Strong Indecomposability of the Outer Automorphism Groups of Nonabelian Free Profinite Groups

Arata Minamide and Shota Tsujimura

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Abstract

In the present paper, we prove that the outer automorphism groups of nonabelian [topologically finitely generated] free profinite groups satisfy strong indecomposability [i.e., the property that every open subgroup has no nontrivial direct product decomposition].

Key words and phrases: free profinite group; outer automorphism group; strong indecomposability; étale fundamental group; hyperbolic curve; anabelian geometry.

Contents

Introduction

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One of the most fundamental objects in the category of infinite nonabelian profinite groups is a free profinite group

$$F_r$$

of rank $r \geq 2$. With regard to the noncommutativity of F_r, the following fact is well-known [cf. [1], Proposition 8]:

Fact 1: The center of F_r is trivial.

Here, we note that the "discrete analogue" of Fact 1 — i.e., the center-freeness of free [discrete] groups of rank r — is relatively easy to prove by using explicit descriptions of elements as "words". On the other hand, since we do not have such nice descriptions of [general] elements $\in F_r$, one cannot apply any similar argument to the argument [using the notion of "words"] applied in the proof of the discrete analogue of Fact 1 to prove Fact 1. [In fact, in the proof of of [1], Proposition 8, cohomology theory of free pro-p groups is applied.] This situation suggests that,

> in general, the study of free profinite groups is much more difficult than the study of free [discrete] groups.

In the present paper, we study the group structure of the outer [continuous] automorphism group

$$\mathrm{Out}(F_r),$$

which is much more complicated than F_r. [We recall that $\mathrm{Out}(F_r)$ admits a natural structure of profinite group — cf. [14], Corollary 4.4.4.] Here, we note that although F_r is topologically finitely generated, $\mathrm{Out}(F_r)$ is not topologically finitely generated [cf. [14], Theorem 4.4.9] — this fact lies in stark contrast to the fact that the outer automorphism group of a free [discrete] group of rank r is finitely generated [cf. [7], §3.5, Theorem N1]. One of the most interesting aspects of the present paper is that,

> in our study of $\mathrm{Out}(F_r)$, we apply results in *anabelian geometry for hyperbolic curves over number fields/finite fields.*

In fact, by applying a result in anabelian geometry for hyperbolic curves over finitely generated fields of characteristic 0, the following fact — which concerns the noncommutativity of $\mathrm{Out}(F_r)$ — has been proved by Tamagawa [cf. [16], Theorem 7.4]:

Fact 2: The center of $\mathrm{Out}(F_r)$ is trivial.

In the following, let $\overline{\mathbb{Q}}$ be an algebraic closure of the field of rational numbers $\mathbb{Q}$; $K \subseteq \overline{\mathbb{Q}}$ a number field; Z a hyperbolic curve of genus 0 over K. Write $\mathfrak{Primes}$ for the set of prime numbers; $G_K \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{\mathbb{Q}}/K)$; $Z_{\overline{\mathbb{Q}}} \stackrel{\mathrm{def}}{=} Z \times_K \overline{\mathbb{Q}}$; $\Pi_{Z_{\overline{\mathbb{Q}}}}$ for the the étale fundamental group of $Z_{\overline{\mathbb{Q}}}$ [relative to a suitable choice of basepoint].

[Here, we recall that $\Pi_{Z_{\overline{\mathbb{Q}}}}$ is a nonabelian free profinite group!] For any nonempty subset $\Sigma \subseteq \mathfrak{Primes}$, write

$$\rho_Z^\Sigma : G_K \to \mathrm{Out}(\Pi_{Z_{\overline{\mathbb{Q}}}}^\Sigma)$$

for the natural pro-Σ outer Galois representation [cf. Definition 2.1] — where we write $(-)^\Sigma$ for the maximal pro-Σ quotient of $(-)$. We are now ready to state our main results [cf. Theorem 3.2; Corollary 3.5], which also concern the noncommutativity of $\mathrm{Out}(F_r)$:

Theorem A. *Let $\Sigma \subseteq \mathfrak{Primes}$ be a subset such that either $\sharp\Sigma = 1$ or $\sharp(\mathfrak{Primes} \setminus \Sigma)$ is finite [where we write $\sharp\Box$ for the cardinality of $\Box$];*

$$G \subseteq \mathrm{Out}(\Pi_{Z_{\overline{\mathbb{Q}}}}^\Sigma)$$

a closed subgroup such that

- *G contains an open subgroup of $\rho_Z^\Sigma(G_K)$;*
- *there exists a prime number $l \in \Sigma$ such that the image of G via the natural surjection [cf. [14], Proposition 4.5.4, (b)]*

$$\phi_l : \mathrm{Out}(\Pi_{Z_{\overline{\mathbb{Q}}}}^\Sigma) \twoheadrightarrow \mathrm{Out}(\Pi_{Z_{\overline{\mathbb{Q}}}}^{\{l\}})$$

  is slim — i.e., the center of every open subgroup of $\phi_l(G)$ is trivial.

Then G is strongly indecomposable — i.e., every open subgroup of G has no nontrivial direct product decomposition.

Corollary B. *Let $m \geq 2$ be an integer; $\Sigma \subseteq \mathfrak{Primes}$ a subset such that either $\sharp\Sigma = 1$ or $\sharp(\mathfrak{Primes} \setminus \Sigma)$ is finite. Then $\mathrm{Aut}(F_m^\Sigma)$ and $\mathrm{Out}(F_m^\Sigma)$ are strongly indecomposable.*

We note that for *arbitrary* nonempty subset $\Sigma \subseteq \mathfrak{Primes}$, F_m^Σ itself is strongly indecomposable [cf. [12], Proposition 3.2]. It is not clear to the authors at the time of writing whether or not the assumption on the cardinality of the subset $\Sigma \subseteq \mathfrak{Primes}$ in Theorem A and Corollary B can be dropped.

Finally, we remark that [in light of a well-known injectivity result of Belyi — cf. [5], Theorem C — together with Remark 3.1.1], as other immediate applications of Theorem A [in the case where one takes "Σ" (respectively, "Z") to be $\mathfrak{Primes}$ (respectively, the projective line minus $\{0, 1, \infty\}$ over K)], one obtains the following assertions:

(i) The [profinite] *Grothendieck-Teichmüller group* $\widehat{GT}$ [cf. [3]; [11], Remark 1.11.1] is strongly indecomposable.

(ii) The *absolute Galois groups of number fields* are strongly indecomposable.

Assertion (i) gives an affirmative answer to an open problem posed in a first author's previous work [cf. [8]]. Assertion (ii) is a special case of the fact that the absolute Galois groups of Hilbertian fields are strongly indecomposable — which follows immediately from a theorem of Haran-Jarden [cf. [4], Corollary 2.5].

Notations and conventions

Numbers: The notation $\mathfrak{Primes}$ will be used to denote the set of all prime numbers. The notation $\mathbb{Q}$ will be used to denote the field of rational numbers. The notation $\mathbb{Z}$ will be used to denote the ring of integers. The notation $\widehat{\mathbb{Z}}$ will be used to denote the profinite completion of the underlying additive group of $\mathbb{Z}$. The notation $\mathbb{Z}_{\geq 1}$ will be used to denote the set of positive integers. We shall refer to a finite extension field of $\mathbb{Q}$ as a *number field*. If p is a prime number, then the notation $\mathbb{Z}_p$ will be used to denote the ring of p-adic integers; the notation $\mathbb{F}_p$ will be used to denote the finite field of cardinality p.

Let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty subset. Then we shall say that an integer $n \in \mathbb{Z}_{\geq 1}$ is a Σ-integer if either $n = 1$ or every prime factor of n is contained in Σ.

Sets: Let S be a set. Then we shall write $\sharp S$ for the cardinality of S; $\mathrm{Sym}(S)$ for the group of automorphisms of S; $\mathfrak{S}_3 \stackrel{\mathrm{def}}{=} \mathrm{Sym}(\{1,2,3\})$.

Rings: Let A be a commutative ring. Then we shall write $\mathrm{char}(A)$ for the characteristic of A; $A^\times$ for the group of units of A. Let F be a perfect field; $\overline{F}$ an algebraic closure of F. Then we shall write $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}/F)$.

Schemes: Let S be a scheme. Then we shall write $\mathrm{Aut}(S)$ for the group of automorphisms of S. Let K be a field; $K \subseteq L$ a field extension; S a scheme over K. Then we shall write $S_L \stackrel{\mathrm{def}}{=} S \times_K L$; $\mathrm{Aut}_K(S)$ for the group of automorphisms of S over K; $\mathbb{P}^1_K$ for the projective line over K.

Profinite groups: Let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty subset; G a profinite group. Then we shall write G^Σ for the maximal pro-Σ quotient of G; G^{ab} for the abelianization of G [i.e., the quotient of G by the closure of the commutator subgroup of G]; $\mathrm{Aut}(G)$ for the group of automorphisms of G [in the category of profinite groups]; $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ for the group of inner automorphisms of G; $\mathrm{Out}(G) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)$. If p is a prime number, then we shall also write $G^p \stackrel{\mathrm{def}}{=} G^{\{p\}}$; $G^{(p)'} \stackrel{\mathrm{def}}{=} G^{\mathfrak{Primes}\setminus\{p\}}$.

