambda_g]{} & \text{Spec } \mathcal{O}_K[[t]] \\ {\scriptstyle \lambda_h}\uparrow\wr & & & & \downarrow{\scriptstyle \iota} \\ \text{Spec } \mathcal{O}_K[[s_{c_1}]] & \xrightarrow[\xi_g]{} & \mathbb{G}_{\mathrm{m}} & \xrightarrow[p^{\mu}]{} & \mathbb{G}_{\mathrm{m}}, \end{array}$$

  where the right-hand lower horizontal arrow denotes the p^{μ}-th power morphism.
- *Write*

$$\tau : \text{Spec } \mathcal{O}_K[[t]] \tilde{\to} \text{Spec } \mathcal{O}_K[[s_{c_1}]]$$

  for the isomorphism over $\mathcal{O}_K$ corresponding to the isomorphism of topological $\mathcal{O}_K$-algebras that maps $s_{c_1} \mapsto t$;

$$\eta(s_{c_2}) \in \mathcal{O}_K[[s_{c_2}]]$$

  for the image of u via the homomorphism $\mathcal{O}_K[u, \frac{1}{u}] \to \mathcal{O}_K[[s_{c_2}]]$ induced by the composite

$$\text{Spec } \mathcal{O}_K[[s_{c_2}]] \underset{\widehat{\phi}_{c_2}}{\longrightarrow} \text{Spec } \mathcal{O}_K[[t]] \underset{\tau}{\xrightarrow{\sim}} \text{Spec } \mathcal{O}_K[[s_{c_1}]] \underset{\xi_g}{\longrightarrow} \mathbb{G}_{\mathrm{m}}.$$

  Then it holds that $\eta(s_{c_2}) - 1 \in \mathfrak{m}_K[[c_2 s_{c_2}]]$, and, moreover,

$$\pi^{-p}(\eta(s_{c_2}) - 1) - s_{c_2}^{i_0} \in \mathfrak{m}_K[s_{c_2}] + \mathfrak{m}_K[[c_2 s_{c_2}]] = \mathfrak{m}_K[s_{c_2}] + \mathfrak{m}_K[[t]].$$

  In particular, there exists a morphism

$$\theta_g : U_{c_2} \longrightarrow U_{\pi^p}$$

  over $\mathcal{O}_K$ that fits into the following commutative diagram

$$\begin{array}{ccccc} U_{c_2} & \xrightarrow[\phi_{c_2}|_{U_{c_2}}]{} & \text{Spec } \mathcal{O}_K[[t]] & \xrightarrow[\tau]{\sim} & \text{Spec } \mathcal{O}_K[[s_{c_1}]] \\ {\scriptstyle \theta_g}\downarrow & & & & \downarrow{\scriptstyle \xi_g} \\ U_{\pi^p} & \xrightarrow[\phi_{\pi^p}|_{U_{\pi^p}}]{} & \text{Spec } \mathcal{O}_K[[t]] & \xrightarrow[\iota]{} & \mathbb{G}_{\mathrm{m}}. \end{array}$$

(ii) Fix a collection of data $(\mu, c_1, c_2, \lambda_h, \xi_g)$ as in (i). Write

$$Y \stackrel{\text{def}}{=} U_{c_2} \times_{U_{\pi^p}} U_\pi$$

for the fiber product determined by the morphism $\theta_g : U_{c_2} \to U_{\pi^p}$ and the morphism $U_\pi \to U_{\pi^p}$ induced by $\tilde{f}$ [cf. Proposition 1.5]. Then the natural morphism

$$Y_s \longrightarrow (U_{c_2})_s = \text{Spec } k[s_{c_2}]$$

induced by the first projection morphism $Y \to U_{c_2}$ corresponds to the natural injective homomorphism

$$k[s_{c_2}] \hookrightarrow k[s_{c_2}, y]/(y^p - y - s_{c_2}^{i_0})$$

over k, where y denotes an indeterminate.

Proof. First, we consider assertion (i). We begin by applying Lemma 1.3, (ii), where we take "$g(s)$" to be $g(t)$ [i.e., so the indeterminate "s" corresponds to t], and we observe that, by replacing K by a suitable finite extension of K, we may assume without loss of generality that there exists an "x" as in Lemma 1.3, (ii). This yields an isomorphism "$h : \mathcal{O}_K[[s]] \stackrel{\sim}{\to} \mathcal{O}_K[[s]]$" as in Lemma 1.3, (ii), whose induced morphism on spectra — where we interpret the indeterminate "s" to be s_{c_1}, and we take "x^j" to be c_1 [so "$x^j s$" corresponds to $c_1 s_{c_1} = t$] — we take to be λ_h. Here, we recall that this isomorphism "h" of Lemma 1.3, (ii), satisfies a condition "$h(g(x^j s)) = 1 + bs^{i_0}$". Next, we would like to apply Lemma 1.3, (iii), where we take "$1 + a_{i_0} s^{i_0}$" to be the "$1 + bs^{i_0}$" of Lemma 1.3, (ii) [i.e., so the indeterminate "s" still corresponds to s_{c_1}]. This yields a power series "$g(s)^{\frac{1}{p^\mu}}$" as in Lemma 1.3, (iii). We then take the "μ" of Lemma 1.3, (iii), to be μ and define ξ_g to be the morphism over $\mathcal{O}_K$ corresponding to the homomorphism that maps u to this power series "$g(s)^{\frac{1}{p^\mu}}$" [i.e., where the indeterminate "s" still corresponds to s_{c_1}]. This yields a collection of data $(\mu, c_1, \lambda_h, \xi_g)$ that satisfies the *first two itemized conditions* of Proposition 1.6, (i). The *third [and final] itemized condition* of Proposition 1.6, (i), now follows by translating the various estimates of Lemma 1.3, (iv), into the notation of the present situation, where we take the "c" of Lemma 1.3, (iv), to be c_2, and we observe that, again by replacing K by a suitable finite extension of K, we may assume without loss of generality that there exist "ζ_p" and "c" as in Lemma 1.3, (iv). Also, we observe that the power series "$g(cs)^{\frac{1}{p^\mu}}$" of Lemma 1.3, (iv), corresponds to $\eta(s_{c_2})$ [i.e., where the indeterminate "s" corresponds to s_{c_2}]. This completes the proof of assertion (i).

Next, we verify assertion (ii). Recall that

- the morphism $\theta_g : U_{c_2} \to U_{\pi^p}$ corresponds to the homomorphism of topological $\mathcal{O}_K$-algebras

$$\mathcal{O}_K[[t]][s_{\pi^p}]/(\pi^p s_{\pi^p} - t) \longrightarrow \mathcal{O}_K[[t]][s_{c_2}]/(c_2 s_{c_2} - t)$$

that maps

$$s_{\pi^p} \mapsto \pi^{-p}(\eta(s_{c_2}) - 1), \quad t \mapsto \eta(s_{c_2}) - 1,$$

while

- the morphism $U_\pi \to U_{\pi^p}$ corresponds to the homomorphism of topological $\mathcal{O}_K$-algebras

$$\mathcal{O}_K[[t]][s_{\pi^p}]/(\pi^p s_{\pi^p} - t) \longrightarrow \mathcal{O}_K[[t]][s_\pi]/(\pi s_\pi - t)$$

that maps

$$s_{\pi^p} \mapsto \pi^{-p}((1+\pi s_\pi)^p - 1), \quad t \mapsto (1+t)^p - 1.$$

Then since

$$\pi^{-p}((1+\pi s_\pi)^p - 1) - (s_\pi^p - s_\pi) \in \mathfrak{m}_K[[s_\pi]]$$

and

$$\pi^{-p}(\eta(s_{c_2}) - 1) - s_{c_2}^{i_0} \in \mathfrak{m}_K[s_{c_2}] + \mathfrak{m}_K[[t]]$$

[cf. Lemma 1.1; Proposition 1.6, (i)], it holds that

- the morphism $(U_{c_2})_s \to (U_{\pi^p})_s$ corresponds to the homomorphism

$$k[s_{\pi^p}] \longrightarrow k[s_{c_2}]$$

over k that maps $s_{\pi^p} \mapsto s_{c_2}^{i_0}$, while

- the morphism $(U_\pi)_s \to (U_{\pi^p})_s$ corresponds to the homomorphism

$$k[s_{\pi^p}] \longrightarrow k[s_\pi]$$

over k that maps $s_{\pi^p} \mapsto s_\pi^p - s_\pi$.

Thus, we conclude that the first projection morphism

$$Y_s = (U_{c_2})_s \times_{(U_{\pi^p})_s} (U_\pi)_s \to (U_{c_2})_s = \text{Spec } k[s_{c_2}]$$

corresponds to the natural injective homomorphism

$$k[s_{c_2}] \hookrightarrow k[s_{c_2}, y]/(y^p - y - s_{c_2}^{i_0})$$

over k. This completes the proof of assertion (ii), hence of Proposition 1.6. $\square$

Remark 1.6.1. In the notation of Proposition 1.6, we observe that the first projection morphism $Y \to U_{c_2}$ fits into a commutative diagram

$$\begin{array}{ccccccc} Y & \xrightarrow[\theta_Y]{} & U_\pi & \xrightarrow[\phi_\pi|_{U_\pi}]{} & \text{Spec } \mathcal{O}_K[[t]] & \xrightarrow[\iota]{} & \mathbb{G}_{\mathrm{m}} \\ {\scriptstyle f_Y}\downarrow & & {\scriptstyle \tilde{f}|_{U_\pi}}\downarrow & & {\scriptstyle f}\downarrow & & {\scriptstyle p}\downarrow \\ U_{c_2} & \xrightarrow[\theta_g]{} & U_{\pi^p} & \xrightarrow[\phi_{\pi^p}|_{U_{\pi^p}}]{} & \text{Spec } \mathcal{O}_K[[t]] & \xrightarrow[\iota]{} & \mathbb{G}_{\mathrm{m}}, \end{array}$$

where the first vertical arrow f_Y denotes the first projection morphism $Y \to U_{c_2}$; the left-hand upper horizontal arrow θ_Y denotes the second projection morphism $Y \to U_\pi$; the vertical arrows are *finite flat morphisms of degree* p [cf. Proposition 1.5, Remark 1.5.1]; the first and third squares are *cartesian*; the second square is *cartesian*, up to taking the *normalization* of the fiber product that would make the second square "truly cartesian".

Remark 1.6.2. Let k be a field of characteristic p; n a positive integer. Write C for the Artin-Schreier curve over k defined by the equation $y^p - y = x^n$, where x and y are indeterminates; g_C for the genus of C. Then it follows immediately from Hurwitz's formula that

$$g_C = \frac{(n' - 1)(p - 1)}{2},$$

where n' denotes the greatest positive integer that divides n and is prime to p. In particular, if n is not a power of p, then $g_C \geq 1$. [Indeed, by considering the Frobenius morphism, one reduces immediately to the case where $n = n'$. Moreover, the computation of g_C is immediate when $n = 1$ in light of the form of the equation $y^p - y = x$. Thus, the computation of g_C reduces to the computation of the genus of a *tamely ramified* cyclic covering of the projective line of degree n whose ramification consists solely of $p+1$ totally ramified points.]

2 Resolution of nonsingularities for arbitrary hyperbolic curves over p-adic local fields

Let p be a prime number. In the present section, we apply certain constructions involving *p-divisible groups* to extend the Artin-Schreier coverings constructed locally in §1 to coverings of an arbitrary hyperbolic curve. As a consequence, we prove that arbitrary hyperbolic curves over p-adic local fields satisfy *RNS*, i.e., "*resolution of nonsingularities*" [cf. Definition 2.2, (vii); Theorem 2.17]. This result may be regarded as a generalization of results obtained by A. Tamagawa and E. Lepage [cf. [Tama2], Theorem 0.2; [Lpg1], Theorem 2.7]. Historically [cf., e.g., the discussion in the Introduction to [Tama2]], the roots of these results of Tamagawa and Lepage may be traced back to the technique of "passing to a covering with *singular reduction* of a given curve with smooth reduction over a p-adic local field" applied in the proof of [PrfGC], Theorem 9.2 [cf. also Proposition 2.3, (xii), below]. Moreover, the techniques of [AbsTopII], §2, may be regarded as a sort of *weak, pro-p version of Tamagawa's RNS* [cf. [AbsTopII], Remark 2.6.1]. In fact, the approach of the present section may be regarded as a sort of *amalgamation* of the techniques of [Lpg1] with the techniques of [AbsTopII], §2. At any rate, from a historical point of view, it is interesting to observe how various RNS results have been *motivated*

by and indeed are *deeply intertwined* with various results in *anabelian geometry* [cf. Corollary 2.5, as well as Theorems 3.12, 3.13 in §3 below].

First, we begin by fixing our conventions concerning models of hyperbolic curves [cf. [DM], Definition 1.1; [Knud], Definition 1.1].

Definition 2.1. Let K be a valuation field; X a hyperbolic curve over K; $\mathcal{X}$ a scheme over $\mathcal{O}_K$. Then:

(i) We shall say that $\mathcal{X}$ is a *compactified model* of X over $\mathcal{O}_K$ if $\mathcal{X}$ is a proper, flat, normal scheme of finite presentation over $\mathcal{O}_K$ whose generic fiber is the [uniquely determined, up to unique isomorphism] smooth compactification of X over K.

(ii) Suppose that the cusps of X are K-rational. Then we shall say that $\mathcal{X}$ is a *compactified semistable model* of X over $\mathcal{O}_K$ if $\mathcal{X}$ is a compactified model of X over $\mathcal{O}_K$ such that the following conditions hold:

- the geometric special fiber of $\mathcal{X}$ is a *semistable curve* [i.e., a reduced, connected curve each of whose nonsmooth points is an ordinary double point];
- the images of the sections $\mathrm{Spec}\ \mathcal{O}_K \to \mathcal{X}$ determined by the cusps of X [which we shall refer to as *cusps* of $\mathcal{X}$] lie in the smooth locus of $\mathcal{X}$ and do not intersect each other.

Suppose that $\mathcal{X}$ is a compactified semistable model of X over $\mathcal{O}_K$. Then we shall say that $\mathcal{X}$ has *split reduction* if $\mathcal{X}_s$ is *split* [i.e., each of the irreducible components and nodes of $\mathcal{X}_s$ is geometrically irreducible].

(iii) Suppose that $\mathcal{X}$ is a compactified semistable model of X over $\mathcal{O}_K$ that has split reduction. Let L be a finite extension of K equipped with a valuation that extends the valuation on K; $\mathcal{X}^*$ a compactified semistable model of X_L over $\mathcal{O}_L$ such that $\mathcal{X}^*$ has split reduction and dominates $\mathcal{X}$. Then we shall say that $\mathcal{X}^*$ is a *toral compactified semistable model* relative to $\mathcal{X}$ if each irreducible component of $\mathcal{X}^*_s$ that maps to a closed point of $\mathcal{X}_s$ [via the uniquely determined morphism $\mathcal{X}^* \to \mathcal{X}$] is normal of genus 0 and has precisely 2 nodes. Suppose that $\mathcal{X}^*$ is a toral compactified semistable model relative to $\mathcal{X}$. Let v be a vertex of the dual graph associated to $\mathcal{X}^*_s$. Then we shall say that v is a *toral semistable vertex* of $\mathcal{X}^*$ if v corresponds to an irreducible component of $\mathcal{X}^*_s$ that maps to a closed point of $\mathcal{X}$.

(iv) We shall say that $\mathcal{X}$ is a *compactified stable model* of X over $\mathcal{O}_K$ if $\mathcal{X}$ is a compactified semistable model of X over $\mathcal{O}_K$ such that $\mathcal{X}$, together with the cusps of $\mathcal{X}$, determines a pointed stable curve.

(v) We shall say that $\mathcal{X}$ is a *semistable model* of X over $\mathcal{O}_K$ if $\mathcal{X}$ is obtained by removing the cusps from a [uniquely determined, by Zariski's Main Theorem, up to unique isomorphism] compactified semistable model of X

over $\mathcal{O}_K$. Suppose that $\mathcal{X}$ is a semistable model of X over $\mathcal{O}_K$. Then we shall say that $\mathcal{X}$ has *split reduction* if $\mathcal{X}_s$ is *split* [i.e., each of the irreducible components and nodes of $\mathcal{X}_s$ is geometrically irreducible].

(vi) We shall say that $\mathcal{X}$ is a *stable model* of X over $\mathcal{O}_K$ if $\mathcal{X}$ is obtained by removing the cusps from a [uniquely determined, by Zariski's Main Theorem, up to unique isomorphism] compactified stable model of X over $\mathcal{O}_K$.

(vii) We shall say that X has *stable reduction* over K if there exists a [necessarily unique, up to unique isomorphism] stable model of X over $\mathcal{O}_K$. Suppose that $\mathcal{X}$ is a stable model of X over $\mathcal{O}_K$. Then we shall say that X has *split stable reduction* over K if $\mathcal{X}_s$ is *split* [i.e., each of the irreducible components and nodes of $\mathcal{X}_s$ is geometrically irreducible].

Remark 2.1.1. It follows from elementary commutative algebra/scheme theory [cf. [EGAIV$_2$], Corollaire 6.1.2; [EGAIV$_3$], Proposition 12.1.1.5] that any *compactified model* as in Definition 2.1, (i), is of *dimension* 2 whenever K is a complete discrete valuation field.

Remark 2.1.2. In the notation of Definition 2.1, suppose that X is a *proper hyperbolic curve* over K. Then it follows immediately from the various definitions involved that the notion of a *compactified semistable model* of X over $\mathcal{O}_K$ coincides with the notion of a *semistable model* of X over $\mathcal{O}_K$.

Remark 2.1.3. In the notation of Definition 2.1, suppose that K is a complete discrete valuation field, and that X has split stable reduction over K. Let $\mathcal{X}$ be a compactified semistable model of X over $\mathcal{O}_K$ that has split reduction; $\phi : \mathcal{Y} \to \mathcal{X}$ a morphism of compactified semistable models over $\mathcal{O}_K$ that restricts to a connected finite étale covering $Y \to X$ over K. Then one verifies immediately — by considering the map induced on irreducible components of the respective special fibers by the necessarily *finite*, hence *surjective* morphism [induced by $\mathcal{Y} \to \mathcal{X}$] between the [*two-dimensional, normal, integral*] spectra of the completions of the local rings of $\mathcal{X}$, $\mathcal{Y}$ at the closed points under consideration — that *ϕ always maps a smooth closed point of $\mathcal{Y}$ that is isolated in the fiber of ϕ to a smooth closed point of $\mathcal{X}$*.

Remark 2.1.4. In the notation of Remark 2.1.3, let $\mathcal{X}^*$ be a toral compactified semistable model relative to $\mathcal{X}$; e an edge of the dual graph associated to $\mathcal{X}_s$; b a branch of the node e; v a toral semistable vertex of $\mathcal{X}^*$ that maps to [i.e., for which the corresponding irreducible component maps to the node corresponding to] e. Recall that the completion $\widehat{\mathcal{O}}_e$ of the local ring of $\mathcal{X}$ at e is isomorphic to $\mathcal{O}_K[[x, y]]/(xy - a)$, where $a \in \mathfrak{m}_K \setminus \{0\}$, and x, y denote indeterminates chosen so that the ideal (x) $(\subseteq \mathcal{O}_K[[x, y]]/(xy - a))$ corresponds to b. Observe that

this ideal (x) is independent of the choice of x, y [cf. [Hur], §3.7, Lemma]. In particular, we obtain a homomorphism of local rings

$$\psi : \mathcal{O}_K[[x, y]]/(xy - a) \longrightarrow \widehat{\mathcal{O}}_v$$

where $\widehat{\mathcal{O}}_v$ denotes the completion of the local ring of $\mathcal{X}^*$ at the generic point of the irreducible component of $\mathcal{X}^*$ corresponding to v. Write $\mathrm{ord}_v(-)$ for the normalized valuation associated to $\widehat{\mathcal{O}}_v$ whose normalization is determined by the condition that $\mathrm{ord}_v(p) = 1 \in \mathbb{R}$. Thus, we obtain a rational number

$$0 \ < \ \rho_{b,v} \ \stackrel{\text{def}}{=} \ \frac{\mathrm{ord}_v(\psi(x))}{\mathrm{ord}_v(\psi(a))} \ < \ 1$$

associated to b and v, which is in fact independent of the normalization of "$\mathrm{ord}_v(-)$". If, moreover, we write b' for the other branch of e, then one verifies immediately that $\rho_{b,v} + \rho_{b',v} = 1$. Finally, we *observe* that

> given any rational number ρ such that $0 < \rho < 1$, there exist, after possibly replacing K by a suitable finite extension field of K, an $\mathcal{X}^*$ and v as above such that $\rho_{b,v} = \rho$.

Indeed, it follows immediately from the theory of *pointed stable curves*, as exposed in [Knud], that, by possibly replacing K by a suitable finite extension field of K and X by a suitable dense open subscheme of X, we may assume that $\mathcal{X}$ is the [unique, up to unique isomorphism] *compactified stable model* of X over $\mathcal{O}_K$, and that ρ may be written as a fraction whose denominator divides the positive integer $v_p(a) \cdot v_p(\pi_K)^{-1}$, where π_K is a uniformizer of $\mathcal{O}_K$. Then it follows again from the theory of *pointed stable curves*, as exposed in [Knud], that, if we take

- $\mathcal{X}^*$ to be the [unique, up to unique isomorphism] *compactified stable model* over $\mathcal{O}_K$ of the hyperbolic curve obtained by removing from X a *suitable K-rational point* of $\mathcal{X}$ with center at e [cf. the construction of the displayed homomorphism "$(\widehat{\mathcal{O}}_{\mathcal{X}^\dagger,x} \stackrel{\sim}{\to}) \ \mathcal{O}_K[[s,t]]/(st - \pi_K^r) \to \mathcal{O}_K$" in the portion of the proof of Proposition 2.3, (iii), below, concerning the case where "x is a *nonsmooth closed point* of $\mathcal{X}^\dagger$"] and
- v to be the *unique irreducible component* of $\mathcal{X}^*_s$ that maps to a *closed point* of $\mathcal{X}_s$ via the natural morphism $\mathcal{X}^* \to \mathcal{X}$ [cf. the morphism "$\mathcal{X}^\dagger[x] \to \mathcal{X}^\dagger$" in the portion of the proof of Proposition 2.3, (iii), below, concerning the case where "x is a *nonsmooth closed point* of $\mathcal{X}^\dagger$"],

then $\rho_{b,v} = \rho$, as desired.

Remark 2.1.5. In the notation of Remark 2.1.4, we *observe* that, for a fixed choice of b,

the assignment

$$v \quad \mapsto \quad \rho_{b,v} \in \mathbb{Q}$$

that assigns to a *toral semistable vertex* v *of* $\mathcal{X}^*$ *that maps to* e the rational number $\rho_{b,v}$ is *injective*.

Indeed, it follows immediately from the theory of *pointed stable curves*, as exposed in [Knud] — i.e., by *adding finitely many suitably positioned cusps* and then considering the various contraction morphisms that arise from eliminating cusps — that, to verify the asserted *injectivity*, it suffices to show that if v and w are the *unique* toral semistable vertices of $\mathcal{X}^*$ that map to e, which implies that there exists an edge e^* of the dual graph associated to $\mathcal{X}^*_s$ that abuts to v and w, then $\rho_{b,v} \neq \rho_{b,w}$. To this end, we recall that $\mathcal{X}$ and $\mathcal{X}^*$ admit *natural log structures* [determined by the respective multiplicative monoids of regular functions invertible outside the respective special fibers $\mathcal{X}_s$, $\mathcal{X}^*_s$] such that the morphism $\mathcal{X}^* \to \mathcal{X}$ extends uniquely to a *morphism of log schemes* [cf., e.g., the subsection in Notations and Conventions entitled "Log schemes"; the discussion of [Hur], §3.7, §3.8, §3.10]. Moreover, the completion $\widehat{\mathcal{O}}_{e^*}$ of the local ring of $\mathcal{X}^*$ at e^* is isomorphic to $\mathcal{O}_K[[x^*, y^*]]/(x^*y^* - a^*)$, where $a^* \in \mathcal{O}_K$, and x^*, y^* denote indeterminates which may be chosen in such a way that the homomorphism of topological $\mathcal{O}_K$-algebras $\widehat{\mathcal{O}}_e \to \widehat{\mathcal{O}}_{e^*}$ induced by $\mathcal{X}^* \to \mathcal{X}$ maps $x \mapsto (x^*)^N \cdot (\pi^*)^M \cdot u$, for some unit $u \in \widehat{\mathcal{O}}^\times_{e^*}$, some uniformizer $\pi^* \in \mathcal{O}_K$, some positive integer N, and some nonnegative integer M. [Indeed, the fact that N is necessarily *positive* follows immediately, in light of our assumptions on v and w, from well-known considerations in *intersection theory* on the $\mathbb{Q}$-*factorial* normal schemes $\mathcal{X}^*$ and $\mathcal{X}$.] Then the desired inequality $\rho_{b,v} \neq \rho_{b,w}$ follows immediately from the fact that, up to a possible permutation of the labels "v" and "w", it holds that x^* is *invertible* at the generic point of [the irreducible component corresponding to] v, but *non-invertible* at the generic point of [the irreducible component corresponding to] w.

Remark 2.1.6. In the notation of Remark 2.1.3, suppose further that $\mathcal{Y}$ has split reduction, and that the morphism $Y \to X$ is an *isomorphism*. Let $\mathcal{X}^*$ be a toral compactified semistable model relative to $\mathcal{X}$, $\mathcal{Y} \to \mathcal{X}^*$ a morphism over $\mathcal{X}$. Then *observe* that it follows immediately from Remark 2.1.3, together with Zariski's Main Theorem, that each normal irreducible component of $\mathcal{Y}_s$ that

- maps to a closed point of $\mathcal{X}_s$,
- is of genus 0, and
- has precisely 1 node

maps to a closed point of $\mathcal{X}^*_s$. In particular, it follows immediately from an iterated application of the above *observation*, together with the theory of *pointed stable curves*, as exposed in [Knud] — i.e., by *adding finitely many suitably positioned cusps* and then considering the various contraction morphisms that arise from eliminating cusps — that

there exists a unique, up to unique isomorphism, toral compactified semistable model $\mathcal{Y}^*$ relative to $\mathcal{X}$, together with a uniquely determined morphism $\mathcal{Y} \to \mathcal{Y}^*$ of compactified semistable models over $\mathcal{X}$, such that the following *universal property* is satisfied: if $\mathcal{X}^\dagger$ is a toral compactified semistable model relative to $\mathcal{X}$ such that the morphism $\mathcal{Y} \to \mathcal{X}$ admits a factorization $\mathcal{Y} \to \mathcal{X}^\dagger \to \mathcal{X}$, then the morphism $\mathcal{Y} \to \mathcal{X}^\dagger$ admits a unique factorization $\mathcal{Y} \to \mathcal{Y}^* \to \mathcal{X}^\dagger$.

That is to say, $\mathcal{Y}^*$ may be thought of as a sort of *"universal toralization [over $\mathcal{X}$]"* of $\mathcal{Y}$. In particular, it follows immediately from the existence of *universal toralizations*, together with the theory of *pointed stable curves*, as exposed in [Knud] — i.e., by *adding finitely many suitably positioned cusps* and then considering the various contraction morphisms that arise from eliminating cusps — that the toral compactified semistable models relative to $\mathcal{X}$ form a *directed inverse system*.

Definition 2.2. Let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty subset; K a mixed characteristic complete discrete valuation field of residue characteristic p; X a hyperbolic curve over K. Write Ω for the p-adic completion of [some fixed] $\overline{K}$. Then:

(i) Let v be a valuation on a field F that contains K. Write $\mathcal{O}_v$ for the ring of integers determined by v; $\mathfrak{m}_v \subseteq \mathcal{O}_v$ for the maximal ideal of $\mathcal{O}_v$. Then we shall say that v is a *p-valuation [over K]* if $\mathcal{O}_K = \mathcal{O}_v \cap K$ [which implies that $p \in \mathfrak{m}_v$]. Here, the phrase *"over K"* will be omitted in situations where the base field K is fixed throughout the discussion. We shall say that v is *primitive* if it is a p-valuation such that the only prime ideal of $\mathcal{O}_v$ that contains p is $\mathfrak{m}_v$. We shall say that v is *residue-transcendental* if it is a p-valuation whose residue field $k_v \stackrel{\text{def}}{=} \mathcal{O}_v/\mathfrak{m}_v$ is a transcendental extension of the residue field of K.

(ii) In the situation of (i), suppose that v is a *p-valuation*, that $\widetilde{X}$ is a *universal geometrically pro-Σ covering* of X, and that F is a *subfield* [in a fashion compatible with the given inclusion $K \hookrightarrow F$] of the *function field* $K(\widetilde{X}_\Omega)$ of $\widetilde{X}_\Omega$. Then we shall say that v is *point-theoretic* if it arises from some point $\tilde{x} \in \widetilde{X}(\Omega)$, i.e., if $\mathcal{O}_v \subseteq F$ is equal to the subring of F consisting of elements $\in F$ that determine rational functions on $\widetilde{X}_\Omega$ that are regular at $\tilde{x}$, and whose value at $\tilde{x}$ is contained in $\mathcal{O}_\Omega \subseteq \Omega$. Thus, every point $\tilde{x} \in \widetilde{X}(\Omega)$ determines a corresponding *point-theoretic valuation* of F.

(iii) In the situation of (ii), let $Z \to X$ be a connected finite étale covering equipped with a factorization $\widetilde{X} \to Z \to X$. Let $\mathcal{Z}$ be a compactified semistable model of Z with split reduction. Then we shall write

$$\mathbb{VE}(\mathcal{Z})$$

for the finite set [equipped with the discrete topology] of vertices and edges of the dual graph associated to $\mathcal{Z}_s$. Note that there is a notion of

specialization/generization among elements of $\mathbb{VE}(\mathcal{Z})$, i.e., we shall say that

- a vertex *specializes* to a node, or, alternatively, that a node *generizes* to a vertex, if the node abuts to the vertex;
- a vertex *specializes/generizes* to a vertex if the two vertices coincide;
- an edge *specializes/generizes* to an edge if the two edges coincide.

For $c_1, c_2 \in \mathbb{VE}(\mathcal{Z})$, if c_1 *specializes* to c_2, or, equivalently, c_2 *generizes* to c_1, then we shall write $c_1 \rightsquigarrow c_2$. By allowing Z and $\mathcal{Z}$ to vary, we thus obtain a topological space

$$\mathbb{VE}(\widetilde{X}) \stackrel{\text{def}}{=} \varprojlim_{\mathcal{Z}} \mathbb{VE}(\mathcal{Z}),$$

where the transition maps in the inverse limit are induced by the corresponding scheme-theoretic morphisms of compactified semistable models [which form a *directed inverse system* — cf. Proposition 2.3, (iii), below], that is to say, by mapping a vertex [i.e., irreducible component] or edge [i.e., node] to the smallest vertex [i.e., irreducible component] or edge [i.e., node] that contains its scheme-theoretic image. [Here, we recall that any such morphism of compactified semistable models always maps a smooth closed point that is isolated in the fiber of the morphism to a smooth closed point — cf. Remark 2.1.3.] We shall refer to an element of $\mathbb{VE}(\widetilde{X})$ as a *VE-chain* of $\widetilde{X}$. Note that the notion of specialization/generization among elements of each $\mathbb{VE}(\mathcal{Z})$ determines [i.e., by considering each constituent set in the above inverse limit] a notion of *specialization/generization* among elements of $\mathbb{VE}(\widetilde{X})$. We shall say that an element $c \in \mathbb{VE}(\widetilde{X})$ is *primitive* if every generization of c is equal to c.

(iv) In the situation of (iii), let $z_{c_1}, z_{c_2} \in \mathbb{VE}(\mathcal{Z})$. Then we shall write

$$\delta(z_{c_1}, z_{c_2}) \in \mathbb{Z}$$

for the integer δ such that the set of vertices contained in a path of minimal length between z_{c_1} and z_{c_2} on the dual graph of $\mathcal{Z}_s$ is of cardinality $\delta + 1$. Let $c_1, c_2 \in \mathbb{VE}(\widetilde{X})$. Then we shall write

$$\delta(c_1, c_2) \stackrel{\text{def}}{=} \sup_{\mathcal{Z}} \delta(w_{c_1}, w_{c_2}) \in \mathbb{Z} \cup \{+\infty\},$$

where $\mathcal{Z}$ ranges over the set of compactified semistable models with split reduction of connected finite étale coverings $Z \to X$ equipped with a factorization $\widetilde{X} \to Z \to X$; $w_{c_i} \in \mathbb{VE}(\mathcal{Z})$ denotes the element determined by c_i for each $i = 1, 2$.

(v) In the situation of (ii), suppose further that F contains the function field $K(\widetilde{X})$ of $\widetilde{X}$. Then observe that v determines, by considering the *centers*

associated to v on the various "$\mathcal{Z}$" in the discussion of (iii), an element $\in \mathbb{VE}(\widetilde{X})$, which we shall refer to as the *center-chain associated to* v. In particular, any point $\tilde{x} \in \widetilde{X}(\Omega)$ determines, by considering the point-theoretic valuation associated to $\tilde{x}$, an element $\in \mathbb{VE}(\widetilde{X})$, which we shall refer to as the *[point-theoretic] center-chain associated to* $\tilde{x}$. Write $x \in X(\Omega)$ for the image of $\tilde{x}$ in $X(\Omega)$. Thus, the $\mathrm{Gal}(\widetilde{X}/X_{\overline{K}})$-orbit of $\tilde{x}$ is completely determined by x. We shall refer to the $\mathrm{Gal}(\widetilde{X}/X_{\overline{K}})$-orbit of the center-chain associated to $\tilde{x}$ as the *[point-theoretic] orbit-center-chain associated to* x [cf. the discussion of Remark 2.2.4 below].

(vi) In the situation of (ii), let $Z \to X$ be a connected finite étale covering equipped with a factorization $\widetilde{X} \to Z \to X$. Let $\mathcal{Z}$ be a compactified semistable model with split reduction of Z. In the remainder of the discussion of the present item (vi), all toral compactified semistable models relative to $\mathcal{Z}$ will be assumed to have generic fibers that are equipped with the structure of a subcovering of the pro-covering $\widetilde{X} \to Z$. Write $\mathcal{V}(\mathcal{Z})$ (respectively, $\mathcal{E}(\mathcal{Z})$) for the set of vertices (respectively, edges) of $\mathcal{Z}_s$. Thus, $\mathbb{VE}(\mathcal{Z}) = \mathcal{V}(\mathcal{Z}) \coprod \mathcal{E}(\mathcal{Z})$. For each $c \in \mathbb{VE}(\mathcal{Z})$, write

$$\mathcal{V}_c$$

for the set of equivalence classes of the set of vertices of toral compactified semistable models relative to $\mathcal{Z}$ that map to the closed subscheme of $\mathcal{Z}_s$ corresponding to c, where we apply the equivalence relation induced by the dominant morphisms over $\mathcal{Z}$ of toral compactified semistable models relative to $\mathcal{Z}$. Let $e \in \mathcal{E}(\mathcal{Z})$. Then we shall write

$$\overline{\mathcal{V}}_e$$

for the union of $\mathcal{V}_e$ and the vertices of $\mathcal{V}(\mathcal{Z})$ that abut to e. Observe that, for each toral compactified semistable model $\mathcal{Z}^\dagger$ relative to $\mathcal{Z}$ and each $c^\dagger \in \mathbb{VE}(\mathcal{Z}^\dagger)$ that maps to the closed subscheme of $\mathcal{Z}_s$ corresponding to c, the set $\mathcal{V}_{c^\dagger}$ may be regarded, in a natural way [i.e., by considering the maps induced by dominant morphisms over $\mathcal{Z}$ of toral compactified semistable models relative to $\mathcal{Z}$], as a subset of $\mathcal{V}_c$. We shall refer to such a subset $\mathcal{V}_{c^\dagger} \subseteq \mathcal{V}_c$ as a *basic open subset* of $\mathcal{V}_c$. Thus, from the point of view of the *natural bijection*, determined by selecting a branch b of the edge e, between $\mathcal{V}_e$ and the set of *rational numbers* ρ such that $0 < \rho < 1$ [cf. Remarks 2.1.4, 2.1.5, 2.1.6], the basic open subsets $\subseteq \mathcal{V}_e$ correspond precisely to the open intervals with rational endpoints of $\mathcal{V}_e$. In particular, it is natural to regard $\mathcal{V}_c$ as being equipped with the *topology* determined by the open basis consisting of the basic open subsets $\subseteq \mathcal{V}_c$. We shall refer to as a *quasi-basic open subset* of $\mathcal{V}_e$ any open subset of $\mathcal{V}_e$ which is a union of a countable collection of basic open subsets $\subseteq \mathcal{V}_e$ for which the relation of inclusion determines a total ordering. We shall refer to as a *Dedekind cut* of $\mathcal{V}_e$ an unordered pair $\{D_1, D_2\}$ of disjoint nonempty

quasi-basic open subsets $D_1, D_2 \subseteq \mathcal{V}_e$ such that $\mathcal{V}_e = D_1 \cup D_2$. Write $\mathcal{D}_e$ for the set of Dedekind cuts of $\mathcal{V}_e$. Note that the topology of the $\mathcal{V}_w$, where $w \in \overline{\mathcal{V}}_e \setminus \mathcal{V}_e$, induces, in a natural way, a *topology* on the set

$$\overline{\mathcal{T}}_e \stackrel{\text{def}}{=} \overline{\mathcal{V}}_e \coprod \mathcal{D}_e$$

[i.e., by taking as an open basis for the topology for $\overline{\mathcal{T}}_e$ the subsets of $\overline{\mathcal{T}}_e$ obtained as the intersections with $\overline{\mathcal{T}}_e$ of unions of an open subset $U \subseteq \mathcal{V}_w$, where $w \in \overline{\mathcal{V}}_e \setminus \mathcal{V}_e$, with the set of Dedekind cuts $\{D_1, D_2\} \in \mathcal{D}_e$ such that both D_1 and D_2 intersect U]. Thus,

$$\mathcal{T}_e \stackrel{\text{def}}{=} \mathcal{V}_e \coprod \mathcal{D}_e.$$

[with the induced topology] is *homeomorphic to the open interval* $(0, 1) \subseteq \mathbb{R}$ of the real line [cf. Remarks 2.1.4, 2.1.5, 2.1.6]. Write

$$\mathbb{VE}(\mathcal{Z})^{\text{tor}} \stackrel{\text{def}}{=} \mathcal{V}(\mathcal{Z}) \cup \left\{ \bigcup_{e \in \mathcal{E}(\mathcal{Z})} \overline{\mathcal{T}}_e \right\}.$$

Thus, the discrete topology on $\mathcal{V}(\mathcal{Z})$, together with the topologies defined above on the $\overline{\mathcal{T}}_e$, determine a *topology* on $\mathbb{VE}(\mathcal{Z})^{\text{tor}}$. Moreover, there exists a *noncontinuous* [cf. Remark 2.2.1 below] *natural surjective map*

$$\epsilon_{\mathcal{Z}} : \mathbb{VE}(\mathcal{Z})^{\text{tor}} \longrightarrow \mathbb{VE}(\mathcal{Z})$$

that maps each $\mathcal{T}_e$ to e. Finally, by allowing Z and $\mathcal{Z}$ to vary, we thus obtain a topological space

$$\mathbb{VE}(\widetilde{X})^{\text{tor}} \stackrel{\text{def}}{=} \varprojlim_{\mathcal{Z}} \mathbb{VE}(\mathcal{Z})^{\text{tor}}$$

[cf. the discussion of (iii)], together with a *natural* [not necessarily surjective — cf. Proposition 2.3, (viii), below!] *map*

$$\epsilon_{\widetilde{X}} : \mathbb{VE}(\widetilde{X})^{\text{tor}} \longrightarrow \mathbb{VE}(\widetilde{X}).$$

(vii) We shall say that X satisfies Σ-*RNS* [i.e., "Σ-*resolution of nonsingularities*" — cf. [Lpg1], Definition 2.1] if the following condition holds:

> Let v be a discrete residue-transcendental p-valuation on the function field $K(X)$ of X. Then there exists a connected geometrically pro-Σ finite étale Galois covering $Y \to X$ such that Y has stable reduction [over *its* base field], and v coincides with the restriction [to $K(X)$] of a discrete valuation on the function field $K(Y)$ of Y that arises from an irreducible component of the special fiber of the stable model of Y.

