GEOMETRIC VERSION OF THE GROTHENDIECK CONJECTURE FOR UNIVERSAL CURVES OVER HURWITZ STACKS

SHOTA TSUJIMURA

Abstract. In this paper, we prove a certain geometric version of the Grothendieck Conjecture for tautological curves over Hurwitz stacks. This result generalizes a similar result obtained by Hoshi and Mochizuki in the case of tautological curves over moduli stacks of pointed smooth curves. In the process of studying this version of the Grothendieck Conjecture, we also examine various fundamental geometric properties of "profiled log Hurwitz stacks", i.e., log algebraic stacks that parametrize Hurwitz coverings for which the marked points are equipped with a certain ordering determined by combinatorial data which we refer to as a "profile".

Contents

Introduction

In [5], the theory of profinite Dehn twists was developed and applied to prove the following "geometric version of the Grothendieck Conjecture for tautological curves over moduli stacks of pointed smooth curves".

Theorem M. *(cf.* [5]*, Theorem D) Let* (g, r) *be a pair of nonnegative integers such that* $2g - 2 + r > 0$*;* Σ *a nonempty set of prime numbers;* k *an algebraically closed field of characteristic zero. Write* $\mathcal{M}_{g,r}$ *for the* **moduli stack** *of* r*-pointed smooth curves of genus* g *whose* r *marked points are equipped with an ordering;* $\mathcal{C}_{g,r} \to \mathcal{M}_{g,r}$ *for the* **tautological curve** *over* $\mathcal{M}_{g,r}$ *[cf. the discussion entitled "Curves" in Notations and Conventions];* $(\mathcal{M}_{g,r})_k \overset{\mathrm{def}}{=} \mathcal{M}_{g,r} \times_{\mathbb{Z}} k$ *[cf. the discussion entitled "Curves" in Notations and Conventions];* $(\mathcal{C}_{g,r})_k \overset{\mathrm{def}}{=} \mathcal{C}_{g,r} \times_{\mathbb{Z}} k$ *[cf. the discussion*

2010 *Mathematics Subject Classification.* Primary 14H30; Secondary 14H10.

Key words and phrases. anabelian geometry; Grothendieck Conjecture; universal curve; Hurwitz stack.

entitled "Curves" in Notations and Conventions]; $\Pi_{\mathcal{M}_{g,r}} \overset{\text{def}}{=} \pi_1((\mathcal{M}_{g,r})_k)$ for the étale fundamental group of the moduli stack $(\mathcal{M}_{g,r})_k$; $\Pi_{g,r}$ for the maximal pro-Σ quotient of the kernel $N_{g,r}$ of the natural surjection $\pi_1((\mathcal{C}_{g,r})_k) \twoheadrightarrow \pi_1((\mathcal{M}_{g,r})_k) = \Pi_{\mathcal{M}_{g,r}}$; $\Pi_{\mathcal{C}_{g,r}}$ for the quotient of the étale fundamental group $\pi_1((\mathcal{C}_{g,r})_k)$ of $(\mathcal{C}_{g,r})_k$ by the kernel of the natural surjection $N_{g,r} \twoheadrightarrow \Pi_{g,r}$; $\mathrm{Out}^{\mathrm{C}}(\Pi_{g,r})$ for the group of outomorphisms [cf. the discussion entitled "Topological groups" in Notations and Conventions] of $\Pi_{g,r}$ which induce bijections on the set of cuspidal inertia subgroups of $\Pi_{g,r}$. Thus, we have a natural sequence of profinite groups

$$1 \longrightarrow \Pi_{g,r} \longrightarrow \Pi_{\mathcal{C}_{g,r}} \longrightarrow \Pi_{\mathcal{M}_{g,r}} \longrightarrow 1$$

which determines an outer representation

$$\rho_{g,r} : \Pi_{\mathcal{M}_{g,r}} \longrightarrow \mathrm{Out}(\Pi_{g,r}).$$

Then the following hold:

(i) Let $H \subseteq \Pi_{\mathcal{M}_{g,r}}$ be an open subgroup of $\Pi_{\mathcal{M}_{g,r}}$. Suppose that one of the following two conditions is satisfied:
   (a) $2g - 2 + r > 1$, i.e., $(g, r) \notin \{(0, 3), (1, 1)\}$;
   (b) $(g, r) = (1, 1), 2 \in \Sigma$, and $H = \Pi_{\mathcal{M}_{g,r}}$.

   Then the composite of natural homomorphisms

$$\begin{aligned}\mathrm{Aut}_{(\mathcal{M}_{g,r})_k}((\mathcal{C}_{g,r})_k) &\longrightarrow \mathrm{Aut}_{\Pi_{\mathcal{M}_{g,r}}}(\Pi_{\mathcal{C}_{g,r}})/\mathrm{Inn}(\Pi_{g,r}) \\ &\xrightarrow{\sim} Z_{\mathrm{Out}(\Pi_{g,r})}(\mathrm{Im}(\rho_{g,r})) \subseteq Z_{\mathrm{Out}(\Pi_{g,r})}(\rho_{g,r}(H))\end{aligned}$$

   [cf. the discussion entitled "Topological groups" in Notations and Conventions] determines an **isomorphism**

$$\mathrm{Aut}_{(\mathcal{M}_{g,r})_k}((\mathcal{C}_{g,r})_k) \xrightarrow{\sim} Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,r})}(\rho_{g,r}(H)).$$

   Here, we recall that $\mathrm{Aut}_{(\mathcal{M}_{g,r})_k}((\mathcal{C}_{g,r})_k)$ is isomorphic to

$$\begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } (g,r) = (0,4); \\ \mathbb{Z}/2\mathbb{Z} & \text{if } (g,r) \in \{(1,1),(1,2),(2,0)\}; \\ \{1\} & \text{if } (g,r) \notin \{(0,4),(1,1),(1,2),(2,0)\}. \end{cases}$$

(ii) Let $H \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{g,r})$ be a closed subgroup of $\mathrm{Out}^{\mathrm{C}}(\Pi_{g,r})$ that contains an open subgroup of $\mathrm{Im}(\rho_{g,r}) \subseteq \mathrm{Out}(\Pi_{g,r})$. Suppose that

$$2g - 2 + r > 1, \text{ i.e.}, (g,r) \notin \{(0,3),(1,1)\}.$$

   Then H is **almost slim** *[cf. the discussion entitled "Topological groups" in Notations and Conventions]. If, moreover,*

$$2g - 2 + r > 2, \text{ i.e.}, (g,r) \notin \{(0,3),(0,4),(1,1),(1,2),(2,0)\},$$

   then H is **slim** *[cf. the discussion entitled "Topological groups" in Notations and Conventions].*

Roughly speaking, this result was obtained in [5] as a consequence of the following two steps.

(1) The $r > 0$ case is reduced to the Grothendieck Conjecture for configuration spaces and then proved by applying the combinatorial Grothendieck Conjecture [i.e., the graphicity of outomorphisms of surface groups satisfying certain combinatorial conditions [cf. [6], Theorem A]] and elementary topological and graph-theoretic considerations.
(2) The $r = 0$ case is reduced to the $r > 0$ case by using the theory of clutching morphisms [cf. [10]] and the theory of profinite Dehn twists [cf. [5]].

In the present paper, we prove a version of Theorem M for tautological curves over (log) Hurwitz stacks [cf. [5], Remark 6.14.1]. In order to carry out steps (1) and (2) in the case of tautological curves over (log) Hurwitz stacks, it is necessary to overcome certain difficulties, as follows:

(1^{Hur}) It is necessary to prove a version of the Grothendieck Conjecture for configuration spaces that applies to certain more combinatorially complicated spaces that arise from (log) Hurwitz stacks. This is done by applying similar techniques to the techniques applied in (1), but these techniques must be applied to spaces that are much more combinatorially complicated than configuration spaces.

(2^{Hur}) Unlike the situation in (2), where one may consider arbitrary deformations and degenerations of pointed stable curves, it is necessary to restrict oneself to deformations and degenerations that are compatible with the covering under consideration. This difficulty is overcome by applying similar techniques to the techniques applied in (2), but, just as in the case of (1^{Hur}), the situation in which these techniques must be applied is considerably more combinatorially complicated than the situation considered in (2).

This paper is organized as follows. In §1, after recalling the definitions of Hurwitz stacks, we define profiled (log) Hurwitz stacks and examine various fundamental geometric properties of profiled (log) Hurwitz stacks such as irreducibility. We also prove the existence of certain natural homotopy exact sequences related to these profiled (log) Hurwitz stacks that will be of use later in the paper. In §2, we define Hurwitz-type log configuration spaces and discuss various objects related to these spaces. In §3, we prove a key result [cf. Proposition 3.1] which asserts that outomorphisms of surface groups that satisfy certain relatively weak conditions are in fact trivial. In §4, after discussing the existence of certain suitable degenerations of simple coverings, i.e., the coverings parametrized by Hurwitz stacks, we prove the main result by applying the theory of profinite Dehn twists, together with the results obtained in previous sections.

Our main result is the following.

Theorem A. *Let Σ be a nonempty set of prime numbers; k an algebraically closed field of characteristic zero; (g,d,r) a triple of nonnegative integers such that*

$$d \geq 2 \ \wedge \ (g,r) \notin \{(0,0),(1,0)\} \ \wedge \ (g,d,r) \notin \{(0,2,1),(0,3,1)\}$$
$$(\Rightarrow 2g-2+dr > 1 \ \wedge \ 2g+2d+r-5 \geq 1).$$

Write $(\mathcal{H}_{g,d,r})_k$ for the r-profiled **Hurwitz stack** *of type (g,d) over k [cf. Definition 1.8; Definition 1.13, (ii)], where $\dim(\mathcal{H}_{g,d,r})_k = 2g-2+2d+r-3 = 2g+2d+r-5 \geq 1$ [cf. Corollary 1.9]; $(\mathcal{C}_{g,d,r})_k \to (\mathcal{H}_{g,d,r})_k$ for the restriction of the* **tautological curve** *over $(\mathcal{M}_{g,dr})_k$ to $(\mathcal{H}_{g,d,r})_k$ via the natural (1-)morphism $(\mathcal{H}_{g,d,r})_k \to (\mathcal{M}_{g,dr})_k$ [cf. Proposition 1.10, (iii)]; $\Pi_{\mathcal{H}_{g,d,r}} \stackrel{\mathrm{def}}{=} \pi_1((\mathcal{H}_{g,d,r})_k)$ for the étale fundamental group of the profiled Hurwitz stack $(\mathcal{H}_{g,d,r})_k$; $\Pi_{g,d,r}$ for the maximal pro-Σ quotient of the kernel $N_{g,d,r}$ of the natural surjection $\pi_1((\mathcal{C}_{g,d,r})_k) \twoheadrightarrow \pi_1((\mathcal{H}_{g,d,r})_k) = \Pi_{\mathcal{H}_{g,d,r}}$; $\Pi_{\mathcal{C}_{g,d,r}}$ for the quotient of the étale fundamental group $\pi_1((\mathcal{C}_{g,d,r})_k)$ of $(\mathcal{C}_{g,d,r})_k$ by the kernel of the natural surjection $N_{g,d,r} \twoheadrightarrow \Pi_{g,d,r}$; $\mathrm{Out}^{\mathrm{C}}(\Pi_{g,d,r})$ for the group of outomorphisms [cf. the discussion entitled "Topological groups" in Notations and Conventions] of $\Pi_{g,d,r}$ which induce bijections on the set of cuspidal inertia subgroups of $\Pi_{g,d,r}$. Thus, we have a natural sequence of profinite groups*

$$1 \longrightarrow \Pi_{g,d,r} \longrightarrow \Pi_{\mathcal{C}_{g,d,r}} \longrightarrow \Pi_{\mathcal{H}_{g,d,r}} \longrightarrow 1$$

which determines an outer representation

$$\rho_{g,d,r} : \Pi_{\mathcal{H}_{g,d,r}} \longrightarrow \mathrm{Out}(\Pi_{g,d,r}).$$

Then the following hold:

(i) *Let* $H \subseteq \Pi_{\mathcal{H}_{g,d,r}}$ *be an open subgroup of* $\Pi_{\mathcal{H}_{g,d,r}}$. *Then the composite of natural homomorphisms*

$$\begin{aligned}\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k) &\longrightarrow \mathrm{Aut}_{\Pi_{\mathcal{H}_{g,d,r}}}(\Pi_{\mathcal{C}_{g,d,r}})/\mathrm{Inn}(\Pi_{g,d,r}) \\ &\xrightarrow{\sim} Z_{\mathrm{Out}(\Pi_{g,d,r})}(\mathrm{Im}(\rho_{g,d,r})) \subseteq Z_{\mathrm{Out}(\Pi_{g,d,r})}(\rho_{g,d,r}(H))\end{aligned}$$

[cf. the discussion entitled "Topological groups" in Notations and Conventions] determines an **isomorphism**

$$\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k) \xrightarrow{\sim} Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,d,r})}(\rho_{g,d,r}(H)).$$

Moreover, $\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k)$ *is isomorphic to*

$$\begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } (g,d,r) \in \{(0,2,2),(0,4,1)\}; \\ \mathbb{Z}/2\mathbb{Z} & \text{if } (g,d,r) \in \{(g,2,r) \mid (g,r) \neq (0,2)\} \cup \{(2,d,0)\}; \\ \{1\} & \text{if } (g,d,r) \notin \{(0,4,1),(g,2,r),(2,d,0)\}. \end{cases}$$

(ii) *Let* $H \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{g,d,r})$ *be a closed subgroup of* $\mathrm{Out}^{\mathrm{C}}(\Pi_{g,d,r})$ *that contains an open subgroup of* $\mathrm{Im}(\rho_{g,d,r}) \subseteq \mathrm{Out}(\Pi_{g,d,r})$. *Then* H *is* **almost slim** *[cf. the discussion entitled "Topological groups" in Notations and Conventions]. If, moreover,*

$$(g,d,r) \notin \{(0,4,1),(g,2,r),(2,d,0)\},$$

then H *is* **slim** *[cf. the discussion entitled "Topological groups" in Notations and Conventions].*

Notations and Conventions

In this paper, we follow the notations and conventions of [5].

Sets : If S is a set, then we shall denote by $S^{\#}$ the *cardinality* of S.

Numbers : The notation $\mathfrak{Primes}$ will be used to denote the set of prime numbers. The notation $\mathbb{N}$ will be used to denote the set or [additive] monoid of nonnegative rational integers. The notation $\mathbb{Z}$ will be used to denote the set, group, or ring of rational integers.

Topological groups : Let G be a topological group and $\mathbf{P}$ a property of topological groups [e.g., "abelian" or "pro-Σ" for some $\Sigma \subseteq \mathfrak{Primes}$]. Then we shall say that G is *almost* $\mathbf{P}$ if there exists an open subgroup of G that is $\mathbf{P}$. Let G be a topological group and $H \subseteq G$ a closed subgroup of G. Then we shall denote by $Z_G(H)$ (respectively, $N_G(H); C_G(H)$) the *centralizer* (respectively, *normalizer; commensurator*) of $H \subseteq G$, i.e.,

$$Z_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid ghg^{-1} = h \text{ for any } h \in H\},$$

$$\text{(respectively, } N_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid g \cdot H \cdot g^{-1} = H\};$$

$$C_G(H) \stackrel{\mathrm{def}}{=} \{g \in G \mid H \cap g \cdot H \cdot g^{-1} \text{ is of finite index in } H \text{ and } g \cdot H \cdot g^{-1}\}).$$

We shall refer to $Z(G) = Z_G(G)$ as the center of G. It is immediate from the definitions that

$$Z_G(H) \subseteq N_G(H) \subseteq C_G(H);\ H \subseteq N_G(H).$$

We shall say that the closed subgroup H is *commensurably terminal* in G if $H = C_G(H)$. We shall say that G is *slim* if $Z_G(U) = \{1\}$ for any open subgroup U of G.

Let G be a topological group. Then we shall write $\mathrm{Aut}(G)$ for the group of [continuous] automorphisms of G, $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ for the group of inner automorphisms of G, and $\mathrm{Out}(G) \stackrel{\mathrm{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)$. We shall refer to an element of $\mathrm{Out}(G)$ as an *outomorphism* of G. Now suppose that G is *center-free* [i.e., $Z_G(G) = \{1\}$]. Then we have an exact sequence of groups

$$1 \longrightarrow G \ \ (\stackrel{\sim}{\to} \mathrm{Inn}(G)) \longrightarrow \mathrm{Aut}(G) \longrightarrow \mathrm{Out}(G) \longrightarrow 1.$$

If J is a group, and $\rho : J \to \mathrm{Out}(G)$ is a homomorphism, then we shall denote by

$$G \stackrel{\mathrm{out}}{\rtimes} J$$

the group obtained by pulling back the above exact sequence of profinite groups via ρ. Thus, we have a *natural exact sequence* of groups

$$1 \longrightarrow G \longrightarrow G \stackrel{\mathrm{out}}{\rtimes} J \longrightarrow J \longrightarrow 1.$$

Suppose further that G is *profinite* and *topologically finitely generated*. Then one verifies immediately that the topology of G admits a basis of *characteristic open subgroups*, which thus induces a *profinite topology* on the groups $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ with respect to which the above exact sequence relating $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ determines an exact sequence of *profinite groups*. In particular, one verifies easily that if, moreover, J is *profinite* and $\rho : J \to \mathrm{Out}(G)$ is *continuous*, then the above exact sequence involving $G \stackrel{\mathrm{out}}{\rtimes} J$ determines an exact sequence of *profinite groups*. Let G, J be *profinite groups*. Suppose that G is *center-free* and *topologically finitely generated*. Let $\rho : J \to \mathrm{Out}(G)$ be a *continuous homomorphism*. Write $\mathrm{Aut}_J(G \stackrel{\mathrm{out}}{\rtimes} J)$ for the group of [continuous] automorphisms of $G \stackrel{\mathrm{out}}{\rtimes} J$ that preserve and induce the identity automorphism on the quotient J. Then one verifies immediately that the operation of restricting to G determines an *isomorphism* of profinite groups

$$\mathrm{Aut}_J(G \stackrel{\mathrm{out}}{\rtimes} J)/\mathrm{Inn}(G) \stackrel{\sim}{\to} Z_{\mathrm{Out}(G)}(\mathrm{Im}(\rho)).$$

Log schemes : For basic notions concerning log schemes, see [8], [9]. When a *scheme* appears in a diagram of log schemes, the scheme is to be understood as the log scheme obtained by equipping the scheme with the *trivial log structure*. If $X^{\log}$ is a log scheme, then we shall refer to the largest open subscheme of the underlying scheme of $X^{\log}$ over which the log structure is trivial as the *interior* of $X^{\log}$. Fiber products of fs log schemes are to be understood as fiber products taken in the category of fs log schemes. Note that in general, the underlying scheme of the fiber product of fs log schemes is *not naturally isomorphic* to the fiber product of the underlying schemes of the given fs log schemes. However, if a morphism $X^{\log} \to Y^{\log}$ between two fs log schemes $X^{\log}$ and $Y^{\log}$ is *strict* [i.e., the pull-back of the log structure of $Y^{\log}$ is naturally isomorphic to the log structure of $X^{\log}$], then for any morphism $Z^{\log} \to Y^{\log}$ between two fs log schemes $Z^{\log}$ and $Y^{\log}$, the underlying scheme of the fiber product $X^{\log} \times_{Y^{\log}} Z^{\log}$ is naturally isomorphic to $X \times_Y Z$.

Curves : We shall use the terms "*hyperbolic curve*", "*cusp*", "*stable log curve*", and "*smooth log curve*" as they are defined in [CmbGC]. We shall denote by $\mathbb{P}^1_{(-)}$

the projective line over $(-)$. If (g,r) is a pair of nonnegative integers such that $2g-2+r>0$, then we shall denote by $\overline{\mathcal{M}}_{g,r}$ the *moduli stack of* r*-pointed stable curves of genus* g over $\mathbb{Z}$ whose r marked points are *equipped with an ordering*, by $\mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ the open substack of $\overline{\mathcal{M}}_{g,r}$ *parametrizing smooth curves*, by $\overline{\mathcal{M}}_{g,r}^{\log}$ the log stack obtained by equipping $\overline{\mathcal{M}}_{g,r}$ with the log structure associated to the divisor with normal crossings $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$, by $\overline{\mathcal{C}}_{g,r} \to \overline{\mathcal{M}}_{g,r}$ the *tautological/universal curve* over $\overline{\mathcal{M}}_{g,r}$, and by $\overline{\mathcal{D}}_{g,r} \subseteq \overline{\mathcal{C}}_{g,r}$ the corresponding *tautological divisor* of *marked points* of $\overline{\mathcal{C}}_{g,r} \to \overline{\mathcal{M}}_{g,r}$. Then the divisor given by the union of $\overline{\mathcal{D}}_{g,r}$ with the inverse image in $\overline{\mathcal{C}}_{g,r}$ of the divisor $\overline{\mathcal{M}}_{g,r} \setminus \mathcal{M}_{g,r} \subseteq \overline{\mathcal{M}}_{g,r}$ determines a log structure on $\overline{\mathcal{C}}_{g,r}$; denote the resulting log stack by $\overline{\mathcal{C}}_{g,r}^{\log}$. Thus, we obtain a (1-)morphism of log stacks $\overline{\mathcal{C}}_{g,r}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$. We shall denote by $\mathcal{C}_{g,r} \subseteq \overline{\mathcal{C}}_{g,r}$ the interior of $\overline{\mathcal{C}}_{g,r}^{\log}$. Thus, we obtain a (1-)morphism of stacks $\mathcal{C}_{g,r} \to \mathcal{M}_{g,r}$. Let S be a scheme. We shall append a subscript "S" to $\overline{\mathcal{M}}_{g,r}$, $\mathcal{M}_{g,r}$, $\overline{\mathcal{M}}_{g,r}^{\log}$, $\overline{\mathcal{C}}_{g,r}$, $\mathcal{C}_{g,r}$, and $\overline{\mathcal{C}}_{g,r}^{\log}$ to denote the result of base-changing to S.

Let n be a positive integer and $X^{\log}$ a stable log curve of type (g,r) over a log scheme $S^{\log}$. Then we shall refer to the log scheme obtained by pulling back the (1-)morphism $\overline{\mathcal{M}}_{g,r+n}^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ given by forgetting the last n points via the classifying (1-)morphism $S^{\log} \to \overline{\mathcal{M}}_{g,r}^{\log}$ of $X^{\log}$ as the *n-th log configuration space* of $X^{\log}$.

1. Basic properties of profiled log Hurwitz stacks

In this section, after reviewing the basic theory of Hurwitz stacks in Definitions 1.1, 1.2, 1.3, 1.4; Theorem 1.5; Lemma 1.6 [cf. [2], §6, and [11]], we define *"profiled"* versions — i.e., versions equipped with various orderings of the marked points — of the notion of a simple admissible covering [cf. Definition 1.7] and of (log) Hurwitz stacks [cf. Definition 1.8]. After defining profiled (log) Hurwitz stacks, we examine various fundamental geometric properties of these stacks in Proposition 1.10 and prove the existence of certain natural homotopy exact sequences related to these stacks in Proposition 1.14.

Definition 1.1. (cf. [11], §1.3) Let (g,d) be a pair of nonnegative integers such that $2g-2+2d \geq 3$ and $d \geq 2$. For any scheme S over Spec $\mathbb{Z}[\frac{1}{d!}]$, write $\mathcal{H}_{g,d}^{\mathrm{ord}}(S)$ for the following groupoid [i.e., a category in which every morphism is invertible]:

- Objects: an object is a collection of arrows
$$(\pi : C \to P; \sigma_1, \ldots, \sigma_{2g-2+2d} : S \to P)$$
in the category of S-schemes such that the following properties hold: there exists an isomorphism of S-schemes $P \xrightarrow{\sim} \mathbb{P}_S^1$; the structure morphism $C \to S$ is a smooth, geometrically connected, proper family of curves of genus g; π is [necessarily finite] flat of degree d with *simple ramification* [i.e., the discriminant divisor of π is étale over the base S] exactly at the [necessarily mutually disjoint] sections $\sigma_1, \ldots, \sigma_{2g-2+2d} : S \to P$.
- Morphisms: a morphism between two objects $(\pi : C \to P; \sigma_1, \ldots, \sigma_{2g-2+2d})$ and $(\pi' : C' \to P'; \sigma_1', \ldots, \sigma_{2g-2+2d}')$ is a pair of isomorphisms $\alpha : C \xrightarrow{\sim} C'$ and $\beta : P \xrightarrow{\sim} P'$ such that $\beta \circ \pi = \pi' \circ \alpha$.

We shall refer to the resulting stack as the *ordered Hurwitz stack* $\mathcal{H}^{\mathrm{ord}}_{g,d}$ *of type* (g,d) [cf. Remark 1.1.1 below]. Note that there is a natural action of the *symmetric group on* $2g-2+2d$ *letters* on $\mathcal{H}^{\mathrm{ord}}_{g,d}$. We shall refer to the *stack-theoretic quotient* of the ordered Hurwitz stack $\mathcal{H}^{\mathrm{ord}}_{g,d}$ of type (g,d) by this action of the symmetric group on $2g-2+2d$ letters as the *Hurwitz stack* $\mathcal{H}_{g,d}$ *of type* (g,d).

Remark 1.1.1. When $d \geq 3$, the stack $\mathcal{H}^{\mathrm{ord}}_{g,d}$ is representable by a scheme [cf. Theorem 1.5 below; [2], Theorem 6.3; [11], §1.3; [11], §3.22]. Here, we remark that a *slight oversight* in the statement of the Theorem of [11], §3.22, is *corrected* in Theorem 1.5 below: That is to say, in the statement of the Theorem of [11], §3.22, the definition of the *morphisms* of the stack under consideration are only explicitly defined in the case where the domain and codomain of the morphism are *identical*; in fact, however, morphisms must be defined in the case where the domain and codomain of the morphism are *not necessarily identical*, i.e., as is done in the statement of Theorem 1.5 given below [where one considers morphisms between *primed* and *un-primed* collections of data].

Next, we recall the notion of admissible coverings introduced in [3], [11] for constructing a compactified version of the Hurwitz stack.

Definition 1.2. (cf. [11], §3.4) Let (g,r) be a pair of nonnegative integers; q a positive integer.

(i) Let S_q denote the symmetric group on q letters. Note that we have a natural action of S_q on $\overline{\mathcal{M}}^{\log}_{g,q+r}$ given by permuting the first q marked points. We shall denote by $\overline{\mathcal{M}}^{\log}_{g,[q]+r}$ the (log) stack-theoretic quotient of $\overline{\mathcal{M}}^{\log}_{g,q+r}$ by S_q. If $r=0$, we simply write $\overline{\mathcal{M}}^{\log}_{g,[q]}$. Note that the universal stable log curve $\overline{\mathcal{C}}^{\log}_{g,q+r} \to \overline{\mathcal{M}}^{\log}_{g,q+r}$ descends to a stable log curve $\overline{\mathcal{C}}^{\log}_{g,[q]+r} \to \overline{\mathcal{M}}^{\log}_{g,[q]+r}$.

(ii) Let $S^{\log}$ be a fine log scheme. A morphism between log stacks $S^{\log} \to \overline{\mathcal{M}}^{\log}_{g,[q]+r}$ will be referred to as the *data for a* $([q]+r)$*-pointed stable log curve of genus* g. Let $C^{\log} \to S^{\log}$ be the pull-back of the universal stable log curve $\overline{\mathcal{C}}^{\log}_{g,[q]+r} \to \overline{\mathcal{M}}^{\log}_{g,[q]+r}$ via such a morphism. By a slight abuse of terminology, we shall refer to such a stable log curve $C^{\log} \to S^{\log}$ as a $([q]+r)$*-pointed stable log curve of genus* g. If we forget the log structures of such a stable log curve, the resulting $(f: C \to S; \mu_f \subseteq C)$ (where $\mu_f \subseteq C$ is the divisor of marked points) will be referred to as a $([q]+r)$*-pointed stable curve of genus* g, or, when $r=0$, simply as a $[q]$*-pointed stable curve of genus* g. When the integers q and g are left *unspecified*, a $[q]$-pointed stable curve of genus g will be referred to as a *symmetrically pointed stable curve [over S]*.

Definition 1.3. (cf. [11], §3.9) Let d be a positive integer; S a scheme; $(f: C \to S; \mu_f \subseteq C)$ and $(h: D \to S; \mu_h \subseteq D)$ symmetrically pointed stable curves over S. A finite morphism $\pi: C \to D$ over S will be called an *admissible covering [of degree d]* if it satisfies the following conditions:

- Each fiber of $h : D \to S$ admits a dense open subset over which π is finite flat of degree d.
- We have inclusions of effective relative (with respect to the morphism f) divisors $\mu_f \subseteq \pi^{-1}(\mu_h) \subseteq d \cdot \mu_f$ on C.
- The morphism f is smooth at $c \in C$ if and only if the morphism h is smooth at $\pi(c)$.
- The morphism π is étale, except
  - over μ_h, where it is tamely ramified;
  - at nodes of the geometric fibers over S: if $\overline{s}$ is a geometric point of S, λ is a node of $f^{-1}(\overline{s})$, and $\nu = \pi(\lambda)$, then there exist $a \in \mathfrak{m}^{\mathrm{sh}}_{S,\overline{s}}$, $x, y \in \mathfrak{m}^{\mathrm{sh}}_{C,\lambda}$, and $u, v \in \mathfrak{m}^{\mathrm{sh}}_{D,\nu}$ such that x, y (respectively, u, v) generate $\mathfrak{m}^{\mathrm{sh}}_{f^{-1}(\overline{s}),\lambda}$ (respectively, $\mathfrak{m}^{\mathrm{sh}}_{h^{-1}(\overline{s}),\nu}$), and $xy = a$, $uv = a^e$, $u = x^e$, $v = y^e$ (for some natural number e such that $e \in (\mathcal{O}^{\mathrm{sh}}_{S,\overline{s}})^\times$).