Suppose that G is topologically finitely generated. Then G admits a basis of characteristic open subgroups [cf. [14], Proposition 2.5.1, (b)], which thus induces a profinite topology on the groups $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$. Let $H \subseteq G$ be a closed subgroup. Write $\mathrm{Aut}^H(G) \subseteq \mathrm{Aut}(G)$ for the subgroup of $\mathrm{Aut}(G)$ consisting of elements that induce automorphisms of H; $\mathrm{Inn}^H(G) \subseteq \mathrm{Aut}^H(G)$ for the image of H via the natural surjection $G \twoheadrightarrow \mathrm{Inn}(G)$. Let J be a profinite group. Then we shall refer to a continuous homomorphism $J \to \mathrm{Aut}^H(G)/\mathrm{Inn}^H(G)$ as an *H-outer action of J on G*.

Fundamental groups: Let S be a connected locally Noetherian scheme. Then we shall write Π_S for the étale fundamental group of S, relative to a suitable choice of basepoint. [Note that, for any perfect field F, $\Pi_{\mathrm{Spec}(F)} \cong G_F$.]

1 Preliminaries

In the present section, we recall some basic notions concerning profinite groups and hyperbolic curves and prove certain auxiliary results [cf. Lemmas 1.3, 1.7] which will be applied in §3.

First, we recall basic notions concerning profinite groups.

Definition 1.1 ([12], Notations and Conventions; [12], Definition 3.1) Let G be a profinite group; $H \subseteq G$ a closed subgroup of G.

(i) We shall write $Z_G(H)$ for the *centralizer* of H in G, i.e., the closed subgroup $\{g \in G \mid ghg^{-1} = h \text{ for any } h \in H\}$; $Z(G) \stackrel{\text{def}}{=} Z_G(G)$.

(ii) We shall say that G is *slim* if $Z_G(U) = \{1\}$ for every open subgroup U of G [or, equivalently, $Z(U) = \{1\}$ for every open subgroup U of G].

(iii) We shall say that G is *decomposable* if there exist nontrivial normal closed subgroups $H_1 \subseteq G$ and $H_2 \subseteq G$ such that $G = H_1 \times H_2$. We shall say that G is *indecomposable* if G is not decomposable. We shall say that G is *strongly indecomposable* if every open subgroup of G is indecomposable.

Remark 1.1.1 Let G be a slim profinite group. Then the following facts are well-known [cf. [12], §0; [8], Lemma 1.6]:

(i) Every finite normal [closed] subgroup of G is trivial.

(ii) Let $H \subseteq G$ be an open subgroup; $\alpha \in \mathrm{Aut}(G)$ an element. Suppose that α induces the identity automorphism on H. Then α is the identity automorphism [on G].

Next, we prove a certain group-theoretic lemma which will be applied in §3.

Lemma 1.2. *Let G be a profinite group; $\{G_i\}_{i \in I}$ a directed subset of the set of characteristic open subgroups of G — where $j \geq i \Leftrightarrow G_j \subseteq G_i$ — such that*

$$\bigcap_{i \in I} G_i = \{1\}.$$

Write $\phi_i : \mathrm{Out}(G) \to \mathrm{Out}(G/G_i)$ for the natural homomorphism. Then

$$\bigcap_{i \in I} \mathrm{Ker}(\phi_i) = \{1\}.$$

Proof. Let $\sigma \in \bigcap_{i \in I} \mathrm{Ker}(\phi_i)$ $(\subseteq \mathrm{Out}(G))$ be an element; $\tilde{\sigma} \in \mathrm{Aut}(G)$ a lifting of $\sigma \in \mathrm{Out}(G)$. For each $i \in I$, write $\tilde{\sigma}_i \in \mathrm{Aut}(G/G_i)$ for the automorphism induced by $\tilde{\sigma}$. Then since $\sigma \in \mathrm{Ker}(\phi_i)$, it holds that $\tilde{\sigma}_i$ is an inner automorphism.

Let $\gamma_i \in G/G_i$ be an element which determines the inner automorphism $\tilde{\sigma}_i$. Write

$$C_i \stackrel{\mathrm{def}}{=} \gamma_i \cdot Z(G/G_i) \subseteq G/G_i.$$

Here, we note that, if $i_1 \geq i_2$ ($i_1, i_2 \in I$), then the natural surjection $G/G_{i_1} \twoheadrightarrow G/G_{i_2}$ induces a map $C_{i_1} \to C_{i_2}$. Observe that since C_i ($i \in I$) is a finite nonempty set, the inverse limit $\varprojlim_{i \in I} C_i$ is nonempty. Let

$$\gamma \in \varprojlim_{i \in I} C_i \quad (\subseteq \varprojlim_{i \in I} G/G_i = G)$$

[cf. [14], Corollary 1.1.6] be an element. Then it follows immediately from the various definitions involved that $\tilde{\sigma}$ is an inner automorphism determined by γ. This completes the proof of Lemma 1.2. □

Lemma 1.3. *Let G be a topologically finitely generated profinite group; $S \subseteq \mathfrak{Primes}$ a finite subset. Then the natural homomorphism*

$$\mathrm{Out}(G) \longrightarrow \prod_{p \in \mathfrak{Primes} \setminus S} \mathrm{Out}(G^{(p)'})$$

is injective.

Proof. Since G is topologically finitely generated, there exists a directed subset $\{G_i\}_{i \in I}$ of the set of characteristic open subgroups of G — where $j \geq i \Leftrightarrow G_j \subseteq G_i$ — such that

$$\bigcap_{i \in I} G_i = \{1\}$$

[cf. [14], Proposition 2.5.1, (b)]. Fix such a family. For each $i \in I$, let $p_i \in \mathfrak{Primes} \setminus S$ be such that p_i does not divide the order of the finite group G/G_i. Then the natural surjection $G \twoheadrightarrow G/G_i$ factors through the natural surjection $G \twoheadrightarrow G^{(p_i)'}$. Thus, Lemma 1.3 follows immediately from Lemma 1.2. □

Next, we recall basic notions concerning hyperbolic curves.

Definition 1.4 Let k be a field; $\overline{k}$ an algebraic closure of k; X a smooth curve [i.e., a one-dimensional, smooth, separated, of finite type, and geometrically connected scheme] over k. Write $\overline{X}_{\overline{k}}$ for the smooth compactification of $X_{\overline{k}}$ over $\overline{k}$. Then we shall say that X is a *smooth curve of type* (g, r) over k if the genus of $X_{\overline{k}}$ is g, and the cardinality of the underlying set — we shall refer to as a *cusp of* $X_{\overline{k}}$ any element of this set — of $\overline{X}_{\overline{k}} \setminus X_{\overline{k}}$ is r. If X is a smooth curve of type (g, r) over k, and $2g - 2 + r > 0$, then we shall say that X is a *hyperbolic curve* over k.

Definition 1.5 Let k be an algebraically closed field; Z a hyperbolic curve over k; Q a profinite group; $q : \Pi_Z \twoheadrightarrow Q$ an epimorphism [in the category of profinite groups].

(i) Write $\overline{Z}$ for the smooth compactification of Z over k; K for the function field of $\overline{Z}$; $\widetilde{K}$ for the maximal Galois extension of K, in a fixed separable closure K^s, unramified over Z; $\widetilde{\overline{Z}}$ for the normalization of $\overline{Z}$ in $\widetilde{K}$; $\widetilde{S}$ for the inverse image of $S \stackrel{\mathrm{def}}{=} \overline{Z} \setminus Z$ in $\widetilde{\overline{Z}}$. [Note that we have a natural action of $\Pi_Z \cong \mathrm{Gal}(\widetilde{K}/K)$ on $\widetilde{\overline{Z}}$.] Then for each point $z \in S$, we shall refer to as a *cuspidal inertia subgroup of* Π_Z *associated to* z the stabilizer subgroup $\subseteq \Pi_Z$ of a point $\in \widetilde{S}$ lying over z; we shall refer to as a *cuspidal inertia subgroup of* Q *associated to* z the image of a cuspidal inertia subgroup of Π_Z associated to z via q.

(ii) Then we shall write

$$\mathrm{Out}^{\mathrm{C}}(Q) \subseteq \mathrm{Out}(Q) \quad (\text{respectively, } \mathrm{Out}^{|\mathrm{C}|}(Q) \subseteq \mathrm{Out}(Q))$$

for the subgroup of outer automorphisms of Q that induce automorphisms (respectively, the identity automorphism) on the set of the conjugacy classes of cuspidal inertia subgroups of Q.

We conclude the present section with a useful lemma [which follows from a similar argument to the argument applied in the proof of [17], Lemma 1.2] concerning outer actions on certain quotients of Π_Z.

Definition 1.6 Let G be a profinite group; Π a topologically finitely generated profinite group such that $Z(\Pi) = \{1\}$; $G \to \mathrm{Out}(\Pi)$ a continuous homomorphism. Then we shall write

$$\Pi \stackrel{\mathrm{out}}{\rtimes} G$$

for the profinite group obtained by pulling-back the continuous homomorphism $G \to \mathrm{Out}(\Pi)$ via the natural surjection $\mathrm{Aut}(\Pi) \twoheadrightarrow \mathrm{Out}(\Pi)$.