Remark 2.2.1. In the notation of Definition 2.2, (vi), the natural surjective map $\epsilon_{\mathcal{Z}}$ is *not continuous* in general. Indeed, to see this, it suffices to observe that the inverse image of the closed subset consisting of a single edge is an open subset of $\mathbb{VE}^{\mathrm{tor}}(\mathcal{Z})$ that is not closed. Finally, we observe that one may also conclude from this noncontinuity of $\epsilon_{\mathcal{Z}}$ that $\epsilon_{\widetilde{X}}$ is not continuous.

Remark 2.2.2. In the notation of Definition 2.2, (vii), suppose that $\Sigma \setminus \{p\}$ is nonempty, and that X satisfies Σ-RNS. Then, by considering a suitable admissible covering of the stable model of "Y" as in Definition 2.2, (vii), one verifies immediately that one may assume that the normalization of the irreducible component that appears in Definition 2.2, (vii), is of genus ≥ 2.

Remark 2.2.3. In the notation of Definition 2.2, (vii), we make the following observations.

(i) Let $L \subseteq \Omega$ be a topological subfield containing K that arises as the [topological] field of fractions of a mixed characteristic complete discrete valuation ring of residue characteristic p. Then let us *observe* that

> any compactified semistable model of X_L over $\mathcal{O}_L$ arises, after possibly replacing K and L, respectively, by suitable finite extension fields of K and L, as the result of base-changing, from $\mathcal{O}_K$ to $\mathcal{O}_L$, some compactified semistable model of X_K over $\mathcal{O}_K$.

Indeed, since every element of L admits arbitrarily close p-adic approximations by elements of finite extension fields of K contained in $\overline{K}$, this *observation* follows immediately by noting that it follows immediately from the well-known theory of *pointed stable curves*, as exposed in [Knud], that, after possibly replacing K and L, respectively, by suitable finite extension fields of K and L and possibly replacing X by some dense open subscheme of X, we may assume without loss of generality that the given compactified semistable model of X_L over $\mathcal{O}_L$ is in fact the [unique, up to unique isomorphism] compactified stable model of X_L, i.e., which necessarily arises as the result of base-changing, from $\mathcal{O}_K$ to $\mathcal{O}_L$, the [unique, up to unique isomorphism] compactified stable model of X_K over $\mathcal{O}_K$.

(ii) We maintain the notation of (i). Then let us *observe* that

> X satisfies Σ-RNS if and only if X_L satisfies Σ-RNS.

Indeed, this *observation* follows immediately, in light of the *observation* of (i), from Proposition 2.3, (ii), (iii), below; Proposition 2.4, (iv), below [cf. also Definition 2.2, (vii), as well as the discussion of the final portion of the subsection in Notations and Conventions entitled "Fundamental groups"].

(iii) Next, let L be a mixed characteristic complete discrete valuation field of residue characteristic p that contains K as a topological subfield. Then *observe* that it follows immediately from the well-known elementary theory of complete discrete valuation fields that

L is isomorphic, as a topological K-algebra, to a field "L" of the sort discussed in (i), (ii) if and only if [the valuation on] L is *not residue-transcendental* [relative to K], i.e., if and only if the residue field of L is an *algebraic* extension of the residue field of K.

(iv) We maintain the notation of (iii). Then let us *observe* that

> if X_L satisfies Σ-RNS, then the residue field of L is an *algebraic* extension of the residue field of K.

Indeed, it suffices to verify this *observation* after replacing K by a finite extension field of K. In particular, we may assume without loss of generality [cf. Proposition 2.3, (iii)] that there exists a compactified semistable model $\mathcal{X}$ of X over $\mathcal{O}_K$, and that the residue field of L is a *non-algebraic* extension of the residue field of K. Then it follows immediately from the uniqueness, up to unique isomorphism, of compactified stable models [cf. also Definition 2.2, (vii), as well as the discussion of the final portion of the subsection in Notations and Conventions entitled "Fundamental groups"; the *stable reduction theorem* of [DM], [Knud]], that if X_L satisfies Σ-RNS, then any closed point of $(\mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_L)_s$ that arises as the *center* of a discrete residue-transcendental p-valuation on the function field of X_L is necessarily defined over some algebraic extension of the residue field of K. On the other hand, this contradicts the existence of discrete residue-transcendental p-valuations on the function field of X_L that arise as the local rings of generic points of exceptional divisors of blow-ups of smooth closed points of $(\mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_L)_s$ that are *not* defined over some algebraic extension of the residue field of K. This completes the proof of the above *observation*.

(v) We maintain the notation of (iii). Then we *observe* further that, under the assumption that X satisfies Σ-RNS, it holds that

> the residue field of L is an *algebraic* extension of the residue field of K if and only if X_L satisfies Σ-RNS.

Indeed, *necessity* follows formally from the observations of (ii), (iii), while *sufficiency* follows formally from the observation of (iv).

Remark 2.2.4. In the context of Definition 2.2, we recall from the *general theory of valuations* the following well-known basic facts. Let L be a *field* equipped with a *valuation* v, M a *finite normal extension field* of L. Write $\mathcal{O}_v$ for the ring of integers of L with respect to v, $\mathfrak{m}_v \subseteq \mathcal{O}_v$ for the maximal ideal of $\mathcal{O}_v$, $\mathcal{O}_M$ for the integral closure of $\mathcal{O}_v$ in M, $v_{M/L}$ for the set of valuations on M that extend v, and $\mathrm{Aut}(M/L)$ for the group of automorphisms of M that restrict to the identity on L. If $w \in v_{M/L}$, then we shall write $\mathcal{O}_w$ for the ring of integers of M with respect to w, $\mathfrak{m}_w \subseteq \mathcal{O}_w$ for the maximal ideal of $\mathcal{O}_w$, $\mathfrak{p}_w \stackrel{\text{def}}{=} \mathcal{O}_M \cap \mathfrak{m}_w$.

Then the set $v_{M/L}$ is *nonempty* [cf. [EP], Theorem 3.1.1], and the natural action of $\mathrm{Aut}(M/L)$ on $v_{M/L}$ is *transitive* [cf. [EP], Theorem 3.2.14]. Moreover,

$$\mathcal{O}_M \quad = \quad \bigcap_{w \in v_{M/L}} \mathcal{O}_w$$

[cf. [EP], Theorem 3.1.3, (2)]; the assignment $v_{M/L} \ni w \mapsto \mathfrak{p}_w$ determines a bijective correspondence between $v_{M/L}$ and the set of prime ideals of $\mathcal{O}_M$ that lie over $\mathfrak{m}_v$, and, for $w \in v_{M/L}$, $\mathcal{O}_w = (\mathcal{O}_M)_{\mathfrak{p}_w}$ [cf. [EP], Theorem 3.1.1; [EP], Theorem 3.2.13]. In this situation, if we assume further that v is *real*, and that L is *complete* with respect to v, then $v_{M/L}$ is of *cardinality* 1 [cf. [Neu], Chapter II, Theorem 4.8]. [Here, we recall that if v is *not real*, then $\mathcal{O}_v$ does *not*, in general, satisfy *Hensel's Lemma*, i.e., even if L is complete with respect to v [cf. [EP], Remark 2.4.6].] More generally, if v is *real*, then L admits a natural *completion* $\widehat{L}$ [cf. [EP], Theorem 1.1.4], which is a *henselian field* [cf. [Neu], Chapter II, Theorem 4.8; the discussion preceding [EP], Lemma 4.1.1] and contains, up to natural isomorphism, the *henselization* L^{h} of L [cf. [EP], the discussion preceding Theorem 5.2.2] as a *subfield*, i.e.,

$$L^{\mathrm{h}} \subseteq \widehat{L}$$

[cf. [EP], Corollary 4.1.5; [EP], Corollary 5.2.3; the discussion of Case 2 in the proof of [EP], Theorem 6.3.1].

The various basic properties stated in the following Proposition 2.3 consist of elementary results that are essentially well-known or implicit in the literature [cf. Remarks 2.3.2, 2.3.3 below], but we give [essentially] self-contained statements and proofs here in the language of the present discussion for the sake of completeness.

Proposition 2.3 (**Basic properties of models of hyperbolic curves**). *Let K be a mixed characteristic complete discrete valuation field of residue characteristic p; X a hyperbolic curve over K. Write $K(X)$ for the function field of X. Then the following hold:*

(i) Let $R \subseteq K(X)$ be a finitely generated normal $\mathcal{O}_K$-subalgebra whose field of fractions coincides with $K(X)$. Then Spec *R arises as an open subscheme of a compactified model of X over $\mathcal{O}_K$.*

(ii) Let v be a discrete residue-transcendental [cf. Remark 2.3.1 below] p-valuation on $K(X)$. Then v arises as the discrete valuation associated to an irreducible component of the special fiber of a compactified model of X over $\mathcal{O}_K$.

(iii) Let $\mathcal{X}$ be a compactified model of X over $\mathcal{O}_K$ equipped with the action of a finite group G by $\mathcal{O}_K$-linear automorphisms [which thus restrict to K-linear automorphisms of X]. Then, after possibly replacing K by a finite field extension of K, there exists a compactified semistable model of X over $\mathcal{O}_K$ that dominates $\mathcal{X}$ and is stabilized by the action of G on X.

(iv) *Let $Y \to X$ be a [connected] finite étale Galois covering of hyperbolic curves over K, $\mathcal{Y}^{\mathrm{sst}}$ a compactified semistable model of Y over $\mathcal{O}_K$ that is stabilized by $G \stackrel{\mathrm{def}}{=} \mathrm{Gal}(Y/X)$. Write $\mathcal{X}$ for the quotient of $\mathcal{Y}^{\mathrm{sst}}$ by the natural action of G on $\mathcal{Y}^{\mathrm{sst}}$. Then $\mathcal{X}$ is a compactified semistable model of X over $\mathcal{O}_K$, and the images of smooth points of $\mathcal{Y}^{\mathrm{sst}}_s$ via the natural morphism $\mathcal{Y}^{\mathrm{sst}}_s \to \mathcal{X}_s$ are smooth points of $\mathcal{X}_s$. Moreover, the image of a node of $\mathcal{Y}^{\mathrm{sst}}_s$ via the natural morphism $\mathcal{Y}^{\mathrm{sst}}_s \to \mathcal{X}_s$ is a node of $\mathcal{X}_s$ if and only if G does not permute the branches of the node. In particular,*

- *the dual graph of $\mathcal{X}_s$ may be reconstructed from the dual graph of $\mathcal{Y}^{\mathrm{sst}}_s$, together with the action of G on the dual graph of $\mathcal{Y}^{\mathrm{sst}}_s$.*

Finally,

- *this reconstruction procedure is functorial, with respect to maps of vertices/edges to vertices/edges [i.e., as in the discussion of Definition 2.2, (iii)], on the category of [connected] finite étale Galois coverings of X over K.*

(v) *Suppose that we are in the situation of Definition 2.2, (ii), (iii), (iv), (v). Then the assignment that maps a p-valuation on $K(\widetilde{X})$ to its associated center-chain determines bijections as follows:*

$$\left\{p\text{-valuations on } K(\widetilde{X})\right\} \stackrel{\sim}{\to} \mathbb{VE}(\widetilde{X}),$$

$$\left\{\text{primitive } p\text{-valuations on } K(\widetilde{X})\right\} \stackrel{\sim}{\to} \mathbb{VE}(\widetilde{X})^{\mathrm{prim}},$$

$$\widetilde{X}(\Omega) \stackrel{\sim}{\to} \mathbb{VE}(\widetilde{X})^{\text{pt-th}},$$

$$X(\Omega) \stackrel{\sim}{\to} \mathbb{VE}(\widetilde{X})^{\text{pt-th}}/\mathrm{Gal}(\widetilde{X}/X_{\overline{K}}),$$

where $\mathbb{VE}(\widetilde{X})^{\mathrm{prim}} \subseteq \mathbb{VE}(\widetilde{X})$ denotes the subset of primitive VE-chains, and $\mathbb{VE}(\widetilde{X})^{\text{pt-th}} \subseteq \mathbb{VE}(\widetilde{X})$ denotes the subset of point-theoretic center-chains.

(vi) *Suppose that we are in the situation of (v). Let $c \in \mathbb{VE}(\widetilde{X})$. Write $R_c \subseteq K(\widetilde{X})$ for the valuation ring of the p-valuation associated to c [cf. (v)]. Then it holds that*

$$R_c \;=\; \varinjlim_{\mathcal{Z}} \mathcal{O}_{\mathcal{Z},z_c},$$

where the direct limit ranges over the set of compactified semistable models with split reduction $\mathcal{Z}$ of the domain curves of connected finite étale coverings $Z \to X$ equipped with a factorization $\widetilde{X} \to Z \to X$; z_c denotes the center on $\mathcal{Z}$ determined by R_c; the transition maps in the direct limit are induced by the corresponding scheme-theoretic morphisms of compactified semistable models [cf. the discussion of Definition 2.2, (iii)].

(vii) *Suppose that we are in the situation of (v). Suppose, moreover, that X is proper. Then a p-valuation of $K(\widetilde{X})$ is primitive if and only if it is either real or point-theoretic. Equivalently,*

- *the set of primitive p-valuations of $K(\widetilde{X})$ consists of the disjoint union of the non-point-theoretic real p-valuations of $K(\widetilde{X})$ and the point-theoretic p-valuations of $K(\widetilde{X})$.*

In particular, if we write $\widetilde{X}^{\mathrm{an}}$ for the topological pro-Berkovich space associated to [i.e., the inverse limit of the underlying topological spaces of the Berkovich spaces associated to the finite subcoverings of] $\widetilde{X}$, then

- *the set of primitive p-valuations of $K(\widetilde{X})$ may be naturally identified with the underlying set of $\widetilde{X}^{\mathrm{an}}$.*

(viii) *Suppose that we are in the situation of (vii). Then there exists a natural commutative diagram of maps of sets*

$$\begin{array}{ccc} \widetilde{X}^{\mathrm{an}} & \xrightarrow[\theta_{\widetilde{X}}]{\sim} & \mathbb{VE}(\widetilde{X})^{\mathrm{tor}} \\ \big\downarrow \wr & & \epsilon_{\widetilde{X}} \big\downarrow \\ \mathbb{VE}(\widetilde{X})^{\mathrm{prim}} & \xrightarrow[\iota_{\widetilde{X}}]{} & \mathbb{VE}(\widetilde{X}), \end{array}$$

where the upper horizontal arrow $\theta_{\widetilde{X}}$ is a homeomorphism [cf. Remark 2.3.3 below]; the lower horizontal arrow $\iota_{\widetilde{X}}$ denotes the natural inclusion; the left-hand vertical arrow denotes the bijection obtained by forming the composite of the natural identification that appears in the statement of (vi) with the second bijection in the display of (v); the right-hand vertical arrow $\epsilon_{\widetilde{X}}$ denotes the natural morphism [cf. Definition 2.2, (vi)]. In particular, $\epsilon_{\widetilde{X}}$ is injective and in fact admits a natural splitting $\tau_{\widetilde{X}} : \mathbb{VE}(\widetilde{X}) \to \mathbb{VE}(\widetilde{X})^{\mathrm{tor}}$ [i.e., such that $\tau_{\widetilde{X}} \circ \epsilon_{\widetilde{X}}$ is the identity on $\mathbb{VE}(\widetilde{X})^{\mathrm{tor}}$]. On the other hand, neither $\iota_{\widetilde{X}}$ nor $\epsilon_{\widetilde{X}}$ is surjective [cf. Remark 2.3.4 below].

(ix) *Suppose that we are in the situation of (v). Let $c_1, c_2 \in \mathbb{VE}(\widetilde{X})$ be distinct elements. Then one of the following conditions holds:*

- *$\delta(c_1, c_2) = +\infty$.*
- *$\delta(c_1, c_2) = 0$, and there exists a unique element $c_3 \in \mathbb{VE}(\widetilde{X})$ such that $c_3 \rightsquigarrow c_1$ and $c_3 \rightsquigarrow c_2$.*

In particular, if c_1 and c_2 are distinct primitive elements, then it holds that $\delta(c_1, c_2) = +\infty$.

(x) *Suppose that we are in the situation of (v). Let $c \in \mathbb{VE}(\widetilde{X})$. Then the cardinality of the set*

$$\{c' \in \mathbb{VE}(\widetilde{X}) \setminus \{c\} \mid c' \rightsquigarrow c\}$$

is at most 1.

(xi) Suppose that we are in the situation of (v). Let $\Sigma \subseteq \mathfrak{Primes}$ *be a subset;* $l \in \Sigma \setminus \{p\}$*;* $H \subseteq G_K$ *a closed subgroup such that the restriction to* H *of the* ***l-adic cyclotomic character*** *of* K *has open image, and, moreover, the intersection* $H \cap I_K$ *of* H *with the inertia subgroup* I_K *of* G_K *admits a surjection to [the profinite group]* $\mathbb{Z}_l$*;* $s : H \to \Pi_X^{(\Sigma)}$ *a section of the restriction to* H *of the natural surjection* $\Pi_X^{(\Sigma)} \twoheadrightarrow G_K$*. Then there exists an element* $c \in \mathbb{VE}(\widetilde{X})^{\mathrm{prim}}$ *that is fixed by the restriction, via* s*, to* H *of the natural action of* $\Pi_X^{(\Sigma)}$ *on* $\mathbb{VE}(\widetilde{X})^{\mathrm{prim}} \subseteq \mathbb{VE}(\widetilde{X})$*. In particular, if* X *is proper, then there exists an element* $c^{\mathrm{an}} \in \widetilde{X}^{\mathrm{an}}$ *[cf. (vii)] that is fixed by the restriction, via* s*, to* H *of the natural action of* $\Pi_X^{(\Sigma)}$ *on the topological pro-Berkovich space* $\widetilde{X}^{\mathrm{an}}$*.*

(xii) Let $\Sigma \subseteq \mathfrak{Primes}$ *be a subset of cardinality* ≥ 2 *that contains* p*. Then there exists a connected geometrically pro-*Σ *finite étale Galois covering* $X^\dagger \to X$ *satisfying the following conditions:*

- $X^\dagger$ *has split stable reduction.*
- *Write* $\mathcal{X}^\dagger$ *for the [unique, up to unique isomorphism] stable model of* $X^\dagger$*. Then* $\mathcal{X}^\dagger_s$ *is singular, and every irreducible component of* $\mathcal{X}^\dagger_s$ *is a smooth curve of genus* ≥ 2*.*

Proof. First, we verify assertion (i). Since R is a finitely generated algebra over $\mathcal{O}_K$ ($\subseteq R$), it follows that Spec R admits an embedding over $\mathcal{O}_K$ into N-dimensional affine space $\mathbb{A}^N_{\mathcal{O}_K}$ for some positive integer N. Write $\mathcal{Z}^\dagger \subseteq \mathbb{P}^N_{\mathcal{O}_K}$ for the *scheme-theoretic closure* of the image of Spec R in $\mathbb{P}^N_{\mathcal{O}_K}(\supseteq \mathbb{A}^N_{\mathcal{O}_K})$; $\mathcal{Z}$ for the normalization of $\mathcal{Z}^\dagger$. Thus, the structure sheaf $\mathcal{O}_{\mathcal{Z}}$ is p-torsion-free, hence *flat* over $\mathcal{O}_K$. Since, moreover, $\mathcal{Z}^\dagger$ is [of finite type over the complete discrete valuation ring $\mathcal{O}_K$, hence] *excellent*, it follows that $\mathcal{Z}$ is a *proper, flat scheme of finite type* over $\mathcal{O}_K$, whose generic fiber may be identified with the [uniquely determined, up to unique isomorphism] smooth compactification of X over K. This completes the proof of assertion (i).

Next, we verify assertion (ii). Write $A \subseteq K(X)$ for the discrete valuation ring associated to v. Note that A may be written as the direct limit [i.e., in fact, union] of a direct system of finitely generated subalgebras $\{A_i \subseteq A\}_{i \in I}$ over $\mathcal{O}_K$. Moreover, since the field extension $K \subseteq K(X)$ is finitely generated, and each A_i is [finitely generated over the complete discrete valuation ring $\mathcal{O}_K$, hence] *excellent*, we may assume without loss of generality, i.e., by replacing A_i by its normalization in $K(X)$, that each A_i is *normal* with field of fractions equal to $K(X)$. Write $\mathfrak{p}_i \subseteq A_i$ for the prime ideal determined by the maximal ideal of A. Thus, since v is a *p-valuation [over K]*, it follows immediately that each of the natural inclusions $\mathcal{O}_K \hookrightarrow (A_i)_{\mathfrak{p}_i} \hookrightarrow A$ is a homomorphism of local rings. Next, let us observe that since the residue field extension determined by the natural inclusion $\mathcal{O}_K \subseteq A$ of local rings is assumed to be transcendental, it follows that there exists an element $i \in I$ such that the residue field $k(\mathfrak{p}_i)$

of $\mathfrak{p}_i$ is a *transcendental* extension of the residue field of $\mathcal{O}_K$. Let $\mathcal{Z}$ be a compactified model of X over $\mathcal{O}_K$ that contains Spec A_i as an open subscheme [cf. Proposition 2.3, (i)]. Then since $\mathcal{Z}$ is of dimension 2 [cf. Remark 2.1.1], it follows that the height of $\mathfrak{p}_i$ is equal to 1 or 2. On the other hand, if $\mathfrak{p}_i$ is of height 2, then it follows that $\mathfrak{p}_i$ corresponds to a closed point of $\mathcal{Z}_s$, hence that $k(\mathfrak{p}_i)$ is a *finite* extension of the residue field of $\mathcal{O}_K$, i.e., in contradiction to our assumption of transcendality. Thus, we conclude that $\mathfrak{p}_i$ is of height 1, hence that $(A_i)_{\mathfrak{p}_i}$ is a discrete valuation ring whose field of fractions is equal to $K(X)$. But this implies that $(A_i)_{\mathfrak{p}_i} = A$. This completes the proof of assertion (ii).

Next, we verify assertion (iii). First, we observe that, after possibly replacing K by a suitable finite extension field of K, there exists a G-equivariant finite morphism $\mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_K}$ to the projective line over $\mathcal{O}_K$ [equipped with the *trivial* action by G]. Indeed, since $\mathcal{O}_K$ is a complete discrete valuation ring, by deforming any [suitably large positive power of a] very ample line bundle on the projective curve $\mathcal{X}_s$, we obtain a very ample line bundle $\mathcal{L}$ on $\mathcal{X}$, hence, after possibly replacing K by a suitable finite extension field of K, a pair of global sections σ_1, σ_2 of the line bundle $\mathcal{L}$ such that the *G-orbit of the zero locus* of σ_1 is *disjoint* from the *G-orbit of the zero locus* of σ_2. Thus, for $i = 1, 2$, the product σ_i^G of the G-translates of σ_i determines a global section of the [still *very ample*!] tensor product $\mathcal{L}^G$ of G-translates of $\mathcal{L}$ such that σ_1^G and σ_2^G still have *disjoint zero loci*, hence determine a *G-equivariant finite morphism* $\mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_K}$ over $\mathcal{O}_K$, as desired. Fix such a finite morphism, and write $f \in K(X)$ for the rational function on X determined by the standard coordinate function on $\mathbb{P}^1_{\mathcal{O}_K}$.

Next, we recall that it follows from the *stable reduction theorem* [cf. [DM], [Knud]] that, after possibly replacing K by a suitable finite extension field of K, we may assume without loss of generality that every closed point in the support $\mathrm{Supp}(f)$ [in the smooth compactification of X] of the principal divisor associated to f is K-rational, and that X has stable reduction over K. Moreover, by replacing X by a suitable G-stable open subscheme of X, we may assume without loss of generality that $\mathrm{Supp}(f)$ is contained in the set of cusps of X. Write $\mathcal{X}^\dagger$ for the compactified stable model of X over $\mathcal{O}_K$; $E \subseteq \mathcal{X}^\dagger$ for the reduced closed subscheme determined by the set of closed points where an irreducible component of the zero divisor of f on $\mathcal{X}^\dagger$ intersects an irreducible component of the divisor of poles of f on $\mathcal{X}^\dagger$. Thus, the action of G on X extends to $\mathcal{X}^\dagger$. Let $x \in E$. Write $\widehat{\mathcal{O}}_{\mathcal{X}^\dagger,x}$ for the completion of the local ring of $\mathcal{X}^\dagger$ at x. Fix a uniformizer $\pi_K \in \mathcal{O}_K$.

Next, suppose that x is a *smooth closed point* of $\mathcal{X}^\dagger$. Then there exist nonzero integers a, b of opposite sign and a unit $u \in (\mathcal{O}_K[[t]])^\times$ [where t denotes an indeterminate], together with an isomorphism of topological $\mathcal{O}_K$-algebras $\widehat{\mathcal{O}}_{\mathcal{X}^\dagger,x} \xrightarrow{\sim} \mathcal{O}_K[[t]]$, such that the image of f in the field of fractions of $\widehat{\mathcal{O}}_{\mathcal{X}^\dagger,x}$ ($\xrightarrow{\sim} \mathcal{O}_K[[t]]$) is of the form $u \cdot t^a \cdot \pi_K^b$. Next, observe that, by replacing K by a suitable finite extension field of K [so it may no longer be the case that the element "π_K" is a uniformizer of $\mathcal{O}_K$!], we may assume without loss of generality that there exists an element $\gamma \in \mathcal{O}_K$ such that $\gamma^a = \pi_K^{-b}$. Write x_η for the K-valued point of the smooth compactification of X determined by the section of the structure

morphism $\mathcal{X}^\dagger \to \mathrm{Spec}\ \mathcal{O}_K$ corresponding to the homomorphism of topological $\mathcal{O}_K$-algebras

$$(\widehat{\mathcal{O}}_{\mathcal{X}^\dagger,x} \xrightarrow{\sim}) \ \mathcal{O}_K[[t]] \longrightarrow \mathcal{O}_K$$

that maps $t \mapsto \gamma \in \mathcal{O}_K$; x'_η for the K-valued point of the smooth compactification of X determined by the section of the structure morphism $\mathcal{X}^\dagger \to \mathrm{Spec}\ \mathcal{O}_K$ corresponding to the homomorphism of topological $\mathcal{O}_K$-algebras

$$(\widehat{\mathcal{O}}_{\mathcal{X}^\dagger,x} \xrightarrow{\sim}) \ \mathcal{O}_K[[t]] \longrightarrow \mathcal{O}_K$$

that maps $t \mapsto 0 \in \mathcal{O}_K$; $\mathcal{X}^\dagger[x]$ for the compactified stable model of $X \setminus \{x_\eta, x'_\eta\}$ over $\mathcal{O}_K$. Thus, it follows immediately from the theory of *pointed stable curves*, as exposed in [Knud], that the natural inclusion $X \setminus \{x_\eta, x'_\eta\} \hookrightarrow X$ determines a natural birational, dominant morphism $\mathcal{X}^\dagger[x] \to \mathcal{X}^\dagger$. Finally, we observe that it follows immediately from the various definitions involved that the rational function f is a *unit* [at x_η, hence] at the generic point of the unique irreducible component of $(\mathcal{X}^\dagger[x])_s$ that maps to a closed point of $\mathcal{X}^\dagger_s$; in particular, the zero divisor of f does *not intersect* the divisor of poles of f in some Zariski neighborhood of this irreducible component.

Next, suppose that x is a *nonsmooth closed point* of $\mathcal{X}^\dagger$. Then since the two irreducible components of $(\mathrm{Spec}\ \widehat{\mathcal{O}}_{\mathcal{X}^\dagger,x})_s$ are $\mathbb{Q}$-*Cartier divisors*, it follows that there exist positive integers a, b, r and a unit $u \in (\mathcal{O}_K[[s,t]]/(st - \pi_K^r))^\times$ [where s, t denote indeterminates], together with an isomorphism of topological $\mathcal{O}_K$-algebras $\widehat{\mathcal{O}}_{\mathcal{X}^\dagger,x} \xrightarrow{\sim} \mathcal{O}_K[[s,t]]/(st - \pi_K^r)$, such that the image of some positive power of f in the field of fractions of $\widehat{\mathcal{O}}_{\mathcal{X}^\dagger,x}$ ($\xrightarrow{\sim} \mathcal{O}_K[[s,t]]/(st - \pi_K^r)$) is of the form $u \cdot s^a \cdot t^{-b}$. Next, observe that, by replacing K by a suitable finite extension field of K [so it may no longer be the case that the element "π_K" is a uniformizer of $\mathcal{O}_K$!], we may assume without loss of generality that there exists an element $\gamma \in \mathcal{O}_K$ such that $\gamma^{a+b} = \pi_K$. Write x_η for the K-valued point of the smooth compactification of X corresponding to the section of the structure morphism $\mathcal{X}^\dagger \to \mathrm{Spec}\ \mathcal{O}_K$ induced by the homomorphism of topological $\mathcal{O}_K$-algebras

$$(\widehat{\mathcal{O}}_{\mathcal{X}^\dagger,x} \xrightarrow{\sim}) \ \mathcal{O}_K[[s,t]]/(st - \pi_K^r) \longrightarrow \mathcal{O}_K$$

that maps $s \mapsto \gamma^{br} \in \mathcal{O}_K$, $t \mapsto \gamma^{ar} \in \mathcal{O}_K$; $x'_\eta \stackrel{\text{def}}{=} x_\eta$; $\mathcal{X}^\dagger[x]$ for the compactified stable model of $X \setminus \{x_\eta, x'_\eta\} = X \setminus \{x_\eta\}$ over $\mathcal{O}_K$. Thus, it follows immediately from the theory of *pointed stable curves*, as exposed in [Knud], that the natural inclusion $X \setminus \{x_\eta, x'_\eta\} = X \setminus \{x_\eta\} \hookrightarrow X$ determines a natural birational, dominant morphism $\mathcal{X}^\dagger[x] \to \mathcal{X}^\dagger$. Finally, we observe that it follows immediately from the various definitions involved that the rational function f is a *unit* [at x_η, hence] at the generic point of the unique irreducible component of $(\mathcal{X}^\dagger[x])_s$ that maps to a closed point of $\mathcal{X}^\dagger_s$; in particular, the zero divisor of f does *not intersect* the divisor of poles of f in some Zariski neighborhood of this irreducible component.

Next, observe that the underlying set of E is finite. Thus, by replacing X by a suitable G-stable open subscheme of X, we may assume without loss of

generality that for each $x \in E$, the K-valued points x_η, x'_η constructed above are contained in the set of cusps of X. Then it follows immediately from the above discussion that the zero divisor of f on $\mathcal{X}^\dagger$ does *not intersect* the divisor of poles of f on $\mathcal{X}^\dagger$. But this implies that f determines a G-equivariant dominant morphism $\mathcal{X}^\dagger \to \mathbb{P}^1_{\mathcal{O}_K}$ over $\mathcal{O}_K$ whose restriction to the respective generic fibers *coincides* with the restriction to the respective generic fibers of the *finite morphism* $\mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_K}$ over $\mathcal{O}_K$ constructed above. Thus, since $\mathcal{X}^\dagger$ is *normal*, we conclude that the morphism $\mathcal{X}^\dagger \to \mathbb{P}^1_{\mathcal{O}_K}$ admits a factorization $\mathcal{X}^\dagger \to \mathcal{X} \to \mathbb{P}^1_{\mathcal{O}_K}$, as desired. This completes the proof of assertion (iii).

Next, we verify assertion (iv). First, we observe that since the operation of forming the quotient of $\mathcal{Y}^{\mathrm{sst}}$ by G commutes with flat base-change, one verifies immediately that it suffices to verify assertion (iv) after performing any finite, faithfully flat base-change from $\mathcal{O}_K$ to the ring of integers in a finite extension field of K. In particular, by replacing K by a suitable finite extension field of K, we may assume without loss of generality that the cusps of X are K-rational, and that X has stable reduction over K [cf. [DM], [Knud]]. [In fact, these conditions are satisfied even if one does not pass to a finite extension field of the original given K, but we omit a proof of this fact since it is not logically necessary for the present discussion.] Write $\mathcal{X}^{\mathrm{st}}$ for the compactified stable model of X over $\mathcal{O}_K$.

Next, let us observe that, after possibly replacing K by a suitable finite extension field of K, one may regard $\mathcal{Y}^{\mathrm{sst}}$ as the compactified stable model associated to the hyperbolic curve Y' obtained by removing from Y a *collection of G-orbits of K-rational points of Y* such that the cardinality of the set of G-orbits of closed points of $\mathcal{Y}^{\mathrm{sst}}_s$ contained in the intersection of any irreducible component of $\mathcal{Y}^{\mathrm{sst}}_s$ with the image of the *corresponding collection of $\mathcal{O}_K$-rational points of* $\mathcal{Y}^{\mathrm{sst}}$ is ≥ 3. In particular, one verifies immediately that, by replacing Y by Y', we may assume without loss of generality that the cardinality of the set of closed points of each irreducible component of $\mathcal{X}_s$ that lie in the image of the cusps of $\mathcal{Y}^{\mathrm{sst}}$ is ≥ 3.

Next, let us observe that it follows immediately from the definition of the natural quotient morphism $\mathcal{Y}^{\mathrm{sst}} \to \mathcal{X}$ that the natural morphism $\mathcal{Y}^{\mathrm{sst}} \to \mathcal{X}^{\mathrm{st}}$ over $\mathcal{O}_K$ induced by the morphism $Y \to X$ [cf. [ExtFam], Theorem A] admits a factorization

$$\mathcal{Y}^{\mathrm{sst}} \longrightarrow \mathcal{X} \longrightarrow \mathcal{X}^{\mathrm{st}},$$

where we note that it follows immediately from the definition of $\mathcal{X}$ that $\mathcal{X}$ is a compactified model of X over $\mathcal{O}_K$.

Thus, it follows immediately from the above discussion of $\mathcal{Y}^{\mathrm{sst}}$, together with the theory of *pointed stable curves*, as exposed in [Knud], that the morphism $\mathcal{X} \to \mathcal{X}^{\mathrm{st}}$ is *birational* and *quasi-finite*, hence, by Zariski's Main Theorem, an *isomorphism*. In particular, we conclude that $\mathcal{X}$ is a compactified semistable model of X over $\mathcal{O}_K$. Moreover, it follows immediately from Remark 2.1.3 that the natural morphism $\mathcal{Y}^{\mathrm{sst}}_s \to \mathcal{X}_s$ maps smooth points of $\mathcal{Y}^{\mathrm{sst}}_s$ to smooth points of $\mathcal{X}_s$.

Next, let $e_Y \in \mathcal{Y}^{\mathrm{sst}}_s$ be a node. Write $e_X \in \mathcal{X}_s$ for the image of e_Y via the

natural morphism $\mathcal{Y}_s^{\mathrm{sst}} \to \mathcal{X}_s$; $G_{e_Y} \subseteq G$ for the stabilizer of e_Y in G; $\mathcal{Y}_{e_Y}^{\mathrm{sst}}$ for the spectrum of the completion of the local ring of $\mathcal{Y}^{\mathrm{sst}}$ at e_Y; $\mathcal{X}_{e_X}$ for the spectrum of the completion of the local ring of $\mathcal{X}$ at e_X. Thus, G_{e_Y} acts naturally on $\mathcal{Y}_{e_Y}^{\mathrm{sst}}$; $\mathcal{X}_{e_X}$ may be identified with the quotient of $\mathcal{Y}_{e_Y}^{\mathrm{sst}}$ by the action of G_{e_Y}; the set B_Y of irrreducible components of $(\mathcal{Y}_{e_Y}^{\mathrm{sst}})_s$ may be identified with the set [of cardinality 2] of branches of e_Y; the set B_X of irrreducible components of $(\mathcal{X}_{e_X})_s$ is of cardinality 1 if and only if e_X is a smooth point of $\mathcal{X}_s$ and may be identified with the set [of cardinality 2] of branches of e_X whenever e_X is a node. On the other hand, it follows immediately from elementary commutative algebra that the set B_X may be naturally identified with the set of G_{e_Y}-orbits of B_Y. The remaining portion of assertion (iv) now follows formally. This completes the proof of assertion (iv).

Next, we verify assertions (v) and (vi). Let $c \in \mathbb{VE}(\widetilde{X})$. Write

$$R_c \stackrel{\mathrm{def}}{=} \varinjlim_{\mathcal{Z}} \mathcal{O}_{\mathcal{Z},z_c},$$

where

- $\mathcal{Z}$ ranges over the compactified semistable models with split reduction of the domain curves of connected finite étale coverings $Z \to X$ equipped with a factorization $\widetilde{X} \to Z \to X$;
- $c_{\mathcal{Z}}$ denotes the irreducible component or node of $\mathcal{Z}_s$ determined by c;
- z_c denotes the generic point of the intersection of the [closed irreducible] images in $\mathcal{Z}_s$ of the $c_{\mathcal{Z}^\dagger}$ associated to compactified semistable models with split reduction of domain curves of connected finite étale coverings $Z^\dagger \to X$ equipped with a factorization $\widetilde{X} \to Z^\dagger \to Z \to X$ such that the morphism $Z^\dagger \to Z$ extends to a morphism $\mathcal{Z}^\dagger \to \mathcal{Z}$;
- the transition maps in the direct limit are the homomorphisms of local rings induced by the corresponding scheme-theoretic morphisms of compactified semistable models [which form a *directed inverse system* — cf. Proposition 2.3, (iii)].

Then it follows immediately from the various definitions involved that the field of fractions of R_c coincides with $K(\widetilde{X})$, that $\mathcal{O}_K \subseteq R_c$, and that R_c is a local domain whose maximal ideal $\mathfrak{m}_{R_c}$ contains p. Let $\widetilde{R}_c \subseteq K(\widetilde{X})$ be a valuation ring that dominates R_c [cf., e.g., [EP], Theorem 3.1.1]. Write $\mathfrak{m}_{\widetilde{R}_c} \subseteq \widetilde{R}_c$ for the maximal ideal of $\widetilde{R}_c$. Thus, since $\mathcal{O}_K \subseteq \mathcal{O}_{\overline{K}} \subseteq R_c \subseteq \widetilde{R}_c$ and $p \in \mathfrak{m}_{R_c} \subseteq \mathfrak{m}_{\widetilde{R}_c}$, we conclude that $\mathcal{O}_K = K \cap \widetilde{R}_c$, i.e., that the valuation determined by the valuation ring $\widetilde{R}_c$ is a *p-valuation*.

Note that $\widetilde{R}_c$ may be written as the direct limit [i.e., in fact, union] of a direct system of finitely generated subalgebras $\{R_i \subseteq \widetilde{R}_c\}_{i \in I}$ over $\mathcal{O}_K$. Moreover, since R_i is [finitely generated over the complete discrete valuation ring $\mathcal{O}_K$, hence] *excellent*, by replacing R_i by its normalization in the subfield of $K(\widetilde{X})$

generated by the field of fractions of R_i and some suitable finite extension field of K, we may assume without loss of generality that R_i is normal, and that there exists a compactified semistable model with split reduction $\mathcal{Z}_i$ of the domain curve of a connected finite étale covering $Z_i \to X$ equipped with a factorization $\widetilde{X} \to Z_i \to X$ such that Spec R_i arises as an open subscheme of $\mathcal{Z}_i$ [cf. Proposition 2.3, (i), (iii)]. Write $\mathfrak{p}_i \stackrel{\text{def}}{=} \mathfrak{m}_{\widetilde{R}_c} \cap R_i \subseteq R_i$. Thus, if we write z_i for the point of $\mathcal{Z}_i$ that corresponds to $\mathfrak{p}_i$, then it holds that

$$\widetilde{R}_c = \varinjlim_{i \in I} \mathcal{O}_{\mathcal{Z}_i, z_i}.$$

On the other hand, observe that, since $\mathfrak{m}_{\widetilde{R}_c} \cap R_c = \mathfrak{m}_{R_c}$, it follows immediately from the various definitions involved that each "$\mathcal{O}_{\mathcal{Z}_i, z_i}$ $(= (R_i)_{\mathfrak{p}_i})$" of the above direct limit appears as one of the "$\mathcal{O}_{\mathcal{Z}, z_c}$" in the direct limit used to define R_c. In particular, we conclude that $\widetilde{R}_c \subseteq R_c \subseteq \widetilde{R}_c$, hence that $R_c = \widetilde{R}_c$, i.e., that R_c is the valuation ring associated to a *p-valuation*. Thus, in summary, we obtain a *natural map*

$$\mathbb{VE}(\widetilde{X}) \longrightarrow \left\{ p\text{-valuations on } K(\widetilde{X}) \right\}$$

that maps $\mathbb{VE}(\widetilde{X}) \ni c \mapsto R_c$. Moreover, one verifies immediately that this map defines an inverse to the *natural map*

$$\left\{ p\text{-valuations on } K(\widetilde{X}) \right\} \longrightarrow \mathbb{VE}(\widetilde{X})$$

in the statement of assertion (v), hence that both of these maps are *bijective*. Since both of these maps are manifestly compatible with specialization/generization, we thus conclude that these induce a bijection

$$\left\{ \text{primitive } p\text{-valuations on } K(\widetilde{X}) \right\} \xrightarrow{\sim} \mathbb{VE}(\widetilde{X})^{\text{prim}}$$

as in the statement of assertion (v).