  Here, "$\mathfrak{m}^{\mathrm{sh}}$" denotes the maximal ideal of the strict henselization "$\mathcal{O}^{\mathrm{sh}}$" at the specified geometric point of the local ring in question.

An admissible covering $\pi : C \to D$ over S will be called a *simple admissible covering* if the discriminant divisor of π is *étale* over S in some neighborhood of μ_h.

Definition 1.4. Let (g, d) be a pair of nonnegative integers such that $2g-2+2d \geq 3$ and $d \geq 2$. For any scheme S over Spec $\mathbb{Z}[\frac{1}{d!}]$, write $\overline{\mathcal{H}}_{g,d}(S)$ for the following groupoid [i.e., a category in which every morphism is invertible]:

- Objects: an object is a *simple admissible covering* $\pi : C \to D$ of degree d from a $[(d-1)(2g-2+2d)]$-pointed stable curve $(f : C \to S; \mu_f \subseteq C)$ of genus g to a $[2g-2+2d]$-pointed stable curve $(h : D \to S; \mu_h \subseteq D)$ of genus 0.
- Morphisms: a morphism between two objects $\pi : C \to D$ and $\pi' : C' \to D'$ is a pair of isomorphisms $\alpha : C \xrightarrow{\sim} C'$ and $\beta : D \xrightarrow{\sim} D'$ that are compatible with the respective divisors of marked points such that $\beta \circ \pi = \pi' \circ \alpha$.

We shall refer to the resulting stack as the *compactified Hurwitz stack* $\overline{\mathcal{H}}_{g,d}$ *of type* (g, d).

Remark 1.4.1. One verifies immediately that the *Hurwitz stack* $\mathcal{H}_{g,d}$ of Definition 1.1 may be regarded as an *open substack* of the *compactified Hurwitz stack* $\overline{\mathcal{H}}_{g,d}$ of Definition 1.4, namely, the substack over which the pointed stable curves that appear in Definition 1.4 are *smooth* over S.

Remark 1.4.2. The stack $\mathcal{H}_{g,d}$ is *geometrically irreducible* over $\mathbb{Z}[\frac{1}{d!}]$ [cf. the assertion concerning "$\mathcal{HUS}_{b,d}$" in [11], §2.9]. Here, we note that whereas in [11], §2.9, one works over $\mathbb{Z}[\frac{1}{b!}]$, where $b = 2g-2+2d$ [cf. [11], §1.3], in the present discussion, we work over $\mathbb{Z}[\frac{1}{d!}]$. On the other hand, one verifies immediately that the asserted geometric irreducibility may be extended to the situation of the present discussion.

One of the main results of [11] is the following.

Theorem 1.5. *(cf.* [11], *§3.22, §3.23, and §3.27) Let* (g, d, r) *be a triple of non-negative integers such that* $2g - 2 + 2d + r \geq 3$ *and* $d \geq 2$*. Write* $\overline{\mathcal{A}}_{g,d,r}$ *for the stack over* $\mathbb{Z}[\frac{1}{d!}]$ *defined as follows: if* S *is a scheme, then we take the objects of* $\overline{\mathcal{A}}_{g,d,r}(S)$ *to be the simple admissible coverings* $\pi : C \to D$ *of degree* d *from a* $[(d-1)(2g-2+2d)+dr]$*-pointed stable curve* $(f : C \to S; \mu_f \subseteq C)$ *of genus* g *to a* $[2g-2+2d+r]$*-pointed stable curve* $(h : D \to S; \mu_h \subseteq D)$ *of genus* 0*; we take the morphisms of* $\overline{\mathcal{A}}_{g,d,r}(S)$ *between two objects* $\pi : C \to D$ *and* $\pi' : C' \to D'$ *to be the pairs of* S*-isomorphisms* $\alpha : C \to C'$ *and* $\beta : D \to D'$ *that are compatible with the respective divisors of marked points such that* $\pi' \circ \alpha = \beta \circ \pi$*. Then* $\overline{\mathcal{A}}_{g,d,r}$ *is a separated algebraic stack of finite type over* $\mathbb{Z}[\frac{1}{d!}]$*. Moreover,* $\overline{\mathcal{A}}_{g,d,r}$ *may be equipped with a natural log structure; denote the resulting log stack by* $\overline{\mathcal{A}}^{\log}_{g,d,r}$*. Finally, there is a natural morphism of log stacks* $\overline{\mathcal{A}}^{\log}_{g,d,r} \to (\overline{\mathcal{M}}^{\log}_{0,[2g-2+2d+r]})_{\mathbb{Z}[\frac{1}{d!}]}$ *(given by mapping* $(C; D; \pi) \mapsto D$*) over* $\mathbb{Z}[\frac{1}{d!}]$ *which is log étale, quasi-finite, and proper.*

Remark 1.5.1. One verifies immediately that, when $r = 0$, the stack $\overline{\mathcal{A}}_{g,d,0}$ may be naturally identified with the stack $\overline{\mathcal{H}}_{g,d}$ of Definition 1.4.

Remark 1.5.2. Write $\mathcal{A}_{g,d,r} \subseteq \overline{\mathcal{A}}_{g,d,r}$ for the open substack over which the curves C and D of Theorem 1.5 are *smooth*. Then a routine explicit computation of the completion of $\overline{\mathcal{A}}_{g,d,r}$ along a point valued in an algebraically closed field shows that the *normalization* $\widetilde{\mathcal{A}}_{g,d,r}$ of $\overline{\mathcal{A}}_{g,d,r}$ contains $\mathcal{A}_{g,d,r}$ as an open substack whose complement in $\widetilde{\mathcal{A}}_{g,d,r}$, equipped with the reduced induced stack structure, is a *relative divisor with normal crossings over* $\mathbb{Z}[\frac{1}{d!}]$, hence determines a log structure on $\widetilde{\mathcal{A}}_{g,d,r}$. Finally, $\widetilde{\mathcal{A}}_{g,d,r}$ is proper, smooth over $\mathbb{Z}[\frac{1}{d!}]$, and $\widetilde{\mathcal{A}}^{\log}_{g,d,r}$ is log smooth over $\mathbb{Z}[\frac{1}{d!}]$ [hence, in particular, log regular] and log étale, quasi-finite, and proper over $(\overline{\mathcal{M}}^{\log}_{0,[2g-2+2d+r]})_{\mathbb{Z}[\frac{1}{d!}]}$ [cf. [11], §3.23].

Lemma 1.6. *Let* (g, q, d, s, t) *be nonnegative integers such that* $d \geq 2$*;* $\pi : C \to D$ *a simple admissible covering of degree* d *from an* $[s]$*-pointed stable curve* $(f : C \to S; \mu_f \subseteq C)$ *of genus* g *to a* $[t]$*-pointed stable curve* $(h : D \to S; \mu_h \subseteq D)$ *of genus* q*. Suppose that* S *is connected. Then, if* $\sigma_f : S \to \mu_f$ *is a section (where we note that such sections always exist étale locally on* S*), then the ramification index of the restriction of* π *to each of the fibers of* f *along* σ_f *is constant on* S*. Moreover, if* π *is unramified (respectively, ramified) over a section* $\sigma_h : S \to \mu_h$*, then the underlying topological space of* $\pi^{-1}(\mathrm{Im}(\sigma_h))$ *is the disjoint union of the images, on underlying topological spaces, of* d *(respectively,* $(d-1)$*) distinct sections* $S \to \mu_f$*.*

Proof. Lemma 1.6 follows immediately from Definition 1.3. □

Next, we introduce the notions of *profiled simple admissible coverings* and *profiled Hurwitz stacks.*

Definition 1.7. Let (g, d, r) be a triple of nonnegative integers such that $2g - 2 + 2d + r \geq 3$ and $d \geq 2$; $\pi : C \to D$ a simple admissible covering of degree d from a $([(d-1)(2g-2+2d)] + dr)$-pointed stable curve $(f : C \to S; \mu_f \subseteq C)$ of genus g to a $([2g-2+2d]+r)$-pointed stable curve $(h : D \to S; \mu_h \subseteq D)$ of genus 0 [cf. Definition 1.2, Definition 1.3]. Then the morphism $\pi : C \to D$, equipped with these partial orderings on the marked points, will be called an *r-profiled simple admissible covering*, if these partial orderings on the marked points satisfy the following conditions [cf. Lemma 1.6]:

- The divisor μ_h consists, étale locally on S, of $2g-2+2d$ unordered sections over which π ramifies and r ordered sections $\sigma_1, \ldots, \sigma_r$ over which π is unramified.
- The divisor μ_f consists, étale locally on S, of $(d-1)(2g-2+2d)$ unordered sections over the sections of μ_h over which π ramifies and dr ordered sections over the sections $\sigma_1, \ldots, \sigma_r$ such that the sections over σ_k $(1 \leq k \leq r)$ are indexed by the natural numbers between $(k-1)d+1$ and kd.

When C and D are *smooth*, we shall, on occasion, omit the word *"admissible"* from this terminology *"r-profiled simple admissible covering"*.

Definition 1.8. Let (g, d, r) be a triple of nonnegative integers such that $2g-2+2d+r \geq 3$ and $d \geq 2$. For any scheme S over Spec $\mathbb{Z}[\frac{1}{d!}]$, write $\overline{\mathcal{H}}_{g,d,r}(S)$ for the following groupoid [i.e., a category in which every morphism is invertible]:

- Objects: an object is an *r-profiled simple admissible covering* $\pi : C \to D$ of degree d from a $([(d-1)(2g-2+2d)] + dr)$-pointed stable curve $(f : C \to S; \mu_f \subseteq C)$ of genus g to a $([2g-2+2d]+r)$-pointed stable curve $(h : D \to S; \mu_h \subseteq D)$ of genus 0.
- Morphisms: a morphism between two objects $\pi : C \to D$ and $\pi' : C' \to D'$ is a pair of isomorphisms $\alpha : C \xrightarrow{\sim} C'$ and $\beta : D \xrightarrow{\sim} D'$ that are compatible with respective divisors of marked points such that $\beta \circ \pi = \pi' \circ \alpha$.

We shall denote by $\mathcal{H}_{g,d,r} \subseteq \overline{\mathcal{H}}_{g,d,r}$ the open substack where the curves C and D of the profiled simple admissible covering $\pi : C \to D$ are smooth. We shall refer to $\mathcal{H}_{g,d,r}$ as the *r-profiled Hurwitz stack of type* (g, d).

Remark 1.8.1. When $r = 0$, the stack $\overline{\mathcal{H}}_{g,d,0}$ may be identified with the stack $\overline{\mathcal{H}}_{g,d}$ of Definition 1.4.

Corollary 1.9. *Let (g, d, r) be a triple of nonnegative integers such that $2g-2+2d+r \geq 3$, $d \geq 2$. Then there exists a natural (1-)morphism $\overline{\mathcal{H}}_{g,d,r} \to \overline{\mathcal{A}}_{g,d,r}$ which is finite étale and surjective. In particular, the relative dimension of $\mathcal{H}_{g,d,r}$ over $\mathbb{Z}[\frac{1}{d!}]$ is equal to $2g-2+2d+r-3 = 2g+2d+r-5$.*

Proof. One verifies immediately from Theorem 1.5 and Definition 1.8 that the only difference between the data parametrized by the stack $\overline{\mathcal{H}}_{g,d,r}$ and the data parametrized by the stack $\overline{\mathcal{A}}_{g,d,r}$ lies in the various *partial orderings* on the marked points. Thus, it follows immediately [cf. Lemma 1.6] that one has a natural (1-)morphism $\overline{\mathcal{H}}_{g,d,r} \to \overline{\mathcal{A}}_{g,d,r}$ that is finite étale and surjective. The final assertion concerning the relative dimension now follows immediately from the final portion of Theorem 1.5. This completes the proof. □

The pull-back of the canonical log structure on $\overline{\mathcal{A}}_{g,d,r}$ [cf. Theorem 1.5] via the finite étale covering

$$\overline{\mathcal{H}}_{g,d,r} \longrightarrow \overline{\mathcal{A}}_{g,d,r}$$

of Corollary 1.9 determines a canonical log structure on $\overline{\mathcal{H}}_{g,d,r}$. Denote the resulting log stack — which we shall refer to as the *r-profiled log Hurwitz stack of type* (g, d) — by $\overline{\mathcal{H}}^{\log}_{g,d,r}$. One verifies immediately that $\mathcal{H}_{g,d,r}$ may be identified with the interior of $\overline{\mathcal{H}}^{\log}_{g,d,r}$.

Proposition 1.10. *Let* (g, d, r) *be a triple of nonnegative integers such that* $2g - 2 + 2d + r \geq 3$, $d \geq 2$.

(i) The normalization $\widetilde{\mathcal{H}}_{g,d,r}$ *of* $\overline{\mathcal{H}}_{g,d,r}$ *is proper, smooth over* $\mathbb{Z}[\frac{1}{d!}]$*. Moreover,* $\mathcal{H}_{g,d,r}$ *may be regarded as an open substack of* $\widetilde{\mathcal{H}}_{g,d,r}$ *[cf. Remark 1.5.2], whose complement [in* $\widetilde{\mathcal{H}}_{g,d,r}$*], equipped with the reduced induced stack structure, is a divisor with normal crossings.*

(ii) The divisor with normal crossings of (i) determines a log structure on $\widetilde{\mathcal{H}}_{g,d,r}$*. Moreover, the resulting log stack* $\widetilde{\mathcal{H}}^{\log}_{g,d,r}$ *is log smooth over* $\mathbb{Z}[\frac{1}{d!}]$*, hence, in particular, log regular.*

(iii) There exists a natural $(1$*-)morphism*

$$\overline{\phi}^{\log}_{g,d,r} : \overline{\mathcal{H}}^{\log}_{g,d,r+1} \longrightarrow \overline{\mathcal{H}}^{\log}_{g,d,r}$$

obtained by forgetting the final d sections (respectively, final section) of the domain curve (respectively, codomain curve) of the covering. Now suppose further that $2g - 2 + dr \geq 1$*. Then there exists a natural* $(1$*-)morphism*

$$\overline{\psi}^{\log}_{g,d,r} : \overline{\mathcal{H}}^{\log}_{g,d,r} \longrightarrow \overline{\mathcal{M}}^{\log}_{g,dr},$$

determined by the domain curve of the covering, equipped with its dr ordered marked points. Moreover, we have a $(1$*-)commutative diagram*

$$\begin{array}{ccc} \overline{\mathcal{H}}^{\log}_{g,d,r+1} & \xrightarrow{\overline{\psi}^{\log}_{g,d,r+1}} & \overline{\mathcal{M}}^{\log}_{g,d(r+1)} \\ {\scriptstyle \overline{\phi}^{\log}_{g,d,r}}\downarrow & & \downarrow \\ \overline{\mathcal{H}}^{\log}_{g,d,r} & \xrightarrow{\overline{\psi}^{\log}_{g,d,r}} & \overline{\mathcal{M}}^{\log}_{g,dr}, \end{array}$$

where the right-hand vertical arrow is the morphism obtained by forgetting the final d sections.

(iv) The $(1$*-)morphism* $\widetilde{\phi}^{\log}_{g,d,r} : \widetilde{\mathcal{H}}^{\log}_{g,d,r+1} \to \widetilde{\mathcal{H}}^{\log}_{g,d,r}$ *induced by the* $(1$*-)morphism* $\overline{\phi}^{\log}_{g,d,r}$ *of (iii) is proper, log smooth, representable.*

(v) The algebraic stacks $\mathcal{H}_{g,d,r}$*,* $\overline{\mathcal{H}}^{\log}_{g,d,r}$*, and* $\widetilde{\mathcal{H}}^{\log}_{g,d,r}$ *are geometrically irreducible over* $\mathbb{Z}[\frac{1}{d!}]$*.*

(vi) The $(1$*-)morphism* $\widetilde{\phi}^{\log}_{g,d,r} : \widetilde{\mathcal{H}}^{\log}_{g,d,r+1} \to \widetilde{\mathcal{H}}^{\log}_{g,d,r}$ *of (iv) is a stable log curve, hence, in particular, has geometrically reduced, geometrically connected fibers.*

Proof. Since the (1-)morphism $\overline{\mathcal{H}}_{g,d,r} \to \overline{\mathcal{A}}_{g,d,r}$ is finite étale [cf. Corollary 1.9], assertions (i) and (ii) follow from the corresponding assertions for $\overline{\mathcal{A}}_{g,d,r}$ [cf. Remark 1.5.2].

Next, we consider assertion (iii). It follows immediately from the well-known uniqueness of the *contraction* morphism that arises by forgetting a marked point of a pointed stable curve [cf. [10], Proposition 2.1] that an r-profiled simple admissible covering of degree d induces [up to canonical isomorphism] a morphism from the curve constructed by contracting the final d sections of the domain curve of the covering to the curve constructed by contracting the final section of the codomain curve of the covering. Assertion (iii) now follows immediately.

Next, we consider assertion (iv). Consider the following (1-)commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{H}}^{\log}_{g,d,r+1} & \longrightarrow & \overline{\mathcal{M}}^{\log}_{0,[2g-2+2d]+r+1} \\ {\scriptstyle \widetilde{\phi}^{\log}_{g,d,r}} \big\downarrow & & \big\downarrow \\ \widetilde{\mathcal{H}}^{\log}_{g,d,r} & \longrightarrow & \overline{\mathcal{M}}^{\log}_{0,[2g-2+2d]+r}, \end{array}$$

where the horizontal arrows are the composites of the normalization morphisms with the (1-)morphisms obtained by sending $(\pi : C \to D) \mapsto D$, and the right-hand vertical arrow is the log smooth morphism obtained by forgetting the final section. Next, recall that it follows from Theorem 1.5, Remark 1.5.2, and Corollary 1.9 that the horizontal arrows of the above diagram are log étale. Since these horizontal arrows are log étale, and the right-hand vertical arrow of the diagram is log smooth [cf. the geometric properties of this morphism discussed in [10]], it then follows formally that the morphism $\widetilde{\phi}^{\log}_{g,d,r}$ is log smooth. The properness of $\widetilde{\phi}^{\log}_{g,d,r}$ follows immediately from the properness of $\widetilde{\mathcal{H}}^{\log}_{g,d,r}$ and $\widetilde{\mathcal{H}}^{\log}_{g,d,r+1}$ over $\mathbb{Z}[\frac{1}{d!}]$ [cf. Proposition 1.10, (i)].

Next, we consider the representability portion of assertion (iv). Consider the following (1-)commutative diagram

$$\begin{array}{ccc} \widetilde{\mathcal{H}}^{\log}_{g,d,r+1} & \longrightarrow & \overline{\mathcal{M}}^{\log}_{g,[(d-1)(2g-2+2d)]+d(r+1)} \times \overline{\mathcal{M}}^{\log}_{0,[2g-2+2d]+r+1} \\ {\scriptstyle \widetilde{\phi}^{\log}_{g,d,r}} \big\downarrow & & \big\downarrow \\ \widetilde{\mathcal{H}}^{\log}_{g,d,r} & \longrightarrow & \overline{\mathcal{M}}^{\log}_{g,[(d-1)(2g-2+2d)]+dr} \times \overline{\mathcal{M}}^{\log}_{0,[2g-2+2d]+r}, \end{array}$$

where the horizontal arrows are the composites of the normalization morphisms with the (1-)morphisms obtained by sending $(\pi : C \to D) \mapsto (C, D)$, and the right-hand vertical arrow is the morphism obtained by forgetting the final d sections on the left-hand factor and the final section on the right-hand factor. Note that the representability of the right-hand vertical arrow is well-known [cf. [10]], and the representability of the horizontal arrows follow immediately from the various constructions involved [cf. [11], the proof of Theorem in §3.22]. Since the horizontal arrows are representable, and the right-hand vertical arrow of the diagram is representable, it then follows formally that the morphism $\widetilde{\phi}^{\log}_{g,d,r}$ is representable.

Next, we consider assertion (v). Since $\mathcal{H}_{g,d,r}$ determines a dense open substack of $\overline{\mathcal{H}}^{\log}_{g,d,r}$ and $\widetilde{\mathcal{H}}^{\log}_{g,d,r}$ on every geometric fiber over $\mathbb{Z}[\frac{1}{d!}]$ [cf. Remark 1.5.2], it suffices to show that $\mathcal{H}_{g,d,r}$ is geometrically irreducible over $\mathbb{Z}[\frac{1}{d!}]$. Observe that when $r = 0$, the desired geometric irreducibility follows from Remarks 1.4.2 and 1.8.1; when $g = 0$, $d = 2$, and $r = 1$, the desired geometric irreducibility follows immediately by noting that $\mathcal{H}_{0,2,1}$ is isomorphic to a stack theoretic quotient of Spec $\mathbb{Z}[\frac{1}{d!}]$. Now write $\phi_{g,d,r} : \mathcal{H}_{g,d,r+1} \to \mathcal{H}_{g,d,r}$ for the (1-)morphism induced by restricting $\widetilde{\phi}^{\log}_{g,d,r}$

to $\mathcal{H}_{g,d,r+1}$. Observe that it follows from assertion (iv) that $\phi_{g,d,r}$ is representable and smooth, hence open. Moreover, it follows from Lemma 1.12 below that $\phi_{g,d,r}$ is geometrically irreducible. Thus, we conclude the desired geometric irreducibility for $\mathcal{H}_{g,d,r+1}$ by applying induction on r, together with Lemma 1.11, applied to the various morphisms obtained by base-changing $\phi_{g,d,r}$ to irreducible $\mathcal{H}_{g,d,r}$-schemes whose structure (1-)morphism to $\mathcal{H}_{g,d,r}$ is étale.

Next, we consider assertion (vi). First, we prove that the geometric fibers of the [proper, by Proposition 1.10, (iv)] (1-)morphism $\widetilde{\phi}^{\log}_{g,d,r}$ are connected. Since $\widetilde{\mathcal{H}}_{g,d,r+1}$ is normal, it follows from well-known properties of the Stein factorization that it suffices to verify that the geometric generic fiber of [the underlying (1-)morphism on algebraic stacks associated to] $\widetilde{\phi}^{\log}_{g,d,r}$ is connected. On the other hand, since the (1-)morphism $\phi_{g,d,r}$ discussed in the proof of assertion (v) is open and geometrically irreducible, this connectedness follows from the irreducibility of $\widetilde{\mathcal{H}}^{\log}_{g,d,r}$ and $\widetilde{\mathcal{H}}^{\log}_{g,d,r+1}$ [cf. Proposition 1.10, (v)], together with the fact that $\mathcal{H}_{g,d,r+1}$ is an open dense substack of $\widetilde{\mathcal{H}}^{\log}_{g,d,r+1}$ [cf. Remark 1.5.2]. This completes the verification of the geometric connectedness of $\widetilde{\phi}^{\log}_{g,d,r}$. In light of this geometric connectedness, it follows immediately from the explicit computation of the local structure of $\widetilde{\phi}^{\log}_{g,d,r}$ discussed in Remark 1.5.2, Corollary 1.9, that $\widetilde{\phi}^{\log}_{g,d,r}$ is a *log curve* in the sense of [7], Definition 1.2.

Thus, it follows from [7], Definition 1.12; [7], Theorem 4.5, that to verify that $\widetilde{\phi}^{\log}_{g,d,r}$ is a *stable log curve*, it suffices to verify that the sheaf of relative logarithmic differentials of $\widetilde{\phi}^{\log}_{g,d,r}$ is *relatively ample*, i.e., with respect to $\widetilde{\phi}^{\log}_{g,d,r}$. To this end, let us recall the (1-)commutative diagram of the first display in the proof of assertion (iv). Observe that it follows from Theorem 1.5, Remark 1.5.2, and Corollary 1.9, that the horizontal arrows of this diagram are log étale, quasi-finite, and proper. Thus, since both the right-hand vertical arrow of this diagram and $\widetilde{\phi}^{\log}_{g,d,r}$ are representable [cf. Proposition 1.10, (iv)], it follows formally that the (1-)morphism $\widetilde{\mathcal{H}}^{\log}_{g,d,r+1} \to \widetilde{\mathcal{H}}^{\log}_{g,d,r} \times_{\overline{\mathcal{M}}^{\log}_{0,[2g-2+2d]+r}} \overline{\mathcal{M}}^{\log}_{0,[2g-2+2d]+r+1}$ induced by the (1-)commutative diagram of the first display in the proof of assertion (iv) is finite, log étale. In particular, the desired relative ampleness of the sheaf of relative logarithmic differentials of $\widetilde{\phi}^{\log}_{g,d,r} : \widetilde{\mathcal{H}}^{\log}_{g,d,r+1} \to \widetilde{\mathcal{H}}^{\log}_{g,d,r}$ follows formally from the [well-known!] relative ampleness of the sheaf of relative logarithmic differentials of the stable log curve $\overline{\mathcal{M}}^{\log}_{0,[2g-2+2d]+r+1} \to \overline{\mathcal{M}}^{\log}_{0,[2g-2+2d]+r}$.

This completes the proof of Proposition 1.10. □

Lemma 1.11. *Let X and Y be topological spaces; $f : X \to Y$ a continuous map satisfying the following conditions:*

(i) Y is an irreducible topological space.

(ii) f is an open map.

(iii) For any $y \in Y$, $f^{-1}(y) \subseteq X$ is an irreducible topological space.

Then X is an irreducible topological space.

Proof. Suppose that X is not irreducible. Then there exist non-empty open subsets U_1 and U_2 of X such that $U_1 \cap U_2$ is empty. Since, by conditions (i) and (ii), $f(U_1)$

and $f(U_2)$ are non-empty open subsets with non-empty intersection, we conclude that there exists an element $y \in f(U_1) \cap f(U_2) \subseteq Y$ such that $f^{-1}(y) \subseteq X$ is not irreducible. But this contradicts condition (iii). $\square$

Lemma 1.12. *Let k be an algebraically closed field; $x : \mathrm{Spec}\ k \to \mathcal{H}_{g,d,r}$ a geometric point of $\mathcal{H}_{g,d,r}$ corresponding to a profiled simple covering $C \to \mathbb{P}^1_k$ of degree d from a $([(d-1)(2g-2+2d)] + dr)$-pointed smooth curve $(f : C \to \mathrm{Spec}\ k; \mu_f \subseteq C)$ of genus g to a $([2g-2+2d]+r)$-pointed projective line $(h : \mathbb{P}^1_k \to \mathrm{Spec}\ k; \mu_h \subseteq \mathbb{P}^1_k)$. Then the geometric fiber of $\phi_{g,d,r} : \mathcal{H}_{g,d,r+1} \to \mathcal{H}_{g,d,r}$ over $x : \mathrm{Spec}\ k \to \mathcal{H}_{g,d,r}$ is isomorphic to*

$$Z \overset{\mathrm{def}}{=} \left\{(C \setminus \mu_f) \times_{(\mathbb{P}^1_k \setminus \mu_h)} (C \setminus \mu_f) \times \cdots \times_{(\mathbb{P}^1_k \setminus \mu_h)} (C \setminus \mu_f)\right\} \setminus \Delta_Z,$$

where the fiber product is the fiber product of d copies of the morphism $C \setminus \mu_f \to \mathbb{P}^1_k \setminus \mu_h$, and Δ_Z denotes the union of the various diagonals associated to pairs of factors in the fiber product. Moreover, Z is the Galois closure of the covering $C \setminus \mu_f \to \mathbb{P}^1_k \setminus \mu_h$, hence, in particular, irreducible.

Proof. The first assertion follows immediately from the various definitions involved. To verify the final assertion, it suffices to verify that the Galois group of the Galois closure of the covering $C \setminus \mu_f \to \mathbb{P}^1_k \setminus \mu_h$ is isomorphic to the symmetric group on d letters S_d. On the other hand, this follows immediately from the well-known elementary fact that any subgroup of S_d that acts transitively on the set $\{1, \ldots, d\}$ and, moreover, is generated by transpositions is in fact equal to S_d. $\square$

Definition 1.13. Let (g, d, r) be a triple of nonnegative integers such that $2g-2+dr \geq 1$, $d \geq 2$ [conditions which imply, as is easily verified, that $2g-2+2d+r-2 \geq 1$]; k an algebraically closed field of characteristic zero.