Lemma 1.7. *In the notation of Definition 1.5, suppose that Q is topologically finitely generated and slim. Let $J \subseteq \mathrm{Out}^{\mathrm{C}}(Q)$ be a closed subgroup; $V \subseteq Q$ an open subgroup. [In particular, $q^{-1}(V) \subseteq \Pi_Z$ may be naturally identified with the étale fundamental group of a hyperbolic curve over k.] Write $\phi_V : \mathrm{Aut}(V) \twoheadrightarrow \mathrm{Out}(V)$, $\phi_Q : \mathrm{Aut}(Q) \twoheadrightarrow \mathrm{Out}(Q)$ for the natural surjections. Then for any sufficiently small open subgroup $M \subseteq J$, there exist an outer action of M on V and an open injection $V \stackrel{\mathrm{out}}{\rtimes} M \hookrightarrow Q \stackrel{\mathrm{out}}{\rtimes} J$ such that*

(a) the outer action of M preserves and induces automorphisms on the set of the conjugacy classes of cuspidal inertia subgroups of V;

(b) *the outer action of M on V extends uniquely [cf. the slimness of Q] to a V-outer action on Q that is compatible with the outer action of J ($\supseteq M$) on Q; the injection $V \overset{\text{out}}{\rtimes} M \hookrightarrow Q \overset{\text{out}}{\rtimes} J$ is the injection determined by the inclusions $V \subseteq Q$ and $M \subseteq J$ and the V-outer actions on V and Q.*

Proof. Write $n \in \mathbb{Z}_{\geq 1}$ for the index of V in Q;

$$\mathrm{Aut}^{[V]}(Q) \subseteq \mathrm{Aut}(Q)$$

for the subgroup of $\mathrm{Aut}(Q)$ consisting of elements that induce automorphisms of V, and, moreover, induce automorphisms on the set of the conjugacy classes of cuspidal inertia subgroups of V. First, we note that since $\mathrm{Inn}^V(Q)$ is an open subgroup of $\mathrm{Inn}(Q)$ [cf. our assumption that V is an open subgroup of Q], there exists an open subgroup $M_1 \subseteq \mathrm{Aut}(Q)$ such that $M_1 \cap \mathrm{Inn}(Q) = \mathrm{Inn}^V(Q)$. Next, we consider the set

$$\mathcal{Q}_n \overset{\text{def}}{=} \{\text{all open subgroups of } Q \text{ of index } n\}.$$

Then since $\mathcal{Q}_n$ is a finite set [cf. our assumption that Q is topologically finitely generated; [14], Proposition 2.5.1, (a)], the kernel — which we denote by M_2 — of the natural homomorphism

$$\mathrm{Aut}(Q) \to \mathrm{Sym}(\mathcal{Q}_n)$$

is an open subgroup of $\mathrm{Aut}(Q)$. In particular, it follows immediately from the various definitions involved that

$$M_{\mathrm{Aut}} \overset{\text{def}}{=} M_1 \cap M_2 \cap \phi_Q^{-1}(J) \ (\subseteq \mathrm{Aut}(Q))$$

is an open subgroup of $\phi_Q^{-1}(J)$ satisfying the following conditions:

(i) $M_{\mathrm{Aut}} \cap \mathrm{Inn}(Q) \subseteq \mathrm{Inn}^V(Q)$;

(ii) $M_{\mathrm{Aut}} \subseteq \mathrm{Aut}^{[V]}(Q)$.

Write

$$M_V \overset{\text{def}}{=} \mathrm{Im}(M_{\mathrm{Aut}} \overset{\text{(ii)}}{\hookrightarrow} \mathrm{Aut}^{[V]}(Q) \to \mathrm{Aut}(V) \overset{\phi_V}{\twoheadrightarrow} \mathrm{Out}(V));$$

$$M \overset{\text{def}}{=} \mathrm{Im}(M_{\mathrm{Aut}} \hookrightarrow \mathrm{Aut}^{[V]}(Q) \hookrightarrow \mathrm{Aut}(Q) \overset{\phi_Q}{\twoheadrightarrow} \mathrm{Out}(Q));$$

$$M_{V,\mathrm{Aut}} \overset{\text{def}}{=} \mathrm{Im}(M_{\mathrm{Aut}} \hookrightarrow \mathrm{Aut}^{[V]}(Q) \twoheadrightarrow \mathrm{Aut}^{[V]}(Q)/\mathrm{Inn}^V(Q)).$$

Then we have a commutative diagram of profinite groups

$$(\star) \qquad \begin{array}{ccccc}
\mathrm{Aut}(V) & \longleftarrow & \mathrm{Aut}^{[V]}(Q) & \longrightarrow & \mathrm{Aut}(Q) \\
\downarrow{\scriptstyle \phi_V} & & \downarrow & & \downarrow{\scriptstyle \phi_Q} \\
\mathrm{Out}(V) & \longleftarrow & \mathrm{Aut}^{[V]}(Q)/\mathrm{Inn}^V(Q) & \longrightarrow & \mathrm{Out}(Q) \\
\uparrow & & \uparrow & & \uparrow \\
M_V & \longleftarrow & M_{V,\mathrm{Aut}} & \longrightarrow & M.
\end{array}$$

— where the horizontal arrows in the third line are surjective [cf. the definitions of M_V, M, $M_{V,\mathrm{Aut}}$]. Now we verify the following assertion:

> Claim 1.7.A: The horizontal arrows in the third line of the diagram ($\star$) are bijective.

Indeed, it follows immediately from the commutativity of the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & V & \longrightarrow & \mathrm{Aut}(V) & \longrightarrow & \mathrm{Out}(V) & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & V & \longrightarrow & \mathrm{Aut}^{[V]}(Q) & \longrightarrow & \mathrm{Aut}^{[V]}(Q)/\mathrm{Inn}^{V}(Q) & \longrightarrow & 1, \end{array}$$

together with the injectivity of $\mathrm{Aut}^{[V]}(Q) \to \mathrm{Aut}(V)$ [cf. our assumption that Q is slim; Remark 1.1.1], that $\mathrm{Aut}^{[V]}(Q)/\mathrm{Inn}^{V}(Q) \to \mathrm{Out}(V)$ is injective, hence that $M_{V,\mathrm{Aut}} \to M_V$ is bijective. The injectivity of $M_{V,\mathrm{Aut}} \to M$ follows immediately from the above condition (i). This completes the proof of Claim 1.7.A. In light of Claim 1.7.A, the assertions of Lemma 1.7 follow formally. $\square$

2 Computations of various Galois centralizers

In the present section, by applying highly nontrivial "Grothendieck Conjecture-type results" [cf. [9], Theorem A; [15], Theorem D] in anabelian geometry, we compute various Galois centralizers. These computations will be applied in §3.

Definition 2.1 Let k be a field; $\overline{k}$ an algebraic closure of k; Z a hyperbolic curve over k. Then we have an exact sequence of profinite groups

$$1 \longrightarrow \Pi_{Z_{\overline{k}}} \longrightarrow \Pi_Z \longrightarrow G_k \longrightarrow 1.$$

We shall write $\rho_Z : G_k \to \mathrm{Out}(\Pi_{Z_{\overline{k}}})$ for the outer representation determined by the above exact sequence. Let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty subset. Then we shall write

$$\rho_Z^{\Sigma} : G_k \to \mathrm{Out}(\Pi_{Z_{\overline{k}}}^{\Sigma})$$

for the outer representation induced by ρ_Z;

$$\Pi_Z^{[\Sigma]} \stackrel{\mathrm{def}}{=} \Pi_Z/\mathrm{Ker}(\Pi_{Z_{\overline{k}}} \twoheadrightarrow \Pi_{Z_{\overline{k}}}^{\Sigma}).$$

Let p be a prime number. If $\Sigma = \{p\}$, then we shall also write $\rho_Z^p \stackrel{\mathrm{def}}{=} \rho_Z^{\Sigma}$.

Lemma 2.2. *Let l be a prime number; $\Sigma \subseteq \mathfrak{Primes}$ a subset such that $l \in \Sigma$; $n \in \mathbb{Z}_{\geq 1}$ a Σ-integer; $K \subseteq \overline{\mathbb{Q}}$ a number field; $Z \subseteq \mathbb{P}^1_K \backslash \{0, 1, \infty\}$ an open subscheme obtained by forming the complement of a finite subset of K-rational points of $\mathbb{P}^1_K \backslash \{0, 1, \infty\}$. [In particular, Z is a hyperbolic curve of genus 0 over K.] Write $(\mathbb{P}^1_{\overline{\mathbb{Q}}} \supseteq)\ Y_{\overline{\mathbb{Q}}} \to Z_{\overline{\mathbb{Q}}}\ (\subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}})$ for the finite étale Galois covering of $Z_{\overline{\mathbb{Q}}}$ of degree n determined by $t \mapsto t^n$;*

$$Q \stackrel{\mathrm{def}}{=} \Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}} / \mathrm{Ker}(\Pi^{\Sigma}_{Y_{\overline{\mathbb{Q}}}} \twoheadrightarrow \Pi^{l}_{Y_{\overline{\mathbb{Q}}}}).$$

Then the following hold:

(i) We have a natural homomorphism

$$\mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}) \to \mathrm{Out}^{|\mathrm{C}|}(Q).$$

(ii) Write ρ for the composite of natural homomorphisms

$$G_K \stackrel{\rho^{\Sigma}_Z}{\to} \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}) \to \mathrm{Out}^{|\mathrm{C}|}(Q)$$

[cf. our assumption that all cusps of Z are K-rational]. Then it holds that

$$Z_{\mathrm{Out}^{|\mathrm{C}|}(Q)}(\mathrm{Im}(\rho)) = \{1\}.$$

(iii) Let

$$G \subseteq \mathrm{Out}^{|\mathrm{C}|}(Q)$$

be a closed subgroup such that G contains an open subgroup of $\mathrm{Im}(\rho)$. Then G is slim.

Proof. We begin by observing that

> the normal open subgroup $\Pi^{\Sigma}_{Y_{\overline{\mathbb{Q}}}} \subseteq \Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}$ [determined by the finite étale Galois covering $Y_{\overline{\mathbb{Q}}} \to Z_{\overline{\mathbb{Q}}}$] may be characterized as the normal open subgroup topologically generated by the cuspidal inertia subgroups of $\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}$ that is not associated to the cusps 0, ∞, and the [unique] closed subgroups of the cuspidal inertia subgroups of $\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}$ associated to the cusps 0, ∞, of index n.