Thus, to complete the proof of assertion (v), it suffices to verify that the natural map

$$\widetilde{X}(\Omega) \longrightarrow \left\{ p\text{-valuations on } K(\widetilde{X}) \right\}$$

[i.e., that assigns to an element of $\widetilde{X}(\Omega)$ the associated point-theoretic valuation on $K(\widetilde{X})$] is *injective*. To this end, let $\tilde{x} \in \widetilde{X}(\Omega)$. Write v for the point-theoretic valuation on $K(\widetilde{X})$ associated to $\tilde{x}$. Then observe that $\tilde{x}$ is defined over $\overline{K}$ if and only if the inclusion $\mathcal{O}_v \cdot \overline{K} \subseteq K(\widetilde{X})$ is *strict*. If this inclusion is strict, then the subring $\mathcal{O}_v \cdot \overline{K} \subseteq K(\widetilde{X})$ is the valuation ring determined [i.e., in the usual sense of the classical theory of one-dimensional function fields over algebraically closed fields] by $\tilde{x}$. In particular, whenever the inclusion $\mathcal{O}_v \cdot \overline{K} \subseteq K(\widetilde{X})$ is *strict*, the point $\tilde{x} \in \widetilde{X}(\overline{K})$ $(\subseteq \widetilde{X}(\Omega))$ is *completely determined* by v. Thus, it remains to

consider the case where $\mathcal{O}_v \cdot \overline{K} = K(\widetilde{X})$. In this case, the valuation v is *real*, and the inclusion $\overline{K} \subseteq K(\widetilde{X})$ induces, by passing to the respective completions, an isomorphism of Ω with the completion of $K(\widetilde{X})$ with respect to v. In particular, we obtain a natural homomorphism $K(\widetilde{X}) \to \Omega$, which *completely determines* the point $\tilde{x} \in \widetilde{X}(\Omega)$. This completes the proof of assertion (v). Assertion (vi) follows immediately from the proof of assertion (v).

Next, we verify assertion (vii). Let $x \in \widetilde{X}(\Omega)$. Write v for the *point-theoretic valuation* on $K(\widetilde{X})$ associated to x; $\phi_x : \mathcal{O}_v \to \mathcal{O}_\Omega$ for the homomorphism obtained by evaluating rational functions at x. Let $\mathfrak{q} \subseteq \mathcal{O}_v$ be a prime ideal that contains p. Then observe that it follows immediately from the construction of v that $\frac{1}{p} \cdot \mathrm{Ker}(\phi_x) \subseteq \mathrm{Ker}(\phi_x)$. Since $p \in \mathfrak{q}$, we thus conclude that $\mathrm{Ker}(\phi_x) \subseteq \mathfrak{q}$. On the other hand, observe that [it follows immediately from the construction of v that] this inclusion implies that $\mathfrak{q}$ contains [hence coincides with] the radical of the ideal $(p, \mathrm{Ker}(\phi_x)) \subseteq \mathcal{O}_v$, which is easily seen to be equal to the maximal ideal $\mathfrak{m}_v$ of $\mathcal{O}_v$. Thus, we conclude that v is *primitive*.

Next, let v be a *real p-valuation* on $K(\widetilde{X})$. Let $a \in \mathfrak{m}_v$. Then since v is real, there exists a positive integer N such that $a^N \in (p)$. In particular, any prime ideal that contains p contains [hence coincides with] $\mathfrak{m}_v$. Thus, we conclude that v is *primitive*.

Next, let v be a *primitive p-valuation* on $K(\widetilde{X})$. For each $z \in \mathcal{O}_v$, write $(\mathcal{O}_v)_z \subseteq K(\widetilde{X})$ for the $\mathcal{O}_v$-subalgebra generated by $\frac{1}{z}$. Thus, if $(\overline{K} \subseteq)\ (\mathcal{O}_v)_p \neq K(\widetilde{X})$, then it follows immediately from the classical theory of one-dimensional function fields over algebraically closed fields that v is a *point-theoretic valuation*. Therefore, we may assume without loss of generality that $(\mathcal{O}_v)_p = K(\widetilde{X})$. Note that this implies that for each $x \in \mathcal{O}_v \setminus \{0\}$, there exist a positive integer N and $y \in \mathcal{O}_v$ such that $p^N = xy$. Moreover, in this situation, it holds that

$$\mathfrak{m}_{\overline{K}} \mathcal{O}_v \ = \ \mathfrak{m}_v.$$

Indeed, since v is a p-valuation, the inclusion $\mathfrak{m}_{\overline{K}} \mathcal{O}_v \subseteq \mathfrak{m}_v$ is immediate. Now suppose that there exists an element $x \in \mathfrak{m}_v \setminus \mathfrak{m}_{\overline{K}} \mathcal{O}_v$. Then it follows that $\frac{1}{p} \notin (\mathcal{O}_v)_x$, hence that there exists a prime ideal $\mathfrak{p}_v$ of $\mathcal{O}_v$ such that $x \notin \mathfrak{p}_v$, and $p \in \mathfrak{p}_v$. On the other hand, since v is primitive, we conclude that $x \in \mathfrak{m}_v = \mathfrak{p}_v$, a contradiction. This completes the proof of the equality in the above display. Note that this equality implies that for each $x \in \mathfrak{m}_v \setminus \{0\}$, there exist a positive integer N and $y \in \mathcal{O}_v$ such that $x^N = py$. In particular, it follows immediately from the various definitions involved that v coincides with the real valuation determined by the assignment

$$\mathcal{O}_v \setminus \{0\} \ni x \ \mapsto \ \sup\{\epsilon \in \mathbb{Q} \mid x \in p^\epsilon \cdot \mathcal{O}_v\} \ = \ \inf\{\epsilon \in \mathbb{Q} \mid x^{-1} \in p^{-\epsilon} \cdot \mathcal{O}_v\} \ \in \mathbb{R}.$$

This completes the proof of assertion (vii).

Next, we verify assertion (viii). First, let us observe that it follows immediately from the final portion of Proposition 2.3, (vii), that the natural map

$$\widetilde{X}^{\mathrm{an}} \longrightarrow \mathbb{VE}(\widetilde{X})$$

— i.e., that assigns to a valuation on $K(\widetilde{X})$ the *center-chain* associated to the valuation — admits a factorization

$$\widetilde{X}^{\mathrm{an}} \stackrel{\sim}{\to} \mathbb{VE}(\widetilde{X})^{\mathrm{prim}} \underset{\iota_{\widetilde{X}}}{\longrightarrow} \mathbb{VE}(\widetilde{X}),$$

where the first arrow is a bijection, and the second arrow $\iota_{\widetilde{X}}$ denotes the natural inclusion. On the other hand, it follows immediately from the discussion of the ratios "$\rho_{b,v}$" in Remark 2.1.4 and the construction of "$\mathbb{VE}(\widetilde{X})^{\mathrm{tor}}$" in Definition 2.2, (vi), that this natural map $\widetilde{X}^{\mathrm{an}} \to \mathbb{VE}(\widetilde{X})$ also admits a factorization

$$\widetilde{X}^{\mathrm{an}} \underset{\theta_{\widetilde{X}}}{\longrightarrow} \mathbb{VE}(\widetilde{X})^{\mathrm{tor}} \underset{\epsilon_{\widetilde{X}}}{\longrightarrow} \mathbb{VE}(\widetilde{X}),$$

where the first map $\theta_{\widetilde{X}}$ is defined by considering ratios "$\rho_{b,v}$" as in Remark 2.1.4 [cf. also the construction of "$\mathbb{VE}(\widetilde{X})^{\mathrm{tor}}$" in Definition 2.2, (vi)], and the second arrow $\epsilon_{\widetilde{X}}$ is the natural map discussed in the final portion of Definition 2.2, (vi). In particular, we obtain a commutative diagram of maps of sets

$$\begin{array}{ccc} \widetilde{X}^{\mathrm{an}} & \underset{\theta_{\widetilde{X}}}{\longrightarrow} & \mathbb{VE}(\widetilde{X})^{\mathrm{tor}} \\ \downarrow \wr & & \epsilon_{\widetilde{X}} \downarrow \\ \mathbb{VE}(\widetilde{X})^{\mathrm{prim}} & \underset{\iota_{\widetilde{X}}}{\longrightarrow} & \mathbb{VE}(\widetilde{X}). \end{array}$$

Note that the commutativity of the diagram already implies that $\theta_{\widetilde{X}}$ is *injective*. Moreover, one verifies immediately — i.e., by considering suitable "v" as in Remark 2.1.4 — that each composite map $\theta_{\mathcal{Z}}$

$$\widetilde{X}^{\mathrm{an}} \underset{\theta_{\widetilde{X}}}{\longrightarrow} \mathbb{VE}(\widetilde{X})^{\mathrm{tor}} \longrightarrow \mathbb{VE}(\mathcal{Z})^{\mathrm{tor}}$$

— where the second arrow is the natural projection arising from the inverse limit in the definition of $\mathbb{VE}(\widetilde{X})^{\mathrm{tor}}$ [cf. Definition 2.2, (vi)] — has *dense image*. Thus, it follows from the *compactness* of $\widetilde{X}^{\mathrm{an}}$ [cf. [Brk], Theorem 1.2.1], together with the easily verified fact [cf. the construction of Definition 2.2, (vi)] that $\mathbb{VE}(\mathcal{Z})^{\mathrm{tor}}$ is *Hausdorff*, that to verify that $\theta_{\widetilde{X}}$ is a *homeomorphism*, it suffices to verify that each map $\theta_{\mathcal{Z}}$ is *continuous*. Moreover, once one knows that $\theta_{\widetilde{X}}$ is a homeomorphism, one may construct a natural splitting $\tau_{\widetilde{X}}$ as in the statement of Proposition 2.3, (viii), by constructing a natural splitting of the natural inclusion $\iota_{\widetilde{X}}$. On the other hand, such a natural splitting $\mathbb{VE}(\widetilde{X}) \to \mathbb{VE}(\widetilde{X})^{\mathrm{prim}}$ of $\iota_{\widetilde{X}}$ is *implicit* in the content of Proposition 2.3, (x) [which will be verified below, independently of the present assertion (viii)], i.e., one assigns to each *nonprimitive* element $c \in \mathbb{VE}(\widetilde{X})$ the *unique generization* $\in \mathbb{VE}(\widetilde{X})^{\mathrm{prim}}$ of c.

Thus, in summary, to complete the proof of assertion (viii), it suffices to verify that each map

$$\theta_{\mathcal{Z}} : \widetilde{X}^{\mathrm{an}} \longrightarrow \mathbb{VE}(\mathcal{Z})^{\mathrm{tor}}$$

as in the above discussion is *continuous*. Let $\mathcal{Z}^{\dagger}$ be a toral compactified semistable model relative to $\mathcal{Z}$. Then we shall refer to an open subscheme U of $\mathcal{Z}^{\dagger}_s$ as a

componentwise open of $\mathcal{Z}^\dagger$ if U is an open subscheme of $\mathcal{Z}^\dagger_s$ whose underlying open subset is the complement of a node or an irreducible component of $\mathcal{Z}^\dagger_s$. Observe that it follows immediately from the construction of $\mathbb{VE}(\mathcal{Z})^{\mathrm{tor}}$ given in Definition 2.2, (vi), that each *componentwise open* of each toral compactified semistable model relative to $\mathcal{Z}$ determines, in a natural way, a *closed subset* of $\mathbb{VE}(\mathcal{Z})^{\mathrm{tor}}$. We shall refer to the closed subsets of $\mathbb{VE}(\mathcal{Z})^{\mathrm{tor}}$ obtained in this way as *componentwise closed subsets* of $\mathbb{VE}(\mathcal{Z})^{\mathrm{tor}}$. Note that it follows immediately from the construction of $\mathbb{VE}(\mathcal{Z})^{\mathrm{tor}}$ given in Definition 2.2, (vi), that the complements of the componentwise closed subsets of $\mathbb{VE}(\mathcal{Z})^{\mathrm{tor}}$ form an *open basis* of the topology of $\mathbb{VE}(\mathcal{Z})^{\mathrm{tor}}$. Thus, to complete the proof of the *contininuity* of $\theta_{\mathcal{Z}}$, it suffices to verify that the inverse image via $\theta_{\mathcal{Z}}$ of any *componentwise closed subset* of $\mathbb{VE}(\mathcal{Z})^{\mathrm{tor}}$ is *closed* in $\widetilde{X}^{\mathrm{an}}$. But this follows immediately from the *definition* of the *topology* of the *Berkovich spaces* [cf. the discussion of [Brk], §1.1, §1.2] that appear in the inverse limit that is used to define "$\widetilde{X}^{\mathrm{an}}$" in the statement of Proposition 2.3, (vii). This completes the proof of assertion (viii).

Next, we verify assertion (ix). In the following, we *assume* that $\delta(c_1, c_2) \neq +\infty$.

Let us first consider the case where $\delta(c_1, c_2) \geq 1$. Then it follows immediately from the definition of $\delta(-,-)$ that there exists a compactified semistable model $\mathcal{Z}$ with split reduction of a connected finite étale covering $Z \to X$ equipped with a factorization $\widetilde{X} \to Z \to X$ such that $\delta(z_{c_1}, z_{c_2}) \geq 1$ [cf. the notation of Proposition 2.3, (vi)]. In particular, there exists a node e of $\mathcal{Z}_s$ that does not coincide with z_{c_1} or z_{c_2}, and whose corresponding edge lies on a path of minimal length between z_{c_1} and z_{c_2}. On the other hand, by considering suitable torally compactified semistable models relative to $\mathcal{Z}$ at e, we conclude that $\delta(c_1, c_2) = +\infty$, in contradiction to our *assumption* that $\delta(c_1, c_2) \neq +\infty$.

Thus, to complete the proof of assertion (ix), it suffices to consider the case where $\delta(c_1, c_2) = 0$. Let us first observe that the condition that $\delta(c_1, c_2) = 0$ implies the existence of an element $c_3 \in \mathbb{VE}(\widetilde{X})$ such that $c_3 \rightsquigarrow c_1$ and $c_3 \rightsquigarrow c_2$. Finally, we verify the uniqueness of such an element $c_3 \in \mathbb{VE}(\widetilde{X})$. Let $c_4 \in \mathbb{VE}(\widetilde{X})$ be such that $c_3 \neq c_4$, $c_4 \rightsquigarrow c_1$, and $c_4 \rightsquigarrow c_2$. Then since $\delta(c_3, c_4) < +\infty$, it follows immediately from the above discussion that there exists an element $c_5 \in \mathbb{VE}(\widetilde{X})$ such that $c_5 \rightsquigarrow c_3$, and $c_5 \rightsquigarrow c_4$. Moreover, since $c_1 \neq c_2$, and $c_3 \neq c_4$, by permuting $\{c_3, c_4\}$ or $\{c_1, c_2\}$ if necessary, we may assume without loss of generality that $c_5 \neq c_3$, and $c_3 \neq c_1$. On the other hand, the resulting nontriviality of the specialization relations $c_5 \rightsquigarrow c_3 \rightsquigarrow c_1$ then contradicts the 1-*dimensionality* of the *special fibers* of the compactified semistable models "$\mathcal{Z}$" that appear in the definition of $\mathbb{VE}(\widetilde{X})$. This completes the proof of assertion (ix).

Next, we verify assertion (x). Write $c_1 \stackrel{\mathrm{def}}{=} c$. Suppose that $c_2 \rightsquigarrow c_1$, $c_2' \rightsquigarrow c_1$ for distinct elements $c_2, c_2' \in \mathbb{VE}(\widetilde{X}) \setminus \{c_1\}$. Then it follows that $\delta(c_2, c_2') < +\infty$. Thus, we conclude from Proposition 2.3, (ix), that there exists an element $c_3 \in \mathbb{VE}(\widetilde{X})$ such that $c_3 \rightsquigarrow c_2$, and $c_3 \rightsquigarrow c_2'$. In particular, by permuting $\{c_2, c_2'\}$ if necessary, we may assume without loss of generality that $c_3 \neq c_2$, and $c_2 \neq c_1$. On the other hand, the resulting nontriviality of the specialization

relations $c_3 \rightsquigarrow c_2 \rightsquigarrow c_1$ then contradicts the uniqueness portion of Proposition 2.3, (ix). This completes the proof of assertion (x).

Next, we verify assertion (xi). Let $Z \to X$ be a [connected] finite étale Galois covering equipped with a factorization $\widetilde{X} \to Z \to X$ such that Z has split stable reduction over K; $\mathcal{Z}^*$ a compactified semistable model with split reduction of Z over $\mathcal{O}_K$ that is stabilized by the natural action of $\mathrm{Gal}(Z/X)$. [Note that it follows immediately from Proposition 2.3, (iii), that such compactified semistable models form a *directed inverse system* that is *cofinal* in the directed inverse system that appears in the definition of $\mathbb{VE}(\widetilde{X})$.] Write $\mathcal{Z}$ for the compactified stable model with split reduction of Z over $\mathcal{O}_K$; Γ for the dual graph of $\mathcal{Z}_s$; Γ^* for the dual graph of $\mathcal{Z}^*_s$. Observe that the natural action of $s(H)$ on Γ^* factors through a finite quotient of $s(H)$. Thus, it follows immediately from [CbTpIV], Corollary 1.15, (iii), that the natural action of $s(H)$ on Γ has a fixed point $c \in \Gamma$. On the other hand, it follows immediately from the well-known theory of stable and semistable models [i.e., which may be reduced, by *adding finitely many suitably positioned cusps*, to the theory of *pointed stable curves* and contraction morphisms that arise from eliminating cusps, as exposed in [Knud]] that the inverse image of [the node or interior of an irreducible component in $\mathcal{Z}^*_s$ corresponding to] c via the dominant morphism $\mathcal{Z}^* \to \mathcal{Z}$ determines a *tree* inside Γ^*. Moreover, we recall that any action of a finite group on a tree has a fixed point [cf., e.g., [SemiAn], Lemma 1.8, (ii)]. Thus, we conclude that the natural action of $s(H)$ on Γ^* has a fixed point. Since any inverse limit of nonempty finite sets is nonempty, we thus conclude that the natural action of $s(H)$ on $\mathbb{VE}(\widetilde{X})$ has a fixed point $\in \mathbb{VE}(\widetilde{X})$, hence from Proposition 2.3, (x), that the natural action of $s(H)$ on $\mathbb{VE}(\widetilde{X})^{\mathrm{prim}}$ has a fixed point $\in \mathbb{VE}(\widetilde{X})^{\mathrm{prim}}$. This completes the proof of assertion (xi).

Next, we verify assertion (xii). Fix a prime number $l \in \Sigma \backslash \{p\}$. Then observe that it follows from the *stable reduction theorem* [cf. [DM], [Knud]] that, after possibly replacing K by a suitable finite extension field of K, we may assume without loss of generality that $\Sigma = \{p, l\}$, and that X has stable reduction over K. Write $\mathcal{X}$ for the [unique, up to unique isomorphism] compactified stable model of X over $\mathcal{O}_K$.

Next, observe that it follows immediately from Hurwitz's formula, together with the well-known structure of geometric fundamental groups of hyperbolic curves over fields of characteristic zero [cf., e.g., [CmbGC], Remark 1.1.3], that, after possibly replacing K by a suitable finite extension field of K, there exists a connected geometrically pro-p finite étale covering $Y \to X$ of hyperbolic curves with split stable reduction over K that satisfies the condition that Y is of genus $g_Y \geq 2$. Write $Y^\dagger$ for the smooth compactification of Y over K; $\mathcal{Y}^\dagger$ for the [unique, up to unique isomorphism] compactified stable model of $Y^\dagger$ over $\mathcal{O}_K$. Then observe that, if $\mathcal{Y}^\dagger$ is *smooth* over $\mathcal{O}_K$, then it follows immediately from the *non-injectivity* of the natural surjective homomorphism

$$\Pi^{\mathrm{ab}}_{Y^\dagger} \otimes \mathbb{Z}/p\mathbb{Z} \twoheadrightarrow \Pi^{\mathrm{ab}}_{\mathcal{Y}^\dagger_s} \otimes \mathbb{Z}/p\mathbb{Z}$$

[where we recall that, since $Y^\dagger$ is of genus $g_{Y^\dagger} \geq 2$, the *domain* of this homo-

morphism is of cardinality $p^{2g_{Y^\dagger}}$, while the *codomain* of this homomorphism is of cardinality $\leq p^{g_{Y^\dagger}}$], together with *Hurwitz's formula* [cf. also *Zariski-Nagata purity*; [ExtFam], Theorem A], that, after possibly replacing K by a suitable finite extension field of K, there exists a [connected] finite étale cyclic covering $Y^\ddagger \to Y^\dagger$ of hyperbolic curves over K that is of degree p and, moreover, satisfies the property that $Y^\ddagger$ has *bad reduction*. In particular, by replacing $Y^\ddagger \times_{Y^\dagger} Y$ by Y, we may assume without loss of generality that Y has *bad reduction*. Moreover, by replacing the connected geometrically pro-p finite étale covering $Y \to X$ by its Galois closure, we may assume without loss of generality that $Y \to X$ is a [connected] geometrically pro-p finite étale Galois covering [cf. Remark 2.1.3; Hurwitz's formula].

Next, observe that it follows immediately from the theory of admissible coverings [cf., e.g., [Hur], §3], together with Hurwitz's formula [and the well-known structure of geometric pro-l fundamental groups of hyperbolic curves over fields of characteristic $p \neq l$ — cf., e.g., [CmbGC], Remark 1.1.3], that, after possibly replacing K by a suitable finite extension field of K, there exists a [connected] geometrically pro-l finite étale Galois covering $Z \to Y$ of hyperbolic curves with split stable reduction over K that satisfies the condition that every irreducible component of the special fiber of the [unique, up to unique isomorphism] stable model of Z is a smooth curve of genus ≥ 2. Here, note that, by replacing $Z \to Y$ by the composite of the $\mathrm{Gal}(Y/X)$-conjugates of the admissible covering $Z \to Y$, we may assume without loss of generality that the composite covering $Z \to Y \to X$ is Galois. Thus, by taking $X^\dagger \stackrel{\text{def}}{=} Z$, we obtain a [connected] geometrically pro-Σ finite étale Galois covering $X^\dagger \to X$ of hyperbolic curves satisfying the conditions in the statement of assertion (xii), as desired. This completes the proof of assertion (xii), hence of Proposition 2.3. $\square$

Remark 2.3.1. We maintain the notation of Proposition 2.3. Then we observe that the statement of Proposition 2.3, (ii), becomes *false* if one *omits* the condition that the p-valuation v is *residue-transcendental*. Indeed, it suffices to construct an example of a discrete p-valuation on $K(X)$ whose residue field is algebraic over the residue field of $\mathcal{O}_K$. Suppose that *no finite extension field* of the residue field of $\mathcal{O}_K$ is *separably closed* [a condition that is satisfied if, for instance, the residue field of $\mathcal{O}_K$ is *finite*]. Then one verifies immediately that $X(\widehat{K}^{\mathrm{ur}}) \setminus X(\overline{K}) \neq \emptyset$, and that for any $x \in X(\widehat{K}^{\mathrm{ur}}) \setminus X(\overline{K})$, the point-theoretic valuation associated to x on $K(X)$ satisfies the desired properties.

Remark 2.3.2. An alternative proof of Proposition 2.3, (iv), may be found in [Ray2], Proposition 5. The proof of Proposition 2.3, (iv), given in the present paper is of interest in that it involves techniques that are closer to the overall approach of the present paper.

Remark 2.3.3. The homeomorphism $\widetilde{X}^{\mathrm{an}} \xrightarrow{\sim} \mathbb{VE}(\widetilde{X})^{\mathrm{tor}}$ of Proposition 2.3, (viii), is essentially the same as the homeomorphism of [Lpg1], Proposition 1.1, but we give [essentially] self-contained statements and proofs here in the language of the present discussion for the sake of completeness.

Remark 2.3.4. We maintain the notation of Proposition 2.3. Let $\mathcal{X}$ be a compactified semistable model of X over $\mathcal{O}_K$; $x \in \mathcal{X}$ a smooth closed point. Write η for the generic point of the unique irreducible component of $\mathcal{X}_s$ that contains x. Then one may construct a *p-valuation* v on $K(X)$ associated to x by taking the ring of integers $\mathcal{O}_v$ to consist of the elements $\in K(X)$ that are integral with respect to the discrete valuation on $K(X)$ associated to η and, moreover, map to an element in the residue field $k(\eta)$ of $\mathcal{X}$ at η that is integral with respect to the discrete valuation on $k(\eta)$ determined by x. Note that η determines a prime ideal of $\mathcal{O}_v$ that contains p. In particular, v is *nonprimitive*.

Proposition 2.4 (**First properties of resolution of nonsingularities**). *Let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty subset; K a mixed characteristic complete discrete valuation field of residue characteristic p; X a hyperbolic curve over K. Then:*

(i) Let $U \subseteq X$ be an open subscheme [so U is a hyperbolic curve over K]. Suppose that X satisfies Σ-RNS. Then it holds that U satisfies Σ-RNS.

(ii) Let $f : Y \to X$ be a connected geometrically pro-Σ finite étale covering over K [so Y is a hyperbolic curve over a finite extension field of K]. Then it holds that X satisfies Σ-RNS if and only if Y satisfies Σ-RNS.

(iii) Suppose that X satisfies the following condition:

> *Let $\mathcal{X}$ be a compactified model of X over $\mathcal{O}_K$; $x \in \mathcal{X}_s$ a closed point. Then, after possibly replacing K by a suitable finite extension field of K, there exist*
>
> - *a connected geometrically pro-Σ finite étale Galois covering $Y \to X$ of hyperbolic curves over K,*
> - *a compactified semistable model $\mathcal{Y}$ of Y over $\mathcal{O}_K$,*
> - *a morphism $\mathcal{Y} \to \mathcal{X}$ of compactified models over $\mathcal{O}_K$ that restricts to the finite étale Galois covering $Y \to X$,*
> - *an irreducible component D of $\mathcal{Y}_s$ whose normalization is of genus ≥ 1, and whose image in $\mathcal{X}_s$ is $x \in \mathcal{X}_s$.*
>
> *Then X satisfies Σ-RNS.*

(iv) Suppose that X satisfies Σ-RNS. Let $\mathcal{X}$ be a compactified model of X over $\mathcal{O}_K$. Then, after possibly replacing K by a suitable finite extension field of K, there exists a connected geometrically pro-Σ finite étale Galois covering $Y \to X$ over K, together with a compactified stable model $\mathcal{Y}$ of Y over $\mathcal{O}_K$, such that the covering $Y \to X$ extends to a morphism $\mathcal{Y} \to \mathcal{X}$.

(v) *Suppose that we are in the situation of Proposition 2.3, (v), and that X satisfies Σ-RNS. Write*

$$\mathbb{VE}(\widetilde{X})^{\mathrm{st}}, \quad \mathbb{VE}(\widetilde{X})^{\mathrm{st,tor}}, \quad \mathbb{VE}(\widetilde{X})^{\mathrm{st,prim}}, \quad \mathbb{VE}(\widetilde{X})^{\mathrm{st,pt\text{-}th}}$$

for the modified versions of $\mathbb{VE}(\widetilde{X})$, $\mathbb{VE}(\widetilde{X})^{\mathrm{tor}}$, $\mathbb{VE}(\widetilde{X})^{\mathrm{prim}}$, $\mathbb{VE}(\widetilde{X})^{\mathrm{pt\text{-}th}}$ obtained by requiring that the compactified semistable models "$\mathcal{Z}$" that appear in the inverse limits used to define these sets be ***compactified stable models****. [Here, we observe that, in light of (iv), the various toral compactified semistable models "$\mathcal{Z}^{\dagger}$" relative to "$\mathcal{Z}$" that appear in the construction of "$\mathbb{VE}(\mathcal{Z})^{\mathrm{tor}}$" in Definition 2.2, (vi), may be understood as being obtained as the result of contracting suitable irreducible components in the special fibers [cf. Remark 2.1.6] of suitable quotients of compactified stable models as in Proposition 2.3, (iv).] Then the natural maps*

$$\begin{aligned}
\mathbb{VE}(\widetilde{X}) &\longrightarrow \mathbb{VE}(\widetilde{X})^{\mathrm{st}}\\
\mathbb{VE}(\widetilde{X})^{\mathrm{tor}} &\longrightarrow \mathbb{VE}(\widetilde{X})^{\mathrm{st,tor}}\\
\mathbb{VE}(\widetilde{X})^{\mathrm{prim}} &\longrightarrow \mathbb{VE}(\widetilde{X})^{\mathrm{st,prim}}\\
\mathbb{VE}(\widetilde{X})^{\mathrm{pt\text{-}th}} &\longrightarrow \mathbb{VE}(\widetilde{X})^{\mathrm{st,pt\text{-}th}}
\end{aligned}$$

are bijective.

(vi) *Suppose that we are in the situation of Proposition 2.3, (v). Then X satisfies Σ-RNS if and only if for each $c \in \mathbb{VE}(\widetilde{X})$, it holds that*

$$R_c \;=\; \varinjlim_{\mathcal{Z}^{\mathrm{st}}} \mathcal{O}_{\mathcal{Z}^{\mathrm{st}},z_c},$$

where $R_c \subseteq K(\widetilde{X})$ denotes the valuation ring of the p-valuation associated to c [cf. Proposition 2.3, (v)]; the direct limit ranges over the set of compactified stable models with split reduction $\mathcal{Z}^{\mathrm{st}}$ of the domain curves of connected finite étale coverings $Z \to X$ equipped with a factorization $\widetilde{X} \to Z \to X$; z_c denotes the center on $\mathcal{Z}^{\mathrm{st}}$ determined by R_c; the transition maps in the direct limit are induced by the corresponding scheme-theoretic morphisms of compactified stable models [which, in light of (iv), form a directed inverse system].

(vii) *Suppose that we are in the situation of Proposition 2.3, (v), and that X satisfies Σ-RNS. Let $l \in \Sigma \setminus \{p\}$; $H \subseteq G_K$ a closed subgroup such that the intersection $H \cap I_K$ of H with the inertia subgroup I_K of G_K admits a surjection to [the profinite group] $\mathbb{Z}_l$; $s : H \to \Pi_X^{(\Sigma)}$ a section of the restriction to H of the natural surjection $\Pi_X^{(\Sigma)} \twoheadrightarrow G_K$. Then there exists at most one element $c \in \mathbb{VE}(\widetilde{X})^{\mathrm{prim}}$ that is fixed by the restriction, via s, to H of the natural action of $\Pi_X^{(\Sigma)}$ on $\mathbb{VE}(\widetilde{X})^{\mathrm{prim}} \subseteq \mathbb{VE}(\widetilde{X})$; if, moreover, the restriction to H of the* ***l-adic cyclotomic character*** *of K has open image, then there exists a unique such element $c \in \mathbb{VE}(\widetilde{X})^{\mathrm{prim}}$. In*

particular, if X is proper, then there exists at most one element $c^{\mathrm{an}} \in \widetilde{X}^{\mathrm{an}}$ [cf. Proposition 2.3, (vii)] that is fixed by the restriction, via s, to H of the natural action of $\Pi_X^{(\Sigma)}$ on the topological pro-Berkovich space $\widetilde{X}^{\mathrm{an}}$; if, moreover, the restriction to H of the ***l-adic cyclotomic character*** *of K has open image, then there exists a unique such element $c^{\mathrm{an}} \in \widetilde{X}^{\mathrm{an}}$.*

Proof. Assertions (i), (ii) follow immediately from the various definitions involved. Next, we verify assertion (iii). Let v be a discrete residue-transcendental p-valuation on $K(X)$. Then it follows immediately from Proposition 2.3, (ii), (iii), that, after possibly replacing K by a suitable finite extension field of K, there exists a compactified semistable model $\mathcal{X}$ of X over $\mathcal{O}_K$ such that v arises from an irreducible component of $\mathcal{X}_s$. Let $\{x_1, \ldots, x_N\} \subseteq \mathcal{X}_s$ be a finite set of distinct closed points in the smooth locus of $\mathcal{X}_s$ such that every irreducible component of $\mathcal{X}_s$ whose normalization is of genus 0 contains three points $\in \{x_1, \ldots, x_N\}$. Then it follows immediately from our assumption on X that, after possibly replacing K by a suitable finite extension field of K, for each positive integer $i \leq N$, there exist

- a connected geometrically pro-Σ finite étale Galois covering $Y_i \to X$ over K,
- a morphism $f_i : \mathcal{Y}_i \to \mathcal{X}$ of compactified semistable models over $\mathcal{O}_K$ that restricts to the finite étale Galois covering $Y_i \to X$,
- an irreducible component D_i of $(\mathcal{Y}_i)_s$ whose normalization is of genus ≥ 1, and whose image in $\mathcal{X}_s$ is x_i.

Write $f_\eta : Y \to X$ for the connected geometrically pro-Σ finite étale Galois covering over K obtained by forming the composite of the finite étale Galois coverings $\{Y_i \to X\}_{1 \leq i \leq N}$ over K. Then it follows immediately from Proposition 2.3, (iii), that, after possibly replacing K by a suitable finite extension field of K, there exists a compactified semistable model $\mathcal{Y}^\dagger$ of Y over $\mathcal{O}_K$ that dominates the respective normalizations of the semistable models $\{\mathcal{Y}_i\}_{1 \leq i \leq N}$ in the function field of Y. In particular, for each positive integer $i \leq N$, there exists an irreducible component $D_i^\dagger$ of $(\mathcal{Y}^\dagger)_s$ whose normalization is of genus ≥ 1, and whose image in $\mathcal{X}_s$ is x_i.

Next, let us observe that it follows immediately from the theory of *pointed stable curves*, as exposed in [Knud], that, after possibly replacing K by a suitable finite extension field of K, we may regard $\mathcal{Y}^\dagger$ as the compactified stable model associated to the hyperbolic curve $Y^\dagger$ obtained by removing from Y a *collection of K-rational points of Y*. In a similar vein, it follows immediately from the theory of *pointed stable curves*, as exposed in [Knud], that, after possibly replacing K by a suitable finite extension field of K and replacing $Y^\dagger$ by a suitable $\mathrm{Gal}(Y/X)$-*stable* dense open subscheme of $Y^\dagger$, we may assume without loss of generality that $\mathcal{Y}^\dagger$ is *stabilized* by the action of $\mathrm{Gal}(Y/X)$.

Write

$$f^\dagger : \mathcal{Y}^\dagger \to \mathcal{X}$$

for the natural dominant morphism that restricts to the finite étale Galois covering $f_\eta : Y \to X$;

$$\kappa^\dagger : \mathcal{Y}^\dagger \to \mathcal{Y}^{\mathrm{st}}$$

for the natural dominant morphism to a compactified stable model $\mathcal{Y}^{\mathrm{st}}$ of Y over $\mathcal{O}_K$ [cf. [ExtFam], Theorem A]. Thus, for each positive integer $i \leq N$, $f^\dagger(D_i^\dagger) = x_i \in \mathcal{X}_s$. In particular, since the covering $f_\eta : Y \to X$ is Galois, and $\mathcal{Y}^\dagger$ is stabilized by the action of $\mathrm{Gal}(Y/X)$, it follows immediately from Zariski's Main Theorem that, for each positive integer $i \leq N$, the inverse image $(f^\dagger)^{-1}(x_i) \subseteq \mathcal{Y}_s^\dagger$ is a closed subscheme that contains $D_i^\dagger$ and is pure of dimension 1. Here, we recall that $D_i^\dagger$ is an irreducible component of $\mathcal{Y}_s^\dagger$ whose normalization is of genus ≥ 1, hence necessarily maps birationally, via $\kappa^\dagger$, to an irreducible component of $\mathcal{Y}_s^{\mathrm{st}}$. In particular, we conclude that each connected component of $(f^\dagger)^{-1}(x_i) \subseteq \mathcal{Y}_s^\dagger$ contains an irreducible component of $\mathcal{Y}_s^\dagger$ that maps birationally, via $\kappa^\dagger$, to an irreducible component of $\mathcal{Y}_s^{\mathrm{st}}$.

Next, let $D^\dagger \subseteq \mathcal{Y}_s^\dagger$ be an irreducible component of $\mathcal{Y}_s^\dagger$ that maps to a closed point $\kappa^\dagger(D^\dagger)$ of $\mathcal{Y}_s^{\mathrm{st}}$ via $\kappa^\dagger : \mathcal{Y}^\dagger \to \mathcal{Y}^{\mathrm{st}}$, but dominates an irreducible component $E \stackrel{\mathrm{def}}{=} f^\dagger(D^\dagger)$ of $\mathcal{X}_s$. Note that these assumptions imply that the normalization of $D^\dagger$ is of genus 0, and hence that E is an irreducible component of $\mathcal{X}_s$ whose normalization is of genus 0. Thus, we conclude [cf. the condition imposed on the subset $\{x_1, \ldots, x_N\} \subseteq \mathcal{X}_s$] that E contains three points $\in \{x_1, \ldots, x_N\}$, i.e., [since $(f^\dagger)^{-1}(x_i)$ is pure of dimension 1] that $D^\dagger$ contains at least 3 nodes [that map to three distinct "x_i"]. On the other hand, this [together with the *birationality* of $\kappa^\dagger$] implies that the closed point $\kappa^\dagger(D^\dagger)$ of $\mathcal{Y}_s^{\mathrm{st}}$ intersects *three distinct irreducible components* of $\mathcal{Y}_s^{\mathrm{st}}$ [i.e., the images of suitable irreducible components of $(f^\dagger)^{-1}(x_i) \subseteq \mathcal{Y}_s^\dagger$, for three distinct "$i$"], that is to say, in *contradiction* to the definition of the notion of a compactified stable model [cf. Definition 2.1, (iv)]. Thus, we conclude that there do not exist any such "$D^\dagger$" [i.e., that map to a closed point of $\mathcal{Y}_s^{\mathrm{st}}$, but dominate an irreducible component of $\mathcal{X}_s$], and hence, by Zariski's Main Theorem, that the morphism $f^\dagger : \mathcal{Y}^\dagger \to \mathcal{X}$ factors as the composite of $\kappa^\dagger$ with a morphism $f^{\mathrm{st}} : \mathcal{Y}^{\mathrm{st}} \to \mathcal{X}$. In particular, it follows from the existence of the morphism $f^{\mathrm{st}} : \mathcal{Y}^{\mathrm{st}} \to \mathcal{X}$ that $\mathcal{Y}_s^{\mathrm{st}}$ contains an irreducible component whose corresponding valuation induces the given valuation v on $K(X)$, i.e., that X satisfies Σ-RNS. This completes the proof of assertion (iii).

Next, we verify assertion (iv). In light of Proposition 2.3, (iii), after possibly replacing K by a suitable finite extension field of K, we may assume without loss of generality that $\mathcal{X}$ is a compactified semistable model with split reduction of X over $\mathcal{O}_K$. Write $\{v_1, \ldots, v_N\}$ for the set of discrete valuations on $K(X)$ that arise from the irreducible components of $\mathcal{X}_s$. Then since X satisfies Σ-RNS, for each positive integer $i \leq N$, there exists a connected geometrically pro-Σ finite étale Galois covering $Y_i \to X$ such that v_i coincides with the restriction of a discrete valuation on the function field of Y_i that arises from an irreducible component of the special fiber of a compactified stable model of Y_i. Write $Y \to X$ for the composite covering of the connected geometrically pro-Σ finite étale Galois coverings $\{Y_i \to X\}_{1 \leq i \leq N}$. Then it follows immediately from Zariski's Main

Theorem [cf. also [ExtFam], Theorem A] that, after possibly replacing K by a suitable finite extension field of K, there exists a compactified stable model $\mathcal{Y}$ of Y over $\mathcal{O}_K$ that dominates $\mathcal{X}$. This completes the proof of assertion (iv).