(i) We shall denote by

$$u_{g,d,r} : \mathcal{C}_{g,d,r} \to \mathcal{H}_{g,d,r} \quad (\text{respectively, } \widetilde{u}^{\log}_{g,d,r} : \widetilde{\mathcal{C}}^{\log}_{g,d,r} \to \widetilde{\mathcal{H}}^{\log}_{g,d,r})$$

the pull-back of the tautological curve $\mathcal{C}_{g,dr} \to \mathcal{M}_{g,dr}$ (respectively, $\overline{\mathcal{C}}^{\log}_{g,dr} \to \overline{\mathcal{M}}^{\log}_{g,dr}$) via $\psi_{g,d,r} : \mathcal{H}_{g,d,r} \to \mathcal{M}_{g,dr}$ (respectively, $\widetilde{\psi}^{\log}_{g,d,r} : \widetilde{\mathcal{H}}^{\log}_{g,d,r} \to \overline{\mathcal{M}}^{\log}_{g,dr}$), where we write $\psi_{g,d,r}$ and $\widetilde{\psi}^{\log}_{g,d,r}$ for the morphisms induced by $\overline{\psi}^{\log}_{g,d,r}$ [cf. Proposition 1.10, (iii)]. We shall refer to $\mathcal{C}_{g,d,r}$ (respectively, $\widetilde{\mathcal{C}}^{\log}_{g,d,r}$) as the *tautological curve* over $\mathcal{H}_{g,d,r}$ (respectively, $\widetilde{\mathcal{H}}^{\log}_{g,d,r}$).

(ii) We shall append a subscript "k" to $\overline{\mathcal{H}}_{g,d,r}$, $\mathcal{H}_{g,d,r}$, $\overline{\mathcal{H}}^{\log}_{g,d,r}$, $\widetilde{\mathcal{H}}^{\log}_{g,d,r}$, $\mathcal{C}_{g,d,r}$, and $\widetilde{\mathcal{C}}^{\log}_{g,d,r}$, as well as to (1-)morphisms between these log stacks, to denote the result of base-changing to k.

Proposition 1.14. *Let (g, d, r) be a triple of nonnegative integers such that $2g-2+2d+r \geq 3$, $d \geq 2$; k an algebraically closed field of characteristic zero.*

(i) Suppose further that $2g-2+dr \geq 1$. Then the tautological curve $(\widetilde{u}^{\log}_{g,d,r})_k : (\widetilde{\mathcal{C}}^{\log}_{g,d,r})_k \to (\widetilde{\mathcal{H}}^{\log}_{g,d,r})_k$ is a proper, log smooth (1-)morphism between log regular log stacks [cf. Definition 1.13, (i), (ii)].

(ii) *Suppose further that* $2g-2+dr \geq 1$. *Let* $\overline{s}$ *(respectively,* $\overline{s}^{\log}$*) be a strict geometric point of* $(\mathcal{H}_{g,d,r})_k$ *(respectively,* $(\widetilde{\mathcal{H}}^{\log}_{g,d,r})_k$*). For suitable choices of basepoints, write*

$$\Pi_{\mathcal{C}_{\overline{s}}} \stackrel{\mathrm{def}}{=} \pi_1((\mathcal{C}_{g,d,r})_k \times_{(\mathcal{H}_{g,d,r})_k} \overline{s})$$
$$\text{(respectively, } \Pi_{\mathcal{C}_{\overline{s}^{\log}}} \stackrel{\mathrm{def}}{=} \pi_1((\widetilde{\mathcal{C}}^{\log}_{g,d,r})_k \times_{(\widetilde{\mathcal{H}}^{\log}_{g,d,r})_k} \overline{s}^{\log})\text{);}$$
$$\Pi_{\mathcal{H}_{g,d,r}} \stackrel{\mathrm{def}}{=} \pi_1((\mathcal{H}_{g,d,r})_k) \text{ (respectively, } \Pi_{\widetilde{\mathcal{H}}^{\log}_{g,d,r}} \stackrel{\mathrm{def}}{=} \pi_1((\widetilde{\mathcal{H}}^{\log}_{g,d,r})_k)\text{);}$$
$$\Pi_{\mathcal{C}_{g,d,r}} \stackrel{\mathrm{def}}{=} \pi_1((\mathcal{C}_{g,d,r})_k) \text{ (respectively, } \Pi_{\widetilde{\mathcal{C}}^{\log}_{g,d,r}} \stackrel{\mathrm{def}}{=} \pi_1((\widetilde{\mathcal{C}}^{\log}_{g,d,r})_k)\text{)}$$

[cf. Definition 1.13, (i), (ii)]. Then, for suitable choices of basepoints, we have a natural commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{\mathcal{C}_{\overline{s}}} & \longrightarrow & \Pi_{\mathcal{C}_{g,d,r}} & \longrightarrow & \Pi_{\mathcal{H}_{g,d,r}} & \longrightarrow & 1 \\ & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 1 & \longrightarrow & \Pi_{\mathcal{C}_{\overline{s}^{\log}}} & \longrightarrow & \Pi_{\widetilde{\mathcal{C}}^{\log}_{g,d,r}} & \longrightarrow & \Pi_{\widetilde{\mathcal{H}}^{\log}_{g,d,r}} & \longrightarrow & 1, \end{array}$$

where the vertical arrows are isomorphisms, and the horizontal sequences are exact.

(iii) *In the notation of assertion (ii), write*

$$\Pi_{\mathcal{H}_{\overline{s}}} \stackrel{\mathrm{def}}{=} \pi_1((\mathcal{H}_{g,d,r+1})_k \times_{(\mathcal{H}_{g,d,r})_k} \overline{s})$$
$$\text{(respectively, } \Pi_{\mathcal{H}_{\overline{s}^{\log}}} \stackrel{\mathrm{def}}{=} \pi_1((\widetilde{\mathcal{H}}^{\log}_{g,d,r+1})_k \times_{(\widetilde{\mathcal{H}}^{\log}_{g,d,r})_k} \overline{s}^{\log})\text{).}$$

Then, for suitable choices of basepoints, we have a natural commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{\mathcal{H}_{\overline{s}}} & \longrightarrow & \Pi_{\mathcal{H}_{g,d,r+1}} & \longrightarrow & \Pi_{\mathcal{H}_{g,d,r}} & \longrightarrow & 1 \\ & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 1 & \longrightarrow & \Pi_{\mathcal{H}_{\overline{s}^{\log}}} & \longrightarrow & \Pi_{\widetilde{\mathcal{H}}^{\log}_{g,d,r+1}} & \longrightarrow & \Pi_{\widetilde{\mathcal{H}}^{\log}_{g,d,r}} & \longrightarrow & 1, \end{array}$$

where the vertical arrows are isomorphisms, and the horizontal sequences are exact.

Proof. Assertion (i) follows immediately from the fact that log smoothness and properness are stable under base change, together with the fact that a log smooth scheme over a log regular scheme is log regular [cf. [9], Theorem 8.2].

Next, we consider assertions (ii) and (iii). First, we observe that the well-known functoriality of the étale fundamental group functor gives rise to natural commutative diagrams of fundamental groups as in the displays of assertions (ii) and (iii). Next, we observe that, in light of the log regularity portion of assertion (i), the fact that the vertical arrows of the diagrams of assertions (ii) and (iii) are isomorphisms follows immediately from the *log purity theorem* [cf., e.g., [12], Theorem B]. Next, we observe that it follows from assertion (i) and Proposition 1.10, (vi), that the (1-)morphisms $(\widetilde{u}^{\log}_{g,d,r})_k : (\widetilde{\mathcal{C}}^{\log}_{g,d,r})_k \to (\widetilde{\mathcal{H}}^{\log}_{g,d,r})_k$ and $(\widetilde{\phi}^{\log}_{g,d,r})_k : (\widetilde{\mathcal{H}}^{\log}_{g,d,r+1})_k \to (\widetilde{\mathcal{H}}^{\log}_{g,d,r})_k$ are *stable log curves*. Thus, the exactness of the horizontal sequences of the diagrams of assertions (ii) and (iii) follows immediately from the fact that these horizontal sequences may be identified with suitable pull-backs of an analogous [exact!] sequence in the universal case, i.e., the sequence in the first display of Theorem M, for suitable "(g, r)".

This completes the proof of Proposition 1.14. □

Remark 1.14.1. Let $m > 0$ be a positive integer. Then, by applying Proposition 1.14, (ii), (iii), successively, we obtain, for suitable choices of basepoints, natural commutative diagrams of profinite groups, as follows:

(1^{succ}) If we write $\Pi_{\mathcal{H}_{g,d,r+m}} \to \Pi_{\mathcal{H}_{g,d,r}}$ and $\Pi_{\widetilde{\mathcal{H}}^{\log}_{g,d,r+m}} \to \Pi_{\widetilde{\mathcal{H}}^{\log}_{g,d,r}}$ for the arrows induced by the composites $\phi_{g,d,r} \circ \phi_{g,d,r+1} \circ \cdots \circ \phi_{g,d,r+m-1} : \mathcal{H}_{g,d,r+m} \to \mathcal{H}_{g,d,r}$ and $\widetilde{\phi}^{\log}_{g,d,r} \circ \widetilde{\phi}^{\log}_{g,d,r+1} \circ \cdots \circ \widetilde{\phi}^{\log}_{g,d,r+m-1} : \widetilde{\mathcal{H}}^{\log}_{g,d,r+m} \to \widetilde{\mathcal{H}}^{\log}_{g,d,r}$, and $\Pi_{\mathcal{H}_{m,\overline{s}}}$ and $\Pi_{\mathcal{H}_{m,\overline{s}^{\log}}}$ for the étale fundamental groups of geometric fibers of these composites, then we obtain a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{\mathcal{H}_{m,\overline{s}}} & \longrightarrow & \Pi_{\mathcal{H}_{g,d,r+m}} & \longrightarrow & \Pi_{\mathcal{H}_{g,d,r}} & \longrightarrow & 1 \\ & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 1 & \longrightarrow & \Pi_{\mathcal{H}_{m,\overline{s}^{\log}}} & \longrightarrow & \Pi_{\widetilde{\mathcal{H}}^{\log}_{g,d,r+m}} & \longrightarrow & \Pi_{\widetilde{\mathcal{H}}^{\log}_{g,d,r}} & \longrightarrow & 1, \end{array}$$

where the vertical arrows are isomorphisms, and the horizontal sequences are exact.

(2^{succ}) First, we observe that since $m \geq 1$, [one verifies easily that] it holds that $2g-2+d(r+m) \geq 1$. If we write $\Pi_{\mathcal{C}_{g,d,r+m}} \to \Pi_{\mathcal{H}_{g,d,r}}$ and $\Pi_{\widetilde{\mathcal{C}}^{\log}_{g,d,r+m}} \to \Pi_{\widetilde{\mathcal{H}}^{\log}_{g,d,r}}$ for the arrows induced by the composites $\phi_{g,d,r} \circ \phi_{g,d,r+1} \circ \cdots \circ \phi_{g,d,r+m-1} \circ u_{g,d,r+m} : \mathcal{C}_{g,d,r+m} \to \mathcal{H}_{g,d,r}$ and $\widetilde{\phi}^{\log}_{g,d,r} \circ \widetilde{\phi}^{\log}_{g,d,r+1} \circ \cdots \circ \widetilde{\phi}^{\log}_{g,d,r+m-1} \circ \widetilde{u}^{\log}_{g,d,r+m} : \widetilde{\mathcal{C}}^{\log}_{g,d,r+m} \to \widetilde{\mathcal{H}}^{\log}_{g,d,r}$, and $\Pi_{\mathcal{C}_{m,\overline{s}}}$ and $\Pi_{\mathcal{C}_{m,\overline{s}^{\log}}}$ for the étale fundamental groups of geometric fibers of these composites, then we obtain a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{\mathcal{C}_{m,\overline{s}}} & \longrightarrow & \Pi_{\mathcal{C}_{g,d,r+m}} & \longrightarrow & \Pi_{\mathcal{H}_{g,d,r}} & \longrightarrow & 1 \\ & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 1 & \longrightarrow & \Pi_{\mathcal{C}_{m,\overline{s}^{\log}}} & \longrightarrow & \Pi_{\widetilde{\mathcal{C}}^{\log}_{g,d,r+m}} & \longrightarrow & \Pi_{\widetilde{\mathcal{H}}^{\log}_{g,d,r}} & \longrightarrow & 1, \end{array}$$

where the vertical arrows are isomorphisms, and the horizontal sequences are exact.

2. Hurwitz-type log configuration spaces

In this section, after defining Hurwitz-type log configuration spaces in Definition 2.2, we examine first properties of various objects related to these spaces [cf. Lemmas 2.3, 2.5, 2.7] that will be of use when we study the *centralizer* of the image of certain *geometric monodromy groups* in §3 and §4. After examining these first properties, we recall [cf. Proposition 2.8, Corollary 2.9, and Proposition 2.10] the existence of simple coverings that satisfy certain conditions; such existence results will be of use in the proof of Theorem 4.6. We also discuss [cf. Lemma 2.11] the existence of degenerations of simple coverings that satisfy certain conditions; this existence result will be of use in the proof of Proposition 3.1.

In this section, we shall apply, without further explanation, the theory and notational conventions concerning *semi-graphs of anabelioids of PSC-type* that are applied in [5], §6.

Definition 2.1. (cf. [16], Definition 1.1, (ii); [5], Definition 6.1) Let Σ be a nonempty set of prime numbers and Π the maximal pro-Σ quotient of the étale fundamental group of a hyperbolic curve over an algebraically closed field of characteristic zero [i.e., a pro-Σ surface group — cf. [17], Definition 1.2]. Then we shall write

$$\mathrm{Out}^{\mathrm{C}}(\Pi)$$

for the group of outomorphisms of Π which induce bijections on the set of cuspidal inertia subgroups of Π. We shall refer to an element of $\mathrm{Out}^{\mathrm{C}}(\Pi)$ as a *C-admissible* outomorphism of Π.

Definition 2.2. Let (g, d, r, m) be nonnegative integers such that $2g-2+2d+r \geq 3$, $d \geq 2$, and $m > 0$; $S^{\log}$ an fs log scheme over $\mathbb{Z}[\frac{1}{d!}]$. We shall refer to a morphism $\pi^{\log} : C^{\log} \to D^{\log}$ obtained by pulling back a (1-)morphism $S^{\log} \to \overline{\mathcal{H}}^{\log}_{g,d}$ — which we shall refer to as the associated *classifying (*1*-)morphism* — as a *simple log admissible covering of degree* d [cf. Definition 1.4]. We shall refer to a morphism $\pi^{\log} : C^{\log} \to D^{\log}$ obtained by pulling back a (1-)morphism $S^{\log} \to \overline{\mathcal{H}}^{\log}_{g,d,r}$ — which we shall refer to as the associated *classifying (*1*-)morphism* — as an *r-profiled simple log admissible covering of degree d* [cf. Definition 1.8]. Let $\pi^{\log} : C^{\log} \to D^{\log}$ be an *r-profiled simple log admissible covering of degree d* from a stable log curve $f^{\log} : C^{\log} \to S^{\log}$ of genus g to a stable log curve $h^{\log} : D^{\log} \to S^{\log}$ of genus 0. Then we shall refer to as the *m-th Hurwitz-type log configuration space* of $\pi^{\log} : C^{\log} \to D^{\log}$ the log scheme [over $S^{\log}$]

$$C^{\log}_m \stackrel{\mathrm{def}}{=} S^{\log} \times_{\widetilde{\mathcal{H}}^{\log}_{g,d,r}} \widetilde{\mathcal{H}}^{\log}_{g,d,r+m},$$

where $S^{\log} \to \widetilde{\mathcal{H}}^{\log}_{g,d,r}$ is the (1-)morphism determined [since $S^{\log}$ is assumed to be an *fs log scheme*] by the classifying morphism associated to the r-profiled simple log admissible covering under consideration, and the (1-)morphism $\widetilde{\mathcal{H}}^{\log}_{g,d,r+m} \to \widetilde{\mathcal{H}}^{\log}_{g,d,r}$ is given by forgetting the final dm sections (respectively, final m sections) of the domain curve (respectively, codomain curve) of the covering.

Lemma 2.3. *(cf.* [5]*, Lemma 6.2) Let (g, d, r, m) be nonnegative integers such that $2g-2+2d+r \geq 3$, $d \geq 2$, $m > 0$; $\Sigma_{\mathrm{F}} \subseteq \mathfrak{Primes}$ a nonempty set of prime numbers; k an algebraically closed field of characteristic zero; $S^{\log} \stackrel{\mathrm{def}}{=} (\mathrm{Spec}\ k)^{\log}$ the log scheme obtained by equipping* Spec k *with the log structure given by the fs chart $\mathbb{N} \to k$ that maps $1 \to 0$; $\pi^{\log} : C^{\log} \to D^{\log}$ an r-profiled simple log admissible covering of degree d from a stable log curve $f^{\log} : C^{\log} \to S^{\log}$ of genus g to a stable log curve $h^{\log} : D^{\log} \to S^{\log}$ of genus* 0*. Write*

$$C^{\log}_m$$

for the m-th Hurwitz-type log configuration space of the r-profiled simple log admissible covering $\pi^{\log} : C^{\log} \to D^{\log}$ [cf. Definition 2.2]; Π_{B} for the kernel of the natural [outer] surjection $\pi_1(C^{\log}_m) \to \pi_1(S^{\log})$; Π^*_{T} for the kernel of the natural [outer] surjection $\pi_1(C^{\log}_m \times_{\widetilde{\mathcal{H}}^{\log}_{g,d,r+m}} \widetilde{\mathcal{C}}^{\log}_{g,d,r+m}) \twoheadrightarrow \pi_1(S^{\log})$ [cf. Definition 1.13, (i)]; Π^*_{F} for the kernel of the natural [outer] surjection $\Pi^*_{\mathrm{T}} \twoheadrightarrow \Pi_{\mathrm{B}}$ induced by the projection $C^{\log}_m \times_{\widetilde{\mathcal{H}}^{\log}_{g,d,r+m}} \widetilde{\mathcal{C}}^{\log}_{g,d,r+m} \to C^{\log}_m$; Π_{F} for the the maximal pro-Σ_{F} quotient of

Π_{F}^{*}; Π_{T} *for the quotient of* Π_{T}^{*} *by the kernel of the natural surjection* $\Pi_{\mathrm{F}}^{*} \twoheadrightarrow \Pi_{\mathrm{F}}$. *Thus, we have a natural exact sequence of profinite groups*

$$1 \longrightarrow \Pi_{\mathrm{F}} \longrightarrow \Pi_{\mathrm{T}} \longrightarrow \Pi_{\mathrm{B}} \longrightarrow 1,$$

which determines an outer representation

$$\rho_m : \Pi_{\mathrm{B}} \longrightarrow \mathrm{Out}(\Pi_{\mathrm{F}}).$$

Then the following hold:

(i) *The isomorphism class of the exact sequence of profinite groups*

$$1 \longrightarrow \Pi_{\mathrm{F}} \longrightarrow \Pi_{\mathrm{T}} \longrightarrow \Pi_{\mathrm{B}} \longrightarrow 1$$

depends only on (g, d, r, m) *and the set* Σ_{F}*, i.e., if* $1 \to \Pi_{\mathrm{F}}^{\bullet} \to \Pi_{\mathrm{T}}^{\bullet} \to \Pi_{\mathrm{B}}^{\bullet} \to 1$ *is the exact sequence "*$1 \to \Pi_{\mathrm{F}} \to \Pi_{\mathrm{T}} \to \Pi_{\mathrm{B}} \to 1$*" associated, with respect to the same* (g, d, r, m) *and* Σ_{F}*, to another* r*-profiled simple log admissible covering of degree* d *from a stable log curve of genus* g *to a stable log curve of genus* 0*, then there exists a commutative diagram of profinite groups*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{\mathrm{F}} & \longrightarrow & \Pi_{\mathrm{T}} & \longrightarrow & \Pi_{\mathrm{B}} & \longrightarrow & 1 \\ & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 1 & \longrightarrow & \Pi_{\mathrm{F}}^{\bullet} & \longrightarrow & \Pi_{\mathrm{T}}^{\bullet} & \longrightarrow & \Pi_{\mathrm{B}}^{\bullet} & \longrightarrow & 1, \end{array}$$

where the vertical arrows are isomorphisms which may be chosen to arise scheme-theoretically [i.e., via specialization and generization], hence, in particular, to be compatible with the respective cuspidal subgroups of Π_{F} *and* $\Pi_{\mathrm{F}}^{\bullet}$ *[cf. Lemma 2.3, (ii)], as well as with the orderings on the ordered cusps [cf. Definition 1.7] of the fibers of* $\widetilde{\mathcal{C}}_{g,d,r+m}^{\log} \to \widetilde{\mathcal{H}}_{g,d,r+m}^{\log}$ *under consideration.*

(ii) Π_{F} *is the maximal pro-*Σ_{F} *quotient of the étale fundamental group of a hyperbolic curve over an algebraically closed field of characteristic zero [i.e., a pro-*Σ_{F} *surface group — cf.* [17]*, Definition 1.2].*

(iii) *The outer representation* $\rho_m : \Pi_{\mathrm{B}} \to \mathrm{Out}(\Pi_{\mathrm{F}})$ *factors through the closed subgroup* $\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}) \subseteq \mathrm{Out}(\Pi_{\mathrm{F}})$ *[cf. Definition 2.1].*

Proof. Assertion (i) follows immediately by considering a suitable specialization isomorphism, i.e., by *varying the basepoint "*$\overline{s}^{\log}$*"* in the exact sequences of Proposition 1.14, (ii) [where we take "r" to be $r+m$ and recall the easily verified fact that, since $m \geq 1$, it holds that $2g - 2 + d(r + m) \geq 1$]; Remark 1.14.1, (1^{succ}). Assertion (ii) follows immediately from assertion (i) and the various definitions involved. Assertion (iii) follows immediately from the various definitions involved. □

Definition 2.4. (cf. [5], Definition 6.3) We apply the notational conventions of Lemma 2.3 in the case where

$$m = 1,\ 2g - 2 + dr \geq 1.$$

Let $x \in C_1(k)$ be a k-valued point of the underlying scheme C_1 of $C_1^{\log} = S^{\log} \times_{\widetilde{\mathcal{H}}_{g,d,r}^{\log}} \widetilde{\mathcal{H}}_{g,d,r+1}^{\log}$ [cf. Definition 2.2]. Write

$$C_*^{\log} \stackrel{\mathrm{def}}{=} S^{\log} \times_{\widetilde{\mathcal{H}}_{g,d,r}^{\log}} \widetilde{\mathcal{C}}_{g,d,r}^{\log}$$

for the stable log curve [cf. Proposition 1.10, (vi)] determined by the (1-)morphism $\widetilde{u}_{g,d,r}^{\log} : \widetilde{\mathcal{C}}_{g,d,r}^{\log} \to \widetilde{\mathcal{H}}_{g,d,r}^{\log}$ of Definition 1.13, (i), and the (1-)morphism $S^{\log} \to \widetilde{\mathcal{H}}_{g,d,r}^{\log}$

determined [since $S^{\log}$ is an fs log scheme] by the classifying (1-)morphism $S^{\log} \to \overline{\mathcal{H}}^{\log}_{g,d,r}$ of the r-profiled simple log admissible covering $\pi^{\log} : C^{\log} \to D^{\log}$;

$$C^{\log}_x \stackrel{\text{def}}{=} x^{\log} \times_{\widetilde{\mathcal{H}}^{\log}_{g,d,r+1}} \widetilde{\mathcal{C}}^{\log}_{g,d,r+1}$$

for the stable log curve [cf. Proposition 1.10, (vi)] over $x^{\log} \stackrel{\text{def}}{=} x \times_{C_1} C^{\log}_1$ determined by the (1-)morphism $\widetilde{u}^{\log}_{g,d,r+1} : \widetilde{\mathcal{C}}^{\log}_{g,d,r+1} \to \widetilde{\mathcal{H}}^{\log}_{g,d,r+1}$ of Definition 1.13, (i). Thus, we have natural contraction morphisms

$$C^{\log}_x \ \to \ C^{\log}_* \ \leftarrow \ C^{\log}$$

of stable log curves over $S^{\log}$.

(i) We shall denote by

$$\mathcal{G}_*$$

the semi-graph of anabelioids of pro-$\mathfrak{Primes}$ PSC-type determined by the stable log curve $C^{\log}_*$; by

$$\mathcal{G}_x$$

the semi-graph of anabelioids of pro-Σ_{F} PSC-type determined by the stable log curve $C^{\log}_x$; by $\Pi_{\mathcal{G}_*}$, $\Pi_{\mathcal{G}_x}$ the [pro-$\mathfrak{Primes}$, pro-Σ_{F}] fundamental groups of $\mathcal{G}_*$, $\mathcal{G}_x$, respectively. Thus, we have a natural $\mathrm{Im}(\rho_1)$ ($\subseteq \mathrm{Out}(\Pi_{\mathrm{F}})$)-torsor of outer isomorphisms

$$\Pi_{\mathrm{F}} \stackrel{\sim}{\longrightarrow} \Pi_{\mathcal{G}_x}.$$

Let us fix an isomorphism $\Pi_{\mathrm{F}} \stackrel{\sim}{\to} \Pi_{\mathcal{G}_x}$ that belongs to this collection of isomorphisms.

(ii) Let $1 \leq i \leq d$ be an integer. Then let us observe that the $(dr+i)$-th tautological section $\overline{\mathcal{M}}_{g,d(r+1)} \hookrightarrow \overline{\mathcal{M}}_{g,d(r+1)+1}$ of the tautological curve $\overline{\mathcal{M}}_{g,d(r+1)+1} \to \overline{\mathcal{M}}_{g,d(r+1)}$ determines, by pull-back via the composite of natural (1-)morphisms

$$C^{\log}_1 \to \widetilde{\mathcal{H}}^{\log}_{g,d,r+1} \stackrel{\widetilde{\psi}^{\log}_{g,d,r+1}}{\to} \overline{\mathcal{M}}^{\log}_{g,d(r+1)}$$

[cf. Proposition 1.10, (iii); the fact that $2g-2+d(r+1) \geq 1$], a section of the underlying morphism of schemes of the natural projection morphism $C^{\log}_1 \times_{\widetilde{\mathcal{H}}^{\log}_{g,d,r+1}} \widetilde{\mathcal{C}}^{\log}_{g,d,r+1} \stackrel{\sim}{\to} C^{\log}_1 \times_{\overline{\mathcal{M}}^{\log}_{g,d(r+1)}} \overline{\mathcal{M}}^{\log}_{g,d(r+1)+1} \to C^{\log}_1$ [cf. Definition 1.13, (i)]. Write

$$D_i$$

for the image in the underlying scheme of

$$C^{\log}_1 \times_{\widetilde{\mathcal{H}}^{\log}_{g,d,r+1}} \widetilde{\mathcal{C}}^{\log}_{g,d,r+1} \stackrel{\sim}{\to} C^{\log}_1 \times_{\overline{\mathcal{M}}^{\log}_{g,d(r+1)}} \overline{\mathcal{M}}^{\log}_{g,d(r+1)+1}$$

of this section. Write

$$\mathrm{pr}_i : C_1 \longrightarrow C_*$$

for the composite of natural (1-)morphisms

$$C_1 \stackrel{\sim}{\to} S \times_{\widetilde{\mathcal{H}}_{g,d,r}} \widetilde{\mathcal{H}}_{g,d,r+1} \to S \times_{\overline{\mathcal{M}}_{g,dr}} \overline{\mathcal{M}}_{g,d(r+1)} \to C_*,$$

where S, C_* are the underlying schemes of $S^{\log}$, $C^{\log}_*$ [cf. Proposition 1.10, (vi)]; the final morphism is the morphism determined by the $(dr+i)$-th marked point of the tautological pointed stable curve parametrized by $\overline{\mathcal{M}}_{g,d(r+1)}$. One verifies easily that $\mathrm{pr}_i : C_1 \longrightarrow C_*$ is *surjective.*

(iii) Denote by
$$c^{\mathrm{F}}_{D_i,x} \in \mathrm{Cusp}(\mathcal{G}_x)$$
the cusp of $\mathcal{G}_x$ [i.e., the cusp of the geometric fiber of $C_1^{\log} \times_{\widetilde{\mathcal{H}}^{\log}_{g,d,r+1}} \widetilde{\mathcal{C}}^{\log}_{g,d,r+1} \to C_1^{\log}$ over $x^{\log}$] determined by the divisor D_i [which lies inside the underlying scheme of $C_1^{\log} \times_{\widetilde{\mathcal{H}}^{\log}_{g,d,r+1}} \widetilde{\mathcal{C}}^{\log}_{g,d,r+1}$] of (ii). For $v \in \mathrm{Vert}(\mathcal{G}_*)$ (respectively, $c \in \mathrm{Cusp}(\mathcal{G}_*)$), denote by
$$v^{\mathrm{F}}_x \in \mathrm{Vert}(\mathcal{G}_x) \text{ (respectively, } c^{\mathrm{F}}_x \in \mathrm{Cusp}(\mathcal{G}_x))$$
the vertex (respectively, cusp) of $\mathcal{G}_x$ that corresponds naturally to $v \in \mathrm{Vert}(\mathcal{G}_*)$ (respectively, $c \in \mathrm{Cusp}(\mathcal{G}_*)$) [cf. the notational conventions of [6], Definition 1.1, (i)].

(iv) Let $y \in C_*(k)$ be a k-valued point of C_*. Let $e \in \mathrm{Edge}(\mathcal{G}_*)$, $v \in \mathrm{Vert}(\mathcal{G}_*)$, $S \subseteq \mathrm{VCN}(\mathcal{G}_*)$, and $z \in \mathrm{VCN}(\mathcal{G}_*)$ [cf. the notational conventions of [6], Definition 1.1, (i), (iii)]. Then we shall say that y lies on e if the image of y is the cusp or node corresponding to $e \in \mathrm{Edge}(\mathcal{G}_*)$. We shall say that y lies on v if y does not lie on any edge of $\mathcal{G}_*$, and, moreover, the image of y is contained in the irreducible component corresponding to $v \in \mathrm{Vert}(\mathcal{G}_*)$. We shall write $y \curvearrowright S$ if y lies on some $s \in S$. We shall write $y \curvearrowright z$ if $y \curvearrowright \{z\}$.