Then assertion (i) follows immediately from this observation, together with the various definitions involved.

Next, we verify assertion (ii). Let $\sigma \in Z_{\mathrm{Out}^{|\mathrm{C}|}(Q)}(\mathrm{Im}(\rho))$ be an element. Then it follows immediately from the above observation that any lifting $\in \mathrm{Aut}(Q)$ of σ induces an automorphism of $\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}$. Let $\tilde{\sigma} \in \mathrm{Aut}(Q)$ be a lifting of σ such that the automorphism $\tilde{\sigma}|_{\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}} \in \mathrm{Aut}(\Pi^{l}_{Y_{\overline{\mathbb{Q}}}})$ induced by $\tilde{\sigma}$ preserves the $\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}$-conjugacy class of cuspidal inertia subgroups of $\Pi^{l}_{Y_{\overline{\mathbb{Q}}}}$ associated to the cusp 1. Here, we note that since $\tilde{\sigma}$ preserves the Q-conjugacy class of cuspidal inertia subgroups of Q

associated to the cusp 0 (respectively, ∞), and the finite étale Galois covering $Y_{\overline{\mathbb{Q}}} \to Z_{\overline{\mathbb{Q}}}$ is totally ramified over the cusp 0 (respectively, ∞), it holds that $\tilde{\sigma}|_{\Pi^l_{Y_{\overline{\mathbb{Q}}}}}$ preserves the $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$-conjugacy class of cuspidal inertia subgroups of $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$ associated to the cusp 0 (respectively, ∞). Write

$$\sigma_Y : \Pi^l_{Y_{\overline{\mathbb{Q}}}} \xrightarrow{\sim} \Pi^l_{Y_{\overline{\mathbb{Q}}}}$$

for the outer automorphism determined by $\tilde{\sigma}|_{\Pi^l_{Y_{\overline{\mathbb{Q}}}}} \in \mathrm{Aut}(\Pi^l_{Y_{\overline{\mathbb{Q}}}})$. Observe that since the outer action of G_K, together with σ_Y, on $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$ preserves the $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$-conjugacy class of cuspidal inertia subgroups of $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$ associated to the cusp 1, it follows from our assumption that $\sigma \in Z_{\mathrm{Out}^{|C|}(Q)}(\mathrm{Im}(\rho))$ that σ_Y commutes with the outer action of G_K on $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$. Then it follows from the Grothendieck Conjecture [cf. [9], Theorem A] that σ_Y arises from a unique isomorphism $f : Y_{\overline{\mathbb{Q}}} \xrightarrow{\sim} Y_{\overline{\mathbb{Q}}}$ of schemes over $\overline{\mathbb{Q}}$. Note that since $\tilde{\sigma}|_{\Pi^l_{Y_{\overline{\mathbb{Q}}}}}$ induces the identity automorphism on the set of the $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$-conjugacy classes of cuspidal inertia subgroups of $\Pi^l_{Y_{\overline{\mathbb{Q}}}}$ associated to the cusps 0, 1, ∞, it holds that f induces the identity automorphism on the subset $\{0, 1, \infty\} \subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}}$. In particular, we conclude that f is the identity automorphism, hence that σ_Y is the identity outer automorphism. Recall that the automorphism $\tilde{\sigma}|_{\Pi^l_{Y_{\overline{\mathbb{Q}}}}} \in \mathrm{Aut}(\Pi^l_{Y_{\overline{\mathbb{Q}}}})$ is the restriction of $\tilde{\sigma} \in \mathrm{Aut}(Q)$. Thus, since Q is slim [cf. [12], Proposition 1.4], it follows from Remark 1.1.1, that $\tilde{\sigma}$ is an inner automorphism, hence that σ is the identity outer automorphism. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). Since every open subgroup of G contains an open subgroup of $\mathrm{Im}(\rho)$, to verify assertion (iii), it suffices to show that $Z(G) = \{1\}$. But this follows immediately from assertion (ii). This completes the proof of assertion (iii), hence of Lemma 2.2. □

The following notion plays important roles in the present paper.

Definition 2.3 Let l be a prime number. We shall say that a profinite group G is *almost* $\mathbb{Z}_l$ if there exists an open subgroup $H \subseteq G$ such that H is isomorphic to $\mathbb{Z}_l$.

Lemma 2.4. *Let p be a prime number; $\Sigma \subseteq \mathfrak{Primes}$ a nonempty subset such that $p \notin \Sigma$; k a finite field of characteristic p. In the notation of Definition 2.1, suppose that Z is a hyperbolic curve of genus 0 over k such that all cusps of Z are k-rational. Write $\rho \overset{\mathrm{def}}{=} \rho^{\Sigma}_{Z}$. Then the following hold:*

(i) Suppose that $\sharp(\mathfrak{Primes} \setminus \Sigma)$ is finite. Then the natural homomorphism $\mathrm{Aut}(Z_{\overline{k}}) \to \mathrm{Out}(\Pi^{\Sigma}_{Z_{\overline{k}}})$ determines an isomorphism

$$\mathrm{Aut}(Z_{\overline{k}}) \xrightarrow{\sim} Z_{\mathrm{Out}(\Pi^{\Sigma}_{Z_{\overline{k}}})}(\rho(G_k)).$$

(ii) Suppose that either $\sharp\Sigma = 1$ or $\sharp(\mathfrak{Primes} \setminus \Sigma)$ is finite. Then, if we write $\chi_\Sigma : \mathrm{Out}^{|\mathrm{C}|}(\Pi^\Sigma_{Z_{\overline{k}}}) \to (\widehat{\mathbb{Z}}^\Sigma)^\times$ for the pro-Σ cyclotomic character [which is obtained by considering the actions on the cuspidal inertia subgroups of $\Pi^\Sigma_{Z_{\overline{k}}}$], then the natural composite

$$Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi^\Sigma_{Z_{\overline{k}}})}(\rho(G_k)) \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^\Sigma_{Z_{\overline{k}}}) \xrightarrow{\chi_\Sigma} (\widehat{\mathbb{Z}}^\Sigma)^\times$$

is injective.

(iii) Let l be a prime number $\neq p$. Then $Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{k}}})}(\rho(G_k))$ is almost $\mathbb{Z}_l$.

Proof. First, we verify assertion (i). Write $\mathrm{Out}_{G_k}(\Pi^{[\Sigma]}_Z)$ for the group of $\Pi^\Sigma_{Z_{\overline{k}}}$-outer automorphisms of $\Pi^{[\Sigma]}_Z$ that lie over G_k [cf. Definition 2.1]. Then since $\Pi^\Sigma_{Z_{\overline{k}}}$ is center-free [cf. [12], Proposition 1.4], it is well-known that the natural homomorphism

$$\mathrm{Out}_{G_k}(\Pi^{[\Sigma]}_Z) \to Z_{\mathrm{Out}(\Pi^\Sigma_{Z_{\overline{k}}})}(\rho(G_k))$$

is an isomorphism [cf. [16], Lemma 7.1]. On the other hand, since G_k is abelian, it follows immediately from [15], Theorem D, together with the definition of $\mathrm{Out}_{G_k}(\Pi^{[\Sigma]}_Z)$, that

$$\mathrm{Aut}(Z_{\overline{k}}/Z) \xrightarrow{\sim} \mathrm{Out}_{G_k}(\Pi^{[\Sigma]}_Z),$$

where $\mathrm{Aut}(Z_{\overline{k}}/Z) \subseteq \mathrm{Aut}(Z_{\overline{k}})$ denotes the subgroup consisting of automorphisms of $Z_{\overline{k}}$ that induce automorphisms of Z compatible with the natural morphism $Z_{\overline{k}} \to Z$.

Next, we verify the following assertion:

Claim 2.4.A: The inclusion $\mathrm{Aut}(Z_{\overline{k}}/Z) \subseteq \mathrm{Aut}(Z_{\overline{k}})$ is bijective.

Indeed, let $\alpha \in \mathrm{Aut}(Z_{\overline{k}})$ be an element; $\sigma \in G_k$ $(\hookrightarrow \mathrm{Aut}(Z_{\overline{k}}))$. Then since G_k is abelian, it follows that

$$\gamma \stackrel{\mathrm{def}}{=} \sigma \circ \alpha \circ \sigma^{-1} \circ \alpha^{-1} \in \mathrm{Aut}_{\overline{k}}(Z_{\overline{k}}).$$

Next, we note that γ induces the identity automorphism on the set of cusps of $Z_{\overline{k}}$. Thus, we conclude that $\gamma = 1$, hence that α induces a unique automorphism $\in \mathrm{Aut}(Z)$ compatible with the natural morphism $Z_{\overline{k}} \to Z$. This completes the proof of Claim 2.4.A.

Thus, by applying Claim 2.4.A, we obtain a natural isomorphism

$$\phi : \mathrm{Aut}(Z_{\overline{k}}) \xrightarrow{\sim} Z_{\mathrm{Out}(\Pi^\Sigma_{Z_{\overline{k}}})}(\rho(G_k)).$$

This completes the proof of assertion (i).