Assertion (v) and the *necessity* portion of assertion (vi) follow immediately from Proposition 2.4, (iv), together with Proposition 2.3, (iv), (vi). Next, we consider the *sufficiency* portion of assertion (vi). Let $c \in \mathbb{VE}(\widetilde{X})$ be an element that corresponds [cf. Proposition 2.3, (v)] to a valuation v on $K(\widetilde{X})$ that extends a discrete residue-transcendental p-valuation on X. [Note that in this situation, v itself is necessarily *residue-transcendental*.] Then it suffices to show that there exists a "z_c" as in the statement of Proposition 2.4, (vi), that is a generic point of "$\mathcal{Z}_s^{\mathrm{st}}$". To this end, we observe that the *nonexistence* of such a "z_c" would imply that all of the "z_c" are closed points of "$\mathcal{Z}_s^{\mathrm{st}}$", hence have residue fields that are *algebraic* over the residue field of $\mathcal{O}_K$. On the other hand, this would imply that the residue field of $R_c = \mathcal{O}_v$ is *algebraic* over the residue field of $\mathcal{O}_K$, in contradiction to the *residue-transcendentality* of v. This completes the proof of assertion (vi).

Finally, we verify assertion (vii). The portion of assertion (vii) concerning the *existence* of an element $c \in \mathbb{VE}(\widetilde{X})^{\mathrm{prim}}$ as in the statement of Proposition 2.4, (vii), follows from Proposition 2.3, (xi). To verify the portion of assertion (vii) concerning the *uniqueness* of such an element $c \in \mathbb{VE}(\widetilde{X})^{\mathrm{prim}}$, it suffices to show the equality of any two elements $c_1, c_2 \in \mathbb{VE}(\widetilde{X})^{\mathrm{prim}}$ that satisfy the condition imposed on element "c" in the statement of Proposition 2.4, (vii). Write c_1', c_2' for the images of c_1, c_2 in $\mathbb{VE}(\widetilde{X})^{\mathrm{st,prim}}$. Then it follows from Proposition 2.4, (iv); [CbTpIV], Corollary 1.15, (iv) [applied to c_1', c_2'], that $\delta(c_1, c_2) < +\infty$, hence from Proposition 2.3, (ix), that $c_1 = c_2$, as desired. This completes the proof of assertion (vii), hence of Proposition 2.4. □

Corollary 2.5 (Constructions associated to geometric tempered fundamental groups). *Let $\Sigma \subseteq \mathfrak{Primes}$ be a subset of cardinality ≥ 2 such that $p \in \Sigma$; $K^\dagger$, $K^\ddagger$ mixed characteristic complete discrete valuation fields of residue characteristic p; $X^\dagger$, $X^\ddagger$ hyperbolic curves over $\overline{K}^\dagger$, $\overline{K}^\ddagger$, respectively. Write $\Omega^\dagger$, $\Omega^\ddagger$ for the p-adic completions of $\overline{K}^\dagger$, $\overline{K}^\ddagger$, respectively. For any hyperbolic curve Z over $\overline{K}^\dagger$, $\overline{K}^\ddagger$, $\Omega^\dagger$, or $\Omega^\ddagger$, write Π_Z^{tp} for the Σ-tempered fundamental group of Z, relative to a suitable choice of basepoint [cf. the subsection in Notations and Conventions entitled "Fundamental groups"]. Write $\widetilde{X}^\dagger \to X^\dagger$, $\widetilde{X}^\ddagger \to X^\ddagger$ for the universal geometrically pro-Σ coverings corresponding to $\Pi_{X^\dagger}^{\mathrm{tp}}$, $\Pi_{X^\ddagger}^{\mathrm{tp}}$, respectively. Suppose that $X^\dagger$ and $X^\ddagger$ satisfy Σ-RNS. Then the following hold [cf. Remark 2.5.1 below]:*

(i) Let $\sigma : \Pi_{X^\dagger}^{\mathrm{tp}} \xrightarrow{\sim} \Pi_{X^\ddagger}^{\mathrm{tp}}$ be an isomorphism of topological groups. Then σ induces homeomorphisms

$$\mathbb{VE}(\widetilde{X}^\dagger) \xrightarrow{\sim} \mathbb{VE}(\widetilde{X}^\ddagger),$$

$$\mathbb{VE}(\widetilde{X}^\dagger)^{\mathrm{tor}} \xrightarrow{\sim} \mathbb{VE}(\widetilde{X}^\ddagger)^{\mathrm{tor}},$$
$$\mathbb{VE}(\widetilde{X}^\dagger)^{\mathrm{prim}} \xrightarrow{\sim} \mathbb{VE}(\widetilde{X}^\ddagger)^{\mathrm{prim}},$$

that are compatible with the respective natural actions of $\Pi^{\mathrm{tp}}_{X^\dagger}$, $\Pi^{\mathrm{tp}}_{X^\ddagger}$. *If, moreover,* $X^\dagger$ *and* $X^\ddagger$ *are proper, then* σ *induces a homeomorphism*

$$(\widetilde{X}^\dagger)^{\mathrm{an}} \xrightarrow{\sim} (\widetilde{X}^\ddagger)^{\mathrm{an}}$$

that is compatible with the respective natural actions of $\Pi^{\mathrm{tp}}_{X^\dagger}$, $\Pi^{\mathrm{tp}}_{X^\ddagger}$.

(ii) Suppose that $K \overset{\mathrm{def}}{=} K^\dagger = K^\ddagger$, $\overline{K} = \overline{K}^\dagger = \overline{K}^\ddagger$, *hence that* $\Omega \overset{\mathrm{def}}{=} \Omega^\dagger = \Omega^\ddagger$. *Let* $x^\dagger \in X^\dagger(\Omega)$, $x^\ddagger \in X^\ddagger(\Omega)$. *Write* $X^\dagger_{x^\dagger}$ *(respectively,* $X^\ddagger_{x^\ddagger}$*) for the hyperbolic curve* $X^\dagger_\Omega \backslash \{x^\dagger\}$ *(respectively,* $X^\ddagger_\Omega \backslash \{x^\ddagger\}$*) over* Ω. *Let* $\widetilde{\sigma} : \Pi^{\mathrm{tp}}_{X^\dagger_{x^\dagger}} \xrightarrow{\sim} \Pi^{\mathrm{tp}}_{X^\ddagger_{x^\ddagger}}$ *be an isomorphism of topological groups that fits into a commutative diagram*

$$\begin{array}{ccc} \Pi^{\mathrm{tp}}_{X^\dagger_{x^\dagger}} & \xrightarrow[\widetilde{\sigma}]{\sim} & \Pi^{\mathrm{tp}}_{X^\ddagger_{x^\ddagger}} \\ \downarrow & & \downarrow \\ \Pi^{\mathrm{tp}}_{X^\dagger} & \xrightarrow[\sigma]{\sim} & \Pi^{\mathrm{tp}}_{X^\ddagger}, \end{array}$$

where the vertical arrows are the natural surjections [determined up to composition with an inner automorphism] induced by the natural open immersions $X^\dagger_{x^\dagger} \hookrightarrow X^\dagger_\Omega$, $X^\ddagger_{x^\ddagger} \hookrightarrow X^\ddagger_\Omega$ *of hyperbolic curves; the lower horizontal arrow* σ *is the isomorphism of topological groups [determined up to composition with an inner automorphism] induced by a(n) [uniquely determined — cf., e.g., [DM], Lemma 1.14] isomorphism* $\sigma_X : X^\dagger \xrightarrow{\sim} X^\ddagger$ *of schemes over* $\overline{K}$. *Then* $x^\ddagger = \sigma_X(x^\dagger)$.

Proof. First, we verify assertion (i). We begin by recalling that [SemiAn], Corollary 3.11, may be generalized/applied to hyperbolic curves over an *arbitrary* mixed characteristic complete discrete valuation field [cf. [AbsTopII], Remark 2.11.1, (i)]. Thus, by applying this generalized version of [SemiAn], Corollary 3.11, we conclude that σ induces a homeomorphism $\mathbb{VE}(\widetilde{X}^\dagger)^{\mathrm{st}} \xrightarrow{\sim} \mathbb{VE}(\widetilde{X}^\ddagger)^{\mathrm{st}}$. On the other hand, it follows from our assumption that $X^\dagger$ and $X^\ddagger$ satisfy Σ-RNS that we may apply the homeomorphisms "$\mathbb{VE}(\widetilde{X}) \xrightarrow{\sim} \mathbb{VE}(\widetilde{X})^{\mathrm{st}}$" and "$\mathbb{VE}(\widetilde{X})^{\mathrm{tor}} \xrightarrow{\sim} \mathbb{VE}(\widetilde{X})^{\mathrm{st,tor}}$" of Proposition 2.4, (v). In particular, we conclude that σ induces a homeomorphism

$$\mathbb{VE}(\widetilde{X}^\dagger)^{\mathrm{tor}} \xrightarrow{\sim} \mathbb{VE}(\widetilde{X}^\ddagger)^{\mathrm{tor}}$$

that is manifestly compatible with the respective natural actions of $\Pi^{\mathrm{tp}}_{X^\dagger}$, $\Pi^{\mathrm{tp}}_{X^\ddagger}$, as well as a homeomorphism

$$\mathbb{VE}(\widetilde{X}^\dagger) \xrightarrow{\sim} \mathbb{VE}(\widetilde{X}^\ddagger)$$

that is manifestly compatible with the respective natural actions of $\Pi^{\mathrm{tp}}_{X^\dagger}$, $\Pi^{\mathrm{tp}}_{X^\ddagger}$ and preserves specialization/generization relations, hence induces a homeomorphism

$$\mathbb{VE}(\widetilde{X}^\dagger)^{\mathrm{prim}} \xrightarrow{\sim} \mathbb{VE}(\widetilde{X}^\ddagger)^{\mathrm{prim}}$$

that is compatible with the respective natural actions of $\Pi^{\mathrm{tp}}_{X^\dagger}$, $\Pi^{\mathrm{tp}}_{X^\ddagger}$. Finally, if, moreover, $X^\dagger$ and $X^\ddagger$ are proper, then we may apply the homeomorphism "$\theta_{\widetilde{X}}$" of Proposition 2.3, (viii), to conclude that σ induces a homeomorphism

$$(\widetilde{X}^\dagger)^{\mathrm{an}} \xrightarrow{\sim} (\widetilde{X}^\ddagger)^{\mathrm{an}}$$

that is compatible with the respective natural actions of $\Pi^{\mathrm{tp}}_{X^\dagger}$, $\Pi^{\mathrm{tp}}_{X^\ddagger}$. This completes the proof of assertion (i).

Next, we verify assertion (ii). We begin by observing that it follows from the generalized version of [SemiAn], Corollary 3.11, discussed above, together with Corollary 2.5, (i), that $\widetilde{\sigma}$ induces a bijection

$$\mathbb{VE}(\widetilde{X}^\dagger) \xrightarrow{\sim} \mathbb{VE}(\widetilde{X}^\ddagger)$$

that maps the *point-theoretic orbit-center-chain associated* to $x^\dagger$ to the *point-theoretic orbit-center-chain associated* to $x^\ddagger$. Since σ arises from σ_X, we thus conclude from the bijection "$X(\Omega) \xrightarrow{\sim} \mathbb{VE}(\widetilde{X})^{\text{pt-th}}/\mathrm{Gal}(\widetilde{X}/X_{\overline{K}})$" of Proposition 2.3, (v), that $\sigma_X(x^\dagger) = x^\ddagger$. This completes the proof of assertion (ii), hence of Corollary 2.5. □

Remark 2.5.1. The homeomorphism "$(\widetilde{X}^\dagger)^{\mathrm{an}} \xrightarrow{\sim} (\widetilde{X}^\ddagger)^{\mathrm{an}}$" of Corollary 2.5, (i), is essentially similar to the homeomorphisms of [Lpg1], Theorem 3.10, but is formulated and proven according to the approach of the present paper. On the other hand, Corollary 2.5, (ii), may be regarded, when taken together with Theorem 2.17 below, as a generalization of [Tsjm], Theorem 2.2; its proof may be regarded as a more sophisticated version of the argument applied in the proof of [Tsjm], Theorem 2.2.

Proposition 2.6 (**Existence of new ordinary parts of certain coverings after Raynaud-Tamagawa).** *Let l be a prime number $\neq p$; K a mixed characteristic complete discrete valuation field of residue characteristic p; X a proper hyperbolic curve over K. Suppose that X has split stable reduction over K. Write $\mathcal{X}^{\mathrm{st}}$ for the [unique, up to unique isomorphism] stable model of X over $\mathcal{O}_K$. Suppose, moreover, that every irreducible component of the special fiber of $\mathcal{X}^{\mathrm{st}}$ is a smooth curve of genus ≥ 2. Write e_X (respectively, v_X) for the cardinality of the set of nodes (respectively, the set of irreducible components) of the stable curve $\mathcal{X}^{\mathrm{st}}_s$. Then:*

(i) For each sufficiently large positive integer m, if we replace K by a finite unramified extension field of K, then there exists a finite étale cyclic covering

$$\mathcal{Y}^{\mathrm{st}} \longrightarrow \mathcal{X}^{\mathrm{st}}$$

over $\mathcal{O}_K$ of degree l^m satisfying the following conditions:

(a) *Write* $(\mathcal{Y}^{\mathrm{st}} \to)\ \mathcal{Z}^{\mathrm{st}} \to \mathcal{X}^{\mathrm{st}}$ *for the finite étale cyclic subcovering over* $\mathcal{O}_K$ *of degree* l^{m-1}*;* Y*,* Z *for the generic fibers of* $\mathcal{Y}^{\mathrm{st}}$*,* $\mathcal{Z}^{\mathrm{st}}$*, respectively. Then* Y *and* Z *have split stable reduction over* K*. Moreover,* $\mathcal{Y}^{\mathrm{st}}$*,* $\mathcal{Z}^{\mathrm{st}}$ *are the stable models of* Y*,* Z*, respectively.*

(b) *The finite étale covering* $\mathcal{Y}^{\mathrm{st}}_s \to \mathcal{X}^{\mathrm{st}}_s$ *determined by the finite étale cyclic covering* $\mathcal{Y}^{\mathrm{st}} \to \mathcal{X}^{\mathrm{st}}$ *induces a bijection between the respective sets of irreducible components.*

(c) *Write* A *for the abelian variety over* K *obtained by forming the cokernel of the natural morphism* $J(Z) \to J(Y)$ *induced by the finite étale cyclic covering* $Y \to Z$ *[of degree* l*]. Then there exists an abelian variety* B *over* K *with good ordinary reduction such that* T_pA *fits into exact sequences of* G_K*-modules [cf. the theory of [FC], especially, [FC], Chapter III, Corollary 7.3]*

$$0 \longrightarrow T_{\mathrm{gd}} \longrightarrow T_pA \longrightarrow T_{\mathrm{cb}} \longrightarrow 0$$

$$0 \longrightarrow T_{\mathrm{tor}} \longrightarrow T_{\mathrm{gd}} \longrightarrow T_pB \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Hom}(T_p\mathcal{B}_s, \mathbb{Z}_p(1)) \longrightarrow T_pB \longrightarrow T_p\mathcal{B}_s \longrightarrow 0,$$

where "(1)*" denotes the Tate twist; the natural action of* G_K *on the "combinatorial quotient"* T_{cb} *[i.e., the inverse limit of the quotients "*$\underline{Y}/n\underline{Y}$*" of [FC], Chapter III, Corollary 7.3, as* n *ranges over the positive integral powers of* p*] of* T_pA *is trivial;* T_{tor} *is isomorphic as a* G_K*-module to the direct sum of a collection of copies of* $\mathbb{Z}_p(1)$*;* $\mathcal{B}$ *denotes the abelian scheme over* $\mathcal{O}_K$ *whose generic fiber is equal to* B*.*

(ii) *Fix a finite étale cyclic covering* $\mathcal{Y}^{\mathrm{st}} \to \mathcal{X}^{\mathrm{st}}$ *as in (i). Write*

$$T_{\mathrm{cb},Y}, \quad T_{\mathrm{cb},Z}$$

for the "combinatorial quotients" [i.e., the inverse limit of the quotients "$\underline{Y}/n\underline{Y}$*" of [FC], Chapter III, Corollary 7.3, as* n *ranges over the positive integral powers of* p*] of* $T_pJ(Y)$*,* $T_pJ(Z)$*, respectively;* h_Y*,* h_Z *for the respective loop-ranks of the dual graphs associated to the stable curves* $\mathcal{Y}^{\mathrm{st}}_s$*,* $\mathcal{Z}^{\mathrm{st}}_s$*. Then it holds that*

$$h_Y = 1 + l^m e_X - v_X, \quad h_Z = 1 + l^{m-1} e_X - v_X.$$

Moreover,

$$\mathrm{rank}_{\mathbb{Z}_p}\ T_{\mathrm{cb},Y} = h_Y; \quad \mathrm{rank}_{\mathbb{Z}_p}\ T_{\mathrm{cb},Z} = h_Z;$$

$$\mathrm{rank}_{\mathbb{Z}_p}\ T_{\mathrm{tor}} = \mathrm{rank}_{\mathbb{Z}_p}\ T_{\mathrm{cb}} = h_Y - h_Z = (l^m - l^{m-1})e_X.$$

Proof. First, we verify assertion (i). Write $\{C_i\}_{1 \leq i \leq v_X}$ for the set of irreducible components of $\mathcal{X}^{\mathrm{st}}_s$. Let m be a positive integer such that, for each positive integer $i \leq v_X$, it holds that

$$l^m > \frac{l^{2g_{C_i}} - l^{2g_{C_i}-1}}{l^{2g_{C_i}} - 1}(p-1)g_{C_i},$$

where $g_{(-)}$ denotes the genus of $(-)$. Then, in light of [Tama1], Lemma 1.9 [i.e., a generalization of [Ray1], Théorème 4.3.1], by replacing K by a finite unramified extension field of K, one may construct finite étale cyclic coverings $\{D_i \to C_i\}_{1 \leq i \leq v_X}$ of degree l^m [of proper hyperbolic curves over the residue field of K] satisfying the following conditions:

- For each positive integer $i \leq v_X$, write $(D_i \to)\ E_i \to C_i$ for the finite étale cyclic subcovering of degree l^{m-1} [of proper hyperbolic curves over the residue field of K]. Then the abelian variety obtained by forming the cokernel of the natural morphism $J(E_i) \to J(D_i)$ induced by the finite étale cyclic covering $D_i \to E_i$ of degree l is *ordinary*.
- The cardinality of the set of closed points of D_i that lie over the closed points of C_i determined by the nodes of $\mathcal{X}_s^{\mathrm{st}}$ is equal to l^m.

Next, one verifies immediately that there exists a finite étale cyclic covering $D \to \mathcal{X}_s^{\mathrm{st}}$ of degree l^m obtained by gluing together the finite étale cyclic coverings $\{D_i \to C_i\}_{1 \leq i \leq v_X}$. Write $\mathcal{Y}^{\mathrm{st}} \to \mathcal{X}^{\mathrm{st}}$ for the finite étale cyclic covering obtained by deforming the finite étale cyclic covering $D \to \mathcal{X}_s^{\mathrm{st}}$. Then it follows immediately from the various definitions involved that conditions (a), (b) hold. Moreover, in light of the theory of Raynaud extensions [cf. [FC], Chapter II, §1; [FC], Chapter III, Corollary 7.3], together with Remark 2.6.1, (i), (ii), below [cf. also [BLR], §9.2, Example 8], one concludes that condition (c) holds. This completes the proof of assertion (i).

Next, we verify assertion (ii). Write e_Y, e_Z for the respective cardinalities of the sets of nodes of the stable curves $\mathcal{Y}_s^{\mathrm{st}}$, $\mathcal{Z}_s^{\mathrm{st}}$; v_Y, v_Z for the respective cardinalities of the sets of irreducible components of the stable curves $\mathcal{Y}_s^{\mathrm{st}}$, $\mathcal{Z}_s^{\mathrm{st}}$. Then it follows immediately from conditions (a), (b), that

$$e_Y = l^m e_X, \quad e_Z = l^{m-1} e_X, \quad v_Y = v_Z = v_X.$$

Thus, we conclude from the well-known computation of the first homology group of a finite graph that

$$h_Y = 1 + e_Y - v_Y = 1 + l^m e_X - v_X, \quad h_Z = 1 + e_Z - v_Z = 1 + l^{m-1} e_X - v_X.$$

Therefore, to complete the proof of assertion (ii), it suffices to prove that $\mathrm{rank}_{\mathbb{Z}_p} T_{\mathrm{cb},Y} = h_Y$, $\mathrm{rank}_{\mathbb{Z}_p} T_{\mathrm{cb},Z} = h_Z$, and $\mathrm{rank}_{\mathbb{Z}_p} T_{\mathrm{tor}} = \mathrm{rank}_{\mathbb{Z}_p} T_{\mathrm{cb}} = h_Y - h_Z$. Recall that the loop-ranks h_Y, h_Z coincide with the *toric ranks* of the Jacobians of the stable curves $\mathcal{Y}_s^{\mathrm{st}}$, $\mathcal{Z}_s^{\mathrm{st}}$, respectively [cf., e.g, [BLR], §9.2, Example 8]. On the other hand, in light of the theory of duality for torsion subgroups of abelian varieties, it holds that these *toric ranks* coincide with the *ranks of the respective corresponding combinatorial quotients* [cf. [FC], Chapter III, Corollary 7.4]. In particular, it follows immediately [cf. Remark 2.6.1, (ii), below; [FC], Chapter III, Corollary 7.4] that $\mathrm{rank}_{\mathbb{Z}_p} T_{\mathrm{cb},Y} = h_Y$, $\mathrm{rank}_{\mathbb{Z}_p} T_{\mathrm{cb},Z} = h_Z$, hence that $\mathrm{rank}_{\mathbb{Z}_p} T_{\mathrm{tor}} = \mathrm{rank}_{\mathbb{Z}_p} T_{\mathrm{cb}} = \mathrm{rank}_{\mathbb{Z}_p} T_{\mathrm{cb},Y} - \mathrm{rank}_{\mathbb{Z}_p} T_{\mathrm{cb},Z} = h_Y - h_Z$. This completes the proof of assertion (ii), hence of Proposition 2.6. □

Remark 2.6.1. We maintain the notation of Proposition 2.6.

(i) Write $\mathcal{A}$ for the identity component of the Néron model of A over $\mathcal{O}_K$ [cf. [BLR], §1.3, Corollary 2]. Then the universal property of the Néron model implies the existence of a surjective homomorphism

$$f : \mathrm{Pic}^0_{\mathcal{Y}^{\mathrm{st}}/\mathcal{O}_K} \twoheadrightarrow \mathcal{A}$$

that extends the natural quotient homomorphism $J(Y) \twoheadrightarrow A$ [cf. [BLR], §1.2, Definition 1; [BLR], §9.4, Theorem 1]. Thus, since $\mathrm{Pic}^0_{\mathcal{Y}^{\mathrm{st}}/\mathcal{O}_K}$ is a semi-abelian scheme over $\mathcal{O}_K$ [cf. [BLR], §9.4, Theorem 1], it follows immediately from the existence of the surjective homomorphism f that $\mathcal{A}$ is also a semi-abelian scheme over $\mathcal{O}_K$.

(ii) Recall that the composite homomorphism $J(Z) \to J(Y) \to J(Z)$ of the norm map $J(Y) \to J(Z)$ with the natural homomorphism $J(Z) \to J(Y)$ coincides with the morphism given by multiplication by l. In particular, the abelian variety $J(Y)$ is isogenous over K to the product abelian variety $J(Z) \times_K A$. Thus, we conclude from [BLR], §7.3, Proposition 6, that the semi-abelian schemes $\mathrm{Pic}^0_{\mathcal{Y}^{\mathrm{st}}/\mathcal{O}_K}$ and $\mathrm{Pic}^0_{\mathcal{Z}^{\mathrm{st}}/\mathcal{O}_K} \times_{\mathcal{O}_K} \mathcal{A}$ over $\mathcal{O}_K$ [cf. [BLR], §9.4, Theorem 1] are isogenous over $\mathcal{O}_K$.

Definition 2.7. In the notation of Remark 2.6.1, let $\mathcal{Z}$ be a semistable model of Z over $\mathcal{O}_K$ that has split reduction. Note that $\mathcal{Z}^{\mathrm{st}}$ satisfies this property, and that, by pulling-back the finite étale cyclic covering $\mathcal{Y}^{\mathrm{st}} \to \mathcal{Z}^{\mathrm{st}}$ via the unique morphism $\mathcal{Z} \to \mathcal{Z}^{\mathrm{st}}$ that extends the identity morphism $Z \to Z$ [cf. [ExtFam], Theorem A], we obtain a finite étale cyclic covering

$$\mathcal{Y} \longrightarrow \mathcal{Z}$$

over $\mathcal{O}_K$ of degree l that extends the finite étale cyclic covering $Y \to Z$ over K. Suppose that

- $\mathcal{X}^{\mathrm{st}}_s$ is a singular curve, and that
- m is sufficiently large that $h_Y \geq h_Z \geq 1$ [cf. Proposition 2.6, (ii)].

(i) Let $y_\eta \in Y(K)$. Write $z_\eta \in Z(K)$ for the image of y_η via the natural map $Y(K) \to Z(K)$. Then y_η and z_η determine embeddings $Y \hookrightarrow J(Y)$, $Z \hookrightarrow J(Z)$ that allow one to regard $J(Y)$, $J(Z)$ as the respective *Albanese varieties* of Y, Z [cf. [AbsTopI], Appendix, Definition A.1, (ii); [Milne], Proposition 6.1]. In particular, we obtain a commutative diagram

$$\begin{array}{ccccc} \Delta_Y & \longrightarrow & T_p J(Y) & \longrightarrow & T_{\mathrm{cb},Y} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta_Z & \longrightarrow & T_p J(Z) & \longrightarrow & T_{\mathrm{cb},Z}, \end{array}$$

where the left-hand vertical arrow denotes the open injection induced by the finite étale cyclic covering $Y \to Z$; the left-hand horizontal arrows denote the natural surjections determined by the Albanese embeddings $Y \hookrightarrow J(Y)$, $Z \hookrightarrow J(Z)$ [cf. [AbsTopI], Appendix, Proposition A.6, (iv)]; the right-hand horizontal arrows denote the natural surjections; the middle and right-hand vertical arrows are *surjections* [cf. the fact that the finite étale cyclic covering $Y \to Z$ is of degree $l \neq p$]; the first square of the diagram commutes in light of the functoriality of the étale fundamental group; the second square of the diagram commutes in light of the functoriality of Raynaud extensions.

(ii) Fix a quotient

$$T_{\mathrm{cb},Z} \twoheadrightarrow \mathbb{Z}_p$$

[cf. our assumption that $h_Z \geq 1$; Proposition 2.6, (ii)]. For each nonnegative integer n, write

$$\mathcal{Z}_n \longrightarrow \mathcal{Z}$$

for the finite étale cyclic ["combinatorial"] covering of degree p^n over $\mathcal{O}_K$ induced by the natural quotient

$$(\Delta_Z \twoheadrightarrow T_p J(Z) \twoheadrightarrow)\ T_{\mathrm{cb},Z} \twoheadrightarrow \mathbb{Z}_p \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z};$$

$$\mathcal{Y}_n \longrightarrow \mathcal{Y}$$

for the finite étale cyclic ["combinatorial"] covering of degree p^n over $\mathcal{O}_K$ induced by the natural quotient

$$(\Delta_Y \twoheadrightarrow T_p J(Y) \twoheadrightarrow)\ T_{\mathrm{cb},Y} \twoheadrightarrow T_{\mathrm{cb},Z} \twoheadrightarrow \mathbb{Z}_p \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z}.$$

Thus, the commutative diagram in (i) induces a cartesian commutative diagram

$$\begin{array}{ccc} \mathcal{Y}_n & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \mathcal{Z}_n & \longrightarrow & \mathcal{Z}, \end{array}$$

where the vertical arrows are finite étale cyclic coverings of degree l over $\mathcal{O}_K$; the horizontal arrows are finite étale cyclic ["combinatorial"] coverings of degree p^n over $\mathcal{O}_K$. Moreover:

(a) Write Y_n, Z_n for the generic fibers of $\mathcal{Y}_n$, $\mathcal{Z}_n$, respectively. Then the finite étale covering $(\mathcal{Y}_n)_s \to (\mathcal{Z}_n)_s$ determined by the finite étale cyclic covering $\mathcal{Y}_n \to \mathcal{Z}_n$ induces a *bijection* between the *sets of irreducible components* that arise from the respective stable models of Y_n and Z_n [cf. Proposition 2.6, (i), (b)].

(b) Write A_n for the abelian variety over K obtained by forming the cokernel of the natural morphism $J(Z_n) \to J(Y_n)$ induced by the finite étale cyclic covering $Y_n \to Z_n$ of degree l [of proper hyperbolic

curves over K]. Then there exists an abelian variety B_n over K with good ordinary reduction such that T_pA_n fits into exact sequences of G_K-modules [cf. the theory of [FC], especially, [FC], Chapter III, Corollary 7.3; the proof of Proposition 2.6, (i), (c)]

$$0 \longrightarrow T_{\mathrm{gd},n} \longrightarrow T_pA_n \longrightarrow T_{\mathrm{cb},n} \longrightarrow 0$$

$$0 \longrightarrow T_{\mathrm{tor},n} \longrightarrow T_{\mathrm{gd},n} \longrightarrow T_pB_n \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Hom}(T_p(\mathcal{B}_n)_s, \mathbb{Z}_p(1)) \longrightarrow T_pB_n \longrightarrow T_p(\mathcal{B}_n)_s \longrightarrow 0,$$

where "(1)" denotes the Tate twist; the natural action of G_K on the "combinatorial quotient" $T_{\mathrm{cb},n}$ of T_pA_n is trivial; $T_{\mathrm{tor},n}$ is isomorphic as a G_K-module to the direct sum of a collection of copies of $\mathbb{Z}_p(1)$; $\mathcal{B}_n$ denotes the abelian scheme over $\mathcal{O}_K$ whose generic fiber is equal to B_n.

(iii) Write

$$\mathcal{A}_n$$

for the identity component of the Néron model of A_n over $\mathcal{O}_K$ [cf. [BLR], §1.3, Corollary 2]. Then the universal property of the Néron model implies the existence of a surjective homomorphism

$$f_n : \mathcal{J}_n \stackrel{\mathrm{def}}{=} \mathrm{Pic}^0_{\mathcal{Y}_n/\mathcal{O}_K} \twoheadrightarrow \mathcal{A}_n$$

that extends the natural quotient homomorphism $J(Y_n) \twoheadrightarrow A_n$ [cf. [BLR], §1.2, Definition 1; [BLR], §9.4, Theorem 1]. Thus, since $\mathcal{J}_n = \mathrm{Pic}^0_{\mathcal{Y}_n/\mathcal{O}_K}$ is a semi-abelian scheme over $\mathcal{O}_K$ [cf. [BLR], §9.4, Theorem 1], it follows immediately from the existence of the surjective homomorphism f_n that $\mathcal{A}_n$ is also a semi-abelian scheme over $\mathcal{O}_K$ [cf. Remark 2.6.1, (i)].

(iv) For each nonnegative integer n, write

$$h_{Y_n}, \quad h_{Z_n}$$

for the loop-ranks of the dual graphs associated to the semistable curves $(\mathcal{Y}_n)_s$, $(\mathcal{Z}_n)_s$, respectively;

$$g_{Y_n}$$

for the [arithmetic] genus of the semistable model $\mathcal{Y}_n$ over $\mathcal{O}_K$. Then a similar argument to the argument applied in Remark 2.6.1, (ii), implies that the semi-abelian schemes $\mathrm{Pic}^0_{\mathcal{Y}_n/\mathcal{O}_K}$ and $\mathrm{Pic}^0_{\mathcal{Z}_n/\mathcal{O}_K} \times_{\mathcal{O}_K} \mathcal{A}_n$ over $\mathcal{O}_K$ are isogenous over $\mathcal{O}_K$. In particular, we obtain equalities

$$\mathrm{rank}_{\mathbb{Z}_p}\ T_{\mathrm{tor},n} = \mathrm{rank}_{\mathbb{Z}_p}\ T_{\mathrm{cb},n} = h_{Y_n} - h_{Z_n}$$

[cf. the proof of Proposition 2.6, (ii)].

Proposition 2.8 (Explicit computations of toric rank and genus). *We maintain the notation of Definition 2.7. Then the following hold:*

(i) It holds that

$$h_{Y_n} = 1 + p^n l^m e_X - p^n v_X, \quad h_{Z_n} = 1 + p^n l^{m-1} e_X - p^n v_X,$$

hence, in particular, that

$$\mathrm{rank}_{\mathbb{Z}_p}\, T_{\mathrm{tor},n} = \mathrm{rank}_{\mathbb{Z}_p}\, T_{\mathrm{cb},n} = h_{Y_n} - h_{Z_n} = p^n(l^m - l^{m-1})e_X$$

[cf. the final display of Definition 2.7, (iv)].

(ii) It holds that

$$g_{Y_n} = p^n(g_{Y_0} - 1) + 1.$$

Proof. First, recall from the well-known theory of stable and semistable models that h_{Y_n}, h_{Z_n}, $\mathrm{rank}_{\mathbb{Z}_p}\, T_{\mathrm{tor},n}$, $\mathrm{rank}_{\mathbb{Z}_p}\, T_{\mathrm{cb},n}$, and g_{Y_n} are *independent* of the choice of the semistable model $\mathcal{Z}$ of Z over $\mathcal{O}_K$. [Indeed, by passing to a suitable finite unramified extension of K and *adding finitely many suitably positioned cusps*, one may, in effect, reduce this "well-known theory of stable and semistable models" to the theory of *pointed stable curves* and contraction morphisms that arise from eliminating cusps, as exposed in [Knud].] In particular, we may assume without loss of generality that $\mathcal{Z}$ is the stable model of Z over $\mathcal{O}_K$. Assertion (i) then follows immediately from a similar argument to the argument applied in the proof of Proposition 2.6, (ii). Assertion (ii) follows immediately from Hurwitz's formula. This completes the proof of Proposition 2.8. □

Definition 2.9. We maintain the notation of Definition 2.7. Then we shall write

$$\widehat{T}_{\mathrm{cnn},n}$$

for the connected p-divisible group over $\mathcal{O}_K$ that arises as the connected part of the p-divisible group [cf. the discussion preceding [Tate], §2.2, Proposition 2] associated to the Raynaud extension [cf. [FC], Chapter II, §1] of A_n;

$$\widehat{T}_{\mathrm{tor},n}$$

for the connected p-divisible group over $\mathcal{O}_K$ associated to the torus that appears in the Raynaud extension of A_n;

$$\widetilde{T}_{\mathrm{cnn},n}, \quad \widetilde{T}_{\mathrm{tor},n}$$

for the respective generic fibers of $\widehat{T}_{\mathrm{cnn},n}$, $\widehat{T}_{\mathrm{tor},n}$;

$$T_{\mathrm{cnn},n} \stackrel{\mathrm{def}}{=} T_p(\widetilde{T}_{\mathrm{cnn},n}), \quad T_{\mathrm{tor},n} = T_p(\widetilde{T}_{\mathrm{tor},n})$$

for the respective p-adic Tate modules of $\widetilde{T}_{\mathrm{cnn},n}$, $\widetilde{T}_{\mathrm{tor},n}$ [cf. the subsection in Notations and Conventions entitled "Schemes"; Definition 2.7, (ii), (b)]. Note that $T_{\mathrm{cnn},n}$ and $T_{\mathrm{tor},n}$ may be regarded as $\mathbb{Z}_p$-submodules of T_pA_n in a natural way. Moreover, we shall write

$$T_{\text{ét},n} \stackrel{\text{def}}{=} T_pA_n/T_{\mathrm{cnn},n}.$$

Note that G_K acts naturally on the $\mathbb{Z}_p$-modules $T_{\mathrm{cnn},n}$, $T_{\mathrm{tor},n}$, $T_{\text{ét},n}$, and $T_{\mathrm{cb},n}$ [cf. Definition 2.7, (ii), (b)]. We shall write

$$T_{\mathrm{cnn},n} \twoheadrightarrow T_{\mathrm{qtr},n}$$

for the maximal torsion-free G_K-stable quotient $\mathbb{Z}_p$-module among the torsion-free G_K-stable quotient $\mathbb{Z}_p$-modules $T_{\mathrm{cnn},n} \twoheadrightarrow T$ such that some open subgroup of G_K acts on T via the p-adic cyclotomic character;

$$T_{\mathrm{qcb},n} \subseteq T_{\text{ét},n}$$

for the maximal G_K-stable $\mathbb{Z}_p$-submodule among the G_K-stable $\mathbb{Z}_p$-submodules $T \subseteq T_{\text{ét},n}$ such that some open subgroup of G_K acts trivially on T. Finally, we observe that [one verifies immediately that] we obtain natural exact sequences of G_K-modules [cf. Definition 2.7, (ii), (b)]

$$0 \longrightarrow T_{\mathrm{tor},n} \longrightarrow T_{\mathrm{cnn},n} \longrightarrow \mathrm{Hom}(T_p(\mathcal{B}_n)_s, \mathbb{Z}_p(1)) \longrightarrow 0$$

$$0 \longrightarrow T_p(\mathcal{B}_n)_s \longrightarrow T_{\text{ét},n} \longrightarrow T_{\mathrm{cb},n} \longrightarrow 0$$

$$0 \longrightarrow T_{\mathrm{cnn},n} \longrightarrow T_pA_n \longrightarrow T_{\text{ét},n} \longrightarrow 0.$$

Lemma 2.10. *We maintain the notation of Definition 2.9. Then:*

(i) The natural action of G_K on T_pA_n induces the trivial action of I_K on $T_{\text{ét},n}$.

(ii) There exist natural compatible G_K-equivariant isomorphisms

$$T_{\text{ét},n} \stackrel{\sim}{\to} \mathrm{Hom}(T_{\mathrm{cnn},n}, \mathbb{Z}_p(1)), \quad T_{\mathrm{cb},n} \stackrel{\sim}{\to} \mathrm{Hom}(T_{\mathrm{tor},n}, \mathbb{Z}_p(1)),$$

$$T_{\mathrm{qcb},n} \stackrel{\sim}{\to} \mathrm{Hom}(T_{\mathrm{qtr},n}, \mathbb{Z}_p(1)).$$

(iii) Suppose that K is a p-adic local field. Then the set of eigenvalues of the $\mathbb{Z}_p$-linear automorphism of $T_p(\mathcal{B}_n)_s$ induced by the Frobenius element of G_K/I_K [cf. (i); the second exact sequence of Definition 2.9] does not contain any roots of unity.

Proof. First, we verify assertion (i). Recall that the quotient of a p-divisible group by its connected part is *étale* [cf., e.g., the discussion preceding [Tate], §2.2, Proposition 2]. Thus, we conclude [cf. the *triviality* of the action of G_K on $T_{\mathrm{cb},n}$ observed in Definition 2.7, (ii), (b); the second exact sequence of the final display of Definition 2.9] from [the second sentence of] [FC], Chapter III, Corollary 7.3, that the natural action of I_K on $T_{\text{ét},n}$ is *trivial*, as desired. This completes the proof of assertion (i). Assertion (ii) follows immediately from the theory of duality for torsion subgroups of abelian varieties [cf. [FC], Chapter III, Corollary 7.4], together with the first and second exact sequences of the final display of Definition 2.9. Assertion (iii) follows immediately from the *finiteness* of the set of rational points of $(\mathcal{B}_n)_s$ over any finite extension field of the [*finite*!] residue field of K. This completes the proof of Lemma 2.10. □

Proposition 2.11 (**Toral quotient of the connected part**). *In the notation of Definition 2.9, write*

$$\chi_n : T_{\mathrm{cnn},n} \twoheadrightarrow T_{\mathrm{qtr},n}$$

for the natural surjection of G_K-modules. Then the following hold:

(i) Suppose that the residue field of K is separably closed. Then

$$T_{\mathrm{cnn},n} = T_{\mathrm{qtr},n} \neq \{0\}, \quad T_{\mathrm{qcb},n} = T_{\text{ét},n} \neq \{0\}.$$

(ii) Suppose that K is a p-adic local field. Then the restriction of χ_n to $T_{\mathrm{tor},n} \subseteq T_{\mathrm{cnn},n}$ induces an injection $T_{\mathrm{tor},n} \hookrightarrow T_{\mathrm{qtr},n}$ with finite cokernel.