Lemma 2.5. *(cf.* [5], *Lemma 6.4) In the notation of Definition 2.4, let $x, x' \in C_1(k)$ be k-valued points of C_1. Then the following hold:*

(i) The isomorphism $\Pi_{\mathcal{G}_x} \xrightarrow{\sim} \Pi_{\mathcal{G}_{x'}}$ obtained by forming the composite of the isomorphisms $\Pi_{\mathcal{G}_x} \xleftarrow{\sim} \Pi_{\mathrm{F}} \xrightarrow{\sim} \Pi_{\mathcal{G}_{x'}}$ [cf. Definition 2.4, (i)] is group-theoretically cuspidal [cf. [15], *Definition 1.4, (iv)].*

(ii) The injection $\mathrm{Cusp}(\mathcal{G}_*) \hookrightarrow \mathrm{Cusp}(\mathcal{G}_x)$ given by mapping $c \mapsto c^{\mathrm{F}}_x$ determines a bijection
$$\mathrm{Cusp}(\mathcal{G}_*) \xrightarrow{\sim} \mathrm{Cusp}(\mathcal{G}_x) \setminus \{c^{\mathrm{F}}_{D_i,x} \ (1 \leq i \leq d)\}$$
[cf. Definition 2.4, (iii)]. Moreover, if we regard $\mathrm{Cusp}(\mathcal{G}_*)$ as a subset of each of the sets $\mathrm{Cusp}(\mathcal{G}_x)$, $\mathrm{Cusp}(\mathcal{G}_{x'})$ by means of the above injections, then the bijection $\mathrm{Cusp}(\mathcal{G}_x) \xrightarrow{\sim} \mathrm{Cusp}(\mathcal{G}_{x'})$ determined by the group-theoretically cuspidal isomorphism $\Pi_{\mathcal{G}_x} \xrightarrow{\sim} \Pi_{\mathcal{G}_{x'}}$ of (i) maps $c^{\mathrm{F}}_{D_i,x} \mapsto c^{\mathrm{F}}_{D_i,x'}$ $(1 \leq i \leq d)$ and induces the identity automorphism on $\mathrm{Cusp}(\mathcal{G}_*)$. Thus, in the remainder of this paper, we shall omit the subscript "x" from the notation "c^{F}_x" and "$c^{\mathrm{F}}_{D_i,x}$".

(iii) Suppose that the log curve $C^{\log}$ is irreducible. [Thus, we have a natural isomorphism $C \xrightarrow{\sim} C_*$.] Then the injection $\mathrm{Vert}(\mathcal{G}_*) \hookrightarrow \mathrm{Vert}(\mathcal{G}_x)$ given by mapping $v \mapsto v^F_x$ [cf. Definition 2.4, (iii)] is bijective if and only if $x_i \stackrel{\mathrm{def}}{=} \mathrm{pr}_i(x) \curvearrowright \mathrm{Vert}(\mathcal{G}_*)$ $(1 \leq i \leq d)$ [cf. Definition 2.4, (ii),(iv)], and the x_i $(1 \leq i \leq d)$ are distinct elements of $C_*(k)$ $(\xleftarrow{\sim} C(k))$.

(iv) Suppose that $C^{\log}$ is a smooth log curve. [Thus, we have a natural isomorphism $C \xrightarrow{\sim} C_*$.] Let (i,j) be integers satisfying $1 \leq i < j \leq d$. Write $v \in \mathrm{Vert}(\mathcal{G}_*)$ for the unique element of $\mathrm{Vert}(\mathcal{G}_*)$. Suppose that $\mathrm{pr}_i(x) = \mathrm{pr}_j(x) \in C_*(k)$. Then $\mathcal{G}_x$ satisfies the following conditions:

- *The complement of the image of $\mathrm{Vert}(\mathcal{G}_*)$ in $\mathrm{Vert}(\mathcal{G}_x)$ is a set of cardinality one whose unique element*

$$v^{\mathrm{F}}_{\mathrm{new},x} \in \mathrm{Vert}(\mathcal{G}_x) \setminus \mathrm{Vert}(\mathcal{G}_*)$$

is of type $(0,3)$. *Moreover,* $\mathcal{C}(v^{\mathrm{F}}_{\mathrm{new},x}) = \{c^{\mathrm{F}}_{D_i}, c^{\mathrm{F}}_{D_j}\}$ *[cf. Lemma 2.5, (ii);* [6], *Definition 1.1, (iv)].*

- $\mathcal{N}(v^{\mathrm{F}}_x) = \mathrm{Node}(\mathcal{G}_x) = \mathcal{N}(v^{\mathrm{F}}_{\mathrm{new},x})$ *[cf.* [6], *Definition 1.1, (iv)] is a set of cardinality one.*
- $\mathcal{C}(v^{\mathrm{F}}_x) \setminus \mathcal{C}(v) = \{c^{\mathrm{F}}_{D_h} (h \neq i,j)\}$.

(v) *Suppose that* $C^{\log}$ *is a smooth log curve. [Thus, we have a natural isomorphism* $C \xrightarrow{\sim} C_*$*.] Let* l *be an integer satisfying* $1 \leq l \leq r$. *Write* $v \in \mathrm{Vert}(\mathcal{G}_*)$ *for the unique element of* $\mathrm{Vert}(\mathcal{G}_*)$. *Suppose that* $x_i \stackrel{\mathrm{def}}{=} \mathrm{pr}_i(x) \curvearrowright \mathrm{Cusp}(\mathcal{G}_*)$ *for some* $i \in \{1,\ldots,d\}$ *[or, equivalently, for all* $i \in \{1,\ldots,d\}$*], and* $\pi^{\log}(x_i)$ *[cf. Lemma 2.3] is the* l*-th cusp of* $D^{\log}$. *Then* $\mathcal{G}_x$ *satisfies the following conditions:*

- *The complement of the image of* $\mathrm{Vert}(\mathcal{G}_*)$ *in* $\mathrm{Vert}(\mathcal{G}_x)$ *is a set of cardinality* d*, each of whose elements is of type* $(0,3)$. *Let* j *be an integer satisfying* $1 \leq j \leq d$. *If we write*
$$v^{\mathrm{F}}_{\mathrm{new},j,x} \in \mathrm{Vert}(\mathcal{G}_x) \setminus \mathrm{Vert}(\mathcal{G}_*)$$
for the unique element of $\mathrm{Vert}(\mathcal{G}_x) \setminus \mathrm{Vert}(\mathcal{G}_*)$ *that abuts to* $c^{\mathrm{F}}_{D_j}$ *[cf. Lemma 2.5, (ii)], then* $\mathcal{C}(v^{\mathrm{F}}_{\mathrm{new},j,x}) = \{c^{\mathrm{F}}_{l,j}, c^{\mathrm{F}}_{D_j}\}$*, where we write* $c_{l,j}$ *for the* $((l-1)d+j)$*-th cusp of* $\mathcal{G}_*$ *[cf. Definition 2.4, (iii)].*
- *Let* j *be an integer satisfying* $1 \leq j \leq d$. $\mathcal{N}(v^{\mathrm{F}}_{\mathrm{new},j,x})$ *is a set of cardinality one. If we write*
$$e^{\mathrm{F}}_j \in \mathcal{N}(v^{\mathrm{F}}_{\mathrm{new},j,x})$$
for the unique element of $\mathcal{N}(v^{\mathrm{F}}_{\mathrm{new},j,x})$*, then* $\mathcal{N}(v^{\mathrm{F}}_x) = \mathrm{Node}(\mathcal{G}_x) = \{e^{\mathrm{F}}_1, e^{\mathrm{F}}_2, \ldots, e^{\mathrm{F}}_d\}$.
- $\mathcal{C}(v^{\mathrm{F}}_x)^{\#} = d(r-1)$.

Proof. Assertion (i) follows immediately from the fact that the cuspidal subgroups in question arise from divisors of the underlying scheme of $C^{\log}_1 \times_{\widetilde{\mathcal{H}}^{\log}_{g,d,r+1}} \widetilde{\mathcal{C}}^{\log}_{g,d,r+1}$. Assertions (ii), (iii), (iv), and (v) follow immediately from the various definitions involved. This completes the proof. □

Definition 2.6. (cf. [5], Definition 6.5) In the notation of Definition 2.4:

(i) Write
$$\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}_*) \stackrel{\mathrm{def}}{=} \mathrm{Cusp}(\mathcal{G}_*) \sqcup \{c^{\mathrm{F}}_{D_i}\ (1 \leq i \leq d)\}$$
[cf. Definition 2.4, (iii); Lemma 2.5, (ii)].

(ii) Let $\alpha \in \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})$ be a C-admissible outomorphism of Π_{F} [cf. Definition 2.1; Lemma 2.3, (ii)]. Then it follows immediately from Lemma 2.5, (i), (ii), that the automorphism of $\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}_*)$ [cf. Definition 2.6, (i)] obtained by conjugating the natural action of α on $\mathrm{Cusp}(\mathcal{G}_x)$ by the natural bijection $\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}_*) \xrightarrow{\sim} \mathrm{Cusp}(\mathcal{G}_x)$ implicit in Lemma 2.5, (ii), does not depend on the choice of x. We shall refer to this automorphism of $\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}_*)$ as the automorphism of $\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}_*)$ determined by α. Thus, we have a natural homomorphism $\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}) \to \mathrm{Aut}(\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}_*))$.

(iii) For $c \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}_*)$ [cf. Definition 2.6, (i)], we shall refer to a closed subgroup of Π_{F} obtained as the image — via the fixed isomorphism $\Pi_{\mathcal{G}_x} \xleftarrow{\sim} \Pi_{\mathrm{F}}$

of Definition 2.4, (i) — of a cuspidal subgroup of $\Pi_{\mathcal{G}_x}$ associated to the cusp of $\mathcal{G}_x$ corresponding to $c \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}_*)$ as a cuspidal subgroup of Π_{F} associated to $c \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}_*)$. Note that it follows immediately from Lemma 2.5, (ii), that the Π_{F}-conjugacy class of a cuspidal subgroup of Π_{F} associated to $c \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}_*)$ depends only on $c \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}_*)$, i.e., does not depend on the choice of x or on the choices of isomorphisms made in Definition 2.4, (i).

Lemma 2.7. *(cf.* [5], *Lemma 6.7) In the notation of Definition 2.4, let* $H \subseteq \Pi_{\mathrm{B}}$ *be an open subgroup of* Π_{B}, $\tilde{\alpha}$ *an automorphism of* $\Pi_{\mathrm{T}}|_H \stackrel{\mathrm{def}}{=} \Pi_{\mathrm{T}} \times_{\Pi_{\mathrm{B}}} H$ *over* H *[i.e., an automorphism that preserves and induces the identity automorphism on the quotient* $\Pi_{\mathrm{T}}|_H \twoheadrightarrow H$*],* $\alpha_{\mathrm{F}} \in \mathrm{Out}(\Pi_{\mathrm{F}})$ *the outomorphism of* Π_{F} *determined by the restriction* $\tilde{\alpha}|_{\Pi_{\mathrm{F}}}$ *of* $\tilde{\alpha}$ *to* $\Pi_{\mathrm{F}} \subseteq \Pi_{\mathrm{T}}|_H$, $\Pi_{c^{\mathrm{F}}_{D_i}} \subseteq \Pi_{\mathrm{F}}$ $(1 \leq i \leq d)$ *a cuspidal subgroup of* Π_{F} *associated to* $c^{\mathrm{F}}_{D_i} \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}_*)$ *[cf. Definition 2.6, (i), (iii)], and* $N_d \subseteq \Pi_{\mathrm{F}}$ *the normal closed subgroup of* Π_{F} *topologically normally generated by the* $\Pi_{c^{\mathrm{F}}_{D_i}}$*, where* $i = 1, \dots, d$*. Then the following hold:*

(i) Suppose that, for each $i = 1, \dots, d$, $\tilde{\alpha}$ *preserves the* Π_{F}*-conjugacy class of* $\Pi_{c^{\mathrm{F}}_{D_i}} \subseteq \Pi_{\mathrm{F}}$*. Then the outomorphism of* Π_{F}/N_d *induced by* $\tilde{\alpha}$ *is the identity outomorphism. If, moreover,* α_{F} *is* C*-admissible [cf. Definition 2.1; Lemma 2.3, (ii)], then the automorphism of* $\mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}_*)$ *induced by* α_{F} *[cf. Definition 2.6, (ii)] is the identity automorphism.*

(ii) Suppose that α_{F} *is* C*-admissible, and that* $C^{\log}$ *is a smooth log curve. Then it holds that* $\alpha_{\mathrm{F}} \in \mathrm{Aut}(\mathcal{G}_x)$ $(\subseteq \mathrm{Out}(\Pi_{\mathcal{G}_x}) \stackrel{\sim}{\leftarrow} \mathrm{Out}(\Pi_{\mathrm{F}}))$*. If, moreover, for each* $i = 1, \dots, d$, $\tilde{\alpha}$ *preserves the* Π_{F}*-conjugacy class of* $\Pi_{c^{\mathrm{F}}_{D_i}} \subseteq \Pi_{\mathrm{F}}$*, then* $\alpha_{\mathrm{F}} \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x) \subseteq \mathrm{Aut}(\mathcal{G}_x)$*, where* $\mathrm{Aut}^{|\mathrm{grph}|}(-)$ *is defined as the subgroup of* $\mathrm{Aut}(-)$ *of automorphisms of* $-$ *which induce the identity automorphism on the underlying semi-graph of* $-$ *[cf. for instance,* [5], *Theorem B].*

Proof. First, we verify assertion (i). By replacing $\tilde{\alpha}$ by a suitable Π_{F}-conjugate, we may assume that $\tilde{\alpha}$ preserves $\Pi_{c^{\mathrm{F}}_1} \subseteq \Pi_{\mathrm{F}}$. Since the decomposition group $D \subseteq \Pi_{\mathrm{T}}|_H$ of $\Pi_{\mathrm{T}}|_H$ associated to the divisor D_1 of $C_1^{\log} \times_{\widetilde{\mathcal{H}}^{\log}_{g,d,r+1}} \widetilde{\mathcal{C}}^{\log}_{g,d,r+1}$ [cf. Definition 2.4, (ii)] is equal to $N_{\Pi_{\mathrm{T}}|_H}(\Pi_{c^{\mathrm{F}}_1})$ [cf. [15], Proposition 1.2, (i), (ii)], $\tilde{\alpha}$ preserves the subgroup $D \subseteq \Pi_{\mathrm{T}}|_H$. Write

$$\mathrm{pr}_{\mathrm{F}} : \Pi_{\mathrm{T}} \twoheadrightarrow \Pi_{\mathrm{F}}/N_d$$

for the surjection induced by the projection to the fiber component, i.e., the composite

$$C_1^{\log} \times_{\widetilde{\mathcal{H}}^{\log}_{g,d,r+1}} \widetilde{\mathcal{C}}^{\log}_{g,d,r+1} \rightarrow C_1^{\log} \times_{\widetilde{\mathcal{H}}^{\log}_{g,d,r}} \widetilde{\mathcal{C}}^{\log}_{g,d,r} \rightarrow S^{\log} \times_{\widetilde{\mathcal{H}}^{\log}_{g,d,r}} \widetilde{\mathcal{C}}^{\log}_{g,d,r} = C_*^{\log}$$

[cf. the contraction morphism $C_x^{\log} \to C_*^{\log}$ discussed at the beginning of Definition 2.4]. Note that the restriction of pr_{F} to Π_{F} is the natural surjection $\Pi_{\mathrm{F}} \twoheadrightarrow \Pi_{\mathrm{F}}/N_d$. Since the morphism $\mathrm{pr}_1 : C_1 \longrightarrow C_*$ is surjective [cf. Definition 2.4, (ii)], and $N_{\Pi_{\mathrm{T}}}(\Pi_{c^{\mathrm{F}}_1})$ may be interpreted as the decomposition group associated to D_1, it follows immediately that the restriction of pr_{F} to $N_{\Pi_{\mathrm{T}}}(\Pi_{c_1^{\mathrm{F}}})$ is open. Now we have a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Pi_{c_1^{\mathrm{F}}} & \longrightarrow & N_{\Pi_{\mathrm{T}}}(\Pi_{c_1^{\mathrm{F}}}) & \longrightarrow & N_{\Pi_{\mathrm{T}}}(\Pi_{c_1^{\mathrm{F}}})/\Pi_{c_1^{\mathrm{F}}} & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \wr\big\downarrow & & \\
1 & \longrightarrow & \Pi_{\mathrm{F}} & \longrightarrow & \Pi_{\mathrm{T}} & \longrightarrow & \Pi_{\mathrm{B}} & \longrightarrow & 1 \\
 & & \big\downarrow & & {\scriptstyle \mathrm{pr}_{\mathrm{F}}}\big\downarrow & & & & \\
 & & \Pi_{\mathrm{F}}/N_d & = \!= \!= & \Pi_{\mathrm{F}}/N_d, & & & &
\end{array}$$

where the arrows $\Pi_{c_1^{\mathrm{F}}} \to \Pi_{\mathrm{F}}$ and $N_{\Pi_{\mathrm{T}}}(\Pi_{c_1^{\mathrm{F}}}) \to \Pi_{\mathrm{T}}$ are the natural inclusions; the arrow $\Pi_{\mathrm{F}} \to \Pi_{\mathrm{F}}/N_d$ is the natural surjection; [one verifies immediately that] the composite $N_{\Pi_{\mathrm{T}}}(\Pi_{c_1^{\mathrm{F}}}) \to \Pi_{\mathrm{T}} \stackrel{\mathrm{pr}_{\mathrm{F}}}{\to} \Pi_{\mathrm{F}}/N_d$ factors through the natural surjection $N_{\Pi_{\mathrm{T}}}(\Pi_{c_1^{\mathrm{F}}}) \twoheadrightarrow N_{\Pi_{\mathrm{T}}}(\Pi_{c_1^{\mathrm{F}}})/\Pi_{c_1^{\mathrm{F}}}$. Since the restriction of pr_{F} to $N_{\Pi_{\mathrm{T}}}(\Pi_{c_1^{\mathrm{F}}})$ is open, and $\tilde{\alpha}$ is an automorphism of $\Pi_{\mathrm{T}}|_H$ over H, we thus conclude that $\tilde{\alpha}$ induces the identity automorphism on some normal open subgroup $J \subseteq \Pi_{\mathrm{F}}/N_d$ of Π_{F}/N_d. Since $2g-2+dr \geq 1$ [cf. Definition 2.4], Π_{F}/N_d is slim [cf., e.g., [17], Proposition 1.4], hence induces an injection $\Pi_{\mathrm{F}}/N_d \hookrightarrow \mathrm{Aut}(J)$. The functoriality of this injection thus implies that $\tilde{\alpha}$ induces the identity automorphism on Π_{F}/N_d. The latter part of assertion (i) follows immediately from the former part of assertion (i), together with the uniqueness of the cusp associated to a given cuspidal inertia subgroup [cf. [15], Proposition 1.2, (i)]. This completes the proof of assertion (i).

Next, we prove assertion (ii). Since $\tilde{\alpha}$ is an automorphism of $\Pi_{\mathrm{T}}|_H$ over H, it holds that $\alpha_{\mathrm{F}} \in Z_{\mathrm{Out}(\Pi_{\mathrm{F}})}(\rho_1(H)) \subseteq \mathrm{Out}(\Pi_{\mathrm{F}})$. Next, observe that each of the stable log curves $C_1^{\log}$, $C_x^{\log}$ over $S^{\log}$ admits at least one cusp. Thus, the fact that $\alpha_{\mathrm{F}} \in \mathrm{Aut}(\mathcal{G}_x)$ follows immediately by applying Theorem A of [6] — or, alternatively, [15], Corollary 2.7, (iii) [cf. the fact that $C^{\log}$ is a smooth log curve; [6], Remark 2.4.2] — to any cuspidal inertia group of H. Now suppose, moreover, that $\tilde{\alpha}$ preserves the Π_{F}-conjugacy class of $\Pi_{c_{D_i}^{\mathrm{F}}} \subseteq \Pi_{\mathrm{F}}$ for each $i = 1, \ldots, d$. Then it follows from assertion (i) that $\alpha_{\mathrm{F}} \in \mathrm{Aut}(\mathcal{G}_x)$ fixes the cusps of $\mathcal{G}_x$. Since $C^{\log}$ is a smooth log curve, it follows that for any vertex $v \in \mathrm{Vert}(\mathcal{G}_x) \setminus \mathrm{Vert}(\mathcal{G}_*)$ of $\mathrm{Vert}(\mathcal{G}_x)$, there exists an integer $i \in \{1, \ldots, d\}$ satisfying $c_{D_i}^{\mathrm{F}} \in \mathcal{C}(v)$ [cf. Lemma 2.5, (iii), (iv), (v)]. In particular, we conclude from the detailed descriptions of Lemma 2.5, (iv), (v), that α_{F} fixes the vertices of $\mathcal{G}_x$, as well as the branches of nodes of $\mathcal{G}_x$. Thus, $\alpha_{\mathrm{F}} \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x) \subseteq \mathrm{Aut}(\mathcal{G}_x)$. This completes the proof of assertion (ii). □

The following result, which is a variant of [2], Proposition 8.1, asserts the existence of coverings that satisfy certain conditions. The proof is similar to [2], Proposition 8.1.

Proposition 2.8. *Let E be a smooth projective curve of genus g over an algebraically closed field k of characteristic zero such that $g \leq 1$; (i, d) nonnegative integers such that $0 \leq i \leq d-1 \geq g$; $x_1, \ldots, x_i, x_{i+1}$ $i+1$ distinct points on E. Then there exists a finite morphism $f : E \to \mathbb{P}_k^1$ of degree d satisfying the following conditions:*

- $x_1, \ldots, x_i, x_{i+1}$ *lie over a single point* y *of* $\mathbb{P}^1_k$, *and the ramification index at* x_{i+1} *is* $d-i$ *[which implies that the ramification index at* $x_1, \ldots, x_i$ *is* 1, *and* $f^{-1}(y) = \{x_1, \ldots, x_i, x_{i+1}\}$*].*
- f *has at most simple ramification except possibly over* y.

Proof. Since the assertion in the case where $d = 1$ is immediate, we may assume without loss of generality that $d \geq 2$. Write S for the d-fold symmetric product of E, $\xi \in S(k)$ for the point determined by the collection of points $\{x_1, \ldots, x_{i+1}\}$, where we take the multiplicity of $x_1, \ldots, x_{i+1}$ to be 1 and the multiplicity of x_{i+1} to be $d - i$. When $d \geq 3$, define morphisms

$$\alpha_1 : E^{d-2} \longrightarrow S,\ \alpha_2 : E^{d-2} \longrightarrow S$$

by the formulas

$$\alpha_1(P_1, P_2, \ldots, P_{d-2}) = 2P_1 + 2P_2 + P_3 + P_4 + \cdots + P_{d-2}$$
$$\alpha_2(P_1, P_2, \ldots, P_{d-2}) = 3P_1 + P_2 + \cdots + P_{d-2}.$$

Write $T \stackrel{\text{def}}{=} \mathrm{Im}(\alpha_1) \cup \mathrm{Im}(\alpha_2)$ when $d \geq 3$ and $T \stackrel{\text{def}}{=} \emptyset$ when $d = 2$. Note that $\dim T \leq d - 2$. Write

$$\phi : S \longrightarrow \mathrm{Pic}^d(E)$$

for the morphism obtained by assigning to a collection of d points of E the line bundle on E determined by the divisor given by the sum of the d points, $\mathcal{M} \stackrel{\text{def}}{=} \phi(\xi)$. For any $\mathcal{L} \in \mathrm{Pic}^d(E)$, write $S_{\mathcal{L}} \stackrel{\text{def}}{=} \phi^{-1}(\mathcal{L})$, $T_{\mathcal{L}} \stackrel{\text{def}}{=} \phi^{-1}(\mathcal{L}) \cap T$. Thus, $S_{\mathcal{L}}$ may be naturally identified with the projective space associated to the dual of the k-vector space $H^0(E, \mathcal{L})$. Since $d - 1 \geq 1 > 0 \geq 2g - 2$, the Riemann-Roch theorem thus implies that $\dim S_{\mathcal{L}} = d - g$, and $\mathcal{L}_P \stackrel{\text{def}}{=} H^0(E, \mathcal{L}(-P)) \subsetneq H^0(E, \mathcal{L})$ for any $P \in E(k)$ and $\mathcal{L} \in \mathrm{Pic}^d(E)$. Next, let us observe that when $d \geq 3$, the composites $\phi \circ \alpha_1$ and $\phi \circ \alpha_2$ are surjective. Thus, since $\dim T \leq d - 2$, and $\dim \mathrm{Pic}^d(E) = g \leq 1$, we conclude that $\dim T_{\mathcal{L}} \leq d - g - 2$, i.e., $T_{\mathcal{L}}$ is of codimension ≥ 2 in $S_{\mathcal{L}}$. In particular, there exists a line in the projective space $S_{\mathcal{M}}$ that contains $\xi \in S_{\mathcal{M}}(k) \subseteq S(k)$ and avoids $T_{\mathcal{M}} \setminus (T_{\mathcal{M}} \cap \{\xi\}) \subseteq S_{\mathcal{M}}$. Such a line determines a morphism $f : E \to \mathbb{P}^1_k$ as desired. □

Corollary 2.9. *Let* (g, d) *be nonnegative integers such that* $g \leq 1$, $d \geq g + 1$, *and* $2g + d \geq 3$; k *an algebraically closed field of characteristic zero. Then,* $(\psi_{g,d,1})_k : (\mathcal{H}_{g,d,1})_k \to (\mathcal{M}_{g,d})_k$ *[cf. Definition 1.13, (i), (ii)] is surjective.*

Proof. This follows immediately from Proposition 2.8, where we take "i" to be $d-1$, together with the various definitions involved. □

Proposition 2.10. *Let* (g, d) *be nonnegative integers such that* $d \geq \frac{g}{2} + 1$, $g \geq 2$; k *an algebraically closed field of characteristic zero. Then the image of* $(\psi_{g,d,0})_k : (\mathcal{H}_{g,d,0})_k \to (\mathcal{M}_{g,0})_k$ *[cf. Definition 1.13, (i), (ii)] is dense in* $(\mathcal{M}_{g,0})_k$.

Proof. Since $(\mathcal{H}_{g,d,0})_k$ is dense in $(\widetilde{\mathcal{H}}^{\log}_{g,d,0})_k$, it suffices to show that the image of $(\widetilde{\psi}^{\log}_{g,d,0})_k : (\widetilde{\mathcal{H}}^{\log}_{g,d,0})_k \to (\overline{\mathcal{M}}^{\log}_{g,0})_k$ [cf. Definition 1.13, (i), (ii)] is dense in $(\overline{\mathcal{M}}^{\log}_{g,0})_k$. Let C be a proper smooth curve of genus g over k. Since $d \geq \frac{g}{2} + 1$, it follows from [1], Chapter VII, Theorem 2.3, that there exists a finite morphism $\pi : C \to \mathbb{P}^1_k$ of degree $d' \leq d$. By constructing from π a similar degenerate covering to the covering illustrated in [11], Pictorial Appendix, Species $3B^*$ [which corresponds to the case where $d' = d - 1$; cf. also Remark 2.10.1 below], we obtain a degenerate covering $\pi' : C' \to D'$ of degree d, where the contraction [obtained by forgetting the ramification points] of C' is isomorphic to C, and the genus of D' is equal to zero. When the covering $\pi' : C' \to D'$ is not simple, by constructing from π' similar degenerate coverings to the coverings illustrated in [11], Pictorial Appendix, Species 1, 2, we obtain a degenerate simple covering $\pi'' : C'' \to D''$ of degree d, where the contraction [obtained by forgetting the ramification points] of C'' is isomorphic to C, and the genus of D'' is equal to zero. This completes the proof. □

Remark 2.10.1. Here, we take the opportunity to point out a minor error in the illustration of [11], Pictorial Appendix, Species $3B^*$: The lowermost irreducible component on the right-hand side of the domain curve of the covering [i.e., the irreducible component marked by the phrase "one copy of $\mathbb{P}^1$"] should be *deleted.*

Lemma 2.11. *In the notation of Definition 2.4 in the case where*

$$d \geq g + 1,\ g \geq 2,\ r = 0,$$

and we take the simple log admissible covering

$$\pi^{\log} : C^{\log} \longrightarrow D^{\log}$$

to be the log admissible covering of degree d obtained by gluing together along the respective points "y" two copies of the covering constructed in Proposition 2.8, where we take "i" to be g and "g" to be 0. *[Thus,* $C^{\log}$ *is of genus g;* $D^{\log}$ *is of genus* 0.*] Write* $\mathrm{Vert}(\mathcal{G}_*) = \{w, w'\}$. *Then, for any g-tuple of integers* $i_1, \ldots, i_g$ *such that* $1 \leq i_1 < \cdots < i_g \leq d$, *there exists a point* $x \in C_1(k)$ *whose associated semi-graph of anabelioids* $\mathcal{G}_x$ *[cf. Definition 2.4, (i)] satisfies the following conditions:*

- $\mathrm{Vert}(\mathcal{G}_x) = \{w^{\mathrm{F}}_x, w'^{\mathrm{F}}_x, v^{\mathrm{F}}_1, v^{\mathrm{F}}_2, \ldots, v^{\mathrm{F}}_{g+1}\}$;
- $\mathrm{Node}(\mathcal{G}_x) = \{e^{\mathrm{F}}_1, e^{\mathrm{F}}_2, \ldots, e^{\mathrm{F}}_{g+1}, e'^{\mathrm{F}}_1, e'^{\mathrm{F}}_2, \ldots, e'^{\mathrm{F}}_{g+1}\}$;
- $\mathcal{N}(w^{\mathrm{F}}_x) = \{e^{\mathrm{F}}_1, e^{\mathrm{F}}_2, \ldots, e^{\mathrm{F}}_{g+1}\}$, $\mathcal{N}(w'^{\mathrm{F}}_x) = \{e'^{\mathrm{F}}_1, e'^{\mathrm{F}}_2, \ldots, e'^{\mathrm{F}}_{g+1}\}$, *and* $\mathcal{C}(w^{\mathrm{F}}_x)^\# = \mathcal{C}(w'^{\mathrm{F}}_x)^\# = 0$;
- *for* $t = 1, \ldots, g+1$, $\mathcal{N}(v^{\mathrm{F}}_t) = \{e^{\mathrm{F}}_t, e'^{\mathrm{F}}_t\}$; *for* $t = 1, \ldots, g$, $\mathcal{C}(v^{\mathrm{F}}_t) = \{c^{\mathrm{F}}_{D_{i_t}}\}$ *[cf. Lemma 2.5, (ii)];*
- $\mathcal{C}(v^{\mathrm{F}}_{g+1}) = \{c^{\mathrm{F}}_{D_{j_1}}, \ldots, c^{\mathrm{F}}_{D_{j_{d-g}}}\}$, *where* $1 \leq j_1 < \cdots < j_{d-g} \leq d$ *are the* $d - g$ *integers such that* $\{1, \cdots, d\} = \{i_1, \ldots, i_g\} \cup \{j_1, \ldots, j_{d-g}\}$;
- *for* $t = 1, \ldots, g$, v^{F}_t *is of type* $(0, 3)$; v^{F}_{g+1} *is of type* $(0, d - g + 2)$.