Next, we verify assertion (ii). If $\sharp\Sigma = 1$, then the desired conclusion follows immediately from the latter half of the proof of [13], Proposition 2.2.4. Thus, we may assume without loss of generality that $\sharp(\mathfrak{Primes} \setminus \Sigma)$ is finite. Write

$\mathrm{Aut}^{|\mathrm{C}|}(Z_{\overline{k}}) \subseteq \mathrm{Aut}(Z_{\overline{k}})$ for the subgroup of automorphisms of $Z_{\overline{k}}$ that induce the identity automorphisms on the set of cusps of $Z_{\overline{k}}$. Then ϕ induces a composite

$$\mathrm{Aut}^{|\mathrm{C}|}(Z_{\overline{k}}) \xrightarrow{\sim} Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{k}}})}(\rho(G_k)) \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{k}}}) \xrightarrow{\chi_{\Sigma}} (\widehat{\mathbb{Z}}^{\Sigma})^{\times}.$$

Observe that this composite factors as the composite of the natural injection $\mathrm{Aut}^{|\mathrm{C}|}(Z_{\overline{k}}) \hookrightarrow G_{\mathbb{F}_p}$ with the pro-Σ cyclotomic character $G_{\mathbb{F}_p} \to (\widehat{\mathbb{Z}}^{\Sigma})^{\times}$ [which is injective — cf. [2], Théorème 1]. Thus, we conclude that the natural composite

$$Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{k}}})}(\rho(G_k)) \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{k}}}) \xrightarrow{\chi_{\Sigma}} (\widehat{\mathbb{Z}}^{\Sigma})^{\times}$$

is injective. This completes the proof of assertion (ii).

Finally, we verify assertion (iii). We begin by observing that since the image of the l-adic cyclotomic character $G_k \to \mathbb{Z}_l^{\times}$ is infinite, it follows from [10], Corollary 2.7, (i), that

$$Z_{\mathrm{Out}(\Pi^{l}_{Z_{\overline{k}}})}(\rho(G_k)) = Z_{\mathrm{Out}^{\mathrm{C}}(\Pi^{l}_{Z_{\overline{k}}})}(\rho(G_k)).$$

Moreover, we observe that since $\mathrm{Out}^{|\mathrm{C}|}(\Pi^{l}_{Z_{\overline{k}}})$ is open in $\mathrm{Out}^{\mathrm{C}}(\Pi^{l}_{Z_{\overline{k}}})$, it holds that $Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi^{l}_{Z_{\overline{k}}})}(\rho(G_k))$ is open in $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi^{l}_{Z_{\overline{k}}})}(\rho(G_k))$. Thus, to verify assertion (iii), it suffices to show that $Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi^{l}_{Z_{\overline{k}}})}(\rho(G_k))$ is almost $\mathbb{Z}_l$. On the other hand, since $\rho(G_k)$ is an infinite abelian group [cf. [8], Lemma 4.2, (iv)], we conclude immediately from assertion (ii) that $Z_{\mathrm{Out}^{|\mathrm{C}|}(\Pi^{l}_{Z_{\overline{k}}})}(\rho(G_k))$ is almost $\mathbb{Z}_l$, as desired. This completes the proof of assertion (iii), hence of Lemma 2.4. □

Remark 2.4.1 It is natural to pose the following question:

> Question: In the notation of Lemma 2.4, (i), (ii), can the assumptions on the cardinality of the subset $\Sigma \subseteq \mathfrak{Primes}$ be dropped?

However, at the time of writing the present paper, the authors do not know whether the answer to this question is affirmative or not.

Lemma 2.5. *Let l be a prime number; $K \subseteq \overline{\mathbb{Q}}$ a number field. In the notation of Definition 2.1, suppose that $k = K$, and Z is a hyperbolic curve over K. Then every open subgroup $U \subseteq \mathrm{Im}(\rho^{l}_{Z})$ is nonabelian.*

Proof. Let us recall that, since the image of the l-adic cyclotomic character $G_K \to \mathbb{Z}_l^{\times}$ is infinite, it holds that $\mathrm{Im}(\rho^{l}_{Z})$ is infinite [cf. [8], Lemma 4.2, (iv)], hence that U is infinite. Write $K' \subseteq \overline{\mathbb{Q}}$ for the finite extension of K determined by U. Suppose that U is abelian. Then since $U \subseteq Z_{\mathrm{Out}(\Pi^{l}_{Z_{\overline{\mathbb{Q}}}})}(U)$, the centralizer $Z_{\mathrm{Out}(\Pi^{l}_{Z_{\overline{\mathbb{Q}}}})}(U)$ is infinite. However, since $\mathrm{Aut}_{K'}(Z_{K'})$ is finite, this contradicts the Grothendieck Conjecture for hyperbolic curves over number fields [cf. [9], Theorem A]. Thus, we conclude that U is nonabelian. This completes the proof of Lemma 2.5. □

3 Strong indecomposability of the outer automorphism groups of nonabelian free profinite groups

In the present section, by applying the results obtained in §2, §3, we prove the main results [cf. Theorem 3.2; Corollary 3.5] of the present paper.

Lemma 3.1. *Let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty subset; $K \subseteq \overline{\mathbb{Q}}$ a number field. In the notation of Definition 2.1, suppose that $k = K$, and Z is a hyperbolic curve of genus 0 over K. Let*

$$G \subseteq \mathrm{Out}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$$

be a closed subgroup such that

- *G contains an open subgroup of $\rho^{\Sigma}_{Z}(G_K)$;*
- *there exists a prime number $l \in \Sigma$ such that the image of G via the natural surjection [cf. [14], Proposition 4.5.4, (b)]*

$$\phi_l : \mathrm{Out}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}) \twoheadrightarrow \mathrm{Out}(\Pi^{l}_{Z_{\overline{\mathbb{Q}}}})$$

  is slim;
- *there exist normal closed subgroups $G_1 \subseteq G$ and $G_2 \subseteq G$ such that $G = G_1 \times G_2$.*

Then

(a) $\phi_l(G_1) = \{1\}$ and $G_1 \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$, or

(b) $\phi_l(G_2) = \{1\}$ and $G_2 \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$

holds.

Proof. First, by replacing K by a finite extension of K, we may assume without loss of generality that $\rho^{\Sigma}_{Z}(G_K) \subseteq G$, and all cusps of Z are K-rational. Let $\mathfrak{p}$ be a maximal ideal of the ring of integers of K such that

- the characteristic of the residue field at $\mathfrak{p}$ is not equal to l, and
- Z has good reduction at $\mathfrak{p}$;

$F \in G_K$ a lifting of the Frobenius element at $\mathfrak{p}$. We shall write,

- for each $i = 1, 2$, $\mathrm{pr}_i : G \twoheadrightarrow G_i$ for the natural projection;
- $J \subseteq G_K$ for the closed subgroup topologically generated by F, where we note that J is isomorphic to $\widehat{\mathbb{Z}}$;

- $I \overset{\text{def}}{=} \rho_Z^{\Sigma}(J) \subseteq G$;
- $I_1 \overset{\text{def}}{=} \mathrm{pr}_1(I) \times \{1\} \subseteq G_1 \times G_2 = G$, $I_2 \overset{\text{def}}{=} \{1\} \times \mathrm{pr}_2(I) \subseteq G_1 \times G_2 = G$.

Here, we note that, since I is abelian, it holds that

$$I \subseteq I_1 \times I_2 \subseteq Z_G(I),$$

hence that

$$\phi_l(I) \subseteq \phi_l(I_1) \cdot \phi_l(I_2) \subseteq Z_{\phi_l(G)}(\phi_l(I)) \subseteq Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\phi_l(I)).$$

Moreover, we note that since Z has good reduction at $\mathfrak{p}$, it follows immediately from the theory of specialization isomorphism, that

- $Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\phi_l(I))$ is almost $\mathbb{Z}_l$ [cf. Lemma 2.4, (iii)], and
- $\phi_l(I)$ is infinite [cf. [8], Lemma 4.2, (iv)].

In particular, it holds that $\phi_l(I_1)$ is infinite, or $\phi_l(I_2)$ is infinite. We may assume without loss of generality that

$$\phi_l(I_2) \text{ is infinite.}$$

Observe that since $Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\phi_l(I))$ is almost $\mathbb{Z}_l$, it holds that $\phi_l(I_2) \cap \phi_l(I) \subseteq \phi_l(I)$ is an open subgroup. Then since $G_1 \subseteq Z_G(I_2)$, there exists an open subgroup ${}^{\dagger}I \subseteq I$ such that

$$\phi_l(G_1) \subseteq Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\phi_l({}^{\dagger}I)).$$

Now suppose that $\phi_l(G_1)$ is infinite. Then since $\phi_l(I) \subseteq Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\phi_l({}^{\dagger}I))$, and, moreover, $Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\phi_l({}^{\dagger}I))$ is almost $\mathbb{Z}_l$ [cf. Lemma 2.4, (iii)], it holds that $\phi_l(G_1) \cap \phi_l(I) \subseteq \phi_l(I)$ is an open subgroup. On the other hand, since $G_2 \subseteq Z_G(G_1)$, there exists an open subgroup ${}^{\ddagger}I \subseteq {}^{\dagger}I$ ($\subseteq I$) such that

$$\phi_l(G_2) \subseteq Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\phi_l({}^{\ddagger}I)).$$

In summary, we have

$$\rho^l_Z(G_K) = \phi_l(\rho^{\Sigma}_Z(G_K)) \subseteq \phi_l(G) = \phi_l(G_1) \cdot \phi_l(G_2) \subseteq Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\phi_l({}^{\ddagger}I)).$$

Then since $Z_{\mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})}(\phi_l({}^{\ddagger}I))$ is almost $\mathbb{Z}_l$ [cf. Lemma 2.4, (iii)], it follows that there exists an open subgroup $U \subseteq \rho^l_Z(G_K)$ such that U is abelian, a contradiction [cf. Lemma 2.5]. Thus, we conclude that $\phi_l(G_1)$ is finite. Therefore, since $\phi_l(G_1) \subseteq \phi_l(G)$ is a finite normal subgroup, it follows from our assumption that $\phi_l(G)$ is slim that $\phi_l(G_1) = \{1\}$ [cf. Remark 1.1.1].