Proof. First, we verify assertion (i). Note that since the residue field of K is separably closed, $G_K = I_K$. Thus, it follows immediately from Lemma 2.10, (i), together with the various definitions involved, that $T_{\mathrm{qcb},n} = T_{\text{ét},n}$. On the other hand, it follows immediately from Proposition 2.8, (i) [cf. also Proposition 2.6; Definition 2.7; the second exact sequence of the final display of Definition 2.9], that $T_{\text{ét},n} \neq \{0\}$. Thus, we conclude from Lemma 2.10, (ii), that $T_{\mathrm{cnn},n} = T_{\mathrm{qtr},n} \neq \{0\}$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Write $N \stackrel{\text{def}}{=} \mathrm{rank}_{\mathbb{Z}_p} T_{\mathrm{qtr},n}$. Note that, in light of the maximality of $T_{\mathrm{qtr},n}$, it suffices to verify that there exists a unique torsion-free G_K-stable quotient $\mathbb{Z}_p$-module

$$T_{\mathrm{cnn},n} \twoheadrightarrow T$$

whose restriction to $T_{\mathrm{tor},n} \subseteq T_{\mathrm{cnn},n}$ induces an injection $T_{\mathrm{tor},n} \hookrightarrow T$ with finite cokernel, and that $\mathrm{rank}_{\mathbb{Z}_p} T \geq N$. Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of $\mathbb{Q}_p$ equipped with the *trivial* action of G_K. Then observe that it follows from a routine argument involving *Galois descent* from $\overline{\mathbb{Q}}_p$ to $\mathbb{Q}_p$ that it suffices to verify that there exists a unique G_K-stable quotient $\overline{\mathbb{Q}}_p$-vector space

$$T_{\mathrm{cnn},n} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p \longrightarrow V$$

whose restriction to $T_{\mathrm{tor},n} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p \subseteq T_{\mathrm{cnn},n} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p$ induces an isomorphism $T_{\mathrm{tor},n} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p \xrightarrow{\sim} V$, and that $\dim_{\overline{\mathbb{Q}}_p} V \geq N$. Write k for the finite residue field of the p-adic local field K [so G_K/I_K may be identified with the absolute Galois group G_k of k]. Then, by applying Lemma 2.10, (i), (ii), we conclude that it suffices to verify that there exists a unique G_k-stable $\overline{\mathbb{Q}}_p$-subspace $V^* \subseteq T_{\text{ét},n} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p$ whose composite with the natural surjection $T_{\text{ét},n} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p \twoheadrightarrow T_{\mathrm{cb},n} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p$ induces an isomorphism $V^* \xrightarrow{\sim} T_{\mathrm{cb},n} \otimes_{\mathbb{Z}_p} \overline{\mathbb{Q}}_p$, and that $\dim_{\overline{\mathbb{Q}}_p} V^* \geq N$. On the other hand, in light of the eigenspace decomposition associated to the natural action of the Frobenius element $\in G_k$, the existence and uniqueness of such a subspace, together with the inequality $\dim_{\overline{\mathbb{Q}}_p} V^* \geq N$ $(= \mathrm{rank}_{\mathbb{Z}_p} T_{\mathrm{qcb},n})$ [cf. Lemma 2.10, (ii)], follows immediately from Lemma 2.10, (iii) [cf. also the *triviality* of the action of G_K on $T_{\mathrm{cb},n}$ observed in Definition 2.7, (ii), (b)]. This completes the proof of assertion (ii), hence of Proposition 2.11. □

Proposition 2.12 (Construction of a certain morphism of formal schemes to the quasi-toral quotient). *In the notation of Proposition 2.11, let $y_n \in \mathcal{Y}_n(\mathcal{O}_K)$ be an $\mathcal{O}_K$-rational point that maps the closed point of* Spec $\mathcal{O}_K$ *to a smooth point $(y_n)_s$ of the semistable curve $(\mathcal{Y}_n)_s$. Write $y_{n,\eta} \in Y_n(K)$ for the K-valued point of Y_n determined by $y_n \in \mathcal{Y}_n(\mathcal{O}_K)$; $C \subseteq (\mathcal{Y}_n)_s$ for the unique irreducible component that contains $(y_n)_s$; $F \subseteq \mathcal{Y}_n$ for the closed subset obtained by forming the union of the irreducible components $\neq C$ of $(\mathcal{Y}_n)_s$; $\mathcal{U}_{y_n} \subseteq \mathcal{Y}_n$ for the open subscheme obtained by forming the complement of $F \subseteq \mathcal{Y}_n$;*

$$h_{n,\eta} : Y_n \longrightarrow J(Y_n)$$

for the Albanese map that maps $y_{n,\eta}$ to the origin [cf. [AbsTopI], Appendix, Definition A.1, (ii); [Milne], Proposition 6.1]. Recall that $\mathcal{J}_n$ is a semi-abelian scheme over $\mathcal{O}_K$ [cf. Definition 2.7, (iii)] whose generic fiber is $J(Y_n)$. In particular, $\mathcal{J}_n$ is isomorphic to the identity component of the Néron model over $\mathcal{O}_K$ of $J(Y_n)$ [cf. [BLR], §7.4, Proposition 3]. Thus, since $\mathcal{U}_{y_n}$ is a connected smooth scheme over $\mathcal{O}_K$ whose generic fiber is Y_n, the universal property of the Néron model implies the existence of a unique morphism

$$h_n : \mathcal{U}_{y_n} \longrightarrow \mathcal{J}_n$$

that extends $h_{n,\eta}$. Next, write $\widehat{\mathcal{J}}_n$, $\widehat{\mathcal{A}}_n$ for the formal completions at the origin of the semi-abelian schemes $\mathcal{J}_n$, $\mathcal{A}_n$ over $\mathcal{O}_K$; $\widehat{\mathcal{O}}_{\mathcal{Y}_n,y_n}$, $\widehat{\mathcal{O}}_{\mathcal{U}_{y_n},y_n}$ for the completions at y_n of $\mathcal{O}_{\mathcal{Y}_n,(y_n)_s}$, $\mathcal{O}_{\mathcal{U}_{y_n},(y_n)_s}$. Then the natural composite map

$$\mathrm{Spf}\ \widehat{\mathcal{O}}_{\mathcal{Y}_n,y_n} = \mathrm{Spf}\ \widehat{\mathcal{O}}_{\mathcal{U}_{y_n},y_n} \longrightarrow \widehat{\mathcal{J}}_n \longrightarrow \widehat{\mathcal{A}}_n$$

induced by h_n and the surjective homomorphism $f_n : \mathcal{J}_n \twoheadrightarrow \mathcal{A}_n$ [cf. Definition 2.7, (iii)] determines a morphism of formal $\mathcal{O}_K$-schemes

$$\mathrm{Spf}\ \widehat{\mathcal{O}}_{\mathcal{Y}_n,y_n} \longrightarrow \widehat{T}_{\mathrm{cnn},n},$$

where we regard the connected p-divisible group $\widehat{T}_{\mathrm{cnn},n}$ as a formal group over $\mathcal{O}_K$ [cf. [Tate], §2.2, Proposition 1]. In particular, by forming the composite with the morphism of formal $\mathcal{O}_K$-schemes induced by χ_n [cf. [Tate], §2.2, Proposition 1; [Tate], §4.2, Corollary 1], we obtain a morphism of formal $\mathcal{O}_K$-schemes

$$\mathrm{Spf}\ \widehat{\mathcal{O}}_{\mathcal{Y}_n,y_n} \longrightarrow \widehat{T}_{\mathrm{qtr},n},$$

where $\widehat{T}_{\mathrm{qtr},n}$ denotes the formal group over $\mathcal{O}_K$ determined by the connected p-divisible group associated to the G_K-module $T_{\mathrm{qtr},n}$ [cf. [Tate], §2.2, Proposition 1; [Tate], §4.2, Corollary 1].

Proof. Proposition 2.12 follows immediately from the various references quoted in the statement of Proposition 2.12. □

Proposition 2.13 (**Coverings associated to characters**). *We maintain the notation of Proposition 2.12. Then the following hold:*

(i) Write $\widehat{\mathbb{G}}_{\mathrm{m}}$ for the formal completion at the origin of the multiplicative group scheme $\mathbb{G}_{\mathrm{m}}$ over $\mathcal{O}_K$; $\mathrm{Hom}_{\mathcal{O}_K}(\widehat{T}_{\mathrm{qtr},n}, \widehat{\mathbb{G}}_{\mathrm{m}})$ for the $\mathbb{Z}_p$-module of homomorphisms over $\mathcal{O}_K$ from $\widehat{T}_{\mathrm{qtr},n}$ to $\widehat{\mathbb{G}}_{\mathrm{m}}$; $\mathrm{Hom}_{G_K}(T_{\mathrm{qtr},n}, \mathbb{Z}_p(1))$ for the $\mathbb{Z}_p$-module of G_K-equivariant homomorphisms of $\mathbb{Z}_p$-modules $T_{\mathrm{qtr},n} \to \mathbb{Z}_p(1)$. Then the natural homomorphism

$$\mathrm{Hom}_{\mathcal{O}_K}(\widehat{T}_{\mathrm{qtr},n}, \widehat{\mathbb{G}}_{\mathrm{m}}) \longrightarrow \mathrm{Hom}_{G_K}(T_{\mathrm{qtr},n}, \mathbb{Z}_p(1))$$

is bijective.

(ii) Let a be a positive integer; $f \in \mathrm{Hom}_{\mathcal{O}_K}(\widehat{T}_{\mathrm{qtr},n}, \widehat{\mathbb{G}}_{\mathrm{m}})$. Consider the composite

$$(\mathrm{Spf}\ \mathcal{O}_K[[t]] \xrightarrow{\sim}) \ \mathrm{Spf}\ \widehat{\mathcal{O}}_{\mathcal{Y}_n,y_n} \longrightarrow \widehat{\mathbb{G}}_{\mathrm{m}}$$

[where t is an indeterminate, and we regard $\mathcal{O}_K[[t]]$ as being equipped with the t-adic topology] of the morphism in the final display of Proposition 2.12 with f. By a slight abuse of notation, we shall also write $f \in \mathcal{O}_K[[t]]^\times$ for the image of the canonical coordinate U of $\widehat{\mathbb{G}}_{\mathrm{m}}$ via the homomorphism of rings induced by the above composite morphism. Then the covering of $\mathrm{Spf}\ \mathcal{O}_K[[t]]$ obtained by extracting a p^a-th root of f is dominated by the covering of $\mathrm{Spf}\ \mathcal{O}_K[[t]]$ obtained by restricting the covering determined by multiplication by p^a on $\widehat{\mathcal{A}}_n$.

Proof. Assertion (i) follows immediately from [Tate], §2.2, Proposition 1; [Tate], §4.2, Corollary 1. Assertion (ii) follows immediately from the various definitions involved. This completes the proof of Proposition 2.13. □

Lemma 2.14. *Let X be a smooth proper curve of genus g_X over a field K; $x \in X(K)$ a K-valued point of X; d_x a nonnegative integer such that $d_x \geq 2g_X - 1$. Then the natural composite map*

$$H^0(X, \Omega_X) \hookrightarrow \Omega_{X,x} \twoheadrightarrow \Omega_{X,x}/\mathfrak{m}_x^{d_x}\Omega_{X,x}$$

is injective.

Proof. First, observe that since $d_x \geq 2g_X - 1 > 2g_X - 2$ [which implies that the degree of the line bundle $\Omega_X(-d_x \cdot x)$ is negative], it follows that $H^0(X, \Omega_X(-d_x \cdot x)) = 0$. Thus, the desired *injectivity* follows immediately by applying the [*left exact*] functor $H^0(X, -)$ to the short exact sequence

$$0 \longrightarrow \Omega_X(-d_x) \longrightarrow \Omega_X \longrightarrow \Omega_{X,x}/\mathfrak{m}_x^{d_x}\Omega_{X,x} \longrightarrow 0.$$

This completes the proof of Lemma 2.14. □

Lemma 2.15. *Let a, b be positive integers; K a p-adic local field of degree $d_K \stackrel{\text{def}}{=} [K : \mathbb{Q}_p]$ over $\mathbb{Q}_p$; $W \subseteq K[T]/(T^{b+1})$ a $\mathbb{Q}_p$-vector subspace of dimension a. For each nonzero element*

$$h = \sum_{0 \leq i \leq b} h_i T^i \ \in K[T]/(T^{b+1}),$$

write $\mathrm{ord}(h)$ $(\leq b)$ *for the smallest integer i such that $h_i \neq 0$. Set* $\mathrm{ord}(0) \stackrel{\text{def}}{=} +\infty$. *Suppose that*

$$W \backslash \{0\} \subseteq F \stackrel{\text{def}}{=} \{h \in K[T]/(T^{b+1}) \mid \mathrm{ord}(h) = p^j - 1 \text{ for some nonnegative integer } j\}.$$

Then it holds that

$$a \leq d_K(\log_p(b+1) + 1) \quad \big(\ll d_K(b+1) = \dim_{\mathbb{Q}_p} K[T]/(T^{b+1}) \big).$$

Proof. For each nonnegative integer j, write

$$F_j \stackrel{\text{def}}{=} W \cap \{h \in K[T]/(T^{b+1}) \mid \mathrm{ord}(h) \geq p^j - 1\}$$

[so F_j is a $\mathbb{Q}_p$-vector space, and $F_{j+1} \subseteq F_j$]. Then it follows immediately from our assumption that $W \setminus \{0\} \subseteq F$ that for each nonnegative integer j,

$$\dim_{\mathbb{Q}_p}(F_j/F_{j+1}) \leq \dim_{\mathbb{Q}_p}(K) = d_K.$$

On the other hand, since $F_j = \{0\}$ for any nonnegative integer j such that $p^j > p^j - 1 \geq b + 1$, we thus conclude that

$$a = \dim_{\mathbb{Q}_p}(W) = \sum_{j=0}^{+\infty} \dim_{\mathbb{Q}_p}(F_j/F_{j+1}) \leq d_K(\log_p(b+1) + 1),$$

as desired. This completes the proof of Lemma 2.15. □

Theorem 2.16 (**Existence of suitable coverings**). *In the notation of Proposition 2.13, suppose further that K is a p-adic local field. Then there exists a real number C_{g_Y,d_K} that depends only on $g_Y \stackrel{\text{def}}{=} g_{Y_0}$ and $d_K \stackrel{\text{def}}{=} [K:\mathbb{Q}_p]$ such that for any positive integer $n \geq C_{g_Y,d_K}$, after possibly replacing K by a finite extension field of K, there exist*

- *a [connected] finite étale Galois covering $W_n \to Y_n$ of proper hyperbolic curves over K of degree a power of p,*
- *a semistable model $\mathcal{W}_n$ of W_n over $\mathcal{O}_K$,*
- *a morphism $\psi : \mathcal{W}_n \to \mathcal{Y}_n$ of semistable models over $\mathcal{O}_K$ that restricts to the finite étale Galois covering $W_n \to Y_n$,*
- *an irreducible component D of $(\mathcal{W}_n)_s$ whose normalization is of genus ≥ 1*

such that $\psi(D) = (y_n)_s \in (\mathcal{Y}_n)_s$.

Proof. Fix a positive integer n. First, we consider the natural homomorphisms of $\mathbb{Z}_p$-modules

$$\begin{aligned}\big(\mathrm{Hom}_{G_K}(T_{\mathrm{qtr},n},\mathbb{Z}_p(1)) \stackrel{\sim}{\leftarrow}\big)\ \mathrm{Hom}_{\mathcal{O}_K}(\widehat{T}_{\mathrm{qtr},n},\widehat{\mathbb{G}}_{\mathrm{m}}) &\hookrightarrow H^0_{\mathrm{inv}}(\widehat{T}_{\mathrm{qtr},n},\Omega_{\widehat{T}_{\mathrm{qtr},n}})\\ &\hookrightarrow H^0_{\mathrm{inv}}(\widehat{T}_{\mathrm{cnn},n},\Omega_{\widehat{T}_{\mathrm{cnn},n}})\\ &\stackrel{\sim}{\leftarrow} H^0(\mathcal{A}_n,\Omega_{\mathcal{A}_n})\\ &\hookrightarrow H^0(\mathcal{U}_{y_n},\Omega_{\mathcal{Y}_n})\\ &\hookrightarrow \Omega_{\mathcal{Y}_n,y_n}\otimes_{\mathcal{O}_{\mathcal{Y}_n,y_n}}\widehat{\mathcal{O}}_{\mathcal{Y}_n,y_n}\\ &\to \Omega_{Y_n,y_{n,\eta}}/\mathfrak{m}_{y_{n,\eta}}^{2g_{Y_n}}\Omega_{Y_n,y_{n,\eta}}\\ &\big(\stackrel{\sim}{\to} K[t]/(t^{2g_{Y_n}})\,dt\big),\end{aligned}$$

where

- the first arrow denotes the natural bijective homomorphism of Proposition 2.13, (i);
- "$H^0_{\mathrm{inv}}(-)$" denotes the $\mathcal{O}_K$-submodule of "$H^0(-)$" that consists of the *invariant differentials* on the p-divisible group in the first argument of "$H^0(-)$";
- the second arrow denotes the injection obtained by pulling back the invariant differential $d\log(U) \stackrel{\text{def}}{=} \frac{dU}{U}$ on $\widehat{\mathbb{G}}_{\mathrm{m}}$;
- the third arrow denotes the injection induced by χ_n [cf. Proposition 2.11; [Tate], §2.2, Proposition 1; [Tate], §4.2, Corollary 1];
- the fourth arrow denotes the natural isomorphism;

- the fifth arrow denotes the homomorphism of $\mathcal{O}_K$-modules obtained by pulling back the differentials via the composite map $\mathcal{U}_{y_n} \hookrightarrow \mathcal{J}_n \twoheadrightarrow \mathcal{A}_n$ that maps y_n to the origin [cf. Definition 2.7, (ii), (b); Definition 2.7, (iii); Proposition 2.12];
- the sixth arrow denotes the natural injection;
- the seventh arrow denotes the natural restriction morphism;
- $\mathfrak{m}_{y_{n,\eta}}$ denotes the maximal ideal of $\mathcal{O}_{Y_n,y_{n,\eta}}$;
- the final arrow denotes the natural isomorphism determined by choosing a "local coordinate" t, i.e., an element of the maximal ideal $\mathfrak{m}_{\mathcal{Y}_{n,y_n}}$ of $\mathcal{O}_{\mathcal{Y}_{n,y_n}}$ such that t and $\mathfrak{m}_K$ generate $\mathfrak{m}_{\mathcal{Y}_{n,y_n}}$.

Write

$$\Psi : \mathrm{Hom}_{\mathcal{O}_K}(\widehat{T}_{\mathrm{qtr},n}, \widehat{\mathbb{G}}_{\mathrm{m}}) \longrightarrow \mathcal{O}_K[[t]]^{\times}$$

for the assignment discussed in Proposition 2.13, (ii) [i.e., relative to the local coordinate t chosen above];

$$\Xi : \mathrm{Hom}_{\mathcal{O}_K}(\widehat{T}_{\mathrm{qtr},n}, \widehat{\mathbb{G}}_{\mathrm{m}}) \hookrightarrow \Omega_{Y_n,y_{n,\eta}}/\mathfrak{m}_{y_{n,\eta}}^{2g_{Y_n}}\Omega_{Y_n,y_{n,\eta}} \ \left(\xrightarrow{\sim} K[t]/(t^{2g_{Y_n}})\ dt \right)$$

for the *injective* [by Lemma 2.14] composite of the second to the seventh arrows in the first display of the present proof. Thus, Ξ may be understood as the result of composing Ψ with the operation of taking the logarithmic derivative with respect to t and then truncating the terms of degree $\geq 2g_{Y_n}$.

Observe that so far we have not applied the assumption that K is a *p-adic local field*. Now we proceed to apply this assumption. Recall from Proposition 2.8, (i), (ii); Proposition 2.11, (ii) [cf. also the initial portions of Proposition 2.6 and Definition 2.7], that

$$\mathrm{rank}_{\mathbb{Z}_p}\ T_{\mathrm{qtr},n} = \mathrm{rank}_{\mathbb{Z}_p}\ T_{\mathrm{tor},n} = p^n(l^m - l^{m-1})e_X;$$

$$2g_{Y_n} = 2(p^n(g_{Y_0} - 1) + 1),$$

where $e_X \geq 1$. Thus, one verifies immediately that there exists a real number C_{g_Y,d_K} that depends only on $g_Y = g_{Y_0}$ and $d_K = [K : \mathbb{Q}_p]$ such that for any positive integer $n \geq C_{g_Y,d_K}$, it holds that

$$p^n(l^m - l^{m-1})e_X > d_K(\log_p(2(p^n(g_{Y_0} - 1) + 1)) + 1).$$

In particular, we conclude from Lemma 2.15 that there exists a homomorphism $f \in \mathrm{Hom}_{\mathcal{O}_K}(\widehat{T}_{\mathrm{qtr},n}, \widehat{\mathbb{G}}_{\mathrm{m}})$ such that $\Xi(f) \neq 0$, and $\mathrm{ord}(\Xi(f)) + 1$ is *not a [non-negative integral] power of* p. Fix such a homomorphism f. Then note that it follows from our choice of f that we may write

$$\Psi(f) = 1 + \sum_{i \geq 1} a_i t^i \in \mathcal{O}_K[[t]]^{\times}$$

[cf. Proposition 2.13, (ii)], where, if we write i_0 for the smallest positive integer i such that $a_i \neq 0$, then i_0 is *not a [nonnegative integral] power of p*.

In the following, we shall apply Proposition 1.6, where we take "$g(t)$" to be $\Psi(f)$ and apply the isomorphism of topological $\mathcal{O}_K$-algebras

$$\mathcal{O}_K[[t]] \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathcal{Y}_n, y_n}$$

determined by t, to complete the proof of Theorem 2.16. Write $N \stackrel{\text{def}}{=} \mu + 1$, where μ is the "μ" that results from applying Proposition 1.6, (i);

$$\psi_\eta : W_n \longrightarrow Y_n$$

for the [connected] finite étale Galois covering over K obtained by pulling-back the morphism induced by multiplication by p^N on $\mathcal{A}_n$ via the composite morphism $Y_n \hookrightarrow \mathcal{U}_{y_n} \to \mathcal{J}_n \twoheadrightarrow \mathcal{A}_n$ [cf. Proposition 2.12].

Next, let us observe that it follows immediately from the various definitions of the morphisms involved that the composite morphism

$$\widehat{\phi}_{c_1} \circ \lambda_h \circ \tau \circ \phi_{c_2}|_{U_{c_2}} : U_{c_2} \to \text{Spec } \mathcal{O}_K[[t]]$$

[cf. the two commutative diagrams of Proposition 1.6, (i)], together with the isomorphism $\mathcal{O}_K[[t]] \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathcal{Y}_n, y_n}$, allow one to regard the *p-adic completion* $\widehat{R}_Y$ *of the localization of* $\Gamma(U_{c_2}, \mathcal{O}_{U_{c_2}})$ *at the generic point of* $(U_{c_2})_s$ as the p-adic completion of the ring of integers R_Y of a certain *discrete residue-transcendental p-valuation* on the function field of Y_n. Write $R_{W/Y}$ for the normalization of R_Y in the function field of W_n. Thus, since R_Y is [a localization of a ring of finite type over the complete discrete valuation ring $\mathcal{O}_K$, hence] *excellent*, it follows that $R_{W/Y}$ is finite over R_Y.

Next, let us observe that it follows from the relations

$$\iota \circ \lambda_g \circ \widehat{\phi}_{c_1} \circ \lambda_h = (p^\mu) \circ \xi_g,$$

$$\xi_g \circ \tau \circ (\phi_{c_2}|_{U_{c_2}}) = \iota \circ (\phi_{\pi^p}|_{U_{\pi^p}}) \circ \theta_g,$$

$$\iota \circ (\phi_{\pi^p}|_{U_{\pi^p}}) \circ \theta_g \circ f_Y = (p) \circ \iota \circ (\phi_\pi|_{U_\pi}) \circ \theta_Y$$

in the first and second commutative diagrams of Proposition 1.6, (i), and the commutative diagram of Remark 1.6.1 that we obtain relations

$$\begin{aligned} \iota \circ \lambda_g \circ \widehat{\phi}_{c_1} \circ \lambda_h \circ \tau \circ (\phi_{c_2}|_{U_{c_2}}) \circ f_Y &= (p^\mu) \circ \xi_g \circ \tau \circ (\phi_{c_2}|_{U_{c_2}}) \circ f_Y \\ &= (p^\mu) \circ \iota \circ (\phi_{\pi^p}|_{U_{\pi^p}}) \circ \theta_g \circ f_Y \\ &= (p^\mu) \circ (p) \circ \iota \circ (\phi_\pi|_{U_\pi}) \circ \theta_Y, \end{aligned}$$

where

- the composite of the *first and second equalities* implies that, after possibly replacing K by a finite extension field of K [which in fact may be taken to be *unramified* — cf. Lemma 2.10, (i)], the *tautological* $(\widehat{R}_Y)_K$*-valued point* y_R of Y_n, where we write $(\widehat{R}_Y)_K \stackrel{\text{def}}{=} \widehat{R}_Y \otimes_{\mathcal{O}_K} K$, *lifts* to an $(\widehat{R}_Y)_K$*-valued point* y_R^* *of a certain intermediate covering* $W_n \to Y_n^*$ of $W_n \to Y_n$ that corresponds to multiplication by p^μ on the codomain of the homomorphism $f \in \mathrm{Hom}_{\mathcal{O}_K}(\widehat{T}_{\mathrm{qtr},n}, \widehat{\mathbb{G}}_{\mathrm{m}})$, while

- the *third equality* implies [cf. also the *"essentially cartesian"* nature of the squares in the commutative diagram of Remark 1.6.1] that y_R^* *lifts* to an $(\widehat{R}_W)_K$*-valued point* w_R of W_n, where $(\widehat{R}_W)_K \stackrel{\text{def}}{=} \widehat{R}_W \otimes_{\mathcal{O}_K} K$, and $\widehat{R}_W$ denotes the p-adic completion of some localization R_W of $R_{W/Y}$ at a maximal ideal of $R_{W/Y}$ such that, if we write $\widehat{R}_{W^\dagger} \subseteq \widehat{R}_W$ for the subring [over which $\widehat{R}_W$ is *finite*] determined by a *certain intermediate covering* of $W_n \to Y_n$ [that corresponds to multiplication by p^N on the codomain of the homomorphism $f \in \mathrm{Hom}_{\mathcal{O}_K}(\widehat{T}_{\mathrm{qtr},n}, \widehat{\mathbb{G}}_{\mathrm{m}})$], then the spectrum of $\widehat{R}_{W^\dagger}$ admits a *tautological isomorphism* over $\widehat{R}_Y$ to the spectrum of the *p-adic completion of the localization of "$\Gamma(Y, \mathcal{O}_Y)$" at the generic point of "Y_s"* [i.e., where the quotation marks refer to the notation of Proposition 1.6, (ii)].

In particular, since i_0 is *not a [nonnegative integral] power of p*, it follows from Proposition 1.6, (ii), and Remark 1.6.2 that the residue field of $\widehat{R}_W$, hence also the residue field of R_W, is the function field of a curve over the residue field of $\mathcal{O}_K$ of *genus* ≥ 1. Thus, we conclude from Proposition 2.3, (ii), (iii), that, after possibly replacing K by a finite extension field of K, there exist a compactified semistable model $\mathcal{W}_n$ of W_n over $\mathcal{O}_K$, together with a dominant morphism

$$\psi : \mathcal{W}_n \longrightarrow \mathcal{Y}_n$$

over $\mathcal{O}_K$, such that

- ψ restricts to the finite étale Galois covering $\psi_\eta : W_n \to Y_n$;
- R_W is the local ring of $\mathcal{W}_n$ at the generic point of an irreducible component D of $(\mathcal{W}_n)_s$ whose normalization is of genus ≥ 1;
- $\psi(D) = (y_n)_s \in (\mathcal{Y}_n)_s$.

This completes the proof of Theorem 2.16. □

Theorem 2.17 (Resolution of nonsingularities for arbitrary hyperbolic curves over p-adic local fields). *Let $\Sigma \subseteq \mathfrak{Primes}$ be a subset of cardinality ≥ 2; K a p-adic local field, for some $p \in \Sigma$; X a hyperbolic curve over K; L a mixed characteristic complete discrete valuation field of residue characteristic p that contains K as a topological subfield. Then X_L satisfies Σ-RNS if and only if the residue field of L is algebraic over the finite field of cardinality p.*

Proof. First, we observe that it follows formally from Remark 2.2.3, (v), that it suffices to verify that X satisfies Σ-RNS. Next, we observe that it follows immediately from the various definitions involved that we may assume without loss of generality that X has stable reduction over K. Write $\mathcal{X}$ for the [unique, up to unique isomorphism] compactified stable model of X over $\mathcal{O}_K$. Then, in light of Propositions 2.3, (xii); 2.4, (i), (ii), by replacing X by the [unique, up to unique isomorphism] smooth compactification of a suitable connected geometrically pro-Σ finite étale covering of X, we may assume without loss of generality that:

- X is a proper hyperbolic curve over K,
- $\mathcal{X}_s$ is split,
- $\mathcal{X}_s$ is singular, and
- every irreducible component of $\mathcal{X}_s$ is a smooth curve of genus ≥ 2.

In particular, X now satisfies the assumptions imposed in the respective initial portions of Proposition 2.6 and Definition 2.7. Next, observe that since the covering $Y_n \to Y$ is *combinatorial* [cf. Definition 2.7, (ii)], it follows immediately that this covering induces a *surjection* $Y_n(K) \twoheadrightarrow Y(K)$ on K-rational points. Thus, it follows immediately from Proposition 2.4, (iii), and Theorem 2.16 that X satisfies Σ-RNS. This completes the proof of Theorem 2.17. □

3 Point-theoreticity, metric-admissibility, and arithmetic cuspidalization

Let p be a prime number. In the present section, we first recall the well-known *classification* of the *points* of the *topological Berkovich space* associated to a proper hyperbolic curve over a mixed characteristic complete discrete valuation field via the notion of *type i points*, where $i \in \{1, 2, 3, 4\}$ [cf. Definition 3.1]. Next, we introduce a certain *combinatorial classification* of the *VE-chains* considered in §2 [cf. Definition 3.2] and observe that this classification of VE-chains leads naturally to a *purely combinatorial characterization* of the well-known classification via type i points mentioned above [cf. Propositions 3.3, 3.4]. This combinatorial classification/characterization [cf. also the approach of Propositions 3.7, 3.8] was motivated by the argument applied in the proof of [CbTpIV], Theorem A.7. We then apply the theory of §2 to give a *group-theoretic characterization*, motivated by [but by no means identical to] the characterization of [Lpg2], §4, of the type i points in terms of the *geometric Σ-tempered fundamental group* of the hyperbolic curve [cf. Propositions 3.5, 3.9]. Then, by combining this group-theoretic characterization with [AbsTopII], Corollary 2.9, we prove an *absolute version of the Grothendieck Conjecture for hyperbolic curves over p-adic local fields* [cf. Theorem 3.12]. This settles one of

the *major open questions* in anabelian geometry. As a corollary of this absolute version of the Grothendieck Conjecture for hyperbolic curves over p-adic local fields, together with [HMM], Theorem A, we also obtain an *absolute version of the Grothendieck Conjecture for configuration spaces* associated to hyperbolic curves over p-adic local fields [cf. Theorem 3.13]. We then switch gears to discuss *metric-admissibility* for p-adic hyperbolic curves. This discussion of metric-admissibility leads to a proof that *all of the various p-adic versions of the Grothendieck-Teichmüller group* that appear in the literature in fact *coincide* [cf. Theorem 3.16]. Moreover, as an application of Corollary 2.5, (i), and the theory developed in the present section, together with the theory of metric-admissibility developed in [CbTpIII], §3, we obtain a construction of a certain type of *arithmetic cuspidalization* of the [$\mathfrak{Primes}$-] tempered fundamental group of a hyperbolic curve over $\overline{\mathbb{Q}}_p$ [cf. Theorem 3.20].

Definition 3.1. Let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty subset; K a mixed characteristic complete discrete valuation field of residue characteristic p; X a proper hyperbolic curve over K; $\widetilde{X} \to X$ a universal geometrically pro-Σ covering of X. Write Ω for the p-adic completion of [some fixed] $\overline{K}$; $(-)^{\mathrm{an}}$ for the topological Berkovich space associated to $(-)$. Let $x \in X^{\mathrm{an}}$ be an element; $\tilde{x} \in \widetilde{X}^{\mathrm{an}}$ [cf. Proposition 2.3, (vii), (viii)] a lifting of x. Then:

(i) We shall say that $\tilde{x}$ is *of type* 1 if $\tilde{x}$ is determined by a point-theoretic p-valuation on the function field $K(\widetilde{X})$ of $\widetilde{X}$ associated to some point $\in \widetilde{X}(\Omega)$ [cf. Definition 2.2, (ii)].

(ii) We shall say that $\tilde{x}$ is *of type* 2 if $\tilde{x}$ is determined by an inverse system of discrete residue-transcendental p-valuations associated to irreducible components of the special fibers of [compactified] semistable models with split reduction of the domain curves of connected finite étale coverings $Z \to X$ equipped with a factorization $\widetilde{X} \to Z \to X$ [cf. Proposition 2.3, (ii), (iii)].

(iii) We shall say that $\tilde{x}$ is *of type* 3 if there exist a finite extension field L of K, a [compactified] semistable model with split reduction $\mathcal{X}$ of X_L over $\mathcal{O}_L$, and a node e of $\mathcal{X}_s$ such that $\tilde{x}$ arises as the inverse image of a lifting $\in \mathbb{VE}(\widetilde{X})^{\mathrm{tor}}$ of some element $\in \mathcal{D}_e \subseteq \mathbb{VE}(\mathcal{X})^{\mathrm{tor}}$ [cf. Definition 2.2, (vi)] via the homeomorphism $\widetilde{X}^{\mathrm{an}} \xrightarrow{\sim} \mathbb{VE}(\widetilde{X})^{\mathrm{tor}}$ [cf. Proposition 2.3, (viii)].

(iv) We shall say that $\tilde{x}$ is *of type* 4 if, for each $i \in \{1, 2, 3\}$, $\tilde{x}$ is not of type i.

(v) For each $i \in \{1, 2, 3, 4\}$, we shall say that x is *of type* i if $\tilde{x}$ is of type i. [One verifies immediately that, for each $i \in \{1, 2, 3, 4\}$, the condition that x is of type i is *independent* of the choices of Σ and $\tilde{x}$.] For each $i \in \{1, 2, 3, 4\}$, we shall write $X^{\mathrm{an}}[i] \subseteq X^{\mathrm{an}}$ (respectively, $\widetilde{X}^{\mathrm{an}}[i] \subseteq \widetilde{X}^{\mathrm{an}}$) for the subset of points of type i of X^{an} (respectively, $\widetilde{X}^{\mathrm{an}}$).

Remark 3.1.1. In the notation of Definition 3.1, we observe that

$$\widetilde{X}^{\mathrm{an}} \;=\; \widetilde{X}^{\mathrm{an}}[1] \;\cup\; \widetilde{X}^{\mathrm{an}}[2] \;\cup\; \widetilde{X}^{\mathrm{an}}[3] \;\cup\; \widetilde{X}^{\mathrm{an}}[4];$$

$$X^{\mathrm{an}} \;=\; X^{\mathrm{an}}[1] \;\cup\; X^{\mathrm{an}}[2] \;\cup\; X^{\mathrm{an}}[3] \;\cup\; X^{\mathrm{an}}[4],$$

and, moreover, for each pair of *distinct* $i, j \in \{1, 2, 3, 4\}$,

$$\widetilde{X}^{\mathrm{an}}[i] \;\cap\; \widetilde{X}^{\mathrm{an}}[j] = \emptyset; \quad X^{\mathrm{an}}[i] \;\cap\; X^{\mathrm{an}}[j] = \emptyset.$$

Indeed, it suffices to verify that $\widetilde{X}^{\mathrm{an}}[i] \;\cap\; \widetilde{X}^{\mathrm{an}}[j] = \emptyset$. First, by considering the *residue fields* of the valuation rings under consideration, we conclude that $\widetilde{X}^{\mathrm{an}}[1] \;\cap\; \widetilde{X}^{\mathrm{an}}[2] = \emptyset$. Next, we observe that it follows immediately from the discussion of the construction of "$\mathbb{VE}(\mathcal{Z})^{\mathrm{tor}}$" in Definition 2.2, (vi), that, for each $i \in \{1, 2\}$, $\widetilde{X}^{\mathrm{an}}[i] \;\cap\; \widetilde{X}^{\mathrm{an}}[3] = \emptyset$. Finally, we observe that it is a *tautology* that, for each $i \in \{1, 2, 3\}$, $\widetilde{X}^{\mathrm{an}}[i] \;\cap\; \widetilde{X}^{\mathrm{an}}[4] = \emptyset$. This completes the proof of relations in the above display.

Definition 3.2. We maintain the notation of Definition 3.1. Let $c = (c_{\mathcal{Z}})_{\mathcal{Z} \in S} \in \mathbb{VE}(\widetilde{X})$, where S denotes the directed set of [compactified] semistable models that appear in the definition of $\mathbb{VE}(\widetilde{X})$ [cf. Definition 2.2, (iii)]. Write $V_c \subseteq S$ (respectively, $E_c \subseteq S$) for the subset of [compactified] semistable models $\mathcal{Z}$ such that $c_{\mathcal{Z}}$ is a vertex (respectively, an edge). Then:

(i) We shall say that c is *asymptotically verticial* (respectively, *asymptotically edge-like*) if the subset $V_c \subseteq S$ (respectively, $E_c \subseteq S$) forms a cofinal subset of S. [In particular, if c is *asymptotically verticial* (respectively, *asymptotically edge-like*), then V_c (respectively, E_c) may be regarded as a directed set in a natural way.]

(ii) Suppose that c is asymptotically verticial. Then we shall say that c is *strongly verticial* if there exists a cofinal subset $S_c \subseteq V_c$ satisfying the following condition:

> Let $\mathcal{Z}_1, \mathcal{Z}_2 \in S_c$ be distinct elements such that $\mathcal{Z}_2$ dominates $\mathcal{Z}_1$. Then the *generic point* of the irreducible component of $(\mathcal{Z}_2)_s$ that corresponds to $c_{\mathcal{Z}_2}$ maps to the *generic point* of the irreducible component of $(\mathcal{Z}_1)_s$ that corresponds to $c_{\mathcal{Z}_1}$ via the dominant morphism $\mathcal{Z}_2 \to \mathcal{Z}_1$.

(iii) Suppose that c is asymptotically verticial. Then we shall say that c is *weakly verticial* if there exists a cofinal subset $S_c \subseteq V_c$ satisfying the following condition:

> Let $\mathcal{Z}_1, \mathcal{Z}_2 \in S_c$ be distinct elements such that $\mathcal{Z}_2$ dominates $\mathcal{Z}_1$. Then the *generic point* of the irreducible component of $(\mathcal{Z}_2)_s$ that corresponds to $c_{\mathcal{Z}_2}$ maps to a *closed point in the interior* of the irreducible component of $(\mathcal{Z}_1)_s$ that corresponds to $c_{\mathcal{Z}_1}$ via the dominant morphism $\mathcal{Z}_2 \to \mathcal{Z}_1$.

(iv) Suppose that c is asymptotically edge-like. Then we shall say that c is *weakly edge-like* if there exists a cofinal subset $S_c \subseteq E_c$ satisfying the following condition:

> Let $\mathcal{Z}_1, \mathcal{Z}_2 \in S_c$ be distinct elements such that $\mathcal{Z}_2$ dominates $\mathcal{Z}_1$. Then there exists a toral compactified semistable model $\mathcal{Z}_1^*$ relative to $\mathcal{Z}_1$ such that the dominant morphism $\mathcal{Z}_2 \to \mathcal{Z}_1$ admits a factorization $\mathcal{Z}_2 \to \mathcal{Z}_1^* \to \mathcal{Z}_1$, and the *node* of $(\mathcal{Z}_2)_s$ that corresponds to $c_{\mathcal{Z}_2}$ maps to a *closed point in the interior* of an irreducible component of $(\mathcal{Z}_1^*)_s$ [that necessarily lies over the node of $(\mathcal{Z}_1)_s$ that corresponds to $c_{\mathcal{Z}_1}$] via the dominant morphism $\mathcal{Z}_2 \to \mathcal{Z}_1^*$.

(v) Suppose that c is asymptotically edge-like. Then we shall say that c is *strongly edge-like* if there exists a cofinal subset $S_c \subseteq E_c$ satisfying the following condition:

> Let $\mathcal{Z}_1, \mathcal{Z}_2 \in S_c$ be distinct elements such that $\mathcal{Z}_2$ dominates $\mathcal{Z}_1$. Then, for each toral compactified semistable model $\mathcal{Z}_1^*$ relative to $\mathcal{Z}_1$ that admits a factorization $\mathcal{Z}_2 \to \mathcal{Z}_1^* \to \mathcal{Z}_1$, the *node* of $(\mathcal{Z}_2)_s$ that corresponds to $c_{\mathcal{Z}_2}$ maps to a *node* of $(\mathcal{Z}_1^*)_s$ [that necessarily lies over the node of $(\mathcal{Z}_1)_s$ that corresponds to $c_{\mathcal{Z}_1}$] via the dominant morphism $\mathcal{Z}_2 \to \mathcal{Z}_1^*$.