Proof. By taking $x \in C_1(k)$ to be a point that corresponds to a 1-profiled simple admissible covering such that the section "σ_1" of Definition 1.7 corresponds to the point "y" that appears in the definition of $\pi^{\log} : C^{\log} \to D^{\log}$, one verifies immediately that one may choose x so that the required conditions are satisfied. □

3. Triviality of certain outomorphisms

In this section, our goal is to prove the following Proposition 3.1.

Proposition 3.1. *In the notation of Definition 2.4, for* $i = 1, \ldots, d$, *let* $\Pi_{c^{\mathrm{F}}_{D_i}} \subseteq \Pi_{\mathrm{F}}$ *be a cuspidal subgroup of* Π_{F} *associated to* $c^{\mathrm{F}}_{D_i} \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}_*)$ *[cf. Definition 2.6, (i), (iii)],* $H \subseteq \Pi_{\mathrm{B}}$ *an open subgroup of* Π_{B}, *and* $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\rho_1(H))$. *Suppose that, for each* $i = 1, \ldots, d$, α *preserves the* Π_{F}*-conjugacy class of* $\Pi_{c^{\mathrm{F}}_{D_i}} \subseteq \Pi_{\mathrm{F}}$. *When* $r = 0$ *[so* $g \geq 2$*], suppose that* $d \geq g + 1$, *and that* $\alpha \in \mathrm{Aut}^{|\{w^{\mathrm{F}}\}|}(\mathcal{G}_{1,2,\ldots,g})$ *[cf. Definition 3.3, (iii);* [5], *Definition 2.6, (i)], relative to some fixed isomorphism between the respective exact sequences "*$1 \to \Pi^{\bullet}_{\mathrm{F}} \to \Pi^{\bullet}_{\mathrm{T}} \to \Pi^{\bullet}_{\mathrm{B}} \to 1$*" of the sort that appears in the discussion at the beginning of Definition 3.3. Then* α *is the identity outomorphism.*

Before proving Proposition 3.1, we discuss certain preparatory aspects of the situation under consideration in Definition 3.2, Definition 3.3, Lemma 3.4, and Lemma 3.5.

Definition 3.2. In the notation of Proposition 3.1, let $N_s \subseteq \Pi_{\mathrm{F}}$ $(1 \leq s \leq d)$ be the normal closed subgroup of Π_{F} topologically normally generated by $\{\Pi_{c^{\mathrm{F}}_{D_i}}\ (d - s + 1 \leq i \leq d)\}$, $\alpha_s \in \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}/N_s)$ an outomorphism of Π_{F}/N_s induced by $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\rho_1(H))$. Write $N_0 = \{1\} \subseteq \Pi_{\mathrm{F}}$ for the trivial subgroup of Π_{F} and $\alpha_0 \stackrel{\mathrm{def}}{=} \alpha$. Note that it follows immediately from Lemma 2.5, (i), that N_s $(\subseteq \Pi_{\mathrm{F}} \subseteq \Pi_{\mathrm{T}})$ is normal in Π_{T} [cf. Lemma 2.3].

Definition 3.3. In the following, we shall consider, relative to *fixed* numerical data g, d, r, various *new choices* of the data "$\pi^{\log} : C^{\log} \to D^{\log}$", "$x \in C_1(k)$" considered in Definition 2.4. The objects "$1 \to \Pi^{\bullet}_{\mathrm{F}} \to \Pi^{\bullet}_{\mathrm{T}} \to \Pi^{\bullet}_{\mathrm{B}} \to 1$" [cf. Lemma 2.3, (i)] that arise from these *new choices* will then be thought as being related to the objects $1 \to \Pi_{\mathrm{F}} \to \Pi_{\mathrm{T}} \to \Pi_{\mathrm{B}} \to 1$ that arise from the original given data of Definition 2.4 [i.e., the data considered, e.g., in Proposition 3.1] by means of the vertical isomorphisms discussed in Lemma 2.3, (i).

(i) Suppose that $C^{\log}$, $x \in C_1(k)$, and i, j are as in Lemma 2.5, (iv). Let s be an integer satisfying $0 \leq s \leq d - j$. Then we shall write
$$\mathcal{G}_{i,j}$$
for the resulting semi-graph of anabelioids of pro-Σ_{F} PSC-type "$\mathcal{G}_x$" of Definition 2.4, (i);
$$\mathcal{G}_{i,j,s} \stackrel{\mathrm{def}}{=} (\mathcal{G}_{i,j})_{\bullet\{c^{\mathrm{F}}_{D_m}(d-s+1 \leq m \leq d)\}}$$
[cf. [5], Definition 2.4];
$$v^{\mathrm{F}}_{i,j,s},\ v^{\mathrm{F}}_{\mathrm{new},i,j,s}\ (\neq v^{\mathrm{F}}_{i,j,s})$$
[cf. the assumption that $0 \leq s \leq d - j$] for the vertices of $\mathcal{G}_{i,j,s}$ determined by the vertices v^{F}_x, $v^{\mathrm{F}}_{\mathrm{new},x}$ [cf. Lemma 2.5, (iv)] of $\mathcal{G}_{i,j}$.

(ii) Suppose that $C^{\log}$, $x \in C_1(k)$, and l, j are as in Lemma 2.5, (v). Let s be an integer satisfying $0 \leq s \leq d - j$. Then we shall write
$$\mathcal{G}_l$$

for the resulting semi-graph of anabelioids of pro-Σ_{F} PSC-type "$\mathcal{G}_x$" of Definition 2.4, (i);

$$\mathcal{G}_{l,s} \stackrel{\mathrm{def}}{=} (\mathcal{G}_l)_{\bullet\{c^{\mathrm{F}}_{D_m}(d-s+1\leq m\leq d)\}}$$

[cf. [5], Definition 2.4];

$$v_{l,s},\ v_{l,j,s}\ (\neq v_{l,s}),\ e_{l,j,s},\ c_{l,j,s}$$

[cf. the assumption that $0 \leq s \leq d-j$] for the vertices, closed edges, and cusps of $\mathcal{G}_{l,s}$ determined by the vertices, closed edges, and cusps v^{F}_x, $v^{\mathrm{F}}_{\mathrm{new},j,x}$, e^{F}_j, and $c^{\mathrm{F}}_{l,j}$ [cf. Lemma 2.5, (v)] of $\mathcal{G}_l$.

(iii) Assume that $d \geq g+1$, $g \geq 2$, $r = 0$. Suppose that $C^{\log}$, $x \in C_1(k)$, and i_t $(1 \leq t \leq g)$ are as in Lemma 2.11. Suppose further that $i_t = t$ $(1 \leq t \leq g)$. Then we shall write

$$\mathcal{G}_{1,2,\ldots,g}$$

for the resulting semi-graph of anabelioids of pro-Σ_{F} PSC-type "$\mathcal{G}_x$" of Definition 2.4, (i). In the remainder of the present §3, we shall omit the subscript "x" from the notation w^{F}_x, w'^{F}_x [cf. Lemma 2.11].

Lemma 3.4. *In the notation of Definition 2.6, Definition 3.2, and Definition 3.3, (i), the following hold:*

(i) Fix i, j. Then there exists a collection of "scheme-theoretic" [in the sense discussed in Lemma 2.3, (i)] outer isomorphisms

$$\left\{\Pi_{\mathrm{F}}/N_s \xrightarrow{\sim} \Pi_{\mathcal{G}_{i,j,s}}\right\}_{s=0,\ldots,d-j}$$

that satisfies the following conditions for each $s \in \{0,\ldots,d-j-1\}$:

- *(Commutativity) We have a natural commutative diagram*

$$\begin{array}{ccc} \Pi_{\mathrm{F}}/N_s & \xrightarrow{\sim} & \Pi_{\mathcal{G}_{i,j,s}} \\ \downarrow & & \downarrow \\ \Pi_{\mathrm{F}}/N_{s+1} & \xrightarrow{\sim} & \Pi_{\mathcal{G}_{i,j,s+1}}, \end{array}$$

  where the vertical arrows are the natural outer surjections.
- *(Injectivity for cuspidal subgroups) Let t be an integer satisfying $1 \leq t \leq d-s-1$. Then the* **composite**
  $$\Pi_{c^{\mathrm{F}}_{D_t}} \to \Pi_{\mathcal{G}_{i,j,s}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_s \twoheadrightarrow \Pi_{\mathrm{F}}/N_{s+1}$$
  [where the first and third arrows are the natural outer homomorphisms] is injective.
- *(Injectivity for non-new verticial subgroups) Suppose that $j = d-s$. Then the* **composite**
  $$\Pi_{v^{\mathrm{F}}_{i,j,s}} \to \Pi_{\mathcal{G}_{i,j,s}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_s \twoheadrightarrow \Pi_{\mathrm{F}}/N_{s+1}$$
  [where the first and third arrows are the natural outer homomorphisms] is injective.
- *(Injectivity for new verticial subgroups) Let j be an integer satisfying $i < j \leq d-s-1$. Then the* **composite**
  $$\Pi_{v^{\mathrm{F}}_{\mathrm{new},i,j,s}} \to \Pi_{\mathcal{G}_{i,j,s}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_s \twoheadrightarrow \Pi_{\mathrm{F}}/N_{s+1}$$
  [where the first and third arrows are the natural outer homomorphisms] is injective.

(ii) *The images of the above* **composites** *are commensurably terminal.*

Proof. Assertion (i) follows immediately from the various definitions involved. Assertion (ii) follows immediately from assertion (i), together with [15], Proposition 1.2, (ii). □

Lemma 3.5. *In the notation of Definition 2.6, Definition 3.2, and Definition 3.3, (ii), the following hold:*

(i) *Fix l. Then there exists a collection of "scheme-theoretic" [in the sense discussed in Lemma 2.3, (i)] outer isomorphisms*

$$\left\{\Pi_{\mathrm{F}}/N_s \xrightarrow{\sim} \Pi_{\mathcal{G}_{l,s}}\right\}_{s=0,\ldots,d}$$

that satisfies the following conditions for each $s \in \{0, \ldots, d-1\}$:

- *(Commutativity) We have a natural commutative diagram*

$$\begin{array}{ccc} \Pi_{\mathrm{F}}/N_s & \xrightarrow{\sim} & \Pi_{\mathcal{G}_{l,s}} \\ \downarrow & & \downarrow \\ \Pi_{\mathrm{F}}/N_{s+1} & \xrightarrow{\sim} & \Pi_{\mathcal{G}_{l,s+1}}, \end{array}$$

  where the vertical arrows are the natural outer surjections.
- *(Injectivity for cuspidal subgroups) Let j be an integer satisfying $1 \leq j \leq d-s-1$. Then the* **composites**

$$\Pi_{c^{\mathrm{F}}_{D_j}} \to \Pi_{\mathcal{G}_{l,s}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_s \twoheadrightarrow \Pi_{\mathrm{F}}/N_{s+1}$$
$$\Pi_{c_{l,j,s}} \to \Pi_{\mathcal{G}_{l,s}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_s \twoheadrightarrow \Pi_{\mathrm{F}}/N_{s+1}$$

  [where the first and third arrows of each line of the display are the natural outer homomorphisms] are injective.
- *(Injectivity for verticial subgroups) Let j be an integer satisfying $1 \leq j \leq d-s-1$. Then the* **composites**

$$\Pi_{v_{l,s}} \to \Pi_{\mathcal{G}_{l,s}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_s \twoheadrightarrow \Pi_{\mathrm{F}}/N_{s+1}$$
$$\Pi_{v_{l,j,s}} \to \Pi_{\mathcal{G}_{l,s}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_s \twoheadrightarrow \Pi_{\mathrm{F}}/N_{s+1}$$

  [where the first and third arrows of each line of the display are the natural outer homomorphisms] are injective.

(ii) *The images of the above* **composites** *are commensurably terminal.*

Proof. Assertion (i) follows immediately from the various definitions involved. Assertion (ii) follows immediately from assertion (i), together with [15], Proposition 1.2, (ii). □

Proof of Proposition 3.1. By Lemma 2.7, (i), $\alpha_d \in \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}/N_d)$ [cf. Definition 3.2] is the identity outomorphism of Π_{F}/N_d. Next, we verify the following assertion:

> Claim 3.1.A: The outomorphism $\alpha_{d-1} \in \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}/N_{d-1})$ [cf. Definition 3.2] of Π_{F}/N_{d-1} is trivial.

By Lemma 2.3, (i), it suffices to verify Claim 3.1.A under the further assumption that $C^{\log}$ is a *smooth log curve*. Since N_{d-1} is normal in Π_{T} [cf. Definition 3.2], we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{\mathrm{F}} & \longrightarrow & \Pi_{\mathrm{T}} & \longrightarrow & \Pi_{\mathrm{B}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Pi_{\mathrm{F}}/N_{d-1} & \longrightarrow & \Pi_{\mathrm{T}}/N_{d-1} & \longrightarrow & \Pi_{\mathrm{B}} & \longrightarrow & 1, \end{array}$$

where the left and middle vertical arrows are the natural outer surjections. Write

- $M_{\mathrm{ram}} \subseteq \mathbb{P}^1_k(k)$ for the branch set of the simple covering $\pi : C \to \mathbb{P}^1_k$ associated to the r-profiled simple log admissible covering $\pi^{\log} : C^{\log} \to (\mathbb{P}^1_k)^{\log}$;
- $M_{\mathrm{unr}} \stackrel{\mathrm{def}}{=} \{z_i \in \mathbb{P}^1_k(k)\ (1 \leq i \leq r)\}$ for the set of ordered marked points; $M \stackrel{\mathrm{def}}{=} M_{\mathrm{ram}} \cup M_{\mathrm{unr}}$; $V_C \stackrel{\mathrm{def}}{=} C \setminus \pi^{-1}(M_{\mathrm{unr}})$; $U_C \stackrel{\mathrm{def}}{=} C \setminus \pi^{-1}(M)$ $(\subseteq V_C)$; $U_P \stackrel{\mathrm{def}}{=} \mathbb{P}^1_k \setminus M$;
- $B \stackrel{\mathrm{def}}{=} (U_C \times_{U_P} U_C \times \cdots \times_{U_P} U_C) \setminus \Delta_B$, where the fiber product is the fiber product of d copies of the morphism $U_C \to U_P$, and Δ_B denotes the union of the various diagonals associated to pairs of factors in the fiber product [cf. Lemma 1.12];
- $T' \stackrel{\mathrm{def}}{=} (B \times_k V_C) \setminus \Delta_{T'}$, $T'' \stackrel{\mathrm{def}}{=} (U_C \times_k V_C) \setminus \Delta_{T''}$, $T''' \stackrel{\mathrm{def}}{=} (V_C \times_k V_C) \setminus \Delta_{T'''}$, where $\Delta_{T'}$ (respectively, $\Delta_{T''}$, $\Delta_{T'''}$) is the graph divisor determined by the composite of the first projection $\mathrm{pr}_1 : B \to U_C$ with the natural inclusion $U_C \hookrightarrow V_C$ (respectively, the natural inclusion $U_C \hookrightarrow V_C$, the identity morphism $V_C \to V_C$);
- F' for a geometric fiber of the first projection $T' \to B$, and $K \stackrel{\mathrm{def}}{=} \mathrm{Ker}(\pi_1(F') \to \pi_1^{\Sigma_{\mathrm{F}}}(F'))$; thus, F' may be regarded as a geometric fiber of either of the first projections $T'' \to U_C$ and $T''' \to V_C$.

Then we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1^{\Sigma_{\mathrm{F}}}(F') & \longrightarrow & \pi_1(T')/K & \longrightarrow & \pi_1(B) & \longrightarrow & 1 \\ & & \| & & \downarrow f_1 & & \downarrow f_2 & & \\ 1 & \longrightarrow & \pi_1^{\Sigma_{\mathrm{F}}}(F') & \longrightarrow & \pi_1(T'')/K & \longrightarrow & \pi_1(U_C) & \longrightarrow & 1 \\ & & \| & & \downarrow f_3 & & \downarrow f_4 & & \\ 1 & \longrightarrow & \pi_1^{\Sigma_{\mathrm{F}}}(F') & \longrightarrow & \pi_1(T''')/K & \longrightarrow & \pi_1(V_C) & \longrightarrow & 1, \end{array}$$

where f_1 is the morphism induced by $\mathrm{pr}_1 \times \mathrm{id} : T' = (B \times_k V_C) \setminus \Delta_{T'} \to T'' = (U_C \times_k V_C) \setminus \Delta_{T''}$; f_2 is the morphism induced by $\mathrm{pr}_1 : B \to U_C$; f_3, f_4 are the morphisms induced by the natural inclusions. Since $\mathrm{pr}_1 : B \to U_C$ is finite étale, and $\Delta_{T'} = (\mathrm{pr}_1 \times \mathrm{id})^{-1}(\Delta_{T''})$, the morphism $\mathrm{pr}_1 \times \mathrm{id} : T' = (B \times_k V_C) \setminus \Delta_{T'} \to T'' = (U_C \times_k V_C) \setminus \Delta_{T''}$ is finite étale. Thus, f_1, f_2 are open injections, and f_3, f_4 are surjections. Since $f_4 \circ f_2 : (\Pi_B \stackrel{\sim}{\leftarrow}) \ \pi_1(B) \to \pi_1(V_C)$ [cf. Proposition 1.10, (vi); Lemma 1.12; [12], Theorem B] is an open homomorphism, the triviality of the outomorphism $\alpha_{d-1} \in \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}/N_{d-1})$ follows from the "Grothendieck Conjecture for configuration spaces" [cf. [5], Theorem 6.12, (i)], together with the *hyperbolicity* of V_C [cf. the condition $2g - 2 + dr \geq 1$ in the first display of Definition 2.4] and our assumption that, for each $i = 1, \ldots, d$, α preserves the Π_{F}-conjugacy class of $\Pi_{c^{\mathrm{F}}_{D_i}} \subseteq \Pi_{\mathrm{F}}$. This completes the proof of Claim 3.1.A.

Next, we verify the following assertion:

> Claim 3.1.B: Let s be an integer such that $0 \leq s \leq d-3$. Suppose that the outomorphism $\alpha_{s+1} \in \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}/N_{s+1})$ is the identity outomorphism of Π_{F}/N_{s+1}. Then the outomorphism $\alpha_s \in \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}/N_s)$ is the identity outomorphism of Π_{F}/N_s [cf. Definition 3.2].

Since $2 \leq d-s-1 \leq d-1$, it makes sense to consider semi-graphs of anabelioids of pro-Σ_{F} PSC-type $\mathcal{G}_{1,d-s-1,s}$ and $\mathcal{G}_{d-s-1,d-s,s}$ as in Definition 3.3, (i), and to fix isomorphisms $\Pi_{\mathcal{G}_{1,d-s-1,s}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_s \xrightarrow{\sim} \Pi_{\mathcal{G}_{d-s-1,d-s,s}}$ that determine outer isomorphisms as in the collections of outer isomorphisms discussed in Lemma 3.4, (i). Since N_s is normal in Π_{T} [cf. Definition 3.2], we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{\mathrm{F}} & \longrightarrow & \Pi_{\mathrm{T}} & \longrightarrow & \Pi_{\mathrm{B}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Pi_{\mathrm{F}}/N_s & \longrightarrow & \Pi_{\mathrm{T}}/N_s & \longrightarrow & \Pi_{\mathrm{B}} & \longrightarrow & 1, \end{array}$$

where the vertical arrows are the natural surjections. Write

$$\rho_{1,s} : \Pi_{\mathrm{B}} \to \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}/N_s)$$

for the outer representation induced by the lower exact sequence of the above commutative diagram. Since $\alpha_0 = \alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\rho_1(H))$ [cf. Definition 3.2], we obtain that $\alpha_s \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}/N_s)}(\rho_{1,s}(H))$. Write $(\Pi_{\mathrm{T}}/N_s)|_H \stackrel{\mathrm{def}}{=} (\Pi_{\mathrm{T}}/N_s) \times_{\Pi_{\mathrm{B}}} H$. Let

$$\tilde{\alpha}_s \in \mathrm{Aut}_H((\Pi_{\mathrm{T}}/N_s)|_H)$$

[cf. the discussion entitled "Topological groups" in Notations and Conventions] be a lifting of

$$\alpha_s \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}/N_s)}(\rho_{1,s}(H)) \subseteq Z_{\mathrm{Out}(\Pi_{\mathrm{F}}/N_s)}(\rho_{1,s}(H))$$
$$\xleftarrow{\sim} \mathrm{Aut}_H((\Pi_{\mathrm{T}}/N_s)|_H)/\mathrm{Inn}(\Pi_{\mathrm{F}}/N_s),$$

where the final isomorphism follows from the center-freeness of Π_{F}/N_s [cf. the inequality $2g-2+dr \geq 1$ in the first display of Definition 2.4; the discussion entitled "Topological groups" in Notations and Conventions]. Since the image $\Pi_{c^{\mathrm{F}}_{D_{1,s+1}}}$ of $\Pi_{c^{\mathrm{F}}_{D_1}}$ in Π_{F}/N_{s+1} is commensurably terminal in Π_{F}/N_{s+1} [cf. Lemma 3.4, (ii)], by replacing $\tilde{\alpha}_s$ by the composite of $\tilde{\alpha}_s$ with a suitable inner automorphism of Π_{F}/N_s, we may assume that the automorphism of Π_{F}/N_{s+1} induced by $\tilde{\alpha}_s$ is the identity automorphism, and that $\tilde{\alpha}_s$ preserves the image $\Pi_{c^{\mathrm{F}}_{D_{1,s}}} \subseteq \Pi_{\mathrm{F}}/N_s$ of $\Pi_{c^{\mathrm{F}}_{D_1}}$ in Π_{F}/N_s. Next, let us fix verticial subgroups

$$\Pi_{c^{\mathrm{F}}_{D_{1,s}}} \subseteq \Pi_{v^{\mathrm{F}}_{\mathrm{new},1,d-s-1,s}} \subseteq \Pi_{\mathcal{G}_{1,d-s-1,s}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_s,$$

$$\Pi_{c^{\mathrm{F}}_{D_{1,s}}} \subseteq \Pi_{v^{\mathrm{F}}_{d-s-1,d-s,s}} \subseteq \Pi_{\mathcal{G}_{d-s-1,d-s,s}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_s$$

containing $\Pi_{c^{\mathrm{F}}_{D_{1,s}}}$ [cf. the notation introduced in Definition 3.3, (i)]. By Lemma 2.7, (ii) [where we take "α_{F}" to be α]; [15], Proposition 1.5, (ii), $\tilde{\alpha}_s$ preserves

$\Pi_{v^{\mathrm{F}}_{\mathrm{new},1,d-s-1,s}}$, $\Pi_{v^{\mathrm{F}}_{d-s-1,d-s,s}}$. Since the composites

$$\Pi_{v^{\mathrm{F}}_{\mathrm{new},1,d-s-1,s}} \hookrightarrow \Pi_{\mathcal{G}_{1,d-s-1,s}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_s \twoheadrightarrow \Pi_{\mathrm{F}}/N_{s+1}$$

$$\Pi_{v^{\mathrm{F}}_{d-s-1,d-s,s}} \hookrightarrow \Pi_{\mathcal{G}_{d-s-1,d-s,s}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_s \twoheadrightarrow \Pi_{\mathrm{F}}/N_{s+1}$$

are injective [cf. Lemma 3.4, (i)], it suffices to show that the images in Π_{F}/N_s of $\Pi_{v^{\mathrm{F}}_{\mathrm{new},1,d-s-1,s}}$ and $\Pi_{v^{\mathrm{F}}_{d-s-1,d-s,s}}$ generate Π_{F}/N_s. However, this follows immediately from the van Kampen theorem [cf. [16], Lemma 1.13, applied to a suitable neighborhood of the cusps labeled 1, $d-s-1$, and $d-s$ in a "topological surface representation" of Π_{F}/N_s, where we take the cusp "a" to be the cusp labeled 1, the cusp "b" to be the cusp labeled $d-s-1$, and the cusp "c" to be the boundary of the neighborhood]. This completes the proof of Claim 3.1.B.

Next, we verify the following assertion:

> Claim 3.1.C: When $r \geq 1$, the outomorphism $\alpha_{d-2} \in \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}/N_{d-2})$ is the identity outomorphism of Π_{F}/N_{d-2} [cf. Definition 3.2].