Finally, we verify the inclusion $G_1 \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$. Write $\chi_l : \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}) \to \mathbb{Z}_l^{\times}$ for the l-adic cyclotomic character [which is obtained by considering the actions on the cuspidal inertia subgroups of $\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}$]. Then since $\chi_l(I)$ is infinite, it follows from [10], Corollary 2.7, (i), that

$$I_1 \times I_2 \subseteq Z_{\mathrm{Out}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})}(I) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}).$$

In particular, since $\phi_l(I_1) \subseteq \phi_l(G_1) = \{1\}$, we have $I_1 \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$ [cf. [10], Proposition 1.2, (i)]. Thus, we obtain an inclusion

$$I \subseteq I_1 \times I_2^{|\mathrm{C}|} \ (\subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})),$$

where we write $I_2^{|\mathrm{C}|} \stackrel{\mathrm{def}}{=} I_2 \cap \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$. Here, we observe that since χ_l factors through [the restriction of] ϕ_l [on $\mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$], it holds that

$$\chi_l(I) \subseteq \chi_l(I_2^{|\mathrm{C}|}) \ (\subseteq \mathbb{Z}_l^{\times}),$$

hence that $\chi_l(I_2^{|\mathrm{C}|})$ is infinite. Therefore, we conclude from [10], Corollary 2.7, (i), that

$$G_1 \subseteq Z_{\mathrm{Out}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})}(I_2^{|\mathrm{C}|}) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}).$$

In particular, since $\phi_l(G_1) = \{1\}$, we have $G_1 \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$ [cf. [10], Proposition 1.2, (i)]. This completes the proof of Lemma 3.1. $\square$

Remark 3.1.1 In the notation of Lemma 3.1, suppose that

$$G \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}).$$

Then the second assumption on G [concerning the slimness of $\phi_l(G)$] follows automatically from the first assumption on G. Indeed, to verify the slimness of $\phi_l(G)$, by replacing K by a finite extension of K, we may assume without loss of generality that Z is an open subscheme of $\mathbb{P}^1_K \backslash \{0, 1, \infty\}$ obtained by forming the complement of a finite subset of K-rational points of $\mathbb{P}^1_K \backslash \{0, 1, \infty\}$. Then the slimness of $\phi_l(G)$ follows immediately from Lemma 2.2, (iii).

Theorem 3.2. *Let $\Sigma \subseteq \mathfrak{Primes}$ be a subset such that either $\sharp\Sigma = 1$ or $\sharp(\mathfrak{Primes} \backslash \Sigma)$ is finite; $K \subseteq \overline{\mathbb{Q}}$ a number field. In the notation of Definition 2.1, suppose that $k = K$, and Z is a hyperbolic curve of genus 0 over K. Let*

$$G \subseteq \mathrm{Out}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})$$

be a closed subgroup such that

- *G contains an open subgroup of $\rho_Z^\Sigma(G_K)$;*

- *there exists a prime number $l \in \Sigma$ such that the image of G via the natural surjection [cf. [14], Proposition 4.5.4, (b)]*

$$\phi_l : \mathrm{Out}(\Pi^\Sigma_{Z_{\overline{\mathbb{Q}}}}) \twoheadrightarrow \mathrm{Out}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})$$

is slim.

Then G is strongly indecomposable.

Proof. We begin by observing that every open subgroup Γ of G contains an open subgroup of $\rho_Z^\Sigma(G_K)$, and, moreover, $\phi_l(\Gamma)$ is slim. In particular, to verify Theorem 3.2, it suffices to prove that G is indecomposable. If $\sharp\Sigma = 1$, then the indecomposability of G follows immediately from Lemma 3.1. Thus, we may assume without loss of generality that

$$\sharp(\mathfrak{Primes} \setminus \Sigma) \text{ is finite.}$$

Next, by replacing K by a finite extension of K, we may assume without loss of generality that $\rho_Z^\Sigma(G_K) \subseteq G$, and all cusps of Z are K-rational. Moreover, we may assume without loss of generality that Z is an open subscheme of $\mathbb{P}^1_K \backslash \{0, 1, \infty\}$ obtained by forming the complement of a finite subset of K-rational points of $\mathbb{P}^1_K \backslash \{0, 1, \infty\}$.

Suppose that there exist normal closed subgroups $G_1 \subseteq G$ and $G_2 \subseteq G$ such that

$$G = G_1 \times G_2.$$

We shall write,

- for each Σ-integer $n \in \mathbb{Z}_{\geq 1}$, $(\mathbb{P}^1_{\overline{\mathbb{Q}}} \supseteq)\ {}^nY_{\overline{\mathbb{Q}}} \to Z_{\overline{\mathbb{Q}}}\ (\subseteq \mathbb{P}^1_{\overline{\mathbb{Q}}})$ for the finite étale Galois covering of $Z_{\overline{\mathbb{Q}}}$ of degree n determined by $t \mapsto t^n$;

- $Q_{n,l} \stackrel{\mathrm{def}}{=} \Pi^\Sigma_{Z_{\overline{\mathbb{Q}}}} / \mathrm{Ker}(\Pi^\Sigma_{{}^nY_{\overline{\mathbb{Q}}}} \twoheadrightarrow \Pi^l_{{}^nY_{\overline{\mathbb{Q}}}})$;

- $\phi_{n,l} : \mathrm{Out}^{|\mathrm{C}|}(\Pi^\Sigma_{Z_{\overline{\mathbb{Q}}}}) \to \mathrm{Out}^{|\mathrm{C}|}(Q_{n,l})$ for the natural homomorphism [cf. Lemma 2.2, (i)].

Note that ${}^1Y_{\overline{\mathbb{Q}}} = Z_{\overline{\mathbb{Q}}}$, and $Q_{1,l} = \Pi^l_{Z_{\overline{\mathbb{Q}}}}$.

Next, let us observe that, by applying Lemma 3.1, we may asuume without loss of generality that

$$\phi_l(G_1) = \{1\} \quad \text{and} \quad G_1 \subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^\Sigma_{Z_{\overline{\mathbb{Q}}}}).$$

In particular, we have a direct product decomposition

$$G^{|\mathrm{C}|} = G_1 \times G_2^{|\mathrm{C}|}\ (\subseteq \mathrm{Out}^{|\mathrm{C}|}(\Pi^\Sigma_{Z_{\overline{\mathbb{Q}}}})),$$

where we write $G^{|\mathrm{C}|} \stackrel{\mathrm{def}}{=} G \cap \mathrm{Out}^{|\mathrm{C}|}(\Pi^\Sigma_{Z_{\overline{\mathbb{Q}}}})$; $G_2^{|\mathrm{C}|} \stackrel{\mathrm{def}}{=} G_2 \cap \mathrm{Out}^{|\mathrm{C}|}(\Pi^\Sigma_{Z_{\overline{\mathbb{Q}}}})$.

Next, we verify the following assertion:

Claim 3.2.A: For any Σ-integer $n \in \mathbb{Z}_{\geq 1}$, it holds that $\phi_{n,l}(G_1) = \{1\}$.

Indeed, let $H \subseteq G^{|C|}$, $H_1 \subseteq G_1$, and $H_2 \subseteq G_2^{|C|}$ be normal open subgroups such that

- $H = H_1 \times H_2$;
- there exists an injection $H \hookrightarrow \mathrm{Out}^{|C|}(\Pi^{\Sigma}_{{}^nY_{\overline{\mathbb{Q}}}})$;
- there exists an injection $\Pi_{{}^nY_{\overline{\mathbb{Q}}}} \overset{\mathrm{out}}{\rtimes} H \hookrightarrow \Pi_{Z_{\overline{\mathbb{Q}}}} \overset{\mathrm{out}}{\rtimes} G^{|C|}$ [cf. [12], Proposition 1.4] that is compatible with the inclusions between respective subgroups $\Pi^{\Sigma}_{{}^nY_{\overline{\mathbb{Q}}}} \subseteq \Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}$ and quotients $H \subseteq G^{|C|}$

[cf. Lemma 1.7]. Then it follows immediately from Lemma 3.1; Remark 3.1.1, together with [12], Proposition 1.4, that $\phi_{n,l}(H_1) = \{1\}$ or $\phi_{n,l}(H_2) = \{1\}$. Suppose that $\phi_{n,l}(H_2) = \{1\}$. Here, we note that since $Q^l_{n,l} \xrightarrow{\sim} \Pi^l_{Z_{\overline{\mathbb{Q}}}}$, it holds that ϕ_l factors as the composite of $\phi_{n,l}$ with the natural homomorphism $\mathrm{Out}^{|C|}(Q_{n,l}) \to \mathrm{Out}^{|C|}(\Pi^l_{Z_{\overline{\mathbb{Q}}}})$. In particular, $\phi_l(H_2) = \{1\}$. Then our assumption that $\phi_l(G_1) = \{1\}$ implies that $\phi_l(G_1 \times H_2) = \{1\}$, hence that $\phi_l(\rho^{\Sigma}_Z(G_K))$ $(\subseteq \phi_l(G))$ is finite. This is a contradiction [cf. [8], Lemma 4.2, (iv)]. Thus, we conclude that $\phi_{n,l}(H_1) = \{1\}$, hence that $\phi_{n,l}(G_1)$ is finite. Therefore, since $\phi_{n,l}(G_1) \subseteq \phi_{n,l}(G)$ is a finite normal subgroup, it follows from the slimness of $\phi_{n,l}(G)$ [cf. Lemma 2.2, (iii)] that $\phi_{n,l}(G_1) = \{1\}$ [cf. Remark 1.1.1]. This completes the proof of Claim 3.2.A.