(vi) We shall write

$$\mathbb{VE}(\widetilde{X})^{\mathrm{vtc}} \subseteq \mathbb{VE}(\widetilde{X}); \quad \mathbb{VE}(\widetilde{X})^{\mathrm{edg}} \subseteq \mathbb{VE}(\widetilde{X});$$

$$\mathbb{VE}(\widetilde{X})^{\text{str-vtc}} \subseteq \mathbb{VE}(\widetilde{X}); \quad \mathbb{VE}(\widetilde{X})^{\text{wk-vtc}} \subseteq \mathbb{VE}(\widetilde{X});$$

$$\mathbb{VE}(\widetilde{X})^{\text{str-edg}} \subseteq \mathbb{VE}(\widetilde{X}); \quad \mathbb{VE}(\widetilde{X})^{\text{wk-edg}} \subseteq \mathbb{VE}(\widetilde{X}),$$

respectively, for the subsets of asymptotically verticial VE-chains, asymptotically edge-like VE-chains, strongly verticial VE-chains, weakly verticial VE-chains, strongly edge-like VE-chains, and weakly edge-like VE-chains. Also, for each $\square \in \{\mathrm{vtc}, \mathrm{edg}, \text{str-vtc}, \text{wk-vtc}, \text{str-edg}, \text{wk-edg}\}$, we shall write

$$\mathbb{VE}(\widetilde{X})^{\mathrm{prim},\square} \stackrel{\mathrm{def}}{=} \mathbb{VE}(\widetilde{X})^{\mathrm{prim}} \cap \mathbb{VE}(\widetilde{X})^{\square} \quad (\subseteq \mathbb{VE}(\widetilde{X})).$$

(vii) Let $\mathcal{Z} \in S$. Then in the notation of Definition 2.2, (vi), we shall write

$$\mathbb{VE}(\mathcal{Z})^{\mathrm{tor,rat}} \stackrel{\mathrm{def}}{=} \bigcup_{w \in \mathcal{V}(\mathcal{Z})} \mathcal{V}_w \quad (\subseteq \mathbb{VE}(\mathcal{Z})^{\mathrm{tor}});$$

$$\mathbb{VE}(\mathcal{Z})^{\mathrm{tor,irr}} \stackrel{\mathrm{def}}{=} \bigcup_{e \in \mathcal{E}(\mathcal{Z})} \mathcal{D}_e \quad (\subseteq \mathbb{VE}(\mathcal{Z})^{\mathrm{tor}}).$$

Note that one verifies immediately that

$$\mathbb{VE}(\mathcal{Z})^{\mathrm{tor}} \;=\; \mathbb{VE}(\mathcal{Z})^{\mathrm{tor,rat}} \coprod \mathbb{VE}(\mathcal{Z})^{\mathrm{tor,irr}},$$

and that, for any $\mathcal{Z}_1, \mathcal{Z}_2 \in S$ such that $\mathcal{Z}_2$ dominates $\mathcal{Z}_1$, the natural map $\mathbb{VE}(\mathcal{Z}_2)^{\mathrm{tor}} \to \mathbb{VE}(\mathcal{Z}_1)^{\mathrm{tor}}$ induces a map

$$\mathbb{VE}(\mathcal{Z}_2)^{\mathrm{tor,rat}} \to \mathbb{VE}(\mathcal{Z}_1)^{\mathrm{tor,rat}}$$

[cf. the discussion of Definition 2.2, (vi)].

Remark 3.2.1. In the notation of Definition 3.2, we observe that it follows immediately from the various definitions involved [cf. also Remarks 2.1.4, 2.1.5; Definition 2.2, (vi)] that:

$$\mathbb{VE}(\widetilde{X})^{\text{str-vtc}} \,\cap\, \mathbb{VE}(\widetilde{X})^{\text{wk-vtc}} = \emptyset; \quad \mathbb{VE}(\widetilde{X})^{\text{str-edg}} \,\cap\, \mathbb{VE}(\widetilde{X})^{\text{wk-edg}} = \emptyset;$$

$$\mathbb{VE}(\widetilde{X})^{\mathrm{vtc}} \;=\; \mathbb{VE}(\widetilde{X})^{\text{str-vtc}} \,\cup\, \mathbb{VE}(\widetilde{X})^{\text{wk-vtc}};$$

$$\mathbb{VE}(\widetilde{X})^{\mathrm{edg}} \;=\; \mathbb{VE}(\widetilde{X})^{\text{str-edg}} \,\cup\, \mathbb{VE}(\widetilde{X})^{\text{wk-edg}};$$

$$\mathbb{VE}(\widetilde{X}) \;=\; \mathbb{VE}(\widetilde{X})^{\mathrm{vtc}} \,\cup\, \mathbb{VE}(\widetilde{X})^{\mathrm{edg}}.$$

Proposition 3.3 (Elementary properties of the combinatorial classification of VE-chains). *In the notation of Definition 3.2, by forming the inductive limit of the natural log structures on the [compactified] semistable models $\mathcal{Z} \in S$, we obtain an ind-log structure on the pro-scheme $\varprojlim_{\mathcal{Z}\in S} \mathcal{Z}$. Then the following hold:*

(i) Let $c \in \mathbb{VE}(\widetilde{X})$. Write $\tilde{z}_c$ for the center on $\varprojlim_{\mathcal{Z}\in S} \mathcal{Z}$ of the valuation ring R_c associated to c [cf. Proposition 2.3, (vi)]; M_c^{pf} for the perfection of the inductive limit monoid obtained by forming the stalk at $\tilde{z}_c$ of the characteristic of the ind-log structure on the pro-scheme $\varprojlim_{\mathcal{Z}\in S} \mathcal{Z}$. Then, if $c \in \mathbb{VE}(\widetilde{X})^{\mathrm{vtc}}$ (respectively, $c \in \mathbb{VE}(\widetilde{X})^{\text{str-edg}}$), then M_c^{pf} is isomorphic to $\mathbb{Q}_{\geq 0}$ (respectively, $\mathbb{Q}_{\geq 0} \times \mathbb{Q}_{\geq 0}$).

(ii) The following relations hold:

$$\mathbb{VE}(\widetilde{X})^{\text{str-vtc}} \,\cap\, \mathbb{VE}(\widetilde{X})^{\text{str-edg}} = \emptyset; \quad \mathbb{VE}(\widetilde{X})^{\text{wk-vtc}} \,\cap\, \mathbb{VE}(\widetilde{X})^{\text{str-edg}} = \emptyset;$$

$$\mathbb{VE}(\widetilde{X})^{\text{str-vtc}} \,\cap\, \mathbb{VE}(\widetilde{X})^{\text{wk-vtc}} = \emptyset; \quad \mathbb{VE}(\widetilde{X})^{\text{wk-edg}} \subseteq \mathbb{VE}(\widetilde{X})^{\text{wk-vtc}};$$

$$\mathbb{VE}(\widetilde{X}) \;=\; \mathbb{VE}(\widetilde{X})^{\text{str-vtc}} \,\cup\, \mathbb{VE}(\widetilde{X})^{\text{wk-vtc}} \,\cup\, \mathbb{VE}(\widetilde{X})^{\text{str-edg}}.$$

Proof. Assertion (i) follows immediately from the well-known log structure of M_c^{pf} [cf. also the discussion of the subsection in Notations and Conventions entitled "Log schemes"], together with the various definitions involved. Assertion (ii) follows immediately from assertion (i), together with the various definitions involved [cf. also Remark 3.2.1]. This completes the proof of Proposition 3.3. □

Proposition 3.4 (**Characterization of points of type 2 and 3 via the combinatorial classification of VE-chains**). *We maintain the notation of Definition 3.2. Then the following hold:*

(i) Write $\mathbb{VE}(\widetilde{X})^{\mathrm{nonprim}} \stackrel{\mathrm{def}}{=} \mathbb{VE}(\widetilde{X}) \setminus \mathbb{VE}(\widetilde{X})^{\mathrm{prim}}$. *Then*

$$\mathbb{VE}(\widetilde{X})^{\text{str-edg}} \;=\; \mathbb{VE}(\widetilde{X})^{\mathrm{nonprim}} \;\cup\; \mathbb{VE}(\widetilde{X})^{\mathrm{prim,str\text{-}edg}}.$$

Moreover, the unique nontrivial generization of a nonprimitive VE-chain [cf. Proposition 2.3, (x)] is strongly verticial.

(ii) For $\mathcal{Z} \in S$, *write*

$$\tau^{\text{str-edg}}_{\widetilde{X},\mathcal{Z}} : \mathbb{VE}(\widetilde{X})^{\text{str-edg}} \;\subseteq\; \mathbb{VE}(\widetilde{X}) \xrightarrow{\tau_{\widetilde{X}}} \mathbb{VE}(\widetilde{X})^{\mathrm{tor}} \longrightarrow \mathbb{VE}(\mathcal{Z})^{\mathrm{tor}}$$

for the natural composite map [cf. Proposition 2.3, (viii)]. Then for any $\mathcal{Z} \in S$, $\tau^{\text{str-edg}}_{\widetilde{X},\mathcal{Z}}$ *induces a map*

$$\mathbb{VE}(\widetilde{X})^{\mathrm{nonprim}} \to \mathbb{VE}(\mathcal{Z})^{\mathrm{tor,rat}}$$

[cf. (i)]. Moreover, for each $c \in \mathbb{VE}(\widetilde{X})^{\mathrm{prim,str\text{-}edg}}$, *there exists an element* $\mathcal{Z}_c \in S$ *such that for every* $\mathcal{Z} \in S$ *that dominates* $\mathcal{Z}_c$,

$$\tau^{\text{str-edg}}_{\widetilde{X},\mathcal{Z}}(c) \in \mathbb{VE}(\mathcal{Z})^{\mathrm{tor,irr}}$$

[cf. (i)]. Finally, $\mathcal{Z}_c \in S$ *may be taken to be a [compactified] semistable model with split reduction of* X_L *over* $\mathcal{O}_L$ *for some finite extension field* L *of* K.

(iii) The bijection $\mathbb{VE}(\widetilde{X})^{\mathrm{prim}} \xrightarrow{\sim} \widetilde{X}^{\mathrm{an}}$ *[cf. Proposition 2.3, (viii)] determines bijections*

$$\mathbb{VE}(\widetilde{X})^{\mathrm{prim,str\text{-}vtc}} \xrightarrow{\sim} \widetilde{X}^{\mathrm{an}}[2]; \quad \mathbb{VE}(\widetilde{X})^{\mathrm{prim,str\text{-}edg}} \xrightarrow{\sim} \widetilde{X}^{\mathrm{an}}[3].$$

(iv) Let Y *be a proper hyperbolic curve over* K; $f : Y \to X$ *a dominant morphism over* K; $y \in Y^{\mathrm{an}}$. *Write* $x \stackrel{\mathrm{def}}{=} f(y) \in X^{\mathrm{an}}$. *Then, for each* $i \in \{1, 2, 3, 4\}$, y *is of type* i *if and only if* x *is of type* i.

Proof. Assertion (i) follows immediately from the various definitions involved. Next, we verify assertion (ii). Let $c \in \mathbb{VE}(\widetilde{X})^{\text{str-edg}}$. First, suppose that $c \in \mathbb{VE}(\widetilde{X})^{\text{nonprim}}$. Write $c' \in \mathbb{VE}(\widetilde{X})^{\text{prim,str-vtc}}$ for the *unique nontrivial generization* of c [cf. assertion (i)]. Then it follows immediately from the definition of $\tau_{\widetilde{X}}$ in the proof of Proposition 2.3, (viii), that $\tau_{\widetilde{X}}(c) = \tau_{\widetilde{X}}(c')$. In particular, it follows immediately from the fact that $c' \in \mathbb{VE}(\widetilde{X})^{\text{prim,str-vtc}}$ that $\tau_{\widetilde{X}}(c')$ maps to an element of $\mathbb{VE}(\mathcal{Z})^{\text{tor,rat}}$ for any $\mathcal{Z} \in S$, hence that $\tau^{\text{str-edg}}_{\widetilde{X},\mathcal{Z}}(c) \in \mathbb{VE}(\mathcal{Z})^{\text{tor,rat}}$, as desired. Next, suppose that $c \in \mathbb{VE}(\widetilde{X})^{\text{prim,str-edg}}$. Let $\mathcal{Z}_c \in S_c$ for some "S_c" as in Definition 3.2, (v). Let $\mathcal{Z} \in S$ be an element that dominates $\mathcal{Z}_c$. Then one verifies immediately [cf. also the final portion of Definition 3.2, (vii)] that any relation $\tau^{\text{str-edg}}_{\widetilde{X}}(c) \in \mathbb{VE}(\mathcal{Z})^{\text{tor,rat}}$ implies a *contradiction* to the condition of Definition 3.2, (v). Thus, we conclude that $\tau^{\text{str-edg}}_{\widetilde{X},\mathcal{Z}}(c) \in \mathbb{VE}(\mathcal{Z})^{\text{tor,irr}}$, as desired. The fact that $\mathcal{Z}_c \in S$ may be taken to be a [compactified] semistable model with split reduction of X_L over $\mathcal{O}_L$ for some finite extension field L of K follows immediately from Proposition 2.3, (iii), (iv). This completes the proof of assertion (ii). Assertion (iii) follows immediately from assertion (ii), together with the various definitions involved [cf. also the final portion of Definition 3.2, (vii); Proposition 3.3, (ii)]. Finally, we consider assertion (iv). First, we observe that the asserted equivalence follows immediately from the various definitions involved [cf. also Proposition 2.3, (ii), (iii), in the case where $i = 2$] when $i \in \{1, 2\}$. Thus, it suffices to verify the asserted equivalence when $i = 3$. When $i = 3$, *sufficiency* follows immediately, in light of assertion (iii) and Proposition 3.3, (ii), from Definition 3.1, (iii) [cf. also the discussion of Definition 2.2, (vi)]. On the other hand, it follows immediately, by replacing Y by the normalization of Y in the *Galois closure* of the finite extension of function fields determined by f and applying the sufficiency that has already been verified, that to verify *necessity* when $i = 3$, we may assume without loss of generality that the finite extension of function fields determined by f is *Galois*. But then the desired necessity follows immediately from the final portion of assertion (ii), together with Proposition 2.3, (iii), (iv) [cf. also the discussion of Definition 2.2, (vi)]. This completes the proof of Proposition 3.4. □

Proposition 3.5 (Types of points and geometrically pro-l decomposition groups). *In the notation of Definition 3.2, let $l \in \Sigma \setminus \{p\}$. Suppose that X satisfies Σ-RNS. Then the following hold:*

(i) Let

$$c \in \mathbb{VE}(\widetilde{X})^{\text{str-vtc}} \text{ (respectively, } c \in \mathbb{VE}(\widetilde{X})^{\text{wk-vtc}}\text{; } c \in \mathbb{VE}(\widetilde{X})^{\text{str-edg}}\text{).}$$

For each connected geometrically pro-Σ finite étale covering $(\widetilde{X} \to) Z \to X$, write $D_{Z,c} \subseteq \Delta_Z^l \stackrel{\text{def}}{=} \Delta_Z^{\{l\}}$ for the decomposition subgroup associated to c of the geometric pro-l fundamental group of Z [cf. the subsection in

Notations and Conventions entitled "Fundamental groups"]. Then there exists a connected geometrically pro-Σ finite étale covering $(\widetilde{X} \to) Y \to X$ of X such that, for each connected geometrically pro-Σ finite étale covering $(\widetilde{X} \to) Z \to Y$ of Y, $D_{Z,c}$ is isomorphic to a [nonabelian] pro-l surface group [cf. [MT], Definition 1.2] (respectively, the trivial group; $\mathbb{Z}_l$).

(ii) Let $i \in \{2, 3\}$. Then the set $\widetilde{X}^{\mathrm{an}}$ and the subset $\widetilde{X}^{\mathrm{an}}[i] \subseteq \widetilde{X}^{\mathrm{an}}$ may be reconstructed, functorially with respect to isomorphisms of topological groups, from the underlying topological group of the geometric Σ-tempered fundamental group of X [cf. the subsection in Notations and Conventions entitled "Fundamental groups"].

*(iii) Let Y, Z be [not necessarily proper!] hyperbolic curves over K that satisfy Σ-RNS; $\widetilde{Y} \to Y$, $\widetilde{Z} \to Z$ universal geometrically pro-Σ coverings of Y, Z, respectively; $f : Y \to Z$ a dominant morphism over K; $H \subseteq G_K$ a closed subgroup such that the restriction to H of the **l-adic cyclotomic character** of K has open image, and, moreover, the intersection $H \cap I_K$ of H with the inertia subgroup I_K of G_K admits a surjection to [the profinite group] $\mathbb{Z}_l$; $s_Y : H \to \Pi_Y^{(\Sigma)} \stackrel{\text{def}}{=} \mathrm{Gal}(\widetilde{Y}/Y)$, $s_Z : H \to \Pi_Z^{(\Sigma)} \stackrel{\text{def}}{=} \mathrm{Gal}(\widetilde{X}/X)$ sections of the restrictions to H of the respective natural surjections $\Pi_Y^{(\Sigma)} \twoheadrightarrow G_K$, $\Pi_Z^{(\Sigma)} \twoheadrightarrow G_K$ such that s_Y is mapped, up to $\Pi_Z^{(\Sigma)}$-conjugation, by f to s_Z via the map induced by f on geometrically pro-Σ fundamental groups. Write $\Omega^H \subseteq \Omega$ for the subfield of Ω fixed by H. Then s_Y arises from a(n) [necessarily unique] Ω^H-rational point $\in Y(\Omega^H)$ if and only if s_Z arises from a(n) [necessarily unique] Ω^H-rational point $\in Z(\Omega^H)$.*

Proof. Since X satisfies Σ-RNS [cf. Definition 2.2, (vii)], assertion (i) follows immediately from the well-known structure of the maximal pro-l quotient of the admissible fundamental group of a stable curve over a separably closed field of characteristic p [cf. e.g., [SemiAn], Example 2.10], together with the various definitions involved. Assertion (ii) follows immediately from assertion (i), together with Corollary 2.5, (i); Proposition 3.4, (iii) [cf. also Proposition 3.3, (ii)]. Finally, we consider assertion (iii). First, we observe that it follows immediately from the *profinite* nature of the topological group $\Pi_Y^{(\Sigma)}$ that, by replacing Y and Z by the smooth compactifications of the various finite étale coverings of Y and Z corresponding, respectively, to suitable open neighborhoods of the image of s_Y in $\Pi_Y^{(\Sigma)} \times_{G_K} H$ and the image of s_Z in $\Pi_Z^{(\Sigma)} \times_{G_K} H$, we may assume without loss of generality that Y and Z are *proper*. Next, we observe that the various *uniqueness* assertions in the statement of Proposition 3.5, (iii), follow immediately from the final portion of Proposition 2.4, (vii), and that *necessity* follows immediately from the various definitions involved. Thus, it suffices to verify *sufficiency*. On the other hand, sufficiency follows, in light of the equivalences of Proposition 3.4, (iv), formally from the final portion of Proposition 2.4, (vii). This completes the proof of Proposition 3.5. □

Definition 3.6. Let K be a mixed characteristic complete discrete valuation field of residue characteristic p; X a proper hyperbolic curve over K. Write $K(X_{\overline{K}})$ for the function field of $X_{\overline{K}}$. Let v be a p-valuation on $K(X_{\overline{K}})$. Write $(\mathcal{O}_{\overline{K}} \subseteq)\ \mathcal{O}_v \subseteq K(X_{\overline{K}})$ for the valuation ring associated to v. [Note that it follows immediately from the well-known theory of *one-dimensional function fields* that $(\mathcal{O}_v)_K \stackrel{\text{def}}{=} \mathcal{O}_v \cdot K \subseteq K(X_{\overline{K}})$ is equal either to $K(X_{\overline{K}})$ or to the discrete valuation ring associated to a closed point of $X_{\overline{K}}$.] Then:

(i) Let M be an $\mathcal{O}_v$-module. Then we shall say that M is *bounded* if the image of M via the natural morphism $M \to M_K \stackrel{\text{def}}{=} M \otimes_{\mathcal{O}_K} K$ is contained in a finitely generated $\mathcal{O}_v$-submodule of M_K. We shall say that M is *unbounded* if M is not bounded.

(ii) We shall say that the p-valuation v is *differentially bounded* (respectively, *differentially unbounded*) if the $\mathcal{O}_v$-module of relative differentials $\Omega_{\mathcal{O}_v/\mathcal{O}_{\overline{K}}}$ is bounded (respectively, unbounded).

Proposition 3.7 (**Approximation of closed points of the generic fiber via generic points of special fibers**). *Let K be a mixed characteristic complete discrete valuation field of residue characteristic p;*

$$K = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_i \subseteq \cdots$$

an ascending chain of finite extension fields of K contained in $\overline{K}$ and indexed by $\mathbb{N}$. Write Ω for the p-adic completion of $\overline{K}$. Let X be a hyperbolic curve over K. For each $i \in \mathbb{N}$, let

$$\mathcal{X}_i$$

be a compactified semistable model with split reduction of X_{K_i} over $\mathcal{O}_{K_i}$;

$$\phi_{i+1} : \mathcal{X}_{i+1} \longrightarrow \mathcal{X}_i \times_{\mathcal{O}_{K_i}} \mathcal{O}_{K_{i+1}}$$

a dominant morphism over $\mathcal{O}_{K_{i+1}}$ that induces the identity automorphism on the generic fiber; v_i an irreducible component of $(\mathcal{X}_i)_s$. Suppose that, for each $i \in \mathbb{N}$, the projection to $\mathcal{X}_i$ of $\phi_{i+1}(v_{i+1})$ is a closed point $x_i \in (\mathcal{X}_i)_s \subseteq \mathcal{X}_i$ of $(\mathcal{X}_i)_s$ that lies in the smooth locus of v_i. Then the following hold:

(i) For each $i \in \mathbb{N}$, let $\psi_i : \operatorname{Spec} \mathcal{O}_{K_i} \hookrightarrow \mathcal{X}_i$ be a section whose image contains x_i. Then there exists a collection $\{t_i, \gamma_{i+1}, \pi_{i+1}\}_{i \in \mathbb{N}}$ of elements $t_i \in A_i \stackrel{\text{def}}{=} \mathcal{O}_{\mathcal{X}_i, x_i}$ and $\gamma_{i+1}, \pi_{i+1} \in \mathfrak{m}_{K_{i+1}}$ such that, for each $i \in \mathbb{N}$, $t_i \in A_i$ is a generator of the ideal that defines the scheme-theoretic image of ψ_i, and

$$t_i \ = \ \gamma_{i+1} + \pi_{i+1} t_{i+1},$$

where we regard A_i as a subring of A_{i+1} via the injection $A_i \hookrightarrow A_{i+1}$ induced by the composite $\mathcal{X}_{i+1} \to \mathcal{X}_i$ [which maps $x_{i+1} \mapsto x_i$] of ϕ_{i+1} with the projection to $\mathcal{X}_i$.

(ii) We maintain the notation of (i). For each positive integer i, write $l_i \stackrel{\text{def}}{=} v_p(\pi_i)$. Suppose that the equality

$$\sum_{i \geq 1} l_i \;=\; +\infty$$

holds. Then there exists a closed point x_Ω of X_Ω such that, for each $i \in \mathbb{N}$, the center on $\mathcal{X}_i$ of the closed point x_Ω of X_Ω, hence also of the point-theoretic p-valuation on the function field of X_{K_i} determined by the closed point x_Ω of X_Ω, coincides with x_i.

Proof. First, we verify assertion (i). We construct elements $t_i \in A_i$ and $\gamma_{i+1}, \pi_{i+1} \in \mathfrak{m}_{K_{i+1}}$, for $i \in \mathbb{N}$, by induction on $i \in \mathbb{N}$. Let $t_0 \in A_0$ be a generator of the ideal that defines the scheme-theoretic image of ψ_0; $i \in \mathbb{N}$. Suppose that the elements t_{j+1}, γ_{j+1}, and π_{j+1} have been constructed for $j \in \mathbb{N}$ such that $j < i$. Write $\gamma_{i+1} \in \mathfrak{m}_{K_{i+1}}$ for the image of t_i via the composite homomorphism $A_i \subseteq A_{i+1} \to \mathcal{O}_{K_{i+1}}$ induced by ψ_{i+1}. Next, observe that the image in $A_{i+1} \otimes_{\mathcal{O}_{K_{i+1}}} K_{i+1}$ of $t_i - \gamma_{i+1}$ is a generator of the maximal ideal associated to the closed point of $X_{K_{i+1}}$ determined by ψ_{i+1}. Thus, since x_{i+1} lies in the smooth locus of $(\mathcal{X}_{i+1})_s$, and A_{i+1} is a regular local ring, hence a *unique factorization domain*, we conclude that there exists an element $\pi_{i+1} \in \mathfrak{m}_{K_{i+1}}$ such that $t_i - \gamma_{i+1} = \pi_{i+1} t_{i+1}$ for some generator $t_{i+1} \in A_{i+1}$ of the ideal that defines the scheme-theoretic image of ψ_{i+1}. This completes the proof of assertion (i).

Next, we verify assertion (ii). Suppose that $\sum_{i \geq 1} l_i = +\infty$. For each $i \in \mathbb{N}$, fix elements $t_i \in A_i$ and $\gamma_{i+1}, \pi_{i+1} \in \mathfrak{m}_{K_{i+1}}$ as in the statement of assertion (i). For each positive integer j, write

$$s_j \;\stackrel{\text{def}}{=}\; \gamma_j \cdot \prod_{1 \leq i \leq j-1} \pi_i.$$

Then since $\mathcal{O}_\Omega$ is p-adically complete, it follows immediately from our assumption that $\sum_{i \geq 1} l_i = +\infty$ that $\sum_{j \geq 1} s_j$ converges to an element $\gamma \in \mathfrak{m}_\Omega$. Write x_Ω for the closed point of X_Ω determined by the homomorphism $\psi_\Omega : A_0 \to \mathcal{O}_\Omega$ over $\mathcal{O}_{K_0}$ that maps $t_0 \mapsto \gamma$. Now observe that it follows immediately from the definition of γ that, for each $i \in \mathbb{N}$, ψ_Ω extends uniquely to a homomorphism $A_i \to \mathcal{O}_\Omega$ over $\mathcal{O}_{K_i}$. On the other hand, the existence of such unique extensions implies that the center on $\mathcal{X}_i$ of the closed point x_Ω of X_Ω, hence also of the point-theoretic p-valuation on the function field of X_{K_i} determined by the closed point x_Ω of X_Ω, coincides with x_i. This completes the proof of assertion (ii), hence of Proposition 3.7. □

Proposition 3.8 (**Characterization of points of type 1 via differentially unboundedness**). *In the notation of Definition 3.6 [cf. also the notation of Proposition 2.3, (viii)], let $\widetilde{v}$ be a p-valuation of $K(\widetilde{X})$ ($\supseteq K(X_{\overline{K}})$) that restricts to v on $K(X_{\overline{K}})$ [cf. Remark 2.2.4]. Suppose that $\widetilde{v}$ is primitive. Then the point $x_{\widetilde{v}} \in \widetilde{X}^{\mathrm{an}}$ associated to $\widetilde{v}$ [cf. Proposition 2.3, (viii)] is of type 1 if and only if v is differentially unbounded.*

Proof. Write $x_v \in X^{\mathrm{an}}$ for the point determined by $x_{\widetilde{v}} \in \widetilde{X}^{\mathrm{an}}$. Note that, in light of Remark 3.1.1, it suffices to verify that if x_v is of type 1 (respectively, of type $i \in \{2, 3, 4\}$), then v is *differentially unbounded* (respectively, *differentially bounded*).

First, we verify that if x_v is of type 1, then v is *differentially unbounded*. Suppose that x_v is of type 1. Write Ω for the p-adic completion of $\overline{K}$; $e_{\widetilde{v}} : \mathcal{O}_v \to \mathcal{O}_\Omega$ for the natural evaluation homomorphism over $\mathcal{O}_{\overline{K}}$ associated to $x_{\widetilde{v}}$. Next, observe that, since $\mathcal{O}_v \otimes_{\mathcal{O}_K} K$ is a valuation ring [contained in the function field $K(X_{\overline{K}})$ of the hyperbolic curve $X_{\overline{K}}$ over $\overline{K}$] that contains $\overline{K}$, and whose field of fractions coincides with $K(X_{\overline{K}})$ [cf. Definition 2.2, (ii)], it follows immediately from the well-known theory of one-dimensional function fields over an algebraically closed field that $\Omega_{\mathcal{O}_v/\mathcal{O}_{\overline{K}}} \otimes_{\mathcal{O}_K} K$ is a rank one free module over $\mathcal{O}_v \otimes_{\mathcal{O}_K} K$. In particular, there exists an element $t \in \mathcal{O}_v$ such that $e_{\widetilde{v}}(t) \in \mathfrak{m}_\Omega$, and dt is a free generator of $\Omega_{\mathcal{O}_v/\mathcal{O}_{\overline{K}}} \otimes_{\mathcal{O}_K} K$ over $\mathcal{O}_v \otimes_{\mathcal{O}_K} K$. Next, we observe that, for any positive integer N, there exists an element $a_N \in \mathfrak{m}_{\overline{K}}$ such that

$$e_{\widetilde{v}}(t - a_N) = e_{\widetilde{v}}(t) - a_N \in p^N \mathcal{O}_\Omega.$$

Note that the above equation implies that $\frac{t-a_N}{p^N} \in \mathcal{O}_v$. On the other hand, it follows immediately from the definition of the module of relative differentials that

$$dt = d(t - a_N) = p^N \cdot d(\tfrac{t-a_N}{p^N}) \in \Omega_{\mathcal{O}_v/\mathcal{O}_{\overline{K}}}.$$

In particular, we conclude that dt is a nonzero p-divisible element of $\Omega_{\mathcal{O}_v/\mathcal{O}_{\overline{K}}}$. Thus, since dt is a free generator of $\Omega_{\mathcal{O}_v/\mathcal{O}_{\overline{K}}} \otimes_{\mathcal{O}_K} K$ over $\mathcal{O}_v \otimes_{\mathcal{O}_K} K$, the *differential boundedness* of v would imply that *arbitrary negative integral powers* of p are contained in $\mathcal{O}_v$, i.e., in contradiction to our assumption that v is a *p-valuation*. Hence we conclude that v is *differentially unbounded*, as desired.

Next, we verify that, if x_v is of type 2, then v is *differentially bounded*. Suppose that x_v is of type 2. Then it follows immediately from Proposition 3.4, (iii), that there exist a finite extension field L of K, a compactified semistable model $\mathcal{X}$ of X_L over $\mathcal{O}_L$, and a generic point x of $\mathcal{X}_s$ such that $\mathcal{O}_v \xrightarrow{\sim} \mathcal{O}_{\mathcal{X},x} \otimes_{\mathcal{O}_L} \mathcal{O}_{\overline{K}}$. In particular, to verify that v is *differentially bounded*, it suffices to verify that $\Omega_{\mathcal{O}_{\mathcal{X},x}/\mathcal{O}_L}$ is a finitely generated $\mathcal{O}_{\mathcal{X},x}$-module. On the other hand, this follows immediately from the fact that $\mathcal{O}_{\mathcal{X},x}$ is essentially of finite type over $\mathcal{O}_L$.

Next, we verify that, if x_v is of type 3, then v is *differentially bounded*. Suppose that x_v is of type 3. Then it follows from Proposition 3.4, (iii), that the VE-chain associated to the p-valuation $\widetilde{v}$ is *strongly edge-like*. Thus, it follows immediately from Proposition 2.3, (iii), (iv) [cf. also Definition 3.2, (v), [the final portion of] (vii); Proposition 3.4, (ii)] that there exist

- an ascending chain

$$K = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_i \subseteq \cdots$$

of finite extension fields of K contained in $\overline{K}$ and indexed by $\mathbb{N}$

and, for each $i \in \mathbb{N}$,

- a compactified semistable model

$$\mathcal{X}_i$$

with split reduction of X_{K_i} over $\mathcal{O}_{K_i}$,

- a dominant morphism

$$\phi_{i+1} : \mathcal{X}_{i+1} \longrightarrow \mathcal{X}_i \times_{\mathcal{O}_{K_i}} \mathcal{O}_{K_{i+1}}$$

over $\mathcal{O}_{K_{i+1}}$ that induces the identity automorphism on the generic fiber,

- a node e_i of $(\mathcal{X}_i)_s$

satisfying the following conditions:

- For each $i \in \mathbb{N}$, the projection to $\mathcal{X}_i$ of $\phi_{i+1}(e_{i+1})$ coincides with the node $e_i \in (\mathcal{X}_i)_s \subseteq \mathcal{X}_i$.

- The equality

$$\mathcal{O}_v = \varinjlim_{i \in \mathbb{N}} A_i \otimes_{\mathcal{O}_{K_i}} \mathcal{O}_{\overline{K}}$$

— where, for each $i \in \mathbb{N}$, we write $A_i \stackrel{\text{def}}{=} \mathcal{O}_{\mathcal{X}_i, e_i}$; the transition map is the homomorphism $A_i \otimes_{\mathcal{O}_{K_i}} \mathcal{O}_{\overline{K}} \to A_{i+1} \otimes_{\mathcal{O}_{K_{i+1}}} \mathcal{O}_{\overline{K}}$ induced by the composite $\mathcal{X}_{i+1} \to \mathcal{X}_i$ [which maps $e_{i+1} \mapsto e_i$] of ϕ_{i+1} with the projection to $\mathcal{X}_i$ — holds [cf. Proposition 2.3, (vi)].

For each $i \in \mathbb{N}$, write $S_i \stackrel{\text{def}}{=} \operatorname{Spec} \mathcal{O}_{K_i}$; $S_i^{\log}$ for the log scheme determined by the log structure on S_i associated to the closed point of S_i; $\mathcal{X}_i^{\log}$ for the log scheme over $S_i^{\log}$ determined by the natural log structure on $\mathcal{X}_i$ [i.e., the multiplicative monoid of sections of $\mathcal{O}_{\mathcal{X}_i}$ that are *invertible* on the open subscheme of $\mathcal{X}_i$ determined by X_{K_i}]; $\omega_{\mathcal{X}_i^{\log}/S_i^{\log}, e_i}$ for the stalk at e_i of the sheaf of relative logarithmic differentials associated to the proper, log smooth morphism $\mathcal{X}_i^{\log} \to S_i^{\log}$ [cf. the subsection in Notations and Conventions entitled "Log schemes"]. Then it follows immediately from the definitions of the various log structures involved that the morphism $\phi_{i+1} : \mathcal{X}_{i+1} \to \mathcal{X}_i \times_{\mathcal{O}_{K_i}} \mathcal{O}_{K_{i+1}} = \mathcal{X}_i \times_{S_i} S_{i+1}$ extends to a *log étale* morphism of log schemes

$$\mathcal{X}_{i+1}^{\log} \longrightarrow \mathcal{X}_i^{\log} \times_{S_i^{\log}} S_{i+1}^{\log},$$

which induces a *natural isomorphism*

$$\omega_{\mathcal{X}_i^{\log}/S_i^{\log}, e_i} \otimes_{A_i} A_{i+1} \stackrel{\sim}{\to} \omega_{\mathcal{X}_{i+1}^{\log}/S_{i+1}^{\log}, e_{i+1}}$$

of A_{i+1}-modules, hence a *natural homomorphism*

$$\phi : \Omega_{\mathcal{O}_v/\mathcal{O}_{\overline{K}}} = \varinjlim_{i \in \mathbb{N}} \Omega_{A_i \otimes_{\mathcal{O}_{K_i}} \mathcal{O}_{\overline{K}}/\mathcal{O}_{\overline{K}}} \longrightarrow \omega_{\mathcal{X}_0^{\log}/S_0^{\log}, e_0} \otimes_{A_0} \mathcal{O}_v$$

of $\mathcal{O}_v$-modules. Here, we note that ϕ induces a *natural isomorphism*

$$\Omega_{\mathcal{O}_v/\mathcal{O}_{\overline{K}}} \otimes_{\mathcal{O}_K} K \xrightarrow{\sim} \omega_{\mathcal{X}_0^{\log}/S_0^{\log},e_0} \otimes_{A_0} \mathcal{O}_v \otimes_{\mathcal{O}_K} K$$

of free $\mathcal{O}_v \otimes_{\mathcal{O}_K} K$-modules of rank 1. Thus, since $\omega_{\mathcal{X}_0^{\log}/S_0^{\log},e_0}$ is a finitely generated A_0-module, we conclude that v is *differentially bounded*, as desired.

Finally, we verify that, if x_v is of type 4, then v is *differentially bounded*. Suppose that x_v is of type 4. Then it follows from Proposition 3.3, (ii); Proposition 3.4, (iii) [cf. also Remark 3.1.1], that the VE-chain associated to the p-valuation $\widetilde{v}$ [cf. Proposition 2.3, (viii)] is *weakly verticial*. Thus, it follows immediately from Proposition 2.3, (iii), (iv) [cf. also Definition 3.2, (iii); the final portion of Remark 2.1.4] that there exist

- an ascending chain

$$K = K_0 \subseteq K_1 \subseteq \cdots \subseteq K_i \subseteq \cdots$$

  of finite extension fields of K contained in $\overline{K}$ and indexed by $\mathbb{N}$

and, for each $i \in \mathbb{N}$,

- a compactified semistable model

$$\mathcal{X}_i$$

  with split reduction of X_{K_i} over $\mathcal{O}_{K_i}$,

- a dominant morphism

$$\phi_{i+1} : \mathcal{X}_{i+1} \longrightarrow \mathcal{X}_i \times_{\mathcal{O}_{K_i}} \mathcal{O}_{K_{i+1}}$$

  over $\mathcal{O}_{K_{i+1}}$ that induces the identity automorphism on the generic fiber,

- an irreducible component v_i of $(\mathcal{X}_i)_s$

satisfying the following conditions:

- For each $i \in \mathbb{N}$, the projection to $\mathcal{X}_i$ of $\phi_{i+1}(v_{i+1})$ is a closed point $x_i \in (\mathcal{X}_i)_s \subseteq \mathcal{X}_i$ of $(\mathcal{X}_i)_s$ that lies in the smooth locus of v_i.

- For each $i \in \mathbb{N}$, there exists a section $\psi_i : \operatorname{Spec} \mathcal{O}_{K_i} \hookrightarrow \mathcal{X}_i$ whose image contains x_i.

- The equality

$$\mathcal{O}_v = \varinjlim_{i \in \mathbb{N}} A_i \otimes_{\mathcal{O}_{K_i}} \mathcal{O}_{\overline{K}}$$

  — where, for each $i \in \mathbb{N}$, we write $A_i \overset{\text{def}}{=} \mathcal{O}_{\mathcal{X}_i,x_i}$; the transition map is the homomorphism $A_i \otimes_{\mathcal{O}_{K_i}} \mathcal{O}_{\overline{K}} \to A_{i+1} \otimes_{\mathcal{O}_{K_{i+1}}} \mathcal{O}_{\overline{K}}$ induced by the composite $\mathcal{X}_{i+1} \to \mathcal{X}_i$ [which maps $x_{i+1} \mapsto x_i$] of ϕ_{i+1} with the projection to $\mathcal{X}_i$ — holds [cf. Proposition 2.3, (vi)].