We consider semi-graphs of anabelioids of pro-Σ_{F} PSC-type $\mathcal{G}_{1,2,d-2}$ and $\mathcal{G}_{1,d-2}$ as in Definition 3.3, (i), (ii), and fix isomorphisms $\Pi_{\mathcal{G}_{1,2,d-2}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_{d-2} \xrightarrow{\sim} \Pi_{\mathcal{G}_{1,d-2}}$ that determine outer isomorphisms as in the collections of outer isomorphisms discussed in Lemmas 3.4, (i); 3.5, (i). Since N_{d-2} is normal in Π_{T} [cf. Definition 3.2], we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{\mathrm{F}} & \longrightarrow & \Pi_{\mathrm{T}} & \longrightarrow & \Pi_{\mathrm{B}} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Pi_{\mathrm{F}}/N_{d-2} & \longrightarrow & \Pi_{\mathrm{T}}/N_{d-2} & \longrightarrow & \Pi_{\mathrm{B}} & \longrightarrow & 1, \end{array}$$

where the vertical arrows are the natural surjections. Write

$$\rho_{1,d-2} : \Pi_{\mathrm{B}} \to \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}/N_{d-2})$$

for the outer representation induced by the lower exact sequence of the above commutative diagram. Since $\alpha_0 = \alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\rho_1(H))$ [cf. Definition 3.2], we obtain that $\alpha_{d-2} \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}/N_{d-2})}(\rho_{1,d-2}(H))$. Write $(\Pi_{\mathrm{T}}/N_{d-2})|_H \stackrel{\mathrm{def}}{=} (\Pi_{\mathrm{T}}/N_{d-2}) \times_{\Pi_{\mathrm{B}}} H$. Let

$$\tilde{\alpha}_{d-2} \in \mathrm{Aut}_H((\Pi_{\mathrm{T}}/N_{d-2})|_H)$$

[cf. the discussion entitled "Topological groups" in Notations and Conventions] be a lifting of

$$\alpha_{d-2} \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}/N_{d-2})}(\rho_{1,d-2}(H)) \subseteq Z_{\mathrm{Out}(\Pi_{\mathrm{F}}/N_{d-2})}(\rho_{1,d-2}(H))$$

$$\xleftarrow{\sim} \mathrm{Aut}_H((\Pi_{\mathrm{T}}/N_{d-2})|_H)/\mathrm{Inn}(\Pi_{\mathrm{F}}/N_{d-2}),$$

where the final isomorphism follows from the center-freeness of Π_{F}/N_{d-2} [cf. the inequality $2g-2+dr \geq 1$ in the first display of Definition 2.4; the discussion entitled "Topological groups" in Notations and Conventions]. Fix a cuspidal subgroup $\Pi_{c_{1,1,d-2}} \subseteq \Pi_{\mathcal{G}_{1,d-2}}$ associated to $c_{1,1,d-2}$ [cf. Definition 3.3, (ii)]. Since the image of $\Pi_{c_{1,1,d-2}}$ in Π_{F}/N_{d-1} is commensurably terminal in Π_{F}/N_{d-1} [cf. Lemma 3.5, (ii)], it follows from Claim 3.1.A that, by replacing $\tilde{\alpha}_{d-2}$ by the composite of $\tilde{\alpha}_{d-2}$ with a suitable inner automorphism of Π_{F}/N_{d-2}, we may assume that the automorphism of Π_{F}/N_{d-1} induced by $\tilde{\alpha}_{d-2}$ is the identity automorphism, and that $\tilde{\alpha}_{d-2}$

preserves $\Pi_{c_{1,1,d-2}} \subseteq \Pi_{\mathrm{F}}/N_{d-2}$. Write $\Pi_{c_{1,1,2,d-2}}$ for the image of the composite $\Pi_{c_{1,1,d-2}} \subseteq \Pi_{\mathcal{G}_{1,d-2}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_{d-2} \xrightarrow{\sim} \Pi_{\mathcal{G}_{1,2,d-2}}$. Next, let us fix verticial subgroups

$$\Pi_{c_{1,1,2,d-2}} \subseteq \Pi_{v^{\mathrm{F}}_{1,2,d-2}} \subseteq \Pi_{\mathcal{G}_{1,2,d-2}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_{d-2},$$

$$\Pi_{c_{1,1,d-2}} \subseteq \Pi_{v_{1,1,d-2}} \subseteq \Pi_{\mathcal{G}_{1,d-2}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_{d-2}$$

[cf. the notation introduced in Definition 3.3, (i), (ii)] containing $\Pi_{c_{1,1,2,d-2}}$, $\Pi_{c_{1,1,d-2}}$, respectively. By Lemma 2.7, (ii) [where we take "α_{F}" to be α]; [15], Proposition 1.5, (ii), $\tilde{\alpha}_{d-2}$ preserves $\Pi_{v^{\mathrm{F}}_{1,2,d-2}}$, $\Pi_{v_{1,1,d-2}}$. Since the composites

$$\Pi_{v^{\mathrm{F}}_{1,2,d-2}} \hookrightarrow \Pi_{\mathcal{G}_{1,2,d-2}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_{d-2} \twoheadrightarrow \Pi_{\mathrm{F}}/N_{d-1}$$

$$\Pi_{v_{1,1,d-2}} \hookrightarrow \Pi_{\mathcal{G}_{1,d-2}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_{d-2} \twoheadrightarrow \Pi_{\mathrm{F}}/N_{d-1}$$

are injective [cf. Lemmas 3.4, (i); 3.5, (i)], it suffices to show that the images in Π_{F}/N_{d-2} of $\Pi_{v^{\mathrm{F}}_{1,2,d-2}}$ and $\Pi_{v_{1,1,d-2}}$ generate Π_{F}/N_{d-2}. However, this follows immediately from the van Kampen theorem [cf. [16], Lemma 1.13, applied to a suitable neighborhood of the cusps labeled $(1,1)$ [i.e., the label of the cusp $c_{1,1,d-2}$], 1, and 2 in a "topological surface representation" of Π_{F}/N_{d-2}, where we take the cusp "a" to be the cusp labeled $(1,1)$, the cusp "b" to be the cusp labeled 1, and the cusp "c" to be the boundary of the neighborhood]. This completes the proof of Claim 3.1.C and hence the proof of Proposition 3.1 when $r \geq 1$.

Next, we verify the following assertion:

> Claim 3.1.D: When $r = 0$ [so $g \geq 2$], and $d \geq g+1$, the outomorphism $\alpha = \alpha_0 \in \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}/N_0)$ is the identity outomorphism of $\Pi_{\mathrm{F}}/N_0 \xleftarrow{\sim} \Pi_{\mathrm{F}}$ [cf. Definition 3.2].

Here, we consider the semi-graph of anabelioids of pro-Σ_{F} PSC-type

$$(\mathcal{G}_{1,2,\dots,g})_{\bullet} \stackrel{\mathrm{def}}{=} (\mathcal{G}_{1,2,\dots,g})_{\bullet\{c^{\mathrm{F}}_{D_i}(g+2\leq i\leq d)\}}$$

[cf. Definition 3.3, (iii); [5], Definition 2.4]. Observe that the *maximal subgraphs* [cf. [14], §1], hence also the respective sets of *vertices* and *nodes*, of the underlying semi-graphs of $\mathcal{G}_{1,2,\dots,g}$ and $(\mathcal{G}_{1,2,\dots,g})_{\bullet}$ may be *naturally identified* with one another. In the following, we fix an isomorphism $\Pi_{\mathcal{G}_{1,2,\dots,g}} \xleftarrow{\sim} \Pi_{\mathrm{F}}$ as in the statement of Proposition 3.1, which induces an isomorphism $\Pi_{(\mathcal{G}_{1,2,\dots,g})_{\bullet}} \xleftarrow{\sim} \Pi_{\mathrm{F}}/N_{d-g-1}$. Recall from Lemma 2.11 that, relative to this *natural identification*, $(\mathcal{G}_{1,2,\dots,g})_{\bullet}$ satisfies the following conditions:

- $\mathrm{Vert}((\mathcal{G}_{1,2,\dots,g})_{\bullet}) = \{w^{\mathrm{F}}, w'^{\mathrm{F}}, v^{\mathrm{F}}_1, v^{\mathrm{F}}_2, \dots, v^{\mathrm{F}}_{g+1}\}$;
- $\mathrm{Node}((\mathcal{G}_{1,2,\dots,g})_{\bullet}) = \{e^{\mathrm{F}}_1, e^{\mathrm{F}}_2, \dots, e^{\mathrm{F}}_{g+1}, e'^{\mathrm{F}}_1, e'^{\mathrm{F}}_2, \dots, e'^{\mathrm{F}}_{g+1}\}$;
- $\mathcal{N}(w^{\mathrm{F}}) = \{e^{\mathrm{F}}_1, e^{\mathrm{F}}_2, \dots, e^{\mathrm{F}}_{g+1}\}$, $\mathcal{N}(w'^{\mathrm{F}}) = \{e'^{\mathrm{F}}_1, e'^{\mathrm{F}}_2, \dots, e'^{\mathrm{F}}_{g+1}\}$, and $\mathcal{C}(w^{\mathrm{F}})^{\#} = \mathcal{C}(w'^{\mathrm{F}})^{\#} = 0$;
- for $t = 1, \dots, g+1$, $\mathcal{N}(v^{\mathrm{F}}_t) = \{e^{\mathrm{F}}_t, e'^{\mathrm{F}}_t\}$, and $\mathcal{C}(v^{\mathrm{F}}_t) = \{c^{\mathrm{F}}_{D_t}\}$;
- for $t = 1, \dots, g+1$, v^{F}_t is of type $(0,3)$.

By the assumptions imposed in the statement of Proposition 3.1, it holds that $\alpha \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{1,2,\dots,g}) \subseteq \mathrm{Out}(\Pi_{\mathcal{G}_{1,2,\dots,g}}) \xleftarrow{\sim} \mathrm{Out}(\Pi_{\mathrm{F}})$, hence that

$$\alpha_{d-g-1} \in \mathrm{Aut}^{|\mathrm{grph}|}((\mathcal{G}_{1,2,\dots,g})_{\bullet}) \subseteq \mathrm{Out}(\Pi_{(\mathcal{G}_{1,2,\dots,g})_{\bullet}}) \xleftarrow{\sim} \mathrm{Out}(\Pi_{\mathrm{F}}/N_{d-g-1}).$$

For $t = 1, \dots, g+1$, write $N_{t,d-1} \subseteq \Pi_{\mathrm{F}}/N_{d-g-1}$ for the normal closed subgroup of $\Pi_{\mathrm{F}}/N_{d-g-1}$ topologically normally generated by the $\Pi_{c^{\mathrm{F}}_{D_i}}$, where i ranges over the elements of $\{1, \ldots, g+1\} \setminus \{t\}$. Thus, we observe that, by possibly permuting the labels $1, \ldots, d$ for the cusps $c^{\mathrm{F}}_{D_i}$, it follows from Claim 3.1.A that α_{d-g-1} induces the identity outomorphism of $(\Pi_{\mathrm{F}}/N_{d-g-1})/N_{t,d-1}$.

Thus, since $g+1 \geq 2$, by applying analogous injectivity properties concerning verticial subgroups to the properties discussed in Lemmas 3.4, (i); 3.5, (i), we conclude that

$$\alpha_{d-g-1} \in \bigcap_{1 \leq t \leq g+1} \mathrm{Im}\big\{\mathrm{Dehn}((\mathcal{G}_{1,2,\ldots,g})_{\bullet \rightsquigarrow \{e_t, e'_t\}}) \to \mathrm{Dehn}((\mathcal{G}_{1,2,\ldots,g})_{\bullet})\big\}$$

$$\subseteq \mathrm{Dehn}((\mathcal{G}_{1,2,\ldots,g})_{\bullet}) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{(\mathcal{G}_{1,2,\ldots,g})_{\bullet}})$$

[cf. Definition 3.3, (iii); [5], Definitions 2.4, 2.8, 4.4]. On the other hand, [again since $g+1 \geq 2$] it follows from [5], Theorem 4.8, (ii), (iv), that

$$\bigcap_{1 \leq t \leq g+1} \mathrm{Im}\big\{\mathrm{Dehn}((\mathcal{G}_{1,2,\ldots,g})_{\bullet \rightsquigarrow \{e_t, e'_t\}}) \to \mathrm{Dehn}((\mathcal{G}_{1,2,\ldots,g})_{\bullet})\big\} = \{1\}.$$

Thus, we conclude that $\alpha_{d-g-1} \in \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}}/N_{d-g-1})$ is the identity outomorphism of $\Pi_{\mathrm{F}}/N_{d-g-1}$. Since $d-g-1 \leq d-2$, Claim 3.1.B thus implies that $\alpha_0 = \alpha \in \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})$ is the identity outomorphism of Π_{F}. This completes the proof of Claim 3.1.D and hence the proof of Proposition 3.1. □

Finally, in the following Proposition 3.6, we observe that in fact, any element $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\rho_1(H))$ [cf. Proposition 3.1] preserves the Π_{F}-conjugacy class of $\Pi_{c^{\mathrm{F}}_{D_i}} \subseteq \Pi_{\mathrm{F}}$, for each $i = 1, \ldots, d$, in almost all cases under consideration.

Proposition 3.6. *In the notation of Definition 2.4, for $i = 1, \ldots, d$, let $\Pi_{c^{\mathrm{F}}_{D_i}} \subseteq \Pi_{\mathrm{F}}$ be a cuspidal subgroup of Π_{F} associated to $c^{\mathrm{F}}_{D_i} \in \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}_*)$ [cf. Definition 2.6, (i), (iii)], $H \subseteq \Pi_{\mathrm{B}}$ an open subgroup of Π_{B}, and $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\rho_1(H))$. Then when $d \geq 3$ (respectively, $d = 2$), $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\rho_1(H))$ preserves the Π_{F}-conjugacy class of $\Pi_{c^{\mathrm{F}}_{D_i}} \subseteq \Pi_{\mathrm{F}}$ for each $i = 1, \ldots, d$ (respectively, preserves the Π_{F}-conjugacy class of $\Pi_{c^{\mathrm{F}}_{D_i}} \subseteq \Pi_{\mathrm{F}}$ for $i = 1, 2$, up to permutation by the "hyperelliptic involution", i.e., the outomorphism of Π_{F} of order 2 induced by the unique nontrivial covering transformation of the covering $\pi^{\log} : C^{\log} \longrightarrow D^{\log}$).*

Proof. By Lemma 2.7, (ii), $\alpha \in \mathrm{Aut}(\mathcal{G}_{i,j})$ for $1 \leq i < j \leq d$ [cf. Definition 3.3, (i)]. Since $2g - 2 + dr \geq 1$ [cf. Definition 2.4], $v^{\mathrm{F}}_{\mathrm{new},i,j,0}$ is of type $(0, 3)$, while $v^{\mathrm{F}}_{i,j,0}$ is *not* of type $(0, 3)$ [cf. Lemma 2.5, (iv); Definition 3.3, (i)]. Thus, α induces the identity automorphism of $\mathrm{Vert}(\mathcal{G}_{i,j})$ and, in particular, preserves the subset $\{c^{\mathrm{F}}_{D_i}, c^{\mathrm{F}}_{D_j}\} \subseteq \mathrm{Cusp}^{\mathrm{F}}(\mathcal{G}_*)$ [cf. Definition 2.6, (ii)].

Thus, when $d \geq 3$, we obtain the desired conclusion by varying i, j and applying the well-known elementary fact that any automorphism of a set of cardinality $d \geq 3$ that stabilizes every subset of cardinality 2 is necessarily the identity automorphism. When $d = 2$, the desired conclusion follows from the fact that the "hyperelliptic involution" permutes $c^{\mathrm{F}}_{D_1}$ and $c^{\mathrm{F}}_{D_2}$. □

4. The proof of Theorem A

In this section, our goal is to prove Theorem A [cf. Theorem 4.6]. After discussing the existence of degenerations of simple coverings that satisfy certain conditions in Lemmas 4.1, 4.2, 4.3, 4.4, and 4.5, we prove Theorem 4.6.

In Lemmas 4.1, 4.2, 4.3, 4.4, and 4.5, let $\Sigma \subseteq \mathfrak{Primes}$ be a nonempty set of prime numbers; k an algebraically closed field of characteristic zero. If $\pi : C \to D$ is an r-profiled simple admissible covering of degree d from a $([(d-1)(2g-2+2d)]+dr)$-pointed stable curve $(f : C \to \mathrm{Spec}\ k; \mu_f \subseteq C)$ of genus g to a $([2g-2+2d]+r)$-pointed stable curve $(h : D \to \mathrm{Spec}\ k; \mu_h \subseteq D)$ of genus 0 [cf. Definition 1.7], then we shall write

$$\mathcal{G}_C \text{ (respectively, } \mathcal{G}_D\text{)}$$

for the semi-graph of anabelioids of pro-Σ PSC-type determined by C (respectively, D).

Lemma 4.1. *Let (t, m, d) be a triple of integers satisfying one of the following two conditions (i), (ii):*

(i) $t \geq 3$, $0 \leq m \leq d-1$, $d \geq 4$;
(ii) $t = 2$, $0 \leq m \leq d-2$, $d \geq 4$.

Then there exists a 0-profiled simple admissible covering $\pi : C \to D$ of degree d over k from a symmetrically pointed stable curve $(f : C \to \mathrm{Spec}\ k; \mu_f \subseteq C)$ [cf. Definition 1.2, (ii)] to a symmetrically pointed stable curve $(h : D \to \mathrm{Spec}\ k; \mu_h \subseteq D)$ satisfying the following conditions:

- *$\pi : C \to D$ is ramified (i.e., fails to be unramified) over each point of μ_h;*
- *$\pi : C \to D$ is unramified over each node of D;*
- $\mathrm{Vert}(\mathcal{G}_C) = \{v_1, v_2, \ldots, v_{m+1}, w_1, w_2, \ldots, w_t\}$;
- $\mathrm{Node}(\mathcal{G}_C) = \{e_1, e_2, \ldots, e_{td}\}$;
- $\mathcal{N}(v_1) = \{e_1, e_2, \ldots, e_{d-m}, e_{d+1}, \ldots, e_{2d-m}, \ldots, e_{(t-1)d+1}, \ldots, e_{td-m}\}$;
- $\mathcal{N}(v_i) = \{e_l \mid l \equiv d-m+i-1 \pmod d\}$ *for* $i = 2, \ldots, m+1$;
- $\mathcal{N}(w_j) = \{e_{(j-1)d+1}, \cdots, e_{jd}\}$ *for* $j = 1, \ldots, t$;
- *v_1 is of type $(0, (d-m)t + 2(d-m-1)^2)$ [where the "$(d-m)t$" corresponds to the cardinality of $\mathcal{N}(v_1)$];*
- *for $i = 2, \ldots, m+1$ (respectively, $j = 1, \ldots, t$), v_i (respectively, w_j) is of type $(0, t + 2(d-m-1))$ (respectively, $(0, d + 2(d-1)^2)$) [where the "t" (respectively, "d") corresponds to the cardinality of $\mathcal{N}(v_i)$ (respectively, $\mathcal{N}(w_j)$)];*
- $\mathrm{Vert}(\mathcal{G}_D) = \{v', w'_1, w'_2, \ldots, w'_t\}$;
- $\mathrm{Node}(\mathcal{G}_D) = \{e'_1, e'_2, \ldots, e'_t\}$;
- $\mathcal{N}(v') = \{e'_1, e'_2, \ldots, e'_t\}$;
- $\mathcal{N}(w'_j) = \{e'_j\}$ $(1 \leq j \leq t)$;
- *v' is of type $(0, t+2(d-m-1))$ [where the "t" corresponds to the cardinality of $\mathcal{N}(v')$];*
- *for $j = 1, \ldots, t$, w'_j is of type $(0, 1+2d-2)$ [where the "1" corresponds to the cardinality of $\mathcal{N}(w'_j)$];*
- *for $i = 1, \ldots, m+1$ (respectively, $j = 1, \ldots, t$), v_i (respectively, w_j) lies over v' (respectively, w'_j).*

Proof. The desired simple admissible covering may be constructed by gluing together suitable simple coverings of smooth curves [cf. [2], Proposition 8.1] at unramified points of the coverings. Note that the numerical conditions imposed on (t, m, d) imply, in particular, that the resulting "symmetrically pointed curve D" is indeed *stable.* □

Remark 4.1.1. In the situation considered in Lemma 4.1, the genus of C is equal to

$$td - (m + t + 1) + 1 = t(d - 1) - m.$$

In particular, every integer g satisfying $g \geq d$ occurs as the genus of some C, i.e., for a suitable choice of (t, m). We use these coverings in the proof of Claim 4.6.D [cf. the proof of Theorem 4.6, (ii)].

Lemma 4.2. *There exist simple admissible coverings satisfying various conditions as follows:*

(i) Let t be an integer satisfying $t \geq 2$. Then there exists a 0*-profiled simple admissible covering $\pi : C \to D$ of degree* 3 *over k from a symmetrically pointed stable curve* $(f : C \to \mathrm{Spec}\ k; \mu_f \subseteq C)$ *[cf. Definition 1.2, (ii)] to a symmetrically pointed stable curve* $(h : D \to \mathrm{Spec}\ k; \mu_h \subseteq D)$ *satisfying the following conditions:*

- $\pi : C \to D$ *is ramified (i.e., fails to be unramified) over each point of* μ_h*;*
- $\pi : C \to D$ *is unramified over each node of D;*
- $\mathrm{Vert}(\mathcal{G}_C) = \{v_1, w_1, w_2, \ldots, w_t\}$*;*
- $\mathrm{Node}(\mathcal{G}_C) = \{e_1, e_2, \ldots, e_{3t}\}$*;*
- $\mathcal{N}(v_1) = \mathrm{Node}(\mathcal{G}_C)$*;*
- $\mathcal{N}(w_j) = \{e_{3j-2}, e_{3j-1}, e_{3j}\}$ *for* $j = 1, \ldots, t$*;*
- v_1 *is of type* $(0, 3t + 8)$ *[where the "3t" corresponds to the cardinality of* $\mathcal{N}(v_1)$*];*
- *for* $j = 1, \ldots, t$, w_j *is of type* $(1, 3 + 12)$ *[where the "3" corresponds to the cardinality of* $\mathcal{N}(w_j)$*];*
- $\mathrm{Vert}(\mathcal{G}_D) = \{v', w'_1, w'_2, \ldots, w'_t\}$*;*
- $\mathrm{Node}(\mathcal{G}_D) = \{e'_1, e'_2, \ldots, e'_t\}$*;*
- $\mathcal{N}(v') = \{e'_1, e'_2, \ldots, e'_t\}$*;*
- $\mathcal{N}(w'_j) = \{e'_j\}$ *for* $j = 1, \ldots, t$*;*
- v' *is of type* $(0, t + 4)$ *[where the "t" corresponds to the cardinality of* $\mathcal{N}(v')$*];*
- *for* $j = 1, \ldots, t$, w'_j *is of type* $(0, 1 + 6)$ *[where the "1" corresponds to the cardinality of* $\mathcal{N}(w'_j)$*];*
- v_1 *lies over* v'*, and* w_j *lies over* w'_j *for* $j = 1, \ldots, t$*.*

(ii) Let t be an integer satisfying $t \geq 3$. Then there exists a 0*-profiled simple admissible covering $\pi : C \to D$ of degree* 3 *over k from a symmetrically pointed stable curve* $(f : C \to \mathrm{Spec}\ k; \mu_f \subseteq C)$ *[cf. Definition 1.2, (ii)] to a symmetrically pointed stable curve* $(h : D \to \mathrm{Spec}\ k; \mu_h \subseteq D)$ *satisfying the following conditions*

- $\pi : C \to D$ *is ramified (i.e., fails to be unramified) over each point of* μ_h*;*

- *$\pi : C \to D$ is unramified over each node of D;*
- $\mathrm{Vert}(\mathcal{G}_C) = \{v_1, v_2, v_3, w_1, w_2, \ldots, w_t\}$*;*
- $\mathrm{Node}(\mathcal{G}_C) = \{e_1, e_2, \ldots, e_{3t}\}$*;*
- $\mathcal{N}(v_1) = \{e_1, e_4, \ldots, e_{3t-2}\}$*;*
- $\mathcal{N}(v_2) = \{e_2, e_5, \ldots, e_{3t-1}\}$*;*
- $\mathcal{N}(v_3) = \{e_3, e_6, \ldots, e_{3t}\}$*;*
- $\mathcal{N}(w_j) = \{e_{3j-2}, e_{3j-1}, e_{3j}\}$ *for $j = 1, \ldots, t$;*
- *for $i = 1, 2, 3$, v_i is of type $(0, t)$ [where the "t" corresponds to the cardinality of $\mathcal{N}(v_i)$];*
- *for $j = 1, \ldots, t$, w_j is of type $(1, 3 + 12)$ [where the "3" corresponds to the cardinality of $\mathcal{N}(w_j)$];*
- $\mathrm{Vert}(\mathcal{G}_D) = \{v', w'_1, w'_2, \ldots, w'_t\}$*;*
- $\mathrm{Node}(\mathcal{G}_D) = \{e'_1, e'_2, \ldots, e'_t\}$*;*
- $\mathcal{N}(v') = \{e'_1, e'_2, \ldots, e'_t\}$*;*
- $\mathcal{N}(w'_j) = \{e'_j\}$ *for $j = 1, \ldots, t$;*
- *v' is of type $(0, t)$ [where the "t" corresponds to the cardinality of $\mathcal{N}(v')$];*
- *for $j = 1, \ldots, t$, w'_j is of type $(0, 1 + 6)$ [where the "1" corresponds to the cardinality of $\mathcal{N}(w'_j)$];*
- *for $i = 1, 2, 3$ (respectively, $j = 1, \ldots, t$), v_i (respectively, w_j) lies over v' (respectively, w'_j).*

(iii) Let t be an integer satisfying $t \geq 2$. Then there exists a 0-profiled simple admissible covering $\pi : C \to D$ of degree 3 over k from a symmetrically pointed stable curve $(f : C \to \mathrm{Spec}\ k; \mu_f \subseteq C)$ [cf. Definition 1.2, (ii)] to a symmetrically pointed stable curve $(h : D \to \mathrm{Spec}\ k; \mu_h \subseteq D)$ satisfying the following conditions:

- *$\pi : C \to D$ is ramified (i.e., fails to be unramified) over each point of μ_h;*
- *$\pi : C \to D$ is unramified over each node of D;*
- $\mathrm{Vert}(\mathcal{G}_C) = \{v_1, v_2, w_1, w_2, \ldots, w_t\}$*;*
- $\mathrm{Node}(\mathcal{G}_C) = \{e_1, e_2, \ldots, e_{3t}\}$*;*
- $\mathcal{N}(v_1) = \{e_1, e_2, e_4, e_5, \ldots, e_{3t-2}, e_{3t-1}\}$*;*
- $\mathcal{N}(v_2) = \{e_3, e_6, \ldots, e_{3t}\}$*;*
- $\mathcal{N}(w_j) = \{e_{3j-2}, e_{3j-1}, e_{3j}\}$ *for $j = 1, \ldots, t$;*
- *v_1 (respectively, v_2) is of type $(0, 2t+2)$ (respectively, $(0, t+2)$) [where the "$2t$" (respectively, "t") corresponds to the cardinality of $\mathcal{N}(v_1)$ (respectively, $\mathcal{N}(v_2)$)];*
- *for $j = 1, \ldots, t$, w_j is of type $(1, 3 + 12)$ [where the "3" corresponds to the cardinality of $\mathcal{N}(w_j)$];*
- $\mathrm{Vert}(\mathcal{G}_D) = \{v', w'_1, w'_2, \ldots, w'_t\}$*;*
- $\mathrm{Node}(\mathcal{G}_D) = \{e'_1, e'_2, \ldots, e'_t\}$*;*
- $\mathcal{N}(v') = \{e'_1, e'_2, \ldots, e'_t\}$*;*
- $\mathcal{N}(w'_j) = \{e'_j\}$ *for $j = 1, \ldots, t$;*
- *v' is of type $(0, t + 2)$ [where the "t" corresponds to the cardinality of $\mathcal{N}(v')$];*
- *for $j = 1, \ldots, t$, w'_j is of type $(0, 1 + 6)$ [where the "1" corresponds to the cardinality of $\mathcal{N}(w'_j)$];*

- *for* $i = 1, 2$ *(respectively,* $j = 1, \ldots, t$*),* v_i *(respectively,* w_j*) lies over* v' *(respectively,* w'_j*).*

Proof. The desired simple admissible covering may be constructed by gluing together suitable simple coverings of smooth curves [cf. [2], Proposition 8.1] at unramified points of the coverings. Note that the numerical conditions imposed on t imply, in particular, that the resulting "symmetrically pointed curve D" is indeed *stable.* □

Remark 4.2.1. We observe that, as "C" varies over the various curves "C" constructed in the situations considered in Lemma 4.2, (i), (ii), (iii), every integer g satisfying $g \geq 5$ occurs as the genus of some C, i.e., for a suitable choice of t:

(i) In the situation considered in Lemma 4.2, (i), the genus of C is equal to
$$t + 3t - (t + 1) + 1 = 3t \ (t \geq 2).$$
We use these coverings in the proof of Claim 4.6.E.3 [cf. the proof of Theorem 4.6, (ii)].

(ii) In the situation considered in Lemma 4.2, (ii), the genus of C is equal to
$$t + 3t - (t + 3) + 1 = 3t - 2 \ (t \geq 3).$$
We use these coverings in the proof of Claim 4.6.E.4 [cf. the proof of Theorem 4.6, (ii)].

(iii) In the situation considered in Lemma 4.2, (iii), the genus of C is equal to
$$t + 3t - (t + 2) + 1 = 3t - 1 \ (t \geq 2).$$
We use these coverings in the proof of Claim 4.6.E.5 [cf. the proof of Theorem 4.6, (ii)].

Lemma 4.3. *There exists a simple* 0*-profiled admissible covering* $\pi : C \to D$ *of degree* 3 *over* k *from a symmetrically pointed stable curve* $(f : C \to \mathrm{Spec}\ k; \mu_f \subseteq C)$ *of genus* 5 *[cf. Definition 1.2, (ii)] to a symmetrically pointed stable curve* $(h : D \to \mathrm{Spec}\ k; \mu_h \subseteq D)$ *of genus* 0 *satisfying the following conditions:*

- $\pi : C \to D$ *is ramified (i.e., fails to be unramified) over each point of* μ_h*;*
- $\pi : C \to D$ *is unramified over each node of* D*;*
- $\mathrm{Vert}(\mathcal{G}_C) = \{v_1, w_1, w_2\}$*;*
- $\mathrm{Node}(\mathcal{G}_C) = \{e_1, e_2, \ldots, e_6\}$*;*
- $\mathcal{N}(v_1) = \mathrm{Node}(\mathcal{G}_C)$*;*
- $\mathcal{N}(w_1) = \{e_1, e_2, e_3\}$*;*
- $\mathcal{N}(w_2) = \{e_4, e_5, e_6\}$*;*
- v_1 *(respectively,* w_1*,* w_2*) is of type* $(0, 6+8)$ *(respectively,* $(1, 3+12)$*,* $(0, 3+8)$*) [where the "6" (respectively, "3", "3") corresponds to the cardinality of* $\mathcal{N}(v_1)$ *(respectively,* $\mathcal{N}(w_1)$*,* $\mathcal{N}(w_2)$*)];*
- $\mathrm{Vert}(\mathcal{G}_D) = \{v', w'_1, w'_2\}$*;*
- $\mathrm{Node}(\mathcal{G}_D) = \{e'_1, e'_2\}$*;*
- $\mathcal{N}(v') = \mathrm{Node}(\mathcal{G}_D)$*;*
- $\mathcal{N}(w'_j) = \{e'_j\}$ *for* $j = 1, 2$*;*
- v' *(respectively,* w'_1*,* w'_2*) is of type* $(0, 2+4)$ *(respectively,* $(0, 1+6)$*,* $(0, 1+4)$*) [where the "2" (respectively, "1", "1") corresponds to the cardinality of* $\mathcal{N}(v')$ *(respectively,* $\mathcal{N}(w'_1)$*,* $\mathcal{N}(w'_2)$*)];*

- v_1 *lies over* v'*, and* w_j *lies over* w'_j *for* $j = 1, 2$.