Write $\chi_{\Sigma} : \mathrm{Out}^{|C|}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}) \to (\widehat{\mathbb{Z}}^{\Sigma})^{\times}$ for the pro-Σ cyclotomic character [which is obtained by considering the actions on the cuspidal inertia subgroups of $\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}$]. Then it follows immediately from Claim 3.2.A that

$$\chi_{\Sigma}(G_1) = \{1\}.$$

For each $p \in \Sigma$, write

$$\phi_{(p)'} : \mathrm{Out}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}) \twoheadrightarrow \mathrm{Out}(\Pi^{\Sigma \setminus \{p\}}_{Z_{\overline{\mathbb{Q}}}})$$

for the natural surjection.

Next, we verify the following assertion:

Claim 3.2.B: There exists a finite subset $S \subseteq \Sigma$ such that, for each $p \in \Sigma \setminus S$, it holds that $\phi_{(p)'}(G_1) = \{1\}$.

Indeed, let $\mathfrak{p}$ be a maximal ideal of the ring of integers of K such that

- the characteristic — which we denote by p — of the residue field at $\mathfrak{p}$ is contained in Σ, and
- Z has good reduction at $\mathfrak{p}$;

$F \in G_K$ a lifting of the Frobenius element at $\mathfrak{p}$. We shall write,

- for each $i = 1, 2$, $\mathrm{pr}_i : G \twoheadrightarrow G_i$ for the natural projection;
- $J \subseteq G_K$ for the closed subgroup topologically generated by F, where we note that J is isomorphic to $\widehat{\mathbb{Z}}$;
- $I \stackrel{\mathrm{def}}{=} \rho_Z^{\Sigma}(J) \subseteq G^{|\mathsf{C}|}$;
- $I_1 \stackrel{\mathrm{def}}{=} \mathrm{pr}_1(I) \times \{1\} \subseteq G_1 \times G_2^{|\mathsf{C}|} = G^{|\mathsf{C}|}$, $I_2 \stackrel{\mathrm{def}}{=} \{1\} \times \mathrm{pr}_2(I) \subseteq G_1 \times G_2^{|\mathsf{C}|} = G^{|\mathsf{C}|}$.

Then since I is abelian, it holds that

$$I \subseteq I_1 \times I_2 \subseteq Z_{G^{|\mathsf{C}|}}(I).$$

Then it follows immediately from Lemma 2.4, (ii), together with the theory of specialization isomorphism, that our assumption that $\chi_\Sigma(I_1) \subseteq \chi_\Sigma(G_1) = \{1\}$ implies that $\phi_{(p)'}(I_1) = \{1\}$. In particular, $\phi_{(p)'}(I) \subseteq \phi_{(p)'}(I_2)$. Thus, since $\chi_\Sigma(G_1) = \{1\}$, and $G_1 \subseteq Z_G(I_2)$, we conclude from Lemma 2.4, (ii), together with the theory of specialization isomorphism, that $\phi_{(p)'}(G_1) = \{1\}$. Observe that there exists a finite subset $S \subseteq \Sigma$ such that Z has good reduction at any maximal ideal of the ring of integers of K that lies over a prime number $\in \Sigma \setminus S$. Therefore, we obtain the desired conclusion. This completes the proof of Claim 3.2.B.

Finally, if we write $\Sigma' \stackrel{\mathrm{def}}{=} \mathfrak{Primes} \setminus \Sigma$, then, in light of the injectivity of the composite

$$\mathrm{Out}(\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}) \to \prod_{p \in \mathfrak{Primes} \setminus (\Sigma' \cup S)} \mathrm{Out}((\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}})^{(p)'}) \xrightarrow{\sim} \prod_{p \in \Sigma \setminus S} \mathrm{Out}(\Pi^{\Sigma \setminus \{p\}}_{Z_{\overline{\mathbb{Q}}}})$$

[cf. Lemma 1.3; the fact that $\sharp(\Sigma' \cup S)$ is finite], we conclude from Claim 3.2.B that $G_1 = \{1\}$, hence that G is indecomposable. This completes the proof of Theorem 3.2. $\square$

Remark 3.2.1 It is natural to pose the following question:

> Question: In the notation of Theorem 3.2, can the assumption on the cardinality of the subset $\Sigma \subseteq \mathfrak{Primes}$ be dropped?

However, at the time of writing the present paper, the authors do not know whether the answer to this question is affirmative or not.

Corollary 3.3. *In the notation of Theorem 3.2, suppose that G is slim. Then $\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}} \overset{\mathrm{out}}{\rtimes} G$ is strongly indecomposable.*

Proof. First, since $\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}}$ is center-free [cf. [12], Proposition 1.4], we have an exact sequence of profinite groups

$$1 \longrightarrow \Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}} \longrightarrow \Pi_{Z_{\overline{\mathbb{Q}}}} \overset{\text{out}}{\rtimes} G \longrightarrow G \longrightarrow 1.$$

Thus, since G is infinite, we conclude from Theorem 3.2, together with [8], Proposition 1.8, (i); [12], Proposition 1.4; [12], Proposition 3.2, that $\Pi^{\Sigma}_{Z_{\overline{\mathbb{Q}}}} \overset{\text{out}}{\rtimes} G$ is strongly indecomposable. This completes the proof of Corollary 3.3. $\square$

Lemma 3.4. *Let $m \geq 2$ be an integer; $\Sigma \subseteq \mathfrak{Primes}$ a nonempty subset; F_m a free profinite group of rank m. Then* $\mathrm{Out}(F^{\Sigma}_m)$ *is slim.*

Proof. First, we consider the case where $m = 2$. In this case, we claim the following assertion:

> Claim 3.4.A: There exists a closed subgroup $H_1 \subseteq \mathrm{Out}(F^{\Sigma}_m)$ (respectively, $H_2 \subseteq \mathrm{Out}(F^{\Sigma}_m)$) such that for every open subgroup $H'_1 \subseteq H_1$ (respectively, $H'_2 \subseteq H_2$), it holds that
>
> $$Z_{\mathrm{Out}(F^{\Sigma}_m)}(H'_1) \cong \mathfrak{S}_3 \quad \text{(respectively, } Z_{\mathrm{Out}(F^{\Sigma}_m)}(H'_2) \cong \mathbb{Z}/2\mathbb{Z}).$$

Indeed, write C_1 for the projective line minus $\{0, 1, \infty\}$ over a number field $K_1 \subseteq \overline{\mathbb{Q}}$. Let $(E, \{o\})$ be an elliptic curve [where o is the origin of E] over a number field $K_2 \subseteq \overline{\mathbb{Q}}$ such that the j-invariant of $(E, \{o\})$ is not equal to 0 or 1728. Write C_2 for the hyperbolic curve over K_2 obtained by removing $\{o\}$ from E. In particular, [as is well-known] for any finite extension $K_1 \subseteq L_1$ $(\subseteq \overline{\mathbb{Q}})$ (respectively, $K_2 \subseteq L_2$ $(\subseteq \overline{\mathbb{Q}})$), we have

$$\mathrm{Aut}_{L_1}((C_1)_{L_1}) \cong \mathfrak{S}_3 \quad \text{(respectively, } \mathrm{Aut}_{L_2}((C_2)_{L_2}) \cong \mathbb{Z}/2\mathbb{Z}).$$

Thus, since [by applying the Grothendieck Conjecture for hyperbolic curves over number fields — cf. [9], Theorem A] we have a natural isomorphism

$$\mathrm{Aut}_{L_1}((C_1)_{L_1}) \xrightarrow{\sim} Z_{\mathrm{Out}(\Pi^{\Sigma}_{(C_1)_{\overline{\mathbb{Q}}}})}(\rho^{\Sigma}_{C_1}(G_{L_1}))$$

$$\text{(respectively, } \mathrm{Aut}_{L_2}((C_2)_{L_2}) \xrightarrow{\sim} Z_{\mathrm{Out}(\Pi^{\Sigma}_{(C_2)_{\overline{\mathbb{Q}}}})}(\rho^{\Sigma}_{C_2}(G_{L_2}))),$$

we conclude that the image of $\rho^{\Sigma}_{C_1}(G_{K_1})$ (respectively, $\rho^{\Sigma}_{C_2}(G_{K_2})$) via any isomorphism $\mathrm{Out}(\Pi^{\Sigma}_{(C_1)_{\overline{\mathbb{Q}}}}) \xrightarrow{\sim} \mathrm{Out}(F^{\Sigma}_m)$ (respectively, $\mathrm{Out}(\Pi^{\Sigma}_{(C_2)_{\overline{\mathbb{Q}}}}) \xrightarrow{\sim} \mathrm{Out}(F^{\Sigma}_m)$) determines the desired subgroup. This completes the proof of Claim 3.4.A.

Let us recall that to verify Lemma 3.4, it suffices to show that for every normal open subgroup $N \subseteq \mathrm{Out}(F^{\Sigma}_m)$, we have

$$Z_{\mathrm{Out}(F^{\Sigma}_m)}(N) = \{1\}.$$

Now suppose that $Z_{\mathrm{Out}(F_m^\Sigma)}(N) \neq \{1\}$. Then since

$$\begin{aligned} Z_{\mathrm{Out}(F_m^\Sigma)}(N) &\subseteq Z_{\mathrm{Out}(F_m^\Sigma)}(H_1 \cap N) \ \cap \ Z_{\mathrm{Out}(F_m^\Sigma)}(H_2 \cap N) \\ &\subseteq Z_{\mathrm{Out}(F_m^\Sigma)}(H_2 \cap N) \cong \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

[cf. Claim 3.4A], we have

$$\begin{aligned} Z_{\mathrm{Out}(F_m^\Sigma)}(N) &= Z_{\mathrm{Out}(F_m^\Sigma)}(H_1 \cap N) \ \cap \ Z_{\mathrm{Out}(F_m^\Sigma)}(H_2 \cap N) \\ &= Z_{\mathrm{Out}(F_m^\Sigma)}(H_2 \cap N) \cong \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

Thus, since $Z_{\mathrm{Out}(F_m^\Sigma)}(N)$ is normal in $\mathrm{Out}(F_m^\Sigma)$ [cf. our assumption that N is normal in $\mathrm{Out}(F_m^\Sigma)$], we conclude that $Z_{\mathrm{Out}(F_m^\Sigma)}(H_1 \cap N)$ ($\cong \mathfrak{S}_3$) [cf. Claim 3.4A] admits a normal subgroup of order 2, a contradiction. Therefore, we have $Z_{\mathrm{Out}(F_m^\Sigma)}(N) = \{1\}$. This completes the proof of Lemma 3.4 in the case where $m = 2$.