Thus, we are in the situation of Proposition 3.7, (i). In particular, there exists a collection $\{t_i, \gamma_{i+1}, \pi_{i+1}\}_{i \in \mathbb{N}}$ of elements $t_i \in A_i$ and $\gamma_{i+1}, \pi_{i+1} \in \mathfrak{m}_{K_{i+1}}$ as in Proposition 3.7, (i), such that $t_i = \gamma_{i+1} + \pi_{i+1} t_{i+1}$. Next, observe that since x_i lies in the smooth locus of $(\mathcal{X}_i)_s$, it follows that $\Omega_{A_i/\mathcal{O}_{K_i}}$ is a free A_i-module of rank 1 generated by dt_i. In particular, since $t_i = \gamma_{i+1} + \pi_{i+1} t_{i+1}$, we conclude that

$$\Omega_{A_{i+1}/\mathcal{O}_{K_{i+1}}} \;=\; \frac{1}{\pi_{i+1}} \cdot \Omega_{A_i/\mathcal{O}_{K_i}} \otimes_{A_i} A_{i+1}.$$

Thus, it follows immediately from the equality $\mathcal{O}_v = \varinjlim_{i \in \mathbb{N}} A_i \otimes_{\mathcal{O}_{K_i}} \mathcal{O}_{\overline{K}}$ that

$$\Omega_{\mathcal{O}_v/\mathcal{O}_{\overline{K}}} \;=\; \varinjlim_{i \in \mathbb{N}} \frac{1}{\prod_{1 \leq j \leq i+1} \pi_j} \cdot \Omega_{A_0/\mathcal{O}_{K_0}} \otimes_{A_0} \mathcal{O}_v.$$

Now suppose that v is *differentially unbounded*. For each positive integer i, write $l_i \stackrel{\text{def}}{=} v_p(\pi_i)$. Then since v is differentially unbounded, we conclude that the equality

$$\sum_{i \geq 1} l_i \;=\; +\infty$$

holds. Next, we observe that it follows immediately from Proposition 3.7, (ii) [cf. also the equality $\mathcal{O}_v = \varinjlim_{i \in \mathbb{N}} A_i \otimes_{\mathcal{O}_{K_i}} \mathcal{O}_{\overline{K}}$], that v determines a closed point x_Ω of X_Ω such that the valuation ring of the point-theoretic p-valuation on $K(X_{\overline{K}})$ associated to x_Ω dominates $\mathcal{O}_v$, hence coincides with $\mathcal{O}_v$. Thus, we conclude that x_v is the point $\in X^{\mathrm{an}}$ determined by x_Ω, hence that x_v is of type 1, in contradiction to our assumption that x_v is of type 4. This completes the proof of Proposition 3.8. $\square$

Proposition 3.9 (**Characterization of points of type 1 via geometric Σ-tempered decomposition groups**). *In the notation of Definition 3.2, suppose that $p \in \Sigma$. Let $x \in X^{\mathrm{an}}$ be an element; $l \in \Sigma \setminus \{p\}$; D_x a decomposition group in the geometric Σ-tempered fundamental group $\Delta_X^{\Sigma\text{-tp}}$ of X associated to x. Then the following hold:*

- *(i) Suppose that x is of type 1. Then D_x is trivial.*
- *(ii) Suppose that x is of type 4. Then there exists an open subgroup of D_x that admits a continuous surjective homomorphism to $\mathbb{Z}_p$. In particular, D_x is nontrivial.*
- *(iii) Suppose that x is of type $i \in \{2, 3\}$, and that X satisfies Σ-RNS. Then there exists an open subgroup of D_x that admits a continuous surjective homomorphism to $\mathbb{Z}_l$. In particular, D_x is nontrivial.*
- *(iv) Suppose that X satisfies Σ-RNS. Then x is of type 1 if and only if D_x is trivial.*

Proof. Let $\widetilde{x} \in \widetilde{X}^{\mathrm{an}}$ be a lifting of x. First, we observe that assertion (i) follows immediately from the various definitions involved. Next, we verify assertion (iii). Suppose that x [or, equivalently, $\widetilde{x}$] is of type $i \in \{2, 3\}$, and that X satisfies Σ-RNS. Then it follows immediately from Proposition 3.4, (iii); Proposition 3.5, (i), that there exists an open subgroup of D_x that admits a continuous surjective homomorphism to $\mathbb{Z}_l$. This completes the proof of assertion (iii). Assertion (iv) follows immediately from assertions (i), (ii), (iii) [cf. also Remark 3.1.1].

Thus, to complete the proof of Proposition 3.9, it suffices to verify assertion (ii). To verify assertion (ii), by replacing K by a suitable extension field of K contained in Ω, we may assume without loss of generality [cf. Proposition 3.4, (iii)] that the residue field of K is *separably closed*. Suppose that x [or, equivalently, $\widetilde{x}$] is of *type* 4, and that *no open subgroup of* D_x *admits a continuous surjective homomorphism to* $\mathbb{Z}_p$. Write $\widetilde{v}$ for the primitive p-valuation on $K(\widetilde{X})$ associated to $\widetilde{x}$ [cf. Proposition 2.3, (viii)]; v for the p-valuation obtained by restricting $\widetilde{v}$ to $K(X_{\overline{K}})$. Then since x is of type 4, it follows from Proposition 3.8 that $\Omega_{\mathcal{O}_v/\mathcal{O}_{\overline{K}}}$ is *bounded*. In particular, by replacing K by a finite extension field of K, if necessary, we observe that there exist

- a positive integer N and
- a compactified semistable model with split reduction $\mathcal{X}$ of X over $\mathcal{O}_K$

such that the center z on $\mathcal{X}$ of the VE-chain associated to $\widetilde{v}$ lies in the smooth locus of $\mathcal{X}_s \subseteq \mathcal{X}$, arises from a point of $\mathcal{X}$ valued in the residue field of $\mathcal{O}_K$, and satisfies the following condition [cf. the portion of the proof of Proposition 3.8 concerning points of type 4]:

$$(\Omega_{\mathcal{O}_{\mathcal{X},z}/\mathcal{O}_K} \subseteq) \quad \Omega_{\mathcal{O}_{\mathcal{X},z}/\mathcal{O}_K} \otimes_{\mathcal{O}_{\mathcal{X},z}} \mathcal{O}_v \subseteq \Omega_{\mathcal{O}_v/\mathcal{O}_{\overline{K}}} \subseteq \frac{1}{p^N} \cdot \Omega_{\mathcal{O}_{\mathcal{X},z}/\mathcal{O}_K} \otimes_{\mathcal{O}_{\mathcal{X},z}} \mathcal{O}_v$$

— where by a slight abuse of notation, we use the notation "$\subseteq$" to denote the various natural inclusions, and we note that since $\Omega_{\mathcal{O}_{\mathcal{X},z}/\mathcal{O}_K}$ is a free $\mathcal{O}_{\mathcal{X},z}$-module of rank 1, the $\mathcal{O}_v$-module $\Omega_{\mathcal{O}_{\mathcal{X},z}/\mathcal{O}_K} \otimes_{\mathcal{O}_{\mathcal{X},z}} \mathcal{O}_v$ is a free $\mathcal{O}_v$-module of rank 1. In particular, it follows immediately that the second and third inclusions of the above display induce the *injections* on the respective *p-adic completions* [cf. the discussion of Remark 2.2.4].

In the remainder of the proof of assertion (ii), we suppose that we are in the situation of Proposition 2.12. Moreover, by replacing X by a suitable geometrically pro-Σ connected finite étale covering of X [cf. Proposition 2.3, (xii); Definition 2.7], we may assume without loss of generality that

$$X = Y_n, \quad \mathcal{X} = \mathcal{Y}_n, \quad z = (y_n)_s, \quad \widehat{\mathcal{O}}_{\mathcal{X},z} = \widehat{\mathcal{O}}_{\mathcal{Y}_n,y_n},$$

where $\widehat{\mathcal{O}}_{\mathcal{X},z}$ denotes the completion of the local ring $\mathcal{O}_{\mathcal{X},z}$. Write

$$\Psi : \mathrm{Hom}_{\mathcal{O}_K}(\widehat{T}_{\mathrm{qtr},n}, \widehat{\mathbb{G}}_{\mathrm{m}}) \longrightarrow \widehat{\mathcal{O}}_{\mathcal{X},z}^{\times}$$

for the assignment discussed in Proposition 2.13, (ii). Let $f \in \mathrm{Hom}_{\mathcal{O}_K}(\widehat{T}_{\mathrm{qtr},n}, \widehat{\mathbb{G}}_{\mathrm{m}})$ be a *nontrivial* element [which exists by Propositions 2.11, (i); 2.13, (i)]. Thus,

the *logarithmic differential*

$$\theta \stackrel{\text{def}}{=} \frac{d\Psi(f)}{\Psi(f)} \in \Omega_{\widehat{\mathcal{O}}_{\mathcal{X},z}/\mathcal{O}_K} \stackrel{\text{def}}{=} \varprojlim_{m \geq 1} \Omega_{(\mathcal{O}_{\mathcal{X},z}/\mathfrak{m}_z^m)/\mathcal{O}_K}$$

— where $\mathfrak{m}_z$ denotes the maximal ideal of $\mathcal{O}_{\mathcal{X},z}$; m ranges over the positive integers — of $\Psi(f) \in \widehat{\mathcal{O}}^{\times}_{\mathcal{X},z}$ is $\neq 0$ [cf. the first display, as well as the discussion following this first display, in the proof of Theorem 2.16, where we observe that this portion of the proof of Theorem 2.16 may be applied even in the case of the "K" — i.e., with *separably closed residue field* — of the present discussion].

Next, write $\widehat{\mathcal{O}}_v$ for the p-adic completion of $\mathcal{O}_v$; $\widehat{K}_v$ for the field of fractions of $\widehat{\mathcal{O}}_v$. Since v is a *real valuation* [cf. Proposition 2.3, (vii); Remark 3.1.1], it follows immediately from the final portion of Remark 2.2.4 that the *henselization* of $\mathcal{O}_v$ may be regarded as a subring of $\widehat{\mathcal{O}}_v$. Write $H \subseteq \Delta_X^{\Sigma\text{-tp}}$ for the closed subgroup obtained by forming the intersection of the kernels of the continuous surjective homomorphisms $\Delta_X^{\Sigma\text{-tp}} \twoheadrightarrow \mathbb{Z}_p$; $K(\widetilde{X})^H$, $K(\widetilde{X})^{D_x} \subseteq K(\widetilde{X})$ for the subfields fixed by H and D_x, respectively. Then since there does not exist any continuous surjective homomorphism $D_x \twoheadrightarrow \mathbb{Z}_p$, we thus conclude that $D_x \subseteq H$, hence that

$$K(\widetilde{X})^H \ \subseteq \ K(\widetilde{X})^{D_x} \ \subseteq \ \widehat{K}_v.$$

On the other hand, it follows immediately from the definition of the center z on $\mathcal{X}$ that there exists a natural homomorphism $\phi : \mathcal{O}_{\mathcal{X},z} \to \mathcal{O}_v$ of local rings, which thus induces a homomorphism $\widehat{\phi} : \widehat{\mathcal{O}}_{\mathcal{X},z} \to \widehat{\mathcal{O}}_v$ of topological local rings.

Now we claim that $\widehat{\phi}$ is *injective*. Indeed, suppose that $\mathfrak{p} \stackrel{\text{def}}{=} \text{Ker}(\widehat{\phi}) \neq 0$. Then since $\widehat{\mathcal{O}}_{\mathcal{X},z}$ is a regular local ring of dimension 2, and $\widehat{\mathcal{O}}_v$ is p-torsion-free, it follows that $\mathfrak{p}$ is a prime ideal of height 1 such that $\mathfrak{p} \cap \mathcal{O}_K = \{0\}$. Next, observe that $(\widehat{\mathcal{O}}_{\mathcal{X},z}/\mathfrak{p}) \otimes_{\mathcal{O}_K} (\mathcal{O}_K/\mathfrak{m}_K)$ is *finite* over $\mathcal{O}_K/\mathfrak{m}_K$. Thus, since $\widehat{\mathcal{O}}_{\mathcal{X},z}/\mathfrak{p}$ is a complete $\mathcal{O}_K$-module, we conclude that $\widehat{\mathcal{O}}_{\mathcal{X},z}/\mathfrak{p}$ is *finite* over $\mathcal{O}_K$. On the other hand, observe that the composite homomorphism $\mathcal{O}_{\mathcal{X},z} \hookrightarrow \mathcal{O}_v \hookrightarrow \widehat{\mathcal{O}}_v$ is injective, hence that the natural homomorphism $\mathcal{O}_{\mathcal{X},z} \to \widehat{\mathcal{O}}_{\mathcal{X},z}/\mathfrak{p}$ is injective. In particular, we conclude that $\mathcal{O}_{\mathcal{X},z} \otimes_{\mathcal{O}_K} K$ embeds into a finite dimensional K-vector space, a contradiction. This completes the proof of our *claim* that $\widehat{\phi}$ is injective.

Next, we observe that since the image of $\Psi(f)$ is *p-divisible* in the multiplicative group $\{\widehat{\mathcal{O}}_{\mathcal{X},z} \otimes_{\mathcal{O}_{\mathcal{X},z}} K(\widetilde{X})^H\}^{\times}$ [cf. Proposition 2.13, (ii)], the image of $\Psi(f)$ in the multiplicative group $\widehat{K}_v^{\times}$, hence also in the multiplicative group $\widehat{\mathcal{O}}_v^{\times}$, is *$p$-divisible*. Write

$$\Omega_{\widehat{\mathcal{O}}_v/\mathcal{O}_{\overline{K}}} \stackrel{\text{def}}{=} \varprojlim_{m \geq 1} \Omega_{(\mathcal{O}_v/p^m \cdot \mathcal{O}_v)/\mathcal{O}_{\overline{K}}},$$

where m ranges over the positive integers. Then it follows immediately from well-known basic facts concerning modules of differentials, together with the fact that $\Omega_{\mathcal{O}_{\mathcal{X},z}/\mathcal{O}_K}$ is a free $\mathcal{O}_{\mathcal{X},z}$-module of rank 1, that

$$\Omega_{\widehat{\mathcal{O}}_{\mathcal{X},z}/\mathcal{O}_K} \ = \ \varprojlim_{m \geq 1} \Omega_{\mathcal{O}_{\mathcal{X},z}/\mathcal{O}_K} \otimes_{\mathcal{O}_{\mathcal{X},z}} \mathcal{O}_{\mathcal{X},z}/\mathfrak{m}_z^m \ = \ \Omega_{\mathcal{O}_{\mathcal{X},z}/\mathcal{O}_K} \otimes_{\mathcal{O}_{\mathcal{X},z}} \widehat{\mathcal{O}}_{\mathcal{X},z};$$

$$\Omega_{\widehat{\mathcal{O}}_v/\mathcal{O}_{\overline{K}}} \;=\; \varprojlim_{m\geq 1}\; \Omega_{\mathcal{O}_v/\mathcal{O}_{\overline{K}}} \otimes_{\mathbb{Z}}\; \mathbb{Z}/p^m\mathbb{Z}.$$

In particular, since $\widehat{\phi}$ is injective, it thus follows from the discussion of the final portion of the second paragraph of the present proof that

$$\Omega_{\widehat{\mathcal{O}}_{\mathcal{X},z}/\mathcal{O}_K} \;=\; \Omega_{\mathcal{O}_{\mathcal{X},z}/\mathcal{O}_K} \otimes_{\mathcal{O}_{\mathcal{X},z}} \widehat{\mathcal{O}}_{\mathcal{X},z} \;\subseteq\; \Omega_{\mathcal{O}_{\mathcal{X},z}/\mathcal{O}_K} \otimes_{\mathcal{O}_{\mathcal{X},z}} \widehat{\mathcal{O}}_v \;\subseteq\; \Omega_{\widehat{\mathcal{O}}_v/\mathcal{O}_{\overline{K}}},$$

hence that the image $\theta_v \in \Omega_{\widehat{\mathcal{O}}_v/\mathcal{O}_{\overline{K}}}$ of θ in $\Omega_{\widehat{\mathcal{O}}_v/\mathcal{O}_{\overline{K}}}$ is $\neq 0$. On the other hand, since the image of $\Psi(f)$ in $\widehat{\mathcal{O}}_v^{\times}$ is *p-divisible*, and $\Omega_{\widehat{\mathcal{O}}_v/\mathcal{O}_{\overline{K}}}$ is, by definition, *p-adically separated*, we conclude that $\theta_v = 0$, a contradiction. This completes the proof of assertion (ii), hence of Proposition 3.9. □

Corollary 3.10 (**Reconstruction of points of type 1 via geometric tempered fundamental groups**). *Let $\Sigma \subseteq \mathfrak{Primes}$ be a subset of cardinality ≥ 2 that contains p; K a mixed characteristic complete discrete valuation field of residue characteristic p; X a hyperbolic curve over $\overline{K}$. Write Ω for the p-adic completion of $\overline{K}$; $\Pi^{\mathrm{tp}}_{(-)}$ for the Σ-tempered fundamental group of $(-)$; $\widetilde{X} \to X$ for the universal pro-Σ covering corresponding to Π^{tp}_X [so $\mathrm{Gal}(\widetilde{X}/X)$ may be identified with the pro-Σ completion of Π^{tp}_X]. Suppose that X satisfies Σ-RNS. Then the set $\widetilde{X}(\Omega)$ equipped with its natural action by $\mathrm{Gal}(\widetilde{X}/X)$ — hence also, by passing to the set of $\mathrm{Gal}(\widetilde{X}/X)$-orbits, the quotient set $\widetilde{X}(\Omega) \twoheadrightarrow X(\Omega)$ — may be reconstructed, in a purely combinatorial/group-theoretic way and functorially with respect to isomorphisms of topological groups, from the underlying topological group of Π^{tp}_X.*

Proof. Recall that, for any hyperbolic curve Y over $\overline{K}$, the set of cuspidal inertia subgroups of Π^{tp}_Y, hence also the genus of Y, may be reconstructed, in a purely combinatorial/group-theoretic way and functorially with respect to isomorphisms of topological groups, from the underlying topological group of Π^{tp}_Y [cf. the generalized version of [SemiAn], Corollary 3.11, discussed in [AbsTopII], Remark 2.11.1, (i)]. On the other hand, in the case where X is a *proper* hyperbolic curve over $\overline{K}$, we *observe* that Corollary 3.10 follows immediately from Proposition 3.9, (iv), and [the proof of] Corollary 2.5, (i). Thus, by applying this *observation* to the Σ-tempered fundamental groups of the smooth compactifications of the various [connected] geometrically pro-Σ finite étale Galois coverings of X over $\overline{K}$ of genus ≥ 2, we conclude that $\widetilde{X}(\Omega)$ equipped with its natural action by $\mathrm{Gal}(\widetilde{X}/X)$ may be reconstructed, in a purely combinatorial/group-theoretic way and functorially with respect to isomorphisms of topological groups, from the underlying topological group of Π^{tp}_X. This completes the proof of Corollary 3.10. □

Theorem 3.11 (Preservation of decomposition subgroups associated to closed points). *For $\square \in \{\dagger, \ddagger\}$, let $p^{\square}$ be a prime number; $\Sigma^{\square} \subseteq \mathfrak{Primes}$ a subset that contains $p^{\square}$; $l \in (\Sigma^{\dagger} \setminus \{p^{\dagger}\}) \cap (\Sigma^{\ddagger} \setminus \{p^{\ddagger}\})$; $K^{\square}$ a mixed characteristic complete discrete valuation field of residue characteristic $p^{\square}$; $X^{\square}$ a hyperbolic curve over $K^{\square}$; $L^{\square} \subseteq \overline{K}^{\square}$ a tamely ramified [not necessarily finite!] Galois extension of $K^{\square}$ that may be written as a union of finite tamely ramified Galois extensions of $K^{\square}$ in $\overline{K}^{\square}$ of ramification index prime to l. Let*

$$\sigma : \Pi_{X^{\dagger}_{L^{\dagger}}}^{(\Sigma^{\dagger})} \xrightarrow{\sim} \Pi_{X^{\ddagger}_{L^{\ddagger}}}^{(\Sigma^{\ddagger})}$$

be an isomorphism of profinite groups between the geometrically pro-$\Sigma^{\dagger}$ étale fundamental group of $X^{\dagger}_{L^{\dagger}}$ and the geometrically pro-$\Sigma^{\ddagger}$ étale fundamental group of $X^{\ddagger}_{L^{\dagger}}$. For $\square \in \{\dagger, \ddagger\}$, write $\Delta_{X^{\square}_{L^{\square}}}^{\Sigma^{\square}}$ for the geometric pro-$\Sigma^{\square}$ étale fundamental group of $X^{\square}_{L^{\square}}$, $I_{L^{\square}} \subseteq G_{L^{\square}}$ for the inertia subgroup of $G_{L^{\square}}$, and $k_{L^{\square}}$ for the residue field of $L^{\square}$. Then the following hold:

- *(i) We have an equality $p^{\dagger} = p^{\ddagger}$, and σ induces isomorphisms of profinite groups $\Delta_{X^{\dagger}_{L^{\dagger}}}^{\Sigma^{\dagger}} \xrightarrow{\sim} \Delta_{X^{\ddagger}_{L^{\ddagger}}}^{\Sigma^{\ddagger}}$, $G_{L^{\dagger}} \xrightarrow{\sim} G_{L^{\ddagger}}$. In particular, $\Sigma^{\dagger} = \Sigma^{\ddagger}$. Finally, if, for each $\square \in \{\dagger, \ddagger\}$, every pro-$l$ closed subgroup of the kernel of the l-adic cyclotomic character on $G_{k_{L^{\square}}}$ is trivial [cf. Remark 3.11.1 below], then, for all sufficiently small open subgroups $J^{\dagger} \subseteq G_{L^{\dagger}}$, $J^{\ddagger} \subseteq G_{L^{\ddagger}}$ such that σ induces an isomorphism $J^{\dagger} \xrightarrow{\sim} J^{\ddagger}$, σ also induces an isomorphism of profinite groups between the respective images of $J^{\dagger} \cap I_{L^{\dagger}}$, $J^{\ddagger} \cap I_{L^{\ddagger}}$ in the maximal pro-l quotients $J^{\dagger} \twoheadrightarrow (J^{\dagger})^{\{l\}}$, $J^{\ddagger} \twoheadrightarrow (J^{\ddagger})^{\{l\}}$.*
- *(ii) Suppose that, for all sufficiently small open subgroups $J^{\dagger} \subseteq G_{L^{\dagger}}$, $J^{\ddagger} \subseteq G_{L^{\ddagger}}$ such that σ induces an isomorphism $J^{\dagger} \xrightarrow{\sim} J^{\ddagger}$, σ also induces an isomorphism of profinite groups between the respective images of $J^{\dagger} \cap I_{L^{\dagger}}$, $J^{\ddagger} \cap I_{L^{\ddagger}}$ in the maximal pro-l quotients $J^{\dagger} \twoheadrightarrow (J^{\dagger})^{\{l\}}$, $J^{\ddagger} \twoheadrightarrow (J^{\ddagger})^{\{l\}}$. Write $\Sigma \stackrel{\text{def}}{=} \Sigma^{\dagger} = \Sigma^{\ddagger}$ [cf. (i)]. Suppose, moreover, that $X^{\dagger}$ and $X^{\ddagger}$ satisfy Σ-RNS. Then σ induces a bijection between the respective sets of decomposition subgroups associated to closed points of $X^{\dagger}_{\widehat{L}^{\dagger}}$ and $X^{\ddagger}_{\widehat{L}^{\ddagger}}$, where $\widehat{L}^{\dagger}$, $\widehat{L}^{\ddagger}$ denote the respective completions of $L^{\dagger}$, $L^{\ddagger}$.*

Proof. First, we verify assertion (i). Write $\tau^{\dagger} \stackrel{\text{def}}{=} \sigma^{-1}$, $\tau^{\ddagger} \stackrel{\text{def}}{=} \sigma$. For $\square \in \{\dagger, \ddagger\}$, write $\square'$ for the unique element of $\{\dagger, \ddagger\} \setminus \{\square\}$. Then observe that it follows immediately, by applying to $\tau^{\square}(\Delta_{X^{\square'}_{L^{\square'}}}^{\Sigma^{\square'}})$, for both $\square = \dagger$ and $\square = \ddagger$,

- the argument of the proof of [MiTs1], Corollary 4.6 [in the case where the extension $L^{\square}/K^{\square}$ is *finite*; here, we note that in this case, it follows from [MiSaTs], Theorem 3.8, that if $\Pi_{X^{\square}_{L^{\square}}}^{(\Sigma^{\square})}$ is topologically finitely generated, then the extension $L^{\square'}/K^{\square'}$ is also *finite*], and

• [MiSaTs], Theorem 3.8 [in the case where the extension $L^{\square}/K^{\square}$ is *infinite*],

that σ induces isomorphisms of profinite groups

$$\Delta_{X^{\dagger}}^{\Sigma^{\dagger}} \stackrel{\sim}{\to} \Delta_{X^{\ddagger}}^{\Sigma^{\ddagger}}; \quad G_{L^{\dagger}} \stackrel{\sim}{\to} G_{L^{\ddagger}}.$$

Thus, we conclude from [MiTs2], Theorem A, (i), together with the well-known structure of geometric fundamental groups of hyperbolic curves over fields of characteristic zero [cf., e.g., [MT], Remark 1.2.2], that $p^{\dagger} = p^{\ddagger}$, and $\Sigma^{\dagger} = \Sigma^{\ddagger}$. The final portion of assertion (i) follows immediately from the well-known structure, for $\square \in \{\dagger, \ddagger\}$, of the Galois group $\mathrm{Gal}((K^{\square})^{\mathrm{tm}}/K^{\square})$ over $K^{\square}$ of the maximal tamely ramified extension $(K^{\square})^{\mathrm{tm}}$ of $K^{\square}$ [under the assumption that every pro-l closed subgroup of the kernel of the l-adic cyclotomic character on $G_{k_{L^{\square}}}$ is trivial], which implies that, for any sufficiently small open subgroup $J^{\square} \subseteq G_{L^{\square}}$, the image of $J^{\square} \cap I_{L^{\square}}$ in the maximal pro-l quotient $J^{\square} \twoheadrightarrow (J^{\square})^{\{l\}}$ coincides with the *unique maximal abelian normal closed subgroup* of $(J^{\square})^{\{l\}}$. This completes the proof of assertion (i).

Next, we verify assertion (ii). First, we note that it follows from assertion (i) that $p \stackrel{\mathrm{def}}{=} p^{\dagger} = p^{\ddagger}$. Thus, in light of our assumption on σ [cf. also assertion (i)], it follows from [CmbGC], Corollary 2.7, (i) [applied in the case where "l" is taken to be the p of the present discussion], (iii) [applied in the case where "l" is taken to be the l of the present discussion], that the isomorphism $\Delta_{X^{\dagger}_{L^{\dagger}}}^{\Sigma} \stackrel{\sim}{\to} \Delta_{X^{\ddagger}_{L^{\dagger}}}^{\Sigma}$ [cf. (i)] satisfies the condition $(b^{\exists})$ of [CbTpIII], Proposition 3.6. In particular, by applying [CbTpIII], Proposition 3.6, (i), we conclude that the isomorphism $\Delta_{X^{\dagger}_{L^{\dagger}}}^{\Sigma} \stackrel{\sim}{\to} \Delta_{X^{\ddagger}_{L^{\ddagger}}}^{\Sigma}$ arises, up to composition with an inner automorphism, from an isomorphism between the respective geometric Σ-tempered fundamental groups of $X^{\dagger}$ and $X^{\ddagger}$. Thus, by replacing σ by the composite of σ with an inner automorphism arising from $\Delta_{X^{\ddagger}_{L^{\ddagger}}}^{\Sigma}$, we may assume without loss of generality that σ arises from an isomorphism between the respective pull-backs via the natural inclusions $G_{L^{\dagger}} \subseteq G_{K^{\dagger}}$, $G_{L^{\ddagger}} \subseteq G_{K^{\ddagger}}$ of the geometrically Σ-tempered fundamental groups of $X^{\dagger}$ and $X^{\ddagger}$.

Next, write $\widetilde{X}^{\dagger}$, $\widetilde{X}^{\ddagger}$ for the universal geometrically pro-Σ coverings corresponding to $\Pi^{(\Sigma)}_{X^{\dagger}_{L^{\dagger}}}$, $\Pi^{(\Sigma)}_{X^{\ddagger}_{L^{\ddagger}}}$, respectively; $\Omega^{\dagger}$, $\Omega^{\ddagger}$ for the p-adic completions of $\overline{K}^{\dagger}$, $\overline{K}^{\ddagger}$, respectively. Then since σ determines an isomorphism between the respective geometric Σ-tempered fundamental groups of $X^{\dagger}$ and $X^{\ddagger}$, it follows immediately from Corollary 3.10 that σ induces a bijection

$$\widetilde{X}^{\dagger}(\Omega^{\dagger}) \stackrel{\sim}{\to} \widetilde{X}^{\ddagger}(\Omega^{\ddagger})$$

that is compatible with the respective natural actions of $\Pi^{(\Sigma)}_{X^{\dagger}_{L^{\dagger}}}$, $\Pi^{(\Sigma)}_{X^{\ddagger}_{L^{\dagger}}}$. Thus, in light of [Tate], §3.3, Theorem 1 [which, as is easily verified, admits a routine generalization to mixed characteristic complete valuation fields such as $\widehat{L}^{\dagger}$, $\widehat{L}^{\ddagger}$, i.e., whose valuations are not necessarily discrete [but nonetheless tamely ramified over some discrete valuation], and whose residue fields are not necessarily

perfect], we conclude that σ induces a bijection between the respective sets of decomposition subgroups associated to closed points of $X^{\dagger}_{\widehat{L}^{\dagger}}$ and $X^{\ddagger}_{\widehat{L}^{\ddagger}}$. This completes the proof of assertion (ii), hence of Theorem 3.11. $\square$

Remark 3.11.1. In passing, we observe that the condition concerning the kernel of the l-adic cyclotomic character on $G_{k_{L^{\square}}}$ that appears in the final portion of Theorem 3.11, (i), is satisfied if $k_{L^{\square}}$ is either *separably closed* or *algebraic over the finite field of cardinality* p.

We are now in a position to verify an *absolute version of the Grothendieck Conjecture for arbitrary hyperbolic curves over p-adic local fields* [cf. Theorem 3.12 below], which is one of the central open questions in anabelian geometry.

Theorem 3.12 (**Absolute version of the Grothendieck Conjecture for arbitrary hyperbolic curves over p-adic local fields).** *Let $p^{\dagger}$, $p^{\ddagger}$ be prime numbers; $\Sigma \subseteq \mathfrak{Primes}$ a subset of cardinality ≥ 2 that contains $p^{\dagger}$ and $p^{\ddagger}$; $K^{\dagger}$, $K^{\ddagger}$ mixed characteristic local fields of residue characteristic $p^{\dagger}$, $p^{\ddagger}$, respectively; $X^{\dagger}$, $X^{\ddagger}$ hyperbolic curves over $K^{\dagger}$, $K^{\ddagger}$, respectively. Then the natural map*

$$\mathrm{Isom}(X^{\dagger}, X^{\ddagger}) \longrightarrow \mathrm{OutIsom}(\Pi^{(\Sigma)}_{X^{\dagger}}, \Pi^{(\Sigma)}_{X^{\ddagger}})$$

is bijective.

Proof. First, we observe that any isomorphism of schemes between $X^{\dagger}$ and $X^{\ddagger}$ necessarily lies over an isomorphism of fields between $K^{\dagger}$ and $K^{\ddagger}$. [Indeed, this follows immediately by considering subgroups of the groups of units $\Gamma(X^{\dagger}, \mathcal{O}^{\times}_{X^{\dagger}})$, $\Gamma(X^{\ddagger}, \mathcal{O}^{\times}_{X^{\ddagger}})$ whose unions with $\{0\}$ are closed under addition.] Now Theorem 3.12 follows immediately by combining Theorems 2.17; 3.11, (i), (ii) [cf. also Remark 3.11.1], of the present paper with [AbsTopII], Corollary 2.9. $\square$

Remark 3.12.1. Theorem 3.12 may be regarded as a *complete affirmative resolution* of the *absolute version of the Grothendieck Conjecture for hyperbolic curves over p-adic local fields* in the *geometrically pro-*Σ case, where $\Sigma \subseteq \mathfrak{Primes}$ is a subset of cardinality ≥ 2 that contains the residue characteristic of the base field. On the other hand, the following questions remain *open*, to the authors' knowledge, at the time of writing of the present paper:

> Question 1: Can one prove a *geometrically pro-p version* of the absolute Grothendieck Conjecture for hyperbolic curves over p-adic local fields? In this context, we observe that certain partial results in this direction are obtained in [Hgsh].
>
> Question 2: Can one prove an absolute version of the Grothendieck Conjecture for hyperbolic curves over *more general base fields*? For instance, one may consider the case where the base fields are *mixed*

characteristic complete discrete valuation fields whose *residue fields* are *algebraic* over $\mathbb{F}_p$, i.e., a class of fields for which a *relative* version of the Grothendieck Conjecture for hyperbolic curves has been known for some time [cf. [AnabTop], Theorem 4.12].

Theorem 3.13 (**Absolute version of the Grothendieck Conjecture for configuration spaces associated to arbitrary hyperbolic curves over p-adic local fields).** *Let $p^\dagger$, $p^\ddagger$ be prime numbers; $K^\dagger$, $K^\ddagger$ mixed characteristic local fields of residue characteristic $p^\dagger$, $p^\ddagger$, respectively; $X^\dagger$, $X^\ddagger$ hyperbolic curves over $K^\dagger$, $K^\ddagger$, respectively; $n^\dagger$, $n^\ddagger$ positive integers. Write $X^\dagger_{n^\dagger}$ (respectively, $X^\ddagger_{n^\ddagger}$) for the $n^\dagger$-th (respectively, $n^\ddagger$-th) configuration space associated to $X^\dagger$ (respectively, $X^\ddagger$). Then the natural map*

$$\mathrm{Isom}(X^\dagger_{n^\dagger}, X^\ddagger_{n^\ddagger}) \longrightarrow \mathrm{OutIsom}(\Pi_{X^\dagger_{n^\dagger}}, \Pi_{X^\ddagger_{n^\ddagger}})$$

is bijective.

Proof. First, we observe that any isomorphism of schemes between $X^\dagger_{n^\dagger}$ and $X^\ddagger_{n^\ddagger}$ necessarily lies over an isomorphism of fields between $K^\dagger$ and $K^\ddagger$. [Indeed, this follows immediately by a similar argument to the argument applied in the proof of Theorem 3.12.] Now Theorem 3.13 follows immediately from a routine argument via induction on $n \stackrel{\text{def}}{=} n^\dagger = n^\ddagger$ [cf. [AbsTopI], Theorem 2.6, (v); [HMM], Theorem A, (i), (ii)], by combining Theorem 3.12 of the present paper with the *relative* version of the Grothendieck Conjecture given in [LocAn], Theorem A. □

Next, we discuss the functorial behavior of the *lengths of nodes* of special fibers of compactified semistable models [cf. Definition 3.14 below] with respect to finite morphisms between compactified semistable models that extend finite étale Galois coverings of hyperbolic curves over mixed characteristic complete discrete valuation fields.

Definition 3.14. Let K be a mixed characteristic complete discrete valuation field of residue characteristic p; X a hyperbolic curve over K; $\mathcal{X}$ a compactified semistable model with split reduction of X over $\mathcal{O}_K$; e a node of $\mathcal{X}_s$. Recall that the completion of the local ring $\mathcal{O}_{\mathcal{X},e}$ at e is isomorphic to $\mathcal{O}_K[[x,y]]/(xy-a)$, where x, y denote indeterminates; $a \in \mathfrak{m}_K \setminus \{0\}$. Then we shall refer to $v_p(a)$ as the *length* of e. [Note that the length of e is independent of the choice of a, as well as of the isomorphism $\mathcal{O}_{\mathcal{X},e} \stackrel{\sim}{\to} \mathcal{O}_K[[x,y]]/(xy-a)$ over $\mathcal{O}_K$ [cf. [Hur], §3.7].]

Proposition 3.15 (Functorial behavior of the lengths of nodes). *Let K be a mixed characteristic complete discrete valuation field of residue characteristic p; X a hyperbolic curve over K; $Y \to X$ a [connected] finite étale Galois covering of hyperbolic curves over K; $\mathcal{Y}$ a compactified semistable model of Y over $\mathcal{O}_K$ that is stabilized by $G \stackrel{\text{def}}{=} \mathrm{Gal}(Y/X)$. Write $\mathcal{X}$ for the compactified semistable model of X over $\mathcal{O}_K$ obtained by forming the quotient of $\mathcal{Y}$ by the action of G on $\mathcal{Y}$ [cf. Proposition 2.3, (iv)]; $f : \mathcal{Y} \to \mathcal{X}$ for the natural quotient morphism. Suppose that*

- *$\mathcal{Y}$ has split reduction, and that*
- *the natural action of G on $\mathcal{Y}_s$ does not permute the branches of some node e_Y of $\mathcal{Y}_s$.*

Write $e_X \stackrel{\text{def}}{=} f(e_Y)$ for the node of $\mathcal{X}_s$ determined by e_Y [cf. Proposition 2.3, (iv)]; l_X, l_Y for the lengths of the nodes e_X, e_Y [relative to the compactified semistable models $\mathcal{X}$, $\mathcal{Y}$, respectively]. Then there exists a positive integer m such that

$$l_X = m \cdot l_Y.$$

Moreover, the positive integer m may be computed as the cardinality of the decomposition subgroup [i.e., the stabilizer subgroup] of e_Y in G.

Proof. Write $S \stackrel{\text{def}}{=} \mathrm{Spec}\ \mathcal{O}_K$; $S^{\log}$ for the log scheme obtained by equipping S with the log structure determined by the closed point of S; $\mathcal{X}^{\log}$, $\mathcal{Y}^{\log}$ for the log schemes over $S^{\log}$ determined by the compactified semistable models $\mathcal{X}$, $\mathcal{Y}$, respectively [cf. the discussion of the subsection in Notations and Conventions entitled "Log schemes"]. Observe that f naturally determines a finite morphism $f^{\log} : \mathcal{Y}^{\log} \to \mathcal{X}^{\log}$ of log schemes, hence a finite morphism

$$(\mathrm{Spec}\ \widehat{\mathcal{O}}_{\mathcal{Y},e_Y})^{\log} \longrightarrow (\mathrm{Spec}\ \widehat{\mathcal{O}}_{\mathcal{X},e_X})^{\log},$$

where $\widehat{\mathcal{O}}_{\mathcal{X},e_X}$, $\widehat{\mathcal{O}}_{\mathcal{Y},e_Y}$ denote the completions of the respective normal local rings $\mathcal{O}_{\mathcal{X},e_X}$, $\mathcal{O}_{\mathcal{Y},e_Y}$, and the superscripts "log" denote the log structures induced by the respective log structures of $\mathcal{X}^{\log}$, $\mathcal{Y}^{\log}$. Since $\mathcal{Y}$ is assumed to have split reduction, it follows [cf. the discussion of Definition 3.14] that

$$\widehat{\mathcal{O}}_{\mathcal{X},e_X} \cong \mathcal{O}_K[[u_1, u_2]]/(u_1u_2 - a), \quad \widehat{\mathcal{O}}_{\mathcal{Y},e_Y} \cong \mathcal{O}_K[[v_1, v_2]]/(v_1v_2 - b),$$

where u_1, u_2, v_1, v_2 denote indeterminates; $a, b \in \mathfrak{m}_K \backslash \{0\}$ are elements such that $l_X = v_p(a)$, $l_Y = v_p(b)$. Moreover, it follows immediately from the definitions of the log structures involved, together with the geometry of the *irreducible components* of the *special fibers* of $\mathrm{Spec}\ \widehat{\mathcal{O}}_{\mathcal{X},e_X}$ and $\mathrm{Spec}\ \widehat{\mathcal{O}}_{\mathcal{Y},e_Y}$, that, after possibly switching the indices $\in \{1, 2\}$ of [either or both of] the pairs (u_1, u_2) and (v_1, v_2), there exist positive integers m_1, m_2 and units $c_1, c_2 \in \widehat{\mathcal{O}}^{\times}_{\mathcal{Y},e_Y}$ such that $m_1 \geq m_2$, and

$$u_1 = c_1 \cdot v_1^{m_1}, \quad u_2 = c_2 \cdot v_2^{m_2},$$

where we regard u_1, u_2 as elements in $\widehat{\mathcal{O}}_{\mathcal{Y},e_Y}$ via the natural injection $\widehat{\mathcal{O}}_{\mathcal{X},e_X} \hookrightarrow \widehat{\mathcal{O}}_{\mathcal{Y},e_Y}$. In particular, it holds that

$$a = u_1 u_2 = c_1 c_2 v_1^{m_1} v_2^{m_2} = c_1 c_2 v_1^{m_1 - m_2} b^{m_2} \in \mathcal{O}_K[[v_1, v_2]]/(v_1 v_2 - b).$$

On the other hand, such a relation implies, in light of the well-known structure of the log structures involved, i.e., in effect, the geometry of the *irreducible components* of the *special fibers* of Spec $\widehat{\mathcal{O}}_{\mathcal{X},e_X}$ and Spec $\widehat{\mathcal{O}}_{\mathcal{Y},e_Y}$, that $m \stackrel{\text{def}}{=} m_1 = m_2$, hence that $a = c_1 c_2 b^m$. In particular, we conclude that $l_X = m \cdot l_Y$, as desired. Finally, the fact that m may be computed as the cardinality of the decomposition subgroup [i.e., the stabilizer subgroup] of e_Y in G follows immediately from the fact that the *generic degree* of the [finite, generically étale] morphism Spec $\widehat{\mathcal{O}}_{\mathcal{Y},e_Y} \to$ Spec $\widehat{\mathcal{O}}_{\mathcal{X},e_X}$ is [easily computed, via the explicit presentations of $\mathcal{O}_{\mathcal{Y},e_Y}$, $\mathcal{O}_{\mathcal{X},e_X}$ given above, to be] m. This completes the proof of Proposition 3.15. □

Next, we apply Proposition 3.15, together with the theory of *p-adic arithmetic cuspidalizations* developed in [Tsjm], §2, to prove that the various *p-adic versions of the Grothendieck-Teichmüller group* that appear in the literature [cf. [Tsjm], Remark 2.1.2] in fact *coincide*.