Proof. The desired simple admissible covering may be constructed by gluing together suitable simple coverings of smooth curves [cf. [2], Proposition 8.1] at unramified points of the coverings. □

Remark 4.3.1. In the proof of Claim 4.6.E.2 [cf. the proof of Theorem 4.6, (ii)], we use the covering constructed in Lemma 4.3 instead of the covering constructed in Lemma 4.2, (iii), for a technical reason.

Lemma 4.4. *There exists a* 1*-profiled simple admissible covering* $\pi : C \to D$ *of degree* 3 *over* k *from a* $([24] + 3)$*-pointed (respectively,* $([20] + 3)$*-pointed) stable curve* $(f : C \to \mathrm{Spec}\ k; \mu_f \subseteq C)$ *of genus* 4 *(respectively,* 3*) to a* $([12] + 1)$*-pointed (respectively,* $([10] + 1)$*-pointed) stable curve* $(h : D \to \mathrm{Spec}\ k; \mu_h \subseteq D)$ *of genus* 0 *satisfying the following conditions:*

- $\pi : C \to D$ *is ramified (i.e., fails to be unramified) over each unordered point of* μ_h*;*
- $\pi : C \to D$ *is unramified over each node of* D*;*
- $\mathrm{Vert}(\mathcal{G}_C) = \{v_1, v_2, v_3, w_1, w_2\}$*;*
- $\mathrm{Node}(\mathcal{G}_C) = \{e_1, e_2, \ldots, e_6\}$*;*
- $\mathcal{N}(v_1) = \{e_1, e_4\}$*;*
- $\mathcal{N}(v_2) = \{e_2, e_5\}$*;*
- $\mathcal{N}(v_3) = \{e_3, e_6\}$*;*
- $\mathcal{N}(w_1) = \{e_1, e_2, e_3\}$*;*
- $\mathcal{N}(w_2) = \{e_4, e_5, e_6\}$*;*
- v_i *is of type* $(0, 2 + 1)$ *for* $i = 1, 2, 3$ *[where the "*2*" corresponds to the cardinality of* $\mathcal{N}(v_i)$*; the "*1*" corresponds to the cardinality of the set of the ordered marked points on* v_i*];*
- w_1 *is of type* $(1, 3 + 12)$ *[where the "*3*" corresponds to the cardinality of* $\mathcal{N}(w_1)$*];*
- w_2 *is of type* $(1, 3 + 12)$ *(respectively,* $(0, 3 + 8)$*) [where the "*3*" corresponds to the cardinality of* $\mathcal{N}(w_2)$*];*
- $\mathrm{Vert}(\mathcal{G}_D) = \{v', w'_1, w'_2\}$*;*
- $\mathrm{Node}(\mathcal{G}_D) = \{e'_1, e'_2\}$*;*
- $\mathcal{N}(v') = \mathrm{Node}(\mathcal{G}_D)$*;*
- $\mathcal{N}(w'_j) = \{e'_j\}$ *for* $j = 1, 2$*;*
- v' *is of type* $(0, 2+1)$ *[where the "*2*" corresponds to the cardinality of* $\mathcal{N}(v')$*; the "*1*" corresponds to the cardinality of the set of the ordered marked points on* v'*];*
- w'_1 *is of type* $(0, 1 + 6)$ *[where the "*1*" corresponds to the cardinality of* $\mathcal{N}(w'_1)$*];*
- w'_2 *is of type* $(0, 1 + 6)$ *(respectively,* $(0, 1 + 4)$*) [where the "*1*" corresponds to the cardinality of* $\mathcal{N}(w'_2)$*];*
- v_i *lies over* v' *for* $i = 1, 2, 3$*, and* w_j *lies over* w'_j *for* $j = 1, 2$.

Proof. The desired 1-profiled simple admissible covering may be constructed by gluing together suitable simple coverings of smooth curves [cf. [2], Proposition 8.1] at unramified points of the coverings. □

Remark 4.4.1. We use these coverings in the proof of Claim 4.6.E.1 [cf. the proof of Theorem 4.6, (ii)].

Lemma 4.5. *Let t be an integer satisfying $t \geq 3$. Then there exists a* 0*-profiled simple admissible covering $\pi : C \to D$ of degree* 2 *over k from a symmetrically pointed stable curve $(f : C \to \text{Spec } k; \mu_f \subseteq C)$ [cf. Definition 1.2, (ii)] to a symmetrically pointed stable curve $(h : D \to \text{Spec } k; \mu_h \subseteq D)$ satisfying the following conditions:*

- *$\pi : C \to D$ is ramified (i.e., fails to be unramified) over each point of μ_h;*
- *$\pi : C \to D$ is unramified over each node of D;*
- $\text{Vert}(\mathcal{G}_C) = \{v_1, v_2, w_1, w_2, \ldots, w_t\}$*;*
- $\text{Node}(\mathcal{G}_C) = \{e_1, e_2, \ldots, e_{2t}\}$*;*
- $\mathcal{N}(v_1) = \{e_1, e_3, \ldots, e_{2t-1}\}$*;*
- $\mathcal{N}(v_2) = \{e_2, e_4, \ldots, e_{2t}\}$*;*
- $\mathcal{N}(w_j) = \{e_{2j-1}, e_{2j}\}$ *for $j = 1, \ldots, t$;*
- *for $i = 1, 2$ (respectively, $j = 1, \ldots, t$), v_i (respectively, w_j) is of type $(0, t)$ (respectively, $(0, 2+2)$) [where the "t" (respectively, the first "2" of "2+2") corresponds to the cardinality of $\mathcal{N}(v_i)$ (respectively, $\mathcal{N}(w_j)$)];*
- $\text{Vert}(\mathcal{G}_D) = \{v', w'_1, w'_2, \ldots, w'_t\}$*;*
- $\text{Node}(\mathcal{G}_D) = \{e'_1, e'_2, \ldots, e'_t\}$*;*
- $\mathcal{N}(v') = \text{Node}(\mathcal{G}_D)$*;*
- $\mathcal{N}(w'_j) = \{e'_j\}$ *for $j = 1, \ldots, t$;*
- *v' is of type $(0, t)$ [where the "t" corresponds to the cardinality of $\mathcal{N}(v')$];*
- *w'_j is of type $(0, 1 + 2)$ for $j = 1, \ldots, t$ [where the "1" corresponds to the cardinality of $\mathcal{N}(w'_j)$];*
- *for $i = 1, 2$ (respectively, $j = 1, \ldots, t$), v_i (respectively, w_j) lies over v' (respectively, w'_j).*

Proof. The desired simple admissible covering may be constructed by gluing together suitable simple coverings of smooth curves [cf. [2], Proposition 8.1] at unramified points of the coverings. Note that the numerical conditions imposed on t imply, in particular, that the resulting "symmetrically pointed curve D" is indeed *stable.* □

Remark 4.5.1. In the situation considered in Lemma 4.4, the genus of C is equal to

$$2t - (t + 2) + 1 = t - 1.$$

We use these coverings in the proof of Claim 4.6.F [cf. the proof of Theorem 4.6, (ii)].

Theorem 4.6. *Let Σ be a nonempty set of prime numbers; k an algebraically closed field of characteristic zero; (g, d, r) a triple of nonnegative integers such that*

$$d \geq 2 \ \wedge \ (g, r) \notin \{(0, 0), (1, 0)\} \ \wedge \ (g, d, r) \notin \{(0, 2, 1), (0, 3, 1)\}$$
$$(\Rightarrow 2g - 2 + dr > 1 \ \wedge \ 2g + 2d + r - 5 \geq 1).$$

Write $(\mathcal{H}_{g,d,r})_k$ for the r-profiled **Hurwitz stack** *of type (g, d) over k [cf. Definition 1.8; Definition 1.13, (ii)], where $\dim(\mathcal{H}_{g,d,r})_k = 2g - 2 + 2d + r - 3 = 2g + 2d + r - 5 \geq 1$ [cf. Corollary 1.9]; $(\mathcal{C}_{g,d,r})_k \to (\mathcal{H}_{g,d,r})_k$ for the restriction of* *the* **tautological curve** *over* $(\mathcal{M}_{g,dr})_k$ *to* $(\mathcal{H}_{g,d,r})_k$ *via the natural (*1*-)morphism* $(\mathcal{H}_{g,d,r})_k \to (\mathcal{M}_{g,dr})_k$ *[cf. Proposition 1.10, (iii)];* $\Pi_{\mathcal{H}_{g,d,r}} \stackrel{\mathrm{def}}{=} \pi_1((\mathcal{H}_{g,d,r})_k)$ *for the étale fundamental group of the profiled Hurwitz stack* $(\mathcal{H}_{g,d,r})_k$*;* $\Pi_{g,d,r}$ *for the maximal pro-*Σ *quotient of the kernel* $N_{g,d,r}$ *of the natural surjection* $\pi_1((\mathcal{C}_{g,d,r})_k) \twoheadrightarrow \pi_1((\mathcal{H}_{g,d,r})_k) = \Pi_{\mathcal{H}_{g,d,r}}$*;* $\Pi_{\mathcal{C}_{g,d,r}}$ *for the quotient of the étale fundamental group* $\pi_1((\mathcal{C}_{g,d,r})_k)$ *of* $(\mathcal{C}_{g,d,r})_k$ *by the kernel of the natural surjection* $N_{g,d,r} \twoheadrightarrow \Pi_{g,d,r}$*;* $\mathrm{Out}^{\mathrm{C}}(\Pi_{g,d,r})$ *for the group of outomorphisms [cf. the discussion entitled "Topological groups" in Notations and Conventions] of* $\Pi_{g,d,r}$ *which induce bijections on the set of cuspidal inertia subgroups of* $\Pi_{g,d,r}$*. Thus, we have a natural sequence of profinite groups*

$$1 \longrightarrow \Pi_{g,d,r} \longrightarrow \Pi_{\mathcal{C}_{g,d,r}} \longrightarrow \Pi_{\mathcal{H}_{g,d,r}} \longrightarrow 1$$

which determines an outer representation

$$\rho_{g,d,r} : \Pi_{\mathcal{H}_{g,d,r}} \longrightarrow \mathrm{Out}(\Pi_{g,d,r}).$$

Then the following hold:

(i) The profinite group $\Pi_{g,d,r}$ *is the maximal pro-*Σ *quotient of the étale fundamental group of a hyperbolic curve over an algebraically closed field of characteristic zero [i.e., a pro-*Σ *surface group — cf.* [17]*, Definition 1.2] and is naturally isomorphic to the profinite group "*$\Pi_{g,r}$*" of* [5]*, Theorem D, in the case where one takes the "*(g,r)*" of loc. cit. to be* (g,dr) *[in the notation of the present discussion].*

(ii) Let $H \subseteq \Pi_{\mathcal{H}_{g,d,r}}$ *be an open subgroup of* $\Pi_{\mathcal{H}_{g,d,r}}$*. Then the composite of natural homomorphisms*

$$\begin{aligned}\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k) &\longrightarrow \mathrm{Aut}_{\Pi_{\mathcal{H}_{g,d,r}}}(\Pi_{\mathcal{C}_{g,d,r}})/\mathrm{Inn}(\Pi_{g,d,r})\\ &\stackrel{\sim}{\longrightarrow} Z_{\mathrm{Out}(\Pi_{g,d,r})}(\mathrm{Im}(\rho_{g,d,r})) \subseteq Z_{\mathrm{Out}(\Pi_{g,d,r})}(\rho_{g,d,r}(H))\end{aligned}$$

[cf. the discussion entitled "Topological groups" in Notations and Conventions] determines an **isomorphism**

$$\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k) \stackrel{\sim}{\longrightarrow} Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,d,r})}(\rho_{g,d,r}(H)).$$

Moreover, $\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k)$ *is isomorphic to*

$$\begin{cases}\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } (g,d,r) \in \{(0,2,2),(0,4,1)\};\\ \mathbb{Z}/2\mathbb{Z} & \text{if } (g,d,r) \in \{(g,2,r) \mid (g,r) \neq (0,2)\} \cup \{(2,d,0)\};\\ \{1\} & \text{if } (g,d,r) \notin \{(0,4,1),(g,2,r),(2,d,0)\}.\end{cases}$$

(iii) Let $H \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{g,d,r})$ *be a closed subgroup of* $\mathrm{Out}^{\mathrm{C}}(\Pi_{g,d,r})$ *that contains an open subgroup of* $\mathrm{Im}(\rho_{g,d,r}) \subseteq \mathrm{Out}(\Pi_{g,d,r})$*. Then* H *is* **almost slim** *[cf. the discussion entitled "Topological groups" in Notations and Conventions]. If, moreover,*

$$(g,d,r) \notin \{(0,4,1),(g,2,r),(2,d,0)\},$$

then H *is* **slim** *[cf. the discussion entitled "Topological groups" in Notations and Conventions].*

Proof. Assertion (i) follows immediately from Proposition 1.14, (ii), together with the various definitions involved.

Next, we verify assertion (ii). First, we verify the following assertion:

Claim 4.6.A: The composite homomorphism

$$\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k) \longrightarrow Z_{\mathrm{Out}(\Pi_{g,d,r})}(\rho_{g,d,r}(H)) = Z_{\mathrm{Out}(\Pi_{g,dr})}(\rho_{g,d,r}(H))$$

[cf. Theorem 4.6, (i)] is injective.

This follows immediately from the well-known fact that any non-trivial automorphism of a hyperbolic curve over an algebraically closed field of characteristic $\notin \Sigma$ induces a non-trivial outomorphism of the maximal pro-Σ quotient of the étale fundamental group of the hyperbolic curve [cf., e.g., [13], the proof of Theorem 14.1]. This completes the proof of Claim 4.6.A.

Note that it follows immediately from the various definitions involved that the composite homomorphism $\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k) \to Z_{\mathrm{Out}(\Pi_{g,dr})}(\rho_{g,d,r}(H))$ factors through $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,dr})}(\rho_{g,d,r}(H))$, hence determines an injection

$$\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k) \hookrightarrow Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,dr})}(\rho_{g,d,r}(H)).$$

In the remainder of the proof, for each $x \in \widetilde{\mathcal{H}}_{g,d,r}(k)$ [cf. Proposition 1.10, (i)], write

$$\mathcal{G}_x$$

for the semi-graph of anabelioids of pro-Σ PSC-type associated to the geometric fiber of $\widetilde{\mathcal{C}}^{\log}_{g,d,r} \to \widetilde{\mathcal{H}}^{\log}_{g,d,r}$ [cf. Definition 1.13, (i)] over $x^{\log} \stackrel{\mathrm{def}}{=} x \times_{\widetilde{\mathcal{H}}_{g,d,r}} \widetilde{\mathcal{H}}^{\log}_{g,d,r}$. Thus, we have a natural $\mathrm{Im}(\rho_{g,d,r})$-torsor of outer isomorphisms $\Pi_{g,dr} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$. Let us fix an isomorphism $\Pi_{g,dr} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$ that belongs to this collection of isomorphisms.

Next, we verify the following assertion:

Claim 4.6.B: Suppose that

$$\big\{r > 1 \ \wedge \ 2g-2+d(r-1) \geq 1\big\} \vee \big\{r = 1 \ \wedge \ 2g-2 \geq 1 \ \wedge \ d > g\big\}$$
$$\Leftrightarrow$$
$$r > 0 \ \wedge \ \{r = 1 \Rightarrow d > g\} \ \wedge \ (g,r) \notin \{(0,1),(1,1)\} \ \wedge \ (g,d,r) \neq (0,2,2).$$

Then the injection

$$\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k) \hookrightarrow Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,dr})}(\rho_{g,d,r}(H))$$

is surjective. Moreover, the description of $\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k)$ in the statement of Theorem 4.6, (ii), holds.

Indeed, since $2g + 2d + r - 5 \geq 1$, it makes sense to define N to be the kernel of the surjection $\Pi_{\mathcal{H}_{g,d,r}} \twoheadrightarrow \Pi_{\mathcal{H}_{g,d,r-1}}$ [cf. Proposition 1.14, (iii)] determined by the (1-)morphism $\phi_{g,d,r-1} : (\mathcal{H}_{g,d,r})_k \to (\mathcal{H}_{g,d,r-1})_k$ [cf. Proposition 1.10, (iii)] obtained by forgetting the final d sections (respectively, final section) of the domain curve (respectively, codomain curve). Then it follows immediately from the various definitions involved that there exists a commutative diagram of profinite groups

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Pi_{g,dr} & \longrightarrow & E & \longrightarrow & N & \longrightarrow & 1 \\ & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \\ 1 & \longrightarrow & \Pi_{\mathrm{F}} & \longrightarrow & \Pi_{\mathrm{T}} & \longrightarrow & \Pi_{\mathrm{B}} & \longrightarrow & 1, \end{array}$$

where the upper sequence is the exact sequence obtained by pulling back the exact sequence

$$1 \longrightarrow \Pi_{g,dr} \longrightarrow \Pi_{\mathcal{C}_{g,d,r}} \longrightarrow \Pi_{\mathcal{H}_{g,d,r}} \longrightarrow 1$$

[cf. Proposition 1.14, (ii)] by the natural inclusion $N \hookrightarrow \Pi_{\mathcal{H}_{g,d,r}}$; the lower sequence is the exact sequence "$1 \to \Pi_{\mathrm{F}} \to \Pi_{\mathrm{T}} \to \Pi_{\mathrm{B}} \to 1$" obtained by applying the procedure given in the statement of Lemma 2.3 in the case where $\Sigma_{\mathrm{F}} = \Sigma$ and $m = 1$ to an $(r-1)$-profiled simple log admissible covering of degree d whose domain is a stable log curve of genus g over $(\mathrm{Spec}\ k)^{\log}$; the vertical arrows are isomorphisms.

Let $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,dr})}(\rho_{g,d,r}(H))$ be an outomorphism of $\Pi_{g,dr}$. Thus, α naturally determines an element of $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,dr})}(\rho_{g,d,r}(H \cap N)) \xrightarrow{\sim} Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\rho_1(H \cap N))$, where ρ_1 is as in Lemma 2.3.

Next, we claim the following:

Claim 4.6.B.1: When $r-1 = 0$, the condition "$\alpha \in \mathrm{Aut}^{|\{w^{\mathrm{F}}\}|}(\mathcal{G}_{1,2,\ldots,g})$" in Proposition 3.1 is satisfied.

Let $x \in C_1(k) \subseteq \widetilde{\mathcal{H}}_{g,d,1}(k)$ be as in Definition 3.3, (iii), so $\mathcal{G}_x$ may be identified with "$\mathcal{G}_{1,2,\ldots,g}$". Next, let us consider the composite

$$\pi_1(x^{\log}) \longrightarrow \pi_1((\widetilde{\mathcal{H}}^{\log}_{g,d,1})_k) \xleftarrow{\sim} \Pi_{\mathcal{H}_{g,d,1}} \xrightarrow{\rho_{g,d,1}} \mathrm{Out}(\Pi_{g,d}),$$

where the first arrow is the natural outer homomorphism; the second arrow is the outer isomorphism obtained by applying the log purity theorem to the natural (1-)morphism $(\mathcal{H}_{g,d,1})_k \hookrightarrow (\widetilde{\mathcal{H}}^{\log}_{g,d,1})_k$ [cf. Proposition 1.10, (ii); [12], Theorem B]. This composite factors through $\pi_1(x^{\log}) \to \mathrm{Dehn}(\mathcal{G}_x)$ [cf. [5], Definition 4.4; [5], Lemma 5.4, (iii)]. By considering the log structure of $(\widetilde{\mathcal{H}}^{\log}_{g,d,1})_k$ [cf. Theorem 1.5; Corollary 1.9; Proposition 1.10, (ii); [11], §3.23], we conclude that the image of this arrow $\pi_1(x^{\log}) \to \mathrm{Dehn}(\mathcal{G}_x)$ contains a positive definite element [cf. [5], Lemma 5.4, (ii); [5], Definition 5.8, (iii)], hence is IPSC-ample [cf. [5], Definition 5.13]. Thus, it follows from [5], Lemma 5.12, (i); [5], Theorem 5.14, (i), that $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,d})}(\rho_{g,d,1}(H))$ is graphic. Next, let us observe that it follows immediately from the description of the log structure of $(\widetilde{\mathcal{H}}^{\log}_{g,d,1})_k$ given in [11], §3.23 [cf. also Theorem 1.5; Corollary 1.9; Proposition 1.10, (ii); Lemma 2.11], that the deformations parametrized by $(\widetilde{\mathcal{H}}^{\log}_{g,d,1})_k$ of nodes $\in \mathcal{N}(w^{\mathrm{F}})$ are *independent* of the deformations parametrized by $(\widetilde{\mathcal{H}}^{\log}_{g,d,1})_k$ of nodes $\in \mathcal{N}(w'^{\mathrm{F}})$. Thus, the fact that $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,d})}(\rho_{g,d,1}(H))$ implies [cf. [5], Lemma 5.4, (ii)] that α determines an element of $\mathrm{Aut}(\mathcal{G}_x)$ that preserves w^{F}. This completes the proof of Claim 4.6.B.1.

Thus, by Propositions 3.1 and 3.6, the elements of $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathrm{F}})}(\rho_1(H \cap N))$ are geometric, hence $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,dr})}(\rho_{g,d,r}(H))$ is geometric [i.e., in this case, is trivial or arises from the hyperelliptic involution]. This completes the proof of Claim 4.6.B.

Next, we verify the following assertion:

Claim 4.6.C: Suppose that

$$\{r = 0 \Rightarrow d \geq \tfrac{g}{2} + 1\} \ \vee \ (g, r) \in \{(0,1), (1,1)\} \ \vee \ (g, d, r) = (0, 2, 2).$$

Then the injection $\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k) \hookrightarrow Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,dr})}(\rho_{g,d,r}(H))$ is surjective. Moreover, the description of $\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k)$ in the statement of Theorem 4.6, (ii), holds.

Since the image of the arrow $\psi_{g,d,r} : (\mathcal{H}_{g,d,r})_k \longrightarrow (\mathcal{M}_{g,dr})_k$ [cf. Definition 1.13, (i)] is dense in this case [cf. Corollary 2.9; Proposition 2.10; the well-known, elementary structure of double coverings of the projective line over k], the image of the arrow $\Pi_{(\mathcal{H}_{g,d,r})_k} \to \Pi_{(\mathcal{M}_{g,dr})_k}$ is open. Thus, the assertion follows immediately from the corresponding "Grothendieck Conjecture" for the universal curve over $(\mathcal{M}_{g,dr})_k$ [cf. Theorem M, (i)]. This completes the proof of Claim 4.6.C.

Next, we verify the following assertion:

Claim 4.6.D: Suppose that

$$r = 0,\ d \geq 4 \text{ (respectively, } r = 1,\ d \geq 4).$$

Then the injection $\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k) \hookrightarrow Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,dr})}(\rho_{g,d,r}(H))$ is surjective. Moreover, the description of $\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k)$ in the statement of Theorem 4.6, (ii), holds.

By Claim 4.6.C (respectively, Claim 4.6.B), we may assume that $g \geq 2d-1 \geq 7$ (respectively, $g \geq d \geq 4$). Thus, it suffices to show that the centralizer

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,dr})}(\rho_{g,d,r}(H))$$

is trivial. Next, let us observe that by considering the covering obtained by applying Lemma 4.1 in the case where

$$t \stackrel{\mathrm{def}}{=} \lceil \tfrac{g}{d-1} \rceil \ (\Rightarrow t \geq 3 \text{ (respectively, } t \geq 2)),\ \ m \stackrel{\mathrm{def}}{=} t(d-1) - g,$$

one may verify easily that there exists a k-valued point $x \in \widetilde{\mathcal{H}}_{g,d,0}(k)$ [so $\mathcal{G}_x$ has no cusps!] (respectively, $x \in \widetilde{\mathcal{H}}_{g,d,1}(k)$ [so $\mathcal{G}_x$ has precisely d cusps]) satisfying the following conditions:

- $\mathrm{Vert}(\mathcal{G}_x) = \{v_1, v_2, \ldots, v_{m+1}, w_1, w_2, \ldots, w_t\}$;
- $\mathrm{Node}(\mathcal{G}_x) = \{e_1, e_2, \ldots, e_{td}\}$;
- $\mathcal{N}(v_1) = \{e_1, e_2, \ldots, e_{d-m}, e_{d+1}, \ldots, e_{2d-m}, \ldots, e_{(t-1)d+1}, \ldots, e_{td-m}\}$;
- $\mathcal{N}(v_i) = \{e_l \mid l \equiv d - m + i - 1 \pmod{d}\}$ for $i = 2, \ldots, m+1$;
- $\mathcal{N}(w_j) = \{e_{(j-1)d+1}, \cdots, e_{jd}\}$ for $j = 1, \ldots, t$;
- v_1 is of type $(0, (d-m)t)$ (respectively, $(0, (d-m)(t+1))$);
- for $i = 2, \ldots, m+1$, v_i is of type $(0, t)$ (respectively, $(0, t+1)$);
- for $j = 1, \ldots, t$, w_j is of type $(0, d)$.

Thus, let us fix $x \in \widetilde{\mathcal{H}}_{g,d,0}(k)$ (respectively, $x \in \widetilde{\mathcal{H}}_{g,d,1}(k)$) satisfying the above conditions.

Let $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,d,0}(H))$ (respectively, $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,d})}(\rho_{g,d,1}(H))$) be an outomorphism of $\Pi_{g,0}$ (respectively, $\Pi_{g,d}$). Suppose, moreover, that, relative to the isomorphism $\Pi_{g,0} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$ (respectively, $\Pi_{g,d} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$) fixed above [cf. the discussion immediately preceding Claim 4.6.B], $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,d,0}(H))$ (respectively, $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,d})}(\rho_{g,d,1}(H))$) determines an element of $\mathrm{Aut}(\mathcal{G}_x)$ that preserves w_j for each $j = 1, \ldots, t$ [cf. Claim 4.6.D.2]. For $j = 1, \ldots, t$, write

$$(\mathcal{G}_x)_j \stackrel{\mathrm{def}}{=} (\mathcal{G}_x)_{\rightsquigarrow\{e_{(j-1)d+1}, e_{(j-1)d+2}, \ldots, e_{jd}\}}$$

[cf. [5], Definition 2.8]; α_j for the image of α via the natural inclusion $\mathrm{Aut}^{|W|}(\mathcal{G}_x) \hookrightarrow \mathrm{Aut}((\mathcal{G}_x)_j)$ [cf. [5], Definition 2.6, (i); [5], Proposition 2.9, (ii)], where we write $W \stackrel{\mathrm{def}}{=} \{w_1, w_2, \ldots, w_t\}$.

Next, we claim the following:

Claim 4.6.D.1: $\alpha_j \in \mathrm{Dehn}((\mathcal{G}_x)_j)$.

Note that one may verify easily that there exists a k-valued point $y_j \in \widetilde{\mathcal{H}}_{g,d,0}(k)$ (respectively, $y_j \in \widetilde{\mathcal{H}}_{g,d,1}(k)$) such that $\mathcal{G}_{y_j}$ may be identified with $(\mathcal{G}_x)_j$.