Next, we consider the case where $m = 3$. In this case, we claim the following assertion:

> Claim 3.4.B: There exists a closed subgroup $H_1 \subseteq \mathrm{Out}(F_m^\Sigma)$ (respectively, $H_2 \subseteq \mathrm{Out}(F_m^\Sigma)$) satisfying the following conditions:
>
> - For every open subgroup $H_1' \subseteq H_1$ (respectively, $H_2' \subseteq H_2$), it holds that
> $$Z_{\mathrm{Out}(F_m^\Sigma)}(H_1') \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$
> $$(\text{respectively, } Z_{\mathrm{Out}(F_m^\Sigma)}(H_2') \cong \mathbb{Z}/2\mathbb{Z}).$$
> - For any $l \in \Sigma$, every nontrivial element $\alpha_1 \in Z_{\mathrm{Out}(F_m^\Sigma)}(H_1')$ (respectively, the [unique] nontrivial element $\alpha_2 \in Z_{\mathrm{Out}(F_m^\Sigma)}(H_2')$) induces a $\mathbb{Z}_l$-automorphism $\overline{\alpha}_1$ (respectively, $\overline{\alpha}_2$) of
> $$(F_m^l)^{\mathrm{ab}}$$
> such that the rank of the $\mathbb{Z}_l$-submodule $\subseteq (F_m^l)^{\mathrm{ab}}$ consisting of elements $\in (F_m^l)^{\mathrm{ab}}$ fixed by $\overline{\alpha}_1$ (respectively, $\overline{\alpha}_2$) is 1 (respectively, 0).

Indeed, write C_1 for the projective line minus $\{0, 1, 3, \infty\}$ over a number field $K_1 \subseteq \overline{\mathbb{Q}}$. Let $(E, \{o\})$ be an elliptic curve over a number field $K_2 \subseteq \overline{\mathbb{Q}}$ such that the j-invariant of $(E, \{o\})$ is not equal to 0 or 1728, and that E has a non-4-torsion K_2-rational point x. Write C_2 for the hyperbolic curve over K_2 obtained by removing $\{x, -x\}$ from E. In particular, [as is easily verified] for any finite extension $K_1 \subseteq L_1$ ($\subseteq \overline{\mathbb{Q}}$) (respectively, $K_2 \subseteq L_2$ ($\subseteq \overline{\mathbb{Q}}$)), we have

$$\mathrm{Aut}_{L_1}((C_1)_{L_1}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \quad (\text{respectively, } \mathrm{Aut}_{L_2}((C_2)_{L_2}) \cong \mathbb{Z}/2\mathbb{Z})$$

[cf. the fact that the $\overline{\mathbb{Q}}$-automorphism "$[-1]$" of $E_{\overline{\mathbb{Q}}}$ [of order 2] preserves the set $\{x, -x\}$; the fact that if a nontrivial element $f \in \mathrm{Aut}_{\overline{\mathbb{Q}}}(E_{\overline{\mathbb{Q}}})$ satisfies $f(x) = x$, then f coincides with the map

$$z \mapsto 2x - z$$

on the set of $\overline{\mathbb{Q}}$-rational points of $E_{\overline{\mathbb{Q}}}$, so $f(-x) \neq -x$].

Let α_1 be a nontrivial element $\in \mathrm{Aut}_{L_1}((C_1)_{L_1})$ that induces an automorphism $s \in \mathrm{Sym}(\{0,1,3,\infty\})$ of the set of cusps $\{0,1,3,\infty\}$. [Here, we note that s may be written as the product of the transposition of two distinct elements $\in \{0,1,3,\infty\}$ and the transposition of the other two distinct elements $\in \{0,1,3,\infty\}$.] Write α_2 for the [unique] nontrivial element $\in \mathrm{Aut}_{L_2}((C_2)_{L_2})$. Then it follows from the well-known structure of the étale fundamental group of a smooth curve over an algebraically closed field of characteristic 0 that if we identify $(\Pi^l_{(C_1)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}}$ (respectively, $(\Pi^l_{(C_2)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}}$) with the [free] $\mathbb{Z}_l$-module

$$(\langle a_0,\ a_1,\ a_3,\ a_\infty \mid a_0 + a_1 + a_3 + a_\infty = 0\rangle^l)^{\mathrm{ab}}$$

$$\text{(respectively, } (\langle b,\ c,\ d_x,\ d_{-x} \mid d_x + d_{-x} = 0\rangle^l)^{\mathrm{ab}})$$

— where we write $(-)^l$ for the pro-l completion of $(-)$; $a_\circ$ (respectively, $d_\circ$) corresponds to the cusp $\circ \in \{0,1,3,\infty\}$ (respectively, $\circ \in \{x,-x\}$); b corresponds to a meridian of the "complex torus $E(\mathbb{C})$"; c corresponds to a longitude of $E(\mathbb{C})$ — then α_1 (respectively, α_2) induces the $\mathbb{Z}_l$-automorphism

$$\overline{\alpha}_1 : (\Pi^l_{(C_1)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}} \xrightarrow{\sim} (\Pi^l_{(C_1)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}}; \quad a_0 \mapsto a_{s(0)},\ a_1 \mapsto a_{s(1)},\ a_3 \mapsto a_{s(3)},\ a_\infty \mapsto a_{s(\infty)}$$

(respectively,

$$\overline{\alpha}_2 : (\Pi^l_{(C_2)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}} \xrightarrow{\sim} (\Pi^l_{(C_2)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}}; \quad b \mapsto -b,\ c \mapsto -c,\ d_x \mapsto d_{-x},\ d_{-x} \mapsto d_x).$$

Now one verifies easily that the rank of the $\mathbb{Z}_l$-submodule $\subseteq (\Pi^l_{(C_1)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}}$ (respectively, $\subseteq (\Pi^l_{(C_2)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}}$) consisting of elements $\in (\Pi^l_{(C_1)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}}$ (respectively, $\in (\Pi^l_{(C_2)_{\overline{\mathbb{Q}}}})^{\mathrm{ab}}$) fixed by $\overline{\alpha}_1$ (respectively, $\overline{\alpha}_2$) is 1 (respectively, 0). Thus, Claim 3.4.B follows from a similar argument to the argument applied in the final portion of the proof of Claim 3.4.A.

In light of Claim 3.4.B, for every open subgroup $N \subseteq \mathrm{Out}(F^\Sigma_m)$, we have

$$Z_{\mathrm{Out}(F^\Sigma_m)}(N) \ \subseteq\ Z_{\mathrm{Out}(F^\Sigma_m)}(H_1 \cap N) \ \cap\ Z_{\mathrm{Out}(F^\Sigma_m)}(H_2 \cap N) \ =\ \{1\}.$$

This completes the proof of Lemma 3.4 in the case where $m = 3$.

Finally, we consider the case where $m \geq 4$. In this case, by considering the hyperbolic curve C over a number field $K \subseteq \overline{\mathbb{Q}}$ obtained by removing $\{r_1, r_2, \dots, r_{m+1}\}$ — where r_i are distinct rational numbers — from $\mathbb{P}^1_K$ such that $\mathrm{Aut}_{\overline{\mathbb{Q}}}(C_{\overline{\mathbb{Q}}}) = \{1\}$, it follows from a similar argument to the argument applied in the final portion of the proof of Claim 3.4.A that:

> Claim 3.4.C: There exists a closed subgroup $H \subseteq \mathrm{Out}(F^\Sigma_m)$ such that for every open subgroup $H' \subseteq H$, it holds that
>
> $$Z_{\mathrm{Out}(F^\Sigma_m)}(H') = \{1\}.$$

In light of Claim 3.4.C, the slimness of $\mathrm{Out}(F^\Sigma_m)$ [where $m \geq 4$] follows immediately. This completes the proof of Lemma 3.4. $\square$

Remark 3.4.1 Lemma 3.4 in the case where $m \geq 4$ also follows immediately from [6], Theorem A.

Corollary 3.5. *Let $m \geq 2$ be an integer; $\Sigma \subseteq \mathfrak{Primes}$ a subset such that either $\sharp\Sigma = 1$ or $\sharp(\mathfrak{Primes} \setminus \Sigma)$ is finite; F_m a free profinite group of rank m. Then $\mathrm{Aut}(F_m^{\Sigma})$ and $\mathrm{Out}(F_m^{\Sigma})$ are strongly indecomposable.*

Proof. The strong indecomposability of $\mathrm{Out}(F_m^{\Sigma})$ follows immediately from Theorem 3.2 and Lemma 3.4. The strong indecomposability of $\mathrm{Aut}(F_m^{\Sigma})$ follows immediately from Corollary 3.3 and Lemma 3.4. This completes the proof of Corollary 3.5. $\square$

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(Arata Minamide) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

Email address: minamide@kurims.kyoto-u.ac.jp

(Shota Tsujimura) Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan

Email address: stsuji@kurims.kyoto-u.ac.jp