Theorem 3.16 (**Equality of various *p*-adic versions of the Grothendieck--Teichmüller group**). *Write* $X \stackrel{\text{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0, 1, \infty\}$;

$$\mathrm{GT} \subseteq \mathrm{Out}(\Pi_X)$$

for the Grothendieck-Teichmüller group [cf. [CmbCsp], Remark 1.11.1];

$$\mathrm{GT}^{\mathrm{M}} \subseteq \mathrm{GT} \ (\subseteq \mathrm{Out}(\Pi_X))$$

for the metrized Grothendieck-Teichmüller group [cf. [CbTpIII], Remark 3.19.2];

$$\mathrm{GT}_p^{\mathrm{tp}} \stackrel{\text{def}}{=} \mathrm{GT} \cap \mathrm{Out}(\Pi_X^{\mathrm{tp}}) \subseteq \mathrm{Out}(\Pi_X)$$

[cf. the subsection in Notations and Conventions entitled "Fundamental groups"; [Tsjm], Definition 2.1]. Then the natural inclusion

$$\mathrm{GT}^{\mathrm{M}} \subseteq \mathrm{GT}_p^{\mathrm{tp}}$$

of subgroups of GT *is an* **equality**. *In particular, it holds that*

$$\mathrm{GT}^{\mathrm{M}} = \mathrm{GT}_p = \mathrm{GT}^{\mathrm{G}} = \mathrm{GT}_p^{\mathrm{tp}}$$

[cf. [Tsjm], Remark 2.1.2].

Proof. First, we recall that there exists a natural surjection

$$\phi : \mathrm{GT}_p^{\mathrm{tp}} \twoheadrightarrow G_{\mathbb{Q}_p}$$

whose restriction to $G_{\mathbb{Q}_p}$ is the identity automorphism [cf. [Tsjm], Corollary B, as well as Remark 3.16.1 below]. Thus, since $G_{\mathbb{Q}_p} \subseteq \mathrm{GT}^{\mathrm{M}} \subseteq \mathrm{GT}_p^{\mathrm{tp}}$, it suffices to prove that $\mathrm{Ker}(\phi) \subseteq \mathrm{GT}^{\mathrm{M}}$. Let $\sigma \in \mathrm{Ker}(\phi)$. Fix a lifting $\tilde{\sigma} \in \mathrm{Aut}(\Pi_X^{\mathrm{tp}})$ of σ. [Here and in the following discussion, $\Pi_{(-)}^{\mathrm{tp}}$ will always denote the $\mathfrak{Primes}$-tempered fundamental group of $(-)$.]

Then it follows immediately from the construction of ϕ [cf. the discussion, in the proof of [Tsjm], Corollary 2.4, of the two paragraphs following the proof of Claim 2.4.B; the discussion, in the proof of [HMT], Theorem 4.4, of the *observation* immediately following the statement of Claim 4.4.A; [NCBel], Corollary 1.2] that, for any finite subset $S \subseteq \overline{\mathbb{Q}} \setminus \{0, 1\} \subseteq X(\overline{\mathbb{Q}}_p)$, $\tilde{\sigma}$ lifts to an automorphism of $\Pi_{X_S}^{\mathrm{tp}}$ [where we write $X_S \stackrel{\mathrm{def}}{=} X \setminus S$] with respect to the natural surjection $\Pi_{X_S}^{\mathrm{tp}} \twoheadrightarrow \Pi_X^{\mathrm{tp}}$ determined [up to composition with an inner automorphism] by the natural open immersion $X_S \hookrightarrow X$.

Next, let $\psi_Y : Y \to X$ be a connected finite étale covering over $\overline{\mathbb{Q}}_p$. Write $\psi_Z : Z \to X$ for the Galois closure of ψ_Y; $\mathcal{Z}$ for the compactified stable model of Z over $\mathcal{O}_{\overline{\mathbb{Q}}_p}$; $\psi_Y^\sigma : Y^\sigma \to X$, $\psi_Z^\sigma : Z^\sigma \to X$ for the connected finite étale coverings over $\overline{\mathbb{Q}}_p$ that correspond to the open subgroups $\tilde{\sigma}(\Pi_Y^{\mathrm{tp}}) \subseteq \Pi_X^{\mathrm{tp}}$, $\tilde{\sigma}(\Pi_Z^{\mathrm{tp}}) \subseteq \Pi_X^{\mathrm{tp}}$, respectively. For each finite subset $S \subseteq \overline{\mathbb{Q}} \setminus \{0, 1\} \subseteq X(\overline{\mathbb{Q}}_p)$, write

- $\mathcal{Z}_S$ (respectively, $\mathcal{Z}_S^\sigma$) for the compactified stable model of $Z_S \stackrel{\mathrm{def}}{=} Z \setminus \psi_Z^{-1}(S)$ (respectively, $Z_S^\sigma \stackrel{\mathrm{def}}{=} Z^\sigma \setminus (\psi_Z^\sigma)^{-1}(S)$) over $\mathcal{O}_{\overline{\mathbb{Q}}_p}$;
- $\mathcal{X}_S$ (respectively, $\mathcal{Y}_S$, $\mathcal{Y}_S^\sigma$) for the compactified semistable model of X_S (respectively, $Y_S \stackrel{\mathrm{def}}{=} Y \setminus \psi_Y^{-1}(S)$, $Y_S^\sigma \stackrel{\mathrm{def}}{=} Y^\sigma \setminus (\psi_Y^\sigma)^{-1}(S)$) obtained by forming the quotient of $\mathcal{Z}_S$ (respectively, $\mathcal{Z}_S$, $\mathcal{Z}_S^\sigma$) via the natural action of $\mathrm{Gal}(Z/X)$ (respectively, $\mathrm{Gal}(Z/Y)$, $\mathrm{Gal}(Z^\sigma/Y^\sigma)$) [cf. Proposition 2.3, (iv)].

Next, observe that there exists a finite subset $T \subseteq \overline{\mathbb{Q}} \setminus \{0, 1\} \subseteq X(\overline{\mathbb{Q}}_p)$ such that

- the natural action of $\mathrm{Gal}(Z/X)$ on $(\mathcal{Z}_T)_s$ does not permute any branches of nodes, and
- $\mathcal{X}_T$, $\mathcal{Y}_T$, $\mathcal{Y}_T^\sigma$ are the respective compactified stable models of X_T, Y_T, Y_T^σ over $\mathcal{O}_{\overline{\mathbb{Q}}_p}$.

Then since $\tilde{\sigma}$ lifts to an automorphism of $\Pi_{X_T}^{\mathrm{tp}}$ [cf. the above discussion], hence to an isomorphism

$$\Pi_Z^{\mathrm{tp}} \times_{\Pi_X^{\mathrm{tp}}} \Pi_{X_T}^{\mathrm{tp}} \;=\; \Pi_{Z_T}^{\mathrm{tp}} \;\stackrel{\sim}{\to}\; \Pi_{Z_T^\sigma}^{\mathrm{tp}} \;=\; \Pi_{Z^\sigma}^{\mathrm{tp}} \times_{\Pi_X^{\mathrm{tp}}} \Pi_{X_T}^{\mathrm{tp}},$$

it follows immediately from Proposition 2.3, (iv), together with [SemiAn], Corollary 3.11, that $\tilde{\sigma}$ induces a commutative diagram of semi-graphs

$$\begin{array}{ccc} \Gamma_{\mathcal{Z}_T} & \xrightarrow{\sim} & \Gamma_{\mathcal{Z}_T^\sigma} \\ \downarrow & & \downarrow \\ \Gamma_{\mathcal{X}_T} & = & \Gamma_{\mathcal{X}_T}, \end{array}$$

where $\Gamma_{(-)}$ denotes the dual semi-graph associated to $(-)_s$, compatible with the respective natural actions of $\mathrm{Gal}(Z/X)$, $\mathrm{Gal}(Z^\sigma/X)$. Thus, we conclude from Proposition 3.15 that the isomorphism $\Gamma_{\mathcal{Z}_T} \xrightarrow{\sim} \Gamma_{\mathcal{Z}_T^\sigma}$ of dual semi-graphs is compatible with the respective metric structures [cf. [CbTpIII], Definition 3.5, (iii)]. On the other hand, $\tilde{\sigma}$ also induces a commutative diagram of semi-graphs

$$\begin{array}{ccc} \Gamma_{\mathcal{Z}_T} & \xrightarrow{\sim} & \Gamma_{\mathcal{Z}_T^\sigma} \\ \downarrow & & \downarrow \\ \Gamma_{\mathcal{Y}_T} & \xrightarrow{\sim} & \Gamma_{\mathcal{Y}_T^\sigma} \end{array}$$

compatible with the respective natural actions of $\mathrm{Gal}(Z/Y)$, $\mathrm{Gal}(Z^\sigma/Y^\sigma)$ [cf. Proposition 2.3, (iv); [SemiAn], Corollary 3.11]. Thus, since the isomorphism $\Gamma_{\mathcal{Z}_T} \xrightarrow{\sim} \Gamma_{\mathcal{Z}_T^\sigma}$ of dual semi-graphs is compatible with the respective metric structures, we conclude from Proposition 3.15 again that the isomorphism $\Gamma_{\mathcal{Y}_T} \xrightarrow{\sim} \Gamma_{\mathcal{Y}_T^\sigma}$ of dual semi-graphs is also compatible with the respective metric structures. Finally, it follows immediately from the well-known theory of *pointed stable curves* and contraction morphisms that arise from eliminating cusps, as exposed in [Knud] [cf. also Remark 2.1.4], that this implies that, if we write $\mathcal{Y}$, $\mathcal{Y}^\sigma$ for the respective compactified stable models of Y, Y^σ over $\mathcal{O}_{\overline{\mathbb{Q}}_p}$, then the isomorphism $\Gamma_{\mathcal{Y}} \xrightarrow{\sim} \Gamma_{\mathcal{Y}^\sigma}$ of dual semi-graphs induced by $\tilde{\sigma}$ [cf. [SemiAn], Corollary 3.11] is compatible with the respective metric structures. Thus, we conclude from [CbTpIII], Definition 3.7, (ii); [CbTpIII], Remark 3.19.2, that $\mathrm{GT}^{\mathrm{M}} = \mathrm{GT}_p^{\mathrm{tp}}$. This completes the proof of Theorem 3.16. □

Remark 3.16.1. Here, we recall that one of the key ingredients in the proof of [Tsjm], Corollary B, is the theory of *resolution of nonsingularities* developed in [Lpg1].

As a corollary, we obtain the following *affirmative answer* to the question posed in the discussion immediately preceding Theorem E in [CbTpIII], Introduction:

Corollary 3.17 (**Commensurable terminality of various *p*-adic versions of the Grothendieck-Teichmüller group**). *We maintain the notation of Theorem 3.16. Then* $\mathrm{GT}^{\mathrm{M}} = \mathrm{GT}_p = \mathrm{GT}^{\mathrm{G}} = \mathrm{GT}_p^{\mathrm{tp}}$ *is commensurably terminal in* GT*, i.e., the commensurator* $C_{\mathrm{GT}}(\mathrm{GT}^{\mathrm{M}})$ *of* GT^{M} *in* GT *is equal to* GT^{M}.

Proof. It follows immediately from Theorem 3.16, together with [CbTpIII], Theorem E, that $\mathrm{GT}^{\mathrm{M}} \subseteq C_{\mathrm{GT}}(\mathrm{GT}^{\mathrm{M}}) \subseteq \mathrm{GT}^{\mathrm{G}} = \mathrm{GT}^{\mathrm{M}}$. Thus, we conclude that $C_{\mathrm{GT}}(\mathrm{GT}^{\mathrm{M}}) = \mathrm{GT}^{\mathrm{M}}$, as desired. □

Proposition 3.18 (**Reconstruction of the subset of $\overline{\mathbb{Q}}_p$-rational points from the p-adic Grothendieck-Teichmüller group**). *Write* $X \stackrel{\text{def}}{=} \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0, 1, \infty\}$; Π^{tp}_X *for the* $\mathfrak{Primes}$*-tempered fundamental group of* X. *Then the subset* $X(\overline{\mathbb{Q}}_p) \subseteq X(\mathbb{C}_p)$, *where we think of "*$X(\mathbb{C}_p)$*" as the set reconstructed from* Π^{tp}_X *in Corollary 3.10, may be reconstructed, in a purely combinatorial/group-theoretic way, from the data*

$$(\Pi^{\mathrm{tp}}_X,\ \mathrm{GT}^{\mathrm{tp}}_p \subseteq \mathrm{Out}(\Pi^{\mathrm{tp}}_X))$$

— consisting of the underlying topological group of Π^{tp}_X *and the subgroup* $\mathrm{GT}^{\mathrm{tp}}_p \subseteq \mathrm{Out}(\Pi^{\mathrm{tp}}_X)$ *— as the subset of elements fixed by some open subgroup of* $\mathrm{GT}^{\mathrm{tp}}_p$. *Moreover, this reconstruction procedure is functorial with respect to isomorphisms of topological groups for which the induced isomorphism on "*$\mathrm{Out}(-)$*" preserves the given subgroup of "*$\mathrm{Out}(-)$*".*

Proof. First, we observe that it follows immediately from the existence of the natural homeomorphism "$\theta_{\widetilde{X}}$" of Proposition 2.3, (viii), together with the definition of "$\mathbb{VE}(\widetilde{X})^{\mathrm{tor}}$" [cf. Definition 2.2, (vi)], that the subset $X(\overline{\mathbb{Q}}_p) \subseteq X(\mathbb{C}_p)$ is *dense* in $X(\mathbb{C}_p)$, and that the natural action of $\mathrm{GT}^{\mathrm{tp}}_p$ on $X(\mathbb{C}_p)$ is via *self-homeomorphisms* of $X(\mathbb{C}_p)$ [cf. Corollary 3.10 and its proof; Corollary 3.16]. Thus, since the natural action of $\mathrm{GT}^{\mathrm{tp}}_p$ on $X(\overline{\mathbb{Q}}_p)$ factors through the surjection $\mathrm{GT}^{\mathrm{tp}}_p \twoheadrightarrow G_{\mathbb{Q}_p}$ [cf. [Tsjm], Corollary B, and its proof], we conclude that the natural action $\mathrm{GT}^{\mathrm{tp}}_p$ on $X(\mathbb{C}_p)$ factors through this surjection $\mathrm{GT}^{\mathrm{tp}}_p \twoheadrightarrow G_{\mathbb{Q}_p}$, and hence [cf. [Tate], §3.3, Theorem 1] that the subset $X(\overline{\mathbb{Q}}_p) \subseteq X(\mathbb{C}_p)$ may be characterized as the subset of elements fixed by some open subgroup of $\mathrm{GT}^{\mathrm{tp}}_p$. This completes the proof of Proposition 3.18. □

Finally, we apply the theory of *resolution of nonsingularities* and *point-theoreticity* [cf., especially, Corollary 2.5, (i); Corollary 3.10], together with the theory of *metric-admissibility* developed in [CbTpIII], §3, to construct certain *arithmetic cuspidalizations* of the [$\mathfrak{Primes}$-]tempered fundamental groups of hyperbolic curves over $\overline{\mathbb{Q}}_p$ equipped with *"proj-metric structures"* [cf. Definition 3.19 below].

Definition 3.19. Let $\Sigma \subseteq \mathfrak{Primes}$ be a subset of cardinality ≥ 2 that contains p; K a mixed characteristic complete discrete valuation field of residue characteristic p; X a hyperbolic curve over $\overline{K}$. Write Π^{tp}_X for the Σ-tempered fundamental group of X. For each open subgroup $\Pi^* \subseteq \Pi^{\mathrm{tp}}_X$ of finite index, write

- $\mathcal{X}_{\Pi^*}$ for the compactified stable model over $\mathcal{O}_{\overline{K}}$ of the hyperbolic curve over $\overline{K}$ corresponding to the open subgroup $\Pi^* \subseteq \Pi_X^{\mathrm{tp}}$;
- Γ_{Π^*} for the dual graph associated to $(\mathcal{X}_{\Pi^*})_s$;
- $\underline{\mu}_{\Pi^*}$ for the metric structure on Γ_{Π^*} associated to $\mathcal{X}_{\Pi^*}$, considered *up to multiplication by a constant* $\in \mathbb{Q}^\times$ [cf. [CbTpIII], Definition 3.5, (iii)].

Then we shall refer to $\underline{\mu}_{\Pi^*}$ as the *proj-metric structure* on Γ_{Π^*}. We shall refer to the collection of data of proj-metric structures $\{\underline{\mu}_{\Pi^*}\}$ associated to the characteristic open subgroups $\{\Pi^* \subseteq \Pi_X^{\mathrm{tp}}\}$ of finite index as the *proj-metric structure* on Π_X^{tp}.

Theorem 3.20 (**Construction of certain arithmetic cuspidalizations of geometric tempered fundamental groups**). *Let K be a mixed characteristic complete discrete valuation field of residue characteristic p; X a hyperbolic curve over $\overline{K}$; $\widetilde{X}$ a universal pro-$\mathfrak{Primes}$ covering of X. Suppose that X satisfies $\mathfrak{Primes}$-RNS. Write Ω for the p-adic completion of $\overline{K}$. For $n \geq 2$ an integer, write X_n for the n-th configuration space associated to X; $\Pi_1 \stackrel{\mathrm{def}}{=} \Pi_X$; $\Pi_n \stackrel{\mathrm{def}}{=} \Pi_{X_n}$; $\Pi_{2/1}$ for the kernel of the natural surjection $\Pi_2 \twoheadrightarrow \Pi_1$ induced by the first projection $X_2 \to X$, where we regard Π_2 as a quotient of Π_n via the projection $X_n \to X_2$ to the first two factors; Π_1^{tp} for the $\mathfrak{Primes}$-tempered fundamental group of X;*

$$(\mathrm{Out}(\Pi_n) \supseteq)\ \mathrm{Out}(\Pi_n)^{\mathrm{tp}} \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{gF}}(\Pi_n) \times_{\mathrm{Out}(\Pi_1)} \mathrm{Out}(\Pi_1^{\mathrm{tp}}),$$

$$(\mathrm{Out}(\Pi_n) \supseteq)\ \mathrm{Out}^{\mathrm{gFC}}(\Pi_n) \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{gF}}(\Pi_n) \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)\ (\subseteq \mathrm{Out}(\Pi_1)),$$

$$(\mathrm{Out}(\Pi_n) \supseteq)\ \mathrm{Out}^{\mathrm{gFC}}(\Pi_n)^{\mathrm{M}} \stackrel{\mathrm{def}}{=} \mathrm{Out}^{\mathrm{gF}}(\Pi_n) \cap \mathrm{Out}^{\mathrm{FC}}(\Pi_n)^{\mathrm{M}}\ (\subseteq \mathrm{Out}(\Pi_1^{\mathrm{tp}}) \subseteq \mathrm{Out}(\Pi_1))$$

[cf. [HMM], Definition 2.1, (iv); [CbTpI], Theorem A, (i); [CbTpIII], Proposition 3.3, (iv); [CbTpIII], Definition 3.7, (i), (ii), (iii); [NodNon], Theorem B];

$$\mathbb{VE}(\Pi_1^{\mathrm{tp}}), \quad \Pi_1^{\mathrm{tp}}(\Omega)$$

for the respective sets "$\mathbb{VE}(\widetilde{X})$" and "$\widetilde{X}(\Omega)$" equipped with their natural actions by $\mathrm{Aut}(\Pi_1^{\mathrm{tp}})$ and Π_1 constructed in [the proof of] Corollary 2.5, (i), and Corollary 3.10 from the underlying topological group of Π_1^{tp}. Let $\tilde{x} \in \mathbb{VE}(\Pi_1^{\mathrm{tp}})$. Then the following hold:

(i) One may construct an "arithmetic cuspidalization" of Π_1^{tp} associated to $\tilde{x}$ from the data consisting of

- *the topological group Π_n equipped with the quotients $\Pi_n \twoheadrightarrow \Pi_2 \twoheadrightarrow \Pi_1$ and a topology [i.e., the tempered topology] on the subquotient $\Pi_n \twoheadrightarrow \Pi_2 \twoheadrightarrow \Pi_1 \supseteq \Pi_1^{\mathrm{tp}}$*

in a fashion that is functorial with respect to isomorphisms of this data [in the evident sense] as follows: Observe that the subgroup $\mathrm{Out}(\Pi_n)^{\mathrm{tp}} \subseteq \mathrm{Out}(\Pi_n)$ *may be constructed from the given data [cf. [HMM], Theorem A, (ii)]. Write*

$$ {}_nD^{\mathrm{tp}}_{\tilde{x}} \subseteq \Pi_1^{\mathrm{tp}} \overset{\mathrm{out}}{\rtimes} \mathrm{Out}(\Pi_n)^{\mathrm{tp}} \;=\; \mathrm{Aut}(\Pi_1^{\mathrm{tp}}) \times_{\mathrm{Out}(\Pi_1^{\mathrm{tp}})} \mathrm{Out}(\Pi_n)^{\mathrm{tp}} $$

[cf. [CbTpIII], Proposition 3.3, (i), (ii); [MT], Proposition 2.2, (ii)] for the stabilizer subgroup of $\tilde{x}$*. Note that there exists a natural exact sequence [that may be constructed from the given data]*

$$ 1 \longrightarrow \Pi_{2/1} \longrightarrow (\Pi_2 \times_{\Pi_1} \Pi_1^{\mathrm{tp}}) \overset{\mathrm{out}}{\rtimes} \mathrm{Out}(\Pi_n)^{\mathrm{tp}} \longrightarrow \Pi_1^{\mathrm{tp}} \overset{\mathrm{out}}{\rtimes} \mathrm{Out}(\Pi_n)^{\mathrm{tp}} \longrightarrow 1. $$

Thus, by pulling-back the above exact sequence via the inclusion ${}_nD^{\mathrm{tp}}_{\tilde{x}} \subseteq \Pi_1^{\mathrm{tp}} \overset{\mathrm{out}}{\rtimes} \mathrm{Out}(\Pi_n)^{\mathrm{tp}}$*, we obtain an exact sequence*

$$ 1 \longrightarrow \Pi_{2/1} \longrightarrow \Pi_{2/1} \overset{\mathrm{out}}{\rtimes} {}_nD^{\mathrm{tp}}_{\tilde{x}} \longrightarrow {}_nD^{\mathrm{tp}}_{\tilde{x}} \longrightarrow 1. $$

We shall refer to $\Pi_{2/1} \overset{\mathrm{out}}{\rtimes} {}_nD^{\mathrm{tp}}_{\tilde{x}}$ *as the [n-th]* **arithmetic cuspidalization** *of* Π_1^{tp} *associated to* x*.*

(ii) Write

$$ \Pi_1^{\mathrm{tp}}(\Omega)^{n\text{-alg}} \;\subseteq\; \Pi_1^{\mathrm{tp}}(\Omega) $$

for the subset of elements $\xi \in \Pi_1^{\mathrm{tp}}(\Omega)$ *whose* Π_1*-orbit* $\Pi_1 \cdot \xi$ *is stabilized by some open subgroup of* $\mathrm{Out}^{\mathrm{gFC}}(\Pi_n)^{\mathrm{M}}$ $(\subseteq \mathrm{Out}(\Pi_1^{\mathrm{tp}}))$ *[cf. Remark 3.20.1 below]. Suppose that* $\tilde{x}$ *arises from an element* $\in \Pi_1^{\mathrm{tp}}(\Omega)^{n\text{-alg}} \subseteq \Pi_1^{\mathrm{tp}}(\Omega)$*, which, by a slight abuse of notation, we shall also denote by* $\tilde{x}$*. Write* $X_x \overset{\mathrm{def}}{=} X_\Omega \setminus \{x\}$*, where* $x \in X(\Omega)$ *denotes the element determined by* $\tilde{x}$*. Then the [*$\mathfrak{Primes}$*-]tempered fundamental group*

$$ \Pi^{\mathrm{tp}}_{X_x} \;(\subseteq\; \Pi_{2/1}) $$

of X_x *[where we identify* $\Pi_{2/1}$ *with* Π_{X_x}*], together with the proj-metric structure on* $\Pi^{\mathrm{tp}}_{X_x}$*, may be reconstructed, in a purely combinatorial/group-theoretic way, from the following data*

- *the topological group* Π_n *equipped with the quotients* $\Pi_n \twoheadrightarrow \Pi_2 \twoheadrightarrow \Pi_1$ *and a topology [i.e., the tempered topology] and proj-metric structure on the subquotient* $\Pi_n \twoheadrightarrow \Pi_2 \twoheadrightarrow \Pi_1 \supseteq \Pi_1^{\mathrm{tp}}$*;*
- *the subgroup* $\mathrm{Out}^{\mathrm{gFC}}(\Pi_n) \subseteq \mathrm{Out}(\Pi_n)$ *[cf. Remark 3.20.2 below]*

in a fashion that is functorial with respect to isomorphisms of this data [in the evident sense].

Proof. Assertion (i) follows immediately from the various definitions involved. Next, we verify assertion (ii). Write

$$ {}_nD^{\mathrm{M}}_{\tilde{x}} \stackrel{\text{def}}{=} \left\{ \Pi_1^{\mathrm{tp}} \stackrel{\text{out}}{\rtimes} \mathrm{Out}^{\mathrm{gFC}}(\Pi_n)^{\mathrm{M}} \right\} \cap {}_nD^{\mathrm{tp}}_{\tilde{x}} \quad (\subseteq \Pi_1^{\mathrm{tp}} \stackrel{\text{out}}{\rtimes} \mathrm{Out}(\Pi_n)^{\mathrm{tp}}). $$

In particular, it follows immediately from assertion (i), together with the various definitions involved, that one may construct, from the given data, ${}_nD^{\mathrm{M}}_{\tilde{x}}$, together with the natural outer action of ${}_nD^{\mathrm{M}}_{\tilde{x}}$ on $\Pi_{2/1}$. Let $\Pi^{\dagger}_{2/1} \subseteq \Pi_{2/1}$ be an open subgroup that is normal in Π_2; l a prime number $\neq p$. Write $K \subseteq K^{\mathrm{tm}}$ $(\subseteq \overline{K})$ for the maximal tame extension of K; $G_K^{\mathrm{tm}} \stackrel{\text{def}}{=} \mathrm{Gal}(K^{\mathrm{tm}}/K^{\mathrm{ur}})$; $\Pi_{2/1} \twoheadrightarrow \Pi^{*}_{2/1}$ for the maximal almost pro-l quotient associated to [i.e., "with respect to"] $\Pi^{\dagger}_{2/1} \subseteq \Pi_{2/1}$ [cf. [CbTpIII], Definition 1.1]. Let us assume further that the quotient $\Pi_2 \twoheadrightarrow \Pi_2/\mathrm{Ker}(\Pi_{2/1} \twoheadrightarrow \Pi^{*}_{2/1})$ is *F-characteristic* [cf. [CbTpIII], Definition 2.1, (iii)]. Note that this implies that, relative to the *identification* of $\Pi_{2/1}$ with Π_{X_x} [cf. the statement of assertion (ii)], the natural $\Pi_{2/1}$-outer action of G_K on Π_2 [which is well-defined after possibly replacing K by a suitable finite extension field of K] induces a natural outer action of G_K on $\Pi^{*}_{2/1}$. Thus, in order to complete the proof of assertion (ii), it suffices, in light of the argument given in the proof of [CbTpIII], Theorem 3.9 [cf., especially, the *equivalence* stated in the final display of the proof of [CbTpIII], Theorem 3.9], to *reconstruct*, after possibly replacing K by a suitable finite extension field of K, *the image of* G_K^{tm} *in* $\mathrm{Out}(\Pi^{*}_{2/1})$ via the natural outer representation.

Next, we verify the following assertion:

> Claim 3.20.A: Let $(\widetilde{X} \to) Y \to X$ be a [connected] finite étale Galois covering over $\overline{K}$. Then there exist a compactified semistable model $\mathcal{Y}$ of Y over $\mathcal{O}_{\overline{K}}$ and a [connected] finite étale Galois covering $(\widetilde{X} \to) Z \to X$ over $\overline{K}$ that dominates $Y \to X$ and satisfies the following conditions:
>
> - The compactified stable model $\mathcal{Z}$ of Z over $\mathcal{O}_{\overline{K}}$ *dominates* $\mathcal{Y}$.
> - The component of the VE-chain $\tilde{x}$ corresponding to $\mathcal{Z}$ is an *irreducible component* of $\mathcal{Z}_s$ that maps to a *smooth closed point* $\in \mathcal{Y}_s$ that is not a cusp.

Indeed, since X satisfies $\mathfrak{Primes}$-RNS, Claim 3.20.A follows immediately from the fact that the VE-chain $\tilde{x}$ arises from an element $\in \Pi_1^{\mathrm{tp}}(\Omega)$ that is of *type* 1, hence *weakly verticial* [cf. Proposition 2.4, (v); Remark 3.1.1; Proposition 3.3, (ii); Proposition 3.4, (iii); the theory of *pointed stable curves*, as exposed in [Knud]].

Next, we observe that it follows from [CbTpIII], Proposition 2.3, (ii) [cf. conditions (a), (b), (c) below]; [CbTpIII], Corollary 2.10 [cf. condition (c) below], together with Claim 3.20.A [cf. condition (b) below], that, after possibly replacing $\Pi^{\dagger}_{2/1} \subseteq \Pi_{2/1}$ by a smaller open subgroup that satisfies the same conditions as $\Pi^{\dagger}_{2/1}$, there exist *F-characteristic SA-maximal almost pro-l quotients* $\Pi_2 \twoheadrightarrow \Pi_2^{*}$, $\Pi_2 \twoheadrightarrow \Pi_2^{**}$ satisfying the following conditions:

(a) The F-characteristic SA-maximal almost pro-l quotient $\Pi_2 \twoheadrightarrow \Pi_2^{**}$ *dominates* the F-characteristic SA-maximal almost pro-l quotient $\Pi_2 \twoheadrightarrow \Pi_2^{*}$. In particular, we obtain a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{2/1}^{**} & \longrightarrow & \Pi_2^{**} & \longrightarrow & \Pi_1^{**} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Pi_{2/1}^{*} & \longrightarrow & \Pi_2^{*} & \longrightarrow & \Pi_1^{*} & \longrightarrow & 1, \end{array}$$

where the quotients Π_1^{*}, Π_1^{**} of Π_1 induced by Π_2^{*}, Π_2^{**} are the *center-free* [cf. [CbTpIII], Proposition 1.7, (i)] maximal almost pro-l quotients of Π_1 associated to normal open subgroups of Π_1; the quotients $\Pi_{2/1}^{*}$, $\Pi_{2/1}^{**}$ of $\Pi_{2/1}$ induced by Π_2^{*} and Π_2^{**} are the *center-free* [cf. [CbTpIII], Proposition 1.7, (i)] maximal almost pro-l quotients of $\Pi_{2/1}$ associated to normal open subgroups of $\Pi_{2/1}$; the vertical arrows denote surjective homomorphisms.

(b) Fix a normal open subgroup $\Pi_Y \subseteq \Pi_1$ whose associated maximal almost pro-l quotient coincides with $\Pi_1 \twoheadrightarrow \Pi_1^{*}$. Then there exists a normal open subgroup $\Pi_Z \subseteq \Pi_1$ such that:

- It holds that $\Pi_Z \subseteq \Pi_Y$. [In particular, the maximal almost pro-l quotient associated to the normal open subgroup $\Pi_Z \subseteq \Pi_1$ dominates the maximal almost pro-l quotient $\Pi_1 \twoheadrightarrow \Pi_1^{*}$.]
- The maximal almost pro-l quotient $\Pi_1 \twoheadrightarrow \Pi_1^{**}$ *dominates* the maximal almost pro-l quotient associated to the normal open subgroup $\Pi_Z \subseteq \Pi_1$.
- Write $(\widetilde{X} \rightarrow)\ Y \rightarrow X$, $(\widetilde{X} \rightarrow)\ Z \rightarrow X$ for the respective [connected] finite étale Galois coverings over $\overline{K}$ associated to the normal open subgroups $\Pi_Y \subseteq \Pi_1$, $\Pi_Z \subseteq \Pi_1$. Then there exists a compactified semistable model $\mathcal{Y}$ of Y over $\mathcal{O}_{\overline{K}}$ such that the compactified stable model $\mathcal{Z}$ of Z over $\mathcal{O}_{\overline{K}}$ *dominates* $\mathcal{Y}$, and, moreover, the component of the VE-chain $\tilde{x}$ corresponding to $\mathcal{Z}$ is an *irreducible component* of $\mathcal{Z}_s$ that maps to a *smooth closed point* $\in \mathcal{Y}_s$ that is not a cusp.

(c) Every element $\in \mathrm{Out}^{\mathrm{FC}}(\Pi_2^{**} \twoheadrightarrow \Pi_2^{*}) \cap \mathrm{Ker}(\mathrm{Out}^{\mathrm{FC}}(\Pi_2^{**}) \rightarrow \mathrm{Out}(\Pi_1^{**}))$ [cf. [CbTpIII], Definition 2.1, (viii)] induces the trivial outer automorphism of Π_2^{*}.

Write

$$D_{\tilde{x}}^{**}$$

for the image of ${}_nD_{\tilde{x}}^{\mathrm{M}}$ $(\subseteq \Pi_1 \overset{\mathrm{out}}{\rtimes} \mathrm{Out}^{\mathrm{gFC}}(\Pi_n) \subseteq \Pi_1 \overset{\mathrm{out}}{\rtimes} \mathrm{Out}^{\mathrm{gFC}}(\Pi_2))$ [where the second inclusion follows from [NodNon], Theorem B] via the natural homomorphism $\Pi_1 \overset{\mathrm{out}}{\rtimes} \mathrm{Out}^{\mathrm{gFC}}(\Pi_2) \rightarrow \Pi_1^{**} \overset{\mathrm{out}}{\rtimes} \mathrm{Out}^{\mathrm{FC}}(\Pi_2^{**})$;

$$\rho^{**} : D_{\tilde{x}}^{**} \ \subseteq\ \Pi_1^{**} \overset{\mathrm{out}}{\rtimes} \mathrm{Out}^{\mathrm{FC}}(\Pi_2^{**} \twoheadrightarrow \Pi_2^{*}) \longrightarrow \mathrm{Out}(\Pi_1^{**})$$

for the natural composite homomorphism.

Next, we verify the following assertion:

Claim 3.20.B: There exists an open subgroup

$$D^{**} \subseteq D^{**}_{\tilde{x}}$$

such that every element $\in D^{**} \cap \mathrm{Ker}(\rho^{**})$ induces the trivial outer action on $\Pi^{*}_{2/1}$.

Indeed, observe that since Π^{**}_1 is center-free [cf. condition (a)], the natural homomorphism

$$D^{**}_{\tilde{x}} \longrightarrow \Pi^{**}_1 \overset{\mathrm{out}}{\rtimes} \mathrm{Out}(\Pi^{**}_1) = \mathrm{Aut}(\Pi^{**}_1)$$

induces a natural homomorphism

$$\phi : \mathrm{Ker}(\rho^{**}) \longrightarrow \Pi^{**}_1.$$

Thus, it follows immediately from the final portion of condition (b) [cf. also the proof of Claim 3.20.A; our assumption that X satisfies $\mathfrak{Primes}$-RNS; the portion of Proposition 3.5, (i), concerning the *weakly verticial* case], together with the various definitions involved [cf., especially, the definition of $D^{**}_{\tilde{x}}$], that the image of the natural composite homomorphism

$$\mathrm{Ker}(\rho^{**}) \overset{\phi}{\longrightarrow} \Pi^{**}_1 \twoheadrightarrow \Pi^{*}_1$$

is *finite*. In particular, there exists an open subgroup $D^{**} \subseteq D^{**}_{\tilde{x}}$ such that every element $\in D^{**} \cap \mathrm{Ker}(\rho^{**})$ induces the trivial automorphism of Π^{*}_1. On the other hand, this implies, in light of condition (c), that every element $\in D^{**} \cap \mathrm{Ker}(\rho^{**})$ induces the trivial outer automorphism of Π^{*}_2. Thus, since Π^{*}_1 is center-free [cf. condition (a)], we conclude that every element $\in D^{**} \cap \mathrm{Ker}(\rho^{**})$ induces the trivial outer automorphism of $\Pi^{*}_{2/1}$. This completes the proof of Claim 3.20.B.

Next, let us observe that, by applying the argument given in the proof of [CbTpIII], Theorem 3.9 [cf., especially, the *equivalence* stated in the final display of the proof of [CbTpIII], Theorem 3.9], together with condition (a), we conclude that, after possibly replacing K by a suitable finite extension field of K, one may reconstruct, from the *proj-metric structure* on Π^{tp}_1, the image I of G^{tm}_K in $\mathrm{Out}(\Pi^{**}_1)$ via the natural outer representation. On the other hand, observe that $\rho^{**}(D^{**})$ contains an open subgroup of I [cf. Remark 3.20.1 below]. Thus, since $D^{**} \cap \mathrm{Ker}(\rho^{**})$ induces the *trivial outer action* on $\Pi^{*}_{2/1}$ [cf. Claim 3.20.B], by considering the outer action of $D^{**} \cap (\rho^{**})^{-1}(I)$ on $\Pi^{*}_{2/1}$, we conclude that, after possibly replacing K by a suitable finite extension field of K, one may *reconstruct the image of* G^{tm}_K *in* $\mathrm{Out}(\Pi^{*}_{2/1})$, as desired. This completes the proof of assertion (ii), hence of Theorem 3.20. $\square$

Remark 3.20.1. Suppose that we are in the situation of Theorem 3.20, (ii). Then it follows immediately from the definitions [cf. also the natural homomorphism $G_K \to \mathrm{Out}^{\mathrm{gFC}}(\Pi_n)^{\mathrm{M}}$ [which is well-defined after possibly replacing K by a suitable finite extension field of K]; the proof of Theorem 3.11, (ii)] that the subset $\Pi_1^{\mathrm{tp}}(\Omega)^{n\text{-alg}} \subseteq \Pi_1^{\mathrm{tp}}(\Omega)$ is contained in the subset $\Pi_1^{\mathrm{tp}}(\overline{K}) \subseteq \Pi_1^{\mathrm{tp}}(\Omega)$ corresponding to the $\overline{K}$*-rational points* of the set "$\widetilde{X}(\Omega)$" constructed in Corollary 3.10. Moreover, it follows immediately from Proposition 3.18 [cf. also Theorem 3.16; [HMM], Corollaries B, C] that the *inclusion*

$$\Pi_1^{\mathrm{tp}}(\Omega)^{n\text{-alg}} \ \subseteq \ \Pi_1^{\mathrm{tp}}(\overline{K})$$

is in fact an *equality* in the case where $X = \mathbb{P}^1_{\overline{\mathbb{Q}}_p} \setminus \{0, 1, \infty\}$. It is not clear to the authors, however, at the time of writing of the present paper whether or not the inclusion of the above display is an equality in general.

Remark 3.20.2. Suppose that we are in the situation of Theorem 3.20, (ii). Then we observe that the data "$\mathrm{Out}^{\mathrm{gFC}}(\Pi_n) \subseteq \mathrm{Out}(\Pi_n)$" may be *omitted* from the list of data in the statement of Theorem 3.20, (ii), in either of the following situations [cf. [CbTpII], Theorem A, (ii); [HMM], Theorem A, (ii); [HMT], Corollary 2.2]:

- X is of *genus* 0.
- X is *affine*, and $n \geq 3$.
- X is *proper*, and $n \geq 4$.

It is not clear to the authors, however, at the time of writing of the present paper whether or not this data "$\mathrm{Out}^{\mathrm{gFC}}(\Pi_n) \subseteq \mathrm{Out}(\Pi_n)$" may be omitted in general.

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