By gluing together

- a $(t-1)$-profiled (respectively, t-profiled) simple covering of degree d from a smooth curve of genus $d-1-m$ [corresponding to $v_1, v_2, \ldots, v_{m+1}, w_j$] to a smooth curve of genus 0

and

- an ordered collection of $(t-1)$ 1-profiled simple coverings of degree d from smooth curves of genus 0 [corresponding to $w_{j'}$, for $j' \in \{1, 2, \ldots, t\} \setminus \{j\}$] to smooth curves of genus 0

at unramified marked points of

- the domain curves [i.e., for each $p = 1, \ldots, t-1$ and $q = 1, \ldots, d$, we glue the $((p-1)d+q)$-th marked point of the domain curve of the $(t-1)$-profiled (respectively, t-profiled) simple covering to the q-th marked point of the domain curve of the p-th member of the ordered collection of $(t-1)$ 1-profiled simple coverings]

and

- the codomain curves [i.e., for each $p = 1, \ldots, t-1$, we glue the p-th marked point of the codomain curve of the $(t-1)$-profiled (respectively, t-profiled) simple covering to the marked point of the codomain curve of the p-th member of the ordered collection of $(t-1)$ 1-profiled simple coverings],

we obtain a clutching morphism [cf. [10], Definition 3.6]

$$\mathcal{H}_{d-1-m,d,t-1} \times \mathcal{H}_{0,d,1} \times \cdots \times \mathcal{H}_{0,d,1} \longrightarrow \overline{\mathcal{H}}_{g,d,0}$$
$$\text{(respectively, } \mathcal{H}_{d-1-m,d,t} \times \mathcal{H}_{0,d,1} \times \cdots \times \mathcal{H}_{0,d,1} \longrightarrow \overline{\mathcal{H}}_{g,d,1}\text{)},$$

where the number of factors in the above product is t. Since the image of this clutching morphism is contained in the normal locus of $\overline{\mathcal{H}}_{g,d,0}$ (respectively, $\overline{\mathcal{H}}_{g,d,1}$) [cf. Theorem 1.5; Corollary 1.9; [11], §3.23], we thus obtain a clutching morphism

$$\mathcal{H}_{d-1-m,d,t-1} \times \mathcal{H}_{0,d,1} \times \cdots \times \mathcal{H}_{0,d,1} \longrightarrow \widetilde{\mathcal{H}}_{g,d,0}$$
$$\text{(respectively, } \mathcal{H}_{d-1-m,d,t} \times \mathcal{H}_{0,d,1} \times \cdots \times \mathcal{H}_{0,d,1} \longrightarrow \widetilde{\mathcal{H}}_{g,d,1}\text{)}.$$

Note that $y_j \in \widetilde{\mathcal{H}}_{g,d,0}(k)$ (respectively, $y_j \in \widetilde{\mathcal{H}}_{g,d,1}(k)$) is contained in the image of the above clutching morphism, and that

$$\alpha_j \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})}(\rho_{g,d,0}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_j) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})$$
$$\text{(respectively, } \alpha_j \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})}(\rho_{g,d,1}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_j) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})\text{)}$$

naturally determines, by considering the above clutching morphism, an element of

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{d-1-m,(t-1)d})}(\rho_{d-1-m,d,t-1}(H^\dagger))$$
$$\text{(respectively, } Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{d-1-m,td})}(\rho_{d-1-m,d,t}(H^\dagger))\text{)},$$

where $H^\dagger \subseteq \Pi_{\mathcal{H}_{d-1-m,d,t-1}}$ (respectively, $H^\dagger \subseteq \Pi_{\mathcal{H}_{d-1-m,d,t}}$) is an open subgroup of $\Pi_{\mathcal{H}_{d-1-m,d,t-1}}$ (respectively, $\Pi_{\mathcal{H}_{d-1-m,d,t}}$). Since $d \geq 4$ and $t \geq 3$ (respectively, $t \geq 2$), this element is trivial by Claim 4.6.B. In particular, $\alpha_j \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{y_j}) \subseteq \mathrm{Aut}(\mathcal{G}_{y_j}) = \mathrm{Aut}((\mathcal{G}_x)_j)$. On the other hand,

$$\alpha_j \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})}(\rho_{g,d,0}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_j) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})$$
$$\text{(respectively, } \alpha_j \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})}(\rho_{g,d,1}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_j) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})\text{)}$$

naturally determines, by considering the above clutching morphism, an element of

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{0,d})}(\rho_{0,d,1}(H^\ddagger))$$

for each component of the above product $\mathcal{H}_{0,d,1} \times \cdots \times \mathcal{H}_{0,d,1}$, where $H^{\ddagger} \subseteq \Pi_{\mathcal{H}_{0,d,1}}$ is an open subgroup of $\Pi_{\mathcal{H}_{0,d,1}}$. Since $\alpha_j \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{y_j})$, these elements preserve each $\Pi_{0,d}$-conjugacy class of cuspidal subgroups of $\Pi_{0,d}$. Thus, since $d \geq 4$, these elements are trivial by Claim 4.6.C. This completes the proof of Claim 4.6.D.1.

Thus, by varying j, we conclude from Claim 4.6.D.1 and [5], Theorem 4.8, (ii), (iv), that α is trivial. Hence it remains to verify the following:

> Claim 4.6.D.2: α determines an element of $\mathrm{Aut}(\mathcal{G}_x)$ that preserves w_j for each $j = 1, \ldots, t$.

The proof of Claim 4.6.D.2 is similar to the proof of Claim 4.6.B.1, i.e., it suffices to observe that, for $j \neq j'$, the deformations parametrized by $(\widetilde{\mathcal{H}}^{\log}_{g,d,0})_k$ (respectively, $(\widetilde{\mathcal{H}}^{\log}_{g,d,1})_k$) of nodes $\in \mathcal{N}(w_j)$ are *independent* of the deformations parametrized by $(\widetilde{\mathcal{H}}^{\log}_{g,d,0})_k$ (respectively, $(\widetilde{\mathcal{H}}^{\log}_{g,d,1})_k$) of nodes $\in \mathcal{N}(w_{j'})$ [cf. Lemma 4.1]. This completes the proof of Claim 4.6.D.2, hence also the proof of Claim 4.6.D.

Next, we verify the following assertion:

> Claim 4.6.E: Suppose that
> $$r = 0,\ d = 3 \text{ (respectively, } r = 1,\ d = 3).$$
> Then the injection $\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k) \hookrightarrow Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,d_r})}(\rho_{g,d,r}(H))$ is surjective. Moreover, the description of $\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k)$ in the statement of Theorem 4.6, (ii), holds.

Since we are operating under the assumption that $(g,d,r) \neq (0,3,1)$, it follows from Claim 4.6.C (respectively, Claims 4.6.B, 4.6.C) that we may assume that $g \geq 5$ (respectively, $g \geq 3$). Thus, it suffices to show that the centralizer

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,d_r})}(\rho_{g,d,r}(H))$$

is trivial.

> Claim 4.6.E.1: When $g = 4$ (respectively, $g = 3$), $r = 1$, Claim 4.6.E holds. [Note that in the statement and proof of the present Claim 4.6.1, the non-resp'd case corresponds to the case $g = 4$, $r = 1$, while the resp'd case corresponds to the case $g = 3$, $r = 1$. This partition into non-resp'd and resp'd cases *differs* from the partition into non-resp'd and resp'd cases that is adopted in the statement of Claim 4.6.E, as well as in the statements and proofs of Claims 4.6.E.2, 4.6.E.3, 4.6.E.4, 4.6.E.5, i.e., where the non-resp'd case corresponds to the case $r = 0$, while the resp'd case corresponds to the case $r = 1$.]

Let us first observe that by considering the covering obtained by applying Lemma 4.4, one may verify easily that there exists a k-valued point $x \in \widetilde{\mathcal{H}}_{g,3,1}(k)$ [so $\mathcal{G}_x$ has 3 cusps] satisfying the following conditions:

- $\mathrm{Vert}(\mathcal{G}_x) = \{v_1, v_2, v_3, w_1, w_2\}$;
- $\mathrm{Node}(\mathcal{G}_x) = \{e_1, e_2, \ldots, e_6\}$;
- $\mathcal{N}(v_1) = \{e_1, e_4\}$;
- $\mathcal{N}(v_2) = \{e_2, e_5\}$;
- $\mathcal{N}(v_3) = \{e_3, e_6\}$;
- $\mathcal{N}(w_1) = \{e_1, e_2, e_3\}$;
- $\mathcal{N}(w_2) = \{e_4, e_5, e_6\}$;
- v_i is of type $(0,3)$ for $i = 1, 2, 3$;

- w_1 is of type $(1,3)$;
- w_2 is of type $(1,3)$ (respectively, $(0,3)$).

Thus, let us fix $x \in \widetilde{\mathcal{H}}_{4,3,1}(k)$ (respectively, $x \in \widetilde{\mathcal{H}}_{3,3,1}(k)$) satisfying the above conditions.

Let $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{4,3})}(\rho_{4,3,1}(H))$ (respectively, $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{3,3})}(\rho_{3,3,1}(H))$) be an outomorphism of $\Pi_{4,3}$ (respectively, $\Pi_{3,3}$). Suppose, moreover, that, relative to the isomorphism $\Pi_{4,3} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$ (respectively, $\Pi_{3,3} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$) fixed above [cf. the discussion immediately preceding Claim 4.6.B], $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{4,3})}(\rho_{4,3,1}(H))$ (respectively, $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{3,3})}(\rho_{3,3,1}(H))$) determines an element of $\mathrm{Aut}(\mathcal{G}_x)$ that preserves w_j for each $j = 1, 2$ [cf. Claim 4.6.E.1.3]. For $j = 1, 2$, write

$$(\mathcal{G}_x)_j \stackrel{\mathrm{def}}{=} (\mathcal{G}_x)_{\rightsquigarrow\{e_{3j-2}, e_{3j-1}, \ldots, e_{3j}\}}$$

[cf. [5], Definition 2.8]; α_j for the image of α via the natural inclusion $\mathrm{Aut}^{|W|}(\mathcal{G}_x) \hookrightarrow \mathrm{Aut}((\mathcal{G}_x)_j)$ [cf. [5], Definition 2.6, (i); [5], Proposition 2.9, (ii)], where we write $W \stackrel{\mathrm{def}}{=} \{w_1, w_2\}$.

Next, we claim the following:

Claim 4.6.E.1.1: $\alpha_2 \in \mathrm{Dehn}((\mathcal{G}_x)_2)$.

Note that one may verify easily that there exists a k-valued point $y_2 \in \widetilde{\mathcal{H}}_{4,3,1}(k)$ (respectively, $y_2 \in \widetilde{\mathcal{H}}_{3,3,1}(k)$) such that $\mathcal{G}_{y_2}$ may be identified with $(\mathcal{G}_x)_2$.

By gluing together simple coverings at unramified marked points as in the proof of Claim 4.6.D.1, we obtain a clutching morphism [cf. [10], Definition 3.6]

$$\mathcal{H}_{1,3,2} \times \mathcal{H}_{1,3,1} \longrightarrow \overline{\mathcal{H}}_{4,3,1}$$
$$(\text{respectively, } \mathcal{H}_{0,3,2} \times \mathcal{H}_{1,3,1} \longrightarrow \overline{\mathcal{H}}_{3,3,1}),$$

where the first factor in the product corresponds to the irreducible component of $(\mathcal{G}_x)_2$ that arises from v_1, v_2, v_3, and w_2; the second factor corresponds to w_1. Since the image of this clutching morphism is contained in the normal locus of $\overline{\mathcal{H}}_{4,3,1}$ (respectively, $\overline{\mathcal{H}}_{3,3,1}$) [cf. Theorem 1.5; Corollary 1.9; [11], §3.23], we thus obtain a clutching morphism

$$\mathcal{H}_{1,3,2} \times \mathcal{H}_{1,3,1} \longrightarrow \widetilde{\mathcal{H}}_{4,3,1}$$
$$(\text{respectively, } \mathcal{H}_{0,3,2} \times \mathcal{H}_{1,3,1} \longrightarrow \widetilde{\mathcal{H}}_{3,3,1}).$$

Note that $y_2 \in \widetilde{\mathcal{H}}_{4,3,1}(k)$ (respectively, $y_2 \in \widetilde{\mathcal{H}}_{3,3,1}(k)$) is contained in the image of the above clutching morphism, and that

$$\alpha_2 \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_2}})}(\rho_{4,3,1}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_2) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_2}})$$
$$(\text{respectively, } \alpha_2 \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_2}})}(\rho_{3,3,1}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_2) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_2}}))$$

naturally determines, by considering the above clutching morphism, an element of

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{1,6})}(\rho_{1,3,2}(H^\dagger))$$
$$(\text{respectively, } Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{0,6})}(\rho_{0,3,2}(H^\dagger))),$$

where $H^\dagger \subseteq \Pi_{\mathcal{H}_{1,3,2}}$ (respectively, $H^\dagger \subseteq \Pi_{\mathcal{H}_{0,3,2}}$) is an open subgroup of $\Pi_{\mathcal{H}_{1,3,2}}$ (respectively, $\Pi_{\mathcal{H}_{0,3,2}}$). This element is trivial by Claim 4.6.B. In particular, $\alpha_2 \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{y_2}) \subseteq \mathrm{Aut}(\mathcal{G}_{y_2}) = \mathrm{Aut}((\mathcal{G}_x)_2)$. On the other hand,

$$\alpha_2 \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_2}})}(\rho_{4,3,1}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_2) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_2}})$$
$$(\text{respectively, } \alpha_2 \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_2}})}(\rho_{3,3,1}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_2) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_2}}))$$

naturally determines, by considering the above clutching morphism, an element of

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{1,3})}(\rho_{1,3,1}(H^{\ddagger})),$$

where $H^{\ddagger} \subseteq \Pi_{\mathcal{H}_{1,3,1}}$ is an open subgroup of $\Pi_{\mathcal{H}_{1,3,1}}$. This element is trivial by Claim 4.6.C. This completes the proof of Claim 4.6.E.1.1.

Next, we claim the following:

Claim 4.6.E.1.2: The image of $\alpha \in \mathrm{Aut}^{|W|}(\mathcal{G}_x)$ by the natural morphism $\mathrm{Aut}^{|W|}(\mathcal{G}_x) \to \mathrm{Aut}(\mathcal{G}_x|_{\mathbb{H}})$ is trivial, where $\mathbb{H}$ denotes the sub-semi-graph of $\mathcal{G}_x$ determined by the set of vertices $\{v_1, v_2, v_3, w_1\}$ [cf. [5], Definition 2.2, (i)].

Note that one may verify easily that there exists a k-valued point $y_1 \in \widetilde{\mathcal{H}}_{4,3,1}(k)$ (respectively, $y_1 \in \widetilde{\mathcal{H}}_{3,3,1}(k)$) such that $\mathcal{G}_{y_1}$ may be identified with $(\mathcal{G}_x)_1$.

By gluing together simple coverings at unramified marked points as in the proof of Claim 4.6.D.1, we obtain a clutching morphism [cf. [10], Definition 3.6]

$$\mathcal{H}_{1,3,2} \times \mathcal{H}_{1,3,1} \longrightarrow \overline{\mathcal{H}}_{4,3,1}$$
$$\text{(respectively, } \mathcal{H}_{1,3,2} \times \mathcal{H}_{0,3,1} \longrightarrow \overline{\mathcal{H}}_{3,3,1}\text{),}$$

where the first factor in the product corresponds to the irreducible component of $(\mathcal{G}_x)_1$ that arises from v_1, v_2, v_3, and w_1; the second factor corresponds to w_2. Since the image of this clutching morphism is contained in the normal locus of $\overline{\mathcal{H}}_{4,3,1}$ (respectively, $\overline{\mathcal{H}}_{3,3,1}$) [cf. Theorem 1.5; Corollary 1.9; [11], §3.23], we thus obtain a clutching morphism

$$\mathcal{H}_{1,3,2} \times \mathcal{H}_{1,3,1} \longrightarrow \widetilde{\mathcal{H}}_{4,3,1}$$
$$\text{(respectively, } \mathcal{H}_{1,3,2} \times \mathcal{H}_{0,3,1} \longrightarrow \widetilde{\mathcal{H}}_{3,3,1}\text{).}$$

Note that $y_1 \in \widetilde{\mathcal{H}}_{4,3,1}(k)$ (respectively, $y_1 \in \widetilde{\mathcal{H}}_{3,3,1}(k)$) is contained in the image of the above clutching morphism, and that

$$\alpha_1 \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_1}})}(\rho_{4,3,1}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_1) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_1}})$$
$$\text{(respectively, } \alpha_1 \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_1}})}(\rho_{3,3,1}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_1) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_1}})\text{)}$$

naturally determines, by considering the above clutching morphism, an element of

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{1,6})}(\rho_{1,3,2}(H^{\dagger})),$$

where $H^{\dagger} \subseteq \Pi_{\mathcal{H}_{1,3,2}}$ is an open subgroup of $\Pi_{\mathcal{H}_{1,3,2}}$. This element is trivial by Claim 4.6.B. This completes the proof of Claim 4.6.E.1.2.

Thus, we conclude from Claim 4.6.E.1.1, Claim 4.6.E.1.2, and [5], Theorem 4.8, (ii), (iv), that α is trivial. Hence it remains to verify the following:

Claim 4.6.E.1.3: α determines an element of $\mathrm{Aut}(\mathcal{G}_x)$ that preserves w_j for each $j = 1, 2$.

The proof of Claim 4.6.E.1.3 is similar to the proof of Claim 4.6.D.2. This completes the proof of Claim 4.6.E.1.

Claim 4.6.E.2: When $g = 5$, Claim 4.6.E holds.

Let us first observe that by considering the covering obtained by applying Lemma 4.3, one may verify easily that there exists a k-valued point $x \in \widetilde{\mathcal{H}}_{5,3,0}(k)$ [so $\mathcal{G}_x$ has no cusps!] (respectively, $x \in \widetilde{\mathcal{H}}_{5,3,1}(k)$ [so $\mathcal{G}_x$ has 3 cusps]) satisfying the following conditions:

- $\mathrm{Vert}(\mathcal{G}_x) = \{v_1, w_1, w_2\}$;

- $\mathrm{Node}(\mathcal{G}_x) = \{e_1, e_2, \ldots, e_6\}$;
- $\mathcal{N}(v_1) = \mathrm{Node}(\mathcal{G}_x)$;
- $\mathcal{N}(w_1) = \{e_1, e_2, e_3\}$;
- $\mathcal{N}(w_2) = \{e_4, e_5, e_6\}$;
- v_1 is of type $(0, 6)$ (respectively, $(0, 9)$);
- w_1 is of type $(1, 3)$;
- w_2 is of type $(0, 3)$.

Thus, let us fix $x \in \widetilde{\mathcal{H}}_{5,3,0}(k)$ (respectively, $x \in \widetilde{\mathcal{H}}_{5,3,1}(k)$) satisfying the above conditions.

Let $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{5,0})}(\rho_{5,3,0}(H))$ (respectively, $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{5,3})}(\rho_{5,3,1}(H))$) be an outomorphism of $\Pi_{5,0}$ (respectively, $\Pi_{5,3}$). Suppose, moreover, that, relative to the isomorphism $\Pi_{5,0} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$ (respectively, $\Pi_{5,3} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$) fixed above [cf. the discussion immediately preceding Claim 4.6.B], $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{5,0})}(\rho_{5,3,0}(H))$ (respectively, $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{5,3})}(\rho_{5,3,1}(H))$) determines an element of $\mathrm{Aut}(\mathcal{G}_x)$ that preserves w_j for each $j = 1, 2$ [cf. Claim 4.6.E.2.3]. For $j = 1, 2$, write

$$(\mathcal{G}_x)_j \stackrel{\mathrm{def}}{=} (\mathcal{G}_x)_{\rightsquigarrow\{e_{3j-2}, e_{3j-1}, \ldots, e_{3j}\}}$$

[cf. [5], Definition 2.8]; α_j for the image of α via the natural inclusion $\mathrm{Aut}^{|W|}(\mathcal{G}_x) \hookrightarrow \mathrm{Aut}((\mathcal{G}_x)_j)$ [cf. [5], Definition 2.6, (i); [5], Proposition 2.9, (ii)], where we write $W \stackrel{\mathrm{def}}{=} \{w_1, w_2\}$.

Next, we claim the following:

Claim 4.6.E.2.1: $\alpha_2 \in \mathrm{Dehn}((\mathcal{G}_x)_2)$.

Note that one may verify easily that there exists a k-valued point $y_2 \in \widetilde{\mathcal{H}}_{5,3,0}(k)$ (respectively, $y_2 \in \widetilde{\mathcal{H}}_{5,3,1}(k)$) such that $\mathcal{G}_{y_2}$ may be identified with $(\mathcal{G}_x)_2$.

By gluing together simple coverings at unramified marked points as in the proof of Claim 4.6.D.1, we obtain a clutching morphism [cf. [10], Definition 3.6]

$$\mathcal{H}_{2,3,1} \times \mathcal{H}_{1,3,1} \longrightarrow \overline{\mathcal{H}}_{5,3,0}$$
$$\text{(respectively, } \mathcal{H}_{2,3,2} \times \mathcal{H}_{1,3,1} \longrightarrow \overline{\mathcal{H}}_{5,3,1}\text{)},$$

where the first factor in the product corresponds to the irreducible component of $(\mathcal{G}_x)_2$ that arises from v_1, w_2; the second factor corresponds to w_1. Since the image of this clutching morphism is contained in the normal locus of $\overline{\mathcal{H}}_{5,3,0}$ (respectively, $\overline{\mathcal{H}}_{5,3,1}$) [cf. Theorem 1.5; Corollary 1.9; [11], §3.23], we thus obtain a clutching morphism

$$\mathcal{H}_{2,3,1} \times \mathcal{H}_{1,3,1} \longrightarrow \widetilde{\mathcal{H}}_{5,3,0}$$
$$\text{(respectively, } \mathcal{H}_{2,3,2} \times \mathcal{H}_{1,3,1} \longrightarrow \widetilde{\mathcal{H}}_{5,3,1}\text{)}.$$

Note that $y_2 \in \widetilde{\mathcal{H}}_{5,3,0}(k)$ (respectively, $y_2 \in \widetilde{\mathcal{H}}_{5,3,1}(k)$) is contained in the image of the above clutching morphism, and that

$$\alpha_2 \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_2}})}(\rho_{5,3,0}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_2) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_2}})$$
$$\text{(respectively, } \alpha_2 \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_2}})}(\rho_{5,3,1}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_2) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_2}})\text{)}$$

naturally determines, by considering the above clutching morphism, an element of

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{2,3})}(\rho_{2,3,1}(H^{\dagger}))$$
$$\text{(respectively, } Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{2,6})}(\rho_{2,3,2}(H^{\dagger}))\text{)},$$

where $H^\dagger \subseteq \Pi_{\mathcal{H}_{2,3,1}}$ (respectively, $H^\dagger \subseteq \Pi_{\mathcal{H}_{2,3,2}}$) is an open subgroup of $\Pi_{\mathcal{H}_{2,3,1}}$ (respectively, $\Pi_{\mathcal{H}_{2,3,2}}$). This element is trivial by Claim 4.6.B. In particular, $\alpha_2 \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{y_2}) \subseteq \mathrm{Aut}(\mathcal{G}_{y_2}) = \mathrm{Aut}((\mathcal{G}_x)_2)$. On the other hand,

$$\alpha_2 \in Z_{\mathrm{Out^C}(\Pi_{\mathcal{G}_{y_2}})}(\rho_{5,3,0}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_2) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_2}})$$
$$(\text{respectively, } \alpha_2 \in Z_{\mathrm{Out^C}(\Pi_{\mathcal{G}_{y_2}})}(\rho_{5,3,1}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_2) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_2}}))$$

naturally determines, by considering the above clutching morphism, an element of

$$Z_{\mathrm{Out^C}(\Pi_{1,3})}(\rho_{1,3,1}(H^\ddagger)),$$

where $H^\ddagger \subseteq \Pi_{\mathcal{H}_{1,3,1}}$ is an open subgroup of $\Pi_{\mathcal{H}_{1,3,1}}$. This element is trivial by Claim 4.6.C. This completes the proof of Claim 4.6.E.2.1.

Next, we claim the following:

> Claim 4.6.E.2.2: The image of $\alpha \in \mathrm{Aut}^{|W|}(\mathcal{G}_x)$ by the natural morphism $\mathrm{Aut}^{|W|}(\mathcal{G}_x) \to \mathrm{Aut}(\mathcal{G}_x|_{\mathbb{H}})$ is trivial, where $\mathbb{H}$ denotes the sub-semi-graph of $\mathcal{G}_x$ determined by the set of vertices $\{v_1, w_1\}$ [cf. [5], Definition 2.2, (i)].

Note that one may verify easily that there exists a k-valued point $y_1 \in \widetilde{\mathcal{H}}_{5,3,0}(k)$ (respectively, $y_1 \in \widetilde{\mathcal{H}}_{5,3,1}(k)$) such that $\mathcal{G}_{y_1}$ may be identified with $(\mathcal{G}_x)_1$.

By gluing together simple coverings at unramified marked points as in the proof of Claim 4.6.D.1, we obtain a clutching morphism [cf. [10], Definition 3.6]

$$\mathcal{H}_{3,3,1} \times \mathcal{H}_{0,3,1} \longrightarrow \overline{\mathcal{H}}_{5,3,0}$$
$$(\text{respectively, } \mathcal{H}_{3,3,2} \times \mathcal{H}_{0,3,1} \longrightarrow \overline{\mathcal{H}}_{5,3,1}),$$

where the first factor in the product corresponds to the irreducible component of $(\mathcal{G}_x)_1$ that arises from v_1, w_1; the second factor corresponds to w_2. Since the image of this clutching morphism is contained in the normal locus of $\overline{\mathcal{H}}_{5,3,0}$ (respectively, $\overline{\mathcal{H}}_{5,3,1}$) [cf. Theorem 1.5; Corollary 1.9; [11], §3.23], we thus obtain a clutching morphism

$$\mathcal{H}_{3,3,1} \times \mathcal{H}_{0,3,1} \longrightarrow \widetilde{\mathcal{H}}_{5,3,0}$$
$$(\text{respectively, } \mathcal{H}_{3,3,2} \times \mathcal{H}_{0,3,1} \longrightarrow \widetilde{\mathcal{H}}_{5,3,1}).$$

Note that $y_1 \in \widetilde{\mathcal{H}}_{5,3,0}(k)$ (respectively, $y_1 \in \widetilde{\mathcal{H}}_{5,3,1}(k)$) is contained in the image of the above clutching morphism, and that

$$\alpha_1 \in Z_{\mathrm{Out^C}(\Pi_{\mathcal{G}_{y_1}})}(\rho_{5,3,0}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_1) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_1}})$$
$$(\text{respectively, } \alpha_1 \in Z_{\mathrm{Out^C}(\Pi_{\mathcal{G}_{y_1}})}(\rho_{5,3,1}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_1) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_1}}))$$

naturally determines, by considering the above clutching morphism, an element of

$$Z_{\mathrm{Out^C}(\Pi_{3,3})}(\rho_{3,3,1}(H^\dagger))$$
$$(\text{respectively, } Z_{\mathrm{Out^C}(\Pi_{3,6})}(\rho_{3,3,2}(H^\dagger))),$$

where $H^\dagger \subseteq \Pi_{\mathcal{H}_{3,3,1}}$ (respectively, $H^\dagger \subseteq \Pi_{\mathcal{H}_{3,3,2}}$) is an open subgroup of $\Pi_{\mathcal{H}_{3,3,1}}$ (respectively, $\Pi_{\mathcal{H}_{3,3,2}}$). This element is trivial by Claim 4.6.E.1 (respectively, Claim 4.6.B). This completes the proof of Claim 4.6.E.2.2.

Thus, we conclude from Claim 4.6.E.2.1, Claim 4.6.E.2.2, and [5], Theorem 4.8, (ii), (iv), that α is trivial. Hence it remains to verify the following:

> Claim 4.6.E.2.3: α determines an element of $\mathrm{Aut}(\mathcal{G}_x)$ that preserves w_j for each $j = 1, 2$.

The proof of Claim 4.6.E.2.3 is similar to the proof of Claim 4.6.D.2. This completes the proof of Claim 4.6.E.2.

Claim 4.6.E.3: When $g \equiv 0 \pmod 3$, Claim 4.6.E holds.

Let us first observe that by Claim 4.6.E.1, we may assume that $g \geq 5$. By considering the covering obtained by applying Lemma 4.2, (i), in the case where

$$t \stackrel{\text{def}}{=} \frac{g}{3} \geq 2,$$

one may verify easily that there exists a k-valued point $x \in \widetilde{\mathcal{H}}_{g,3,0}(k)$ [so $\mathcal{G}_x$ has no cusps!] (respectively, $x \in \widetilde{\mathcal{H}}_{g,3,1}(k)$ [so $\mathcal{G}_x$ has 3 cusps]) satisfying the following conditions:

- $\mathrm{Vert}(\mathcal{G}_x) = \{v_1, w_1, w_2, \ldots, w_t\}$;
- $\mathrm{Node}(\mathcal{G}_x) = \{e_1, e_2, \ldots, e_{3t}\}$;
- $\mathcal{N}(v_1) = \mathrm{Node}(\mathcal{G}_x)$;
- $\mathcal{N}(w_j) = \{e_{3j-2}, e_{3j-1}, e_{3j}\}$ for $j = 1, \ldots, t$;
- v_1 is of type $(0, 3t)$ (respectively, $(0, 3t+3)$);
- for $j = 1, \ldots, t$, w_j is of type $(1, 3)$.

Thus, let us fix $x \in \widetilde{\mathcal{H}}_{g,3,0}(k)$ (respectively, $x \in \widetilde{\mathcal{H}}_{g,3,1}(k)$) satisfying the above conditions.

Let $\alpha \in Z_{\mathrm{Out^C}(\Pi_{g,0})}(\rho_{g,3,0}(H))$ (respectively, $\alpha \in Z_{\mathrm{Out^C}(\Pi_{g,3})}(\rho_{g,3,1}(H))$) be an outomorphism of $\Pi_{g,0}$ (respectively, $\Pi_{g,3}$). Suppose, moreover, that, relative to the isomorphism $\Pi_{g,0} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$ (respectively, $\Pi_{g,3} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$) fixed above [cf. the discussion immediately preceding Claim 4.6.B], $\alpha \in Z_{\mathrm{Out^C}(\Pi_{g,0})}(\rho_{g,3,0}(H))$ (respectively, $\alpha \in Z_{\mathrm{Out^C}(\Pi_{g,3})}(\rho_{g,3,1}(H))$) determines an element of $\mathrm{Aut}(\mathcal{G}_x)$ that preserves w_j for each $j = 1, \ldots, t$ [cf. Claim 4.6.E.3.2]. For $j = 1, \ldots, t$, write

$$(\mathcal{G}_x)_j \stackrel{\text{def}}{=} (\mathcal{G}_x)_{\rightsquigarrow\{e_{3j-2}, e_{3j-1}, \ldots, e_{3j}\}}$$

[cf. [5], Definition 2.8]; α_j for the image of α via the natural inclusion $\mathrm{Aut}^{|W|}(\mathcal{G}_x) \hookrightarrow \mathrm{Aut}((\mathcal{G}_x)_j)$ [cf. [5], Definition 2.6, (i); [5], Proposition 2.9, (ii)], where we write $W \stackrel{\text{def}}{=} \{w_1, w_2, \ldots, w_t\}$.

Next, we claim the following:

Claim 4.6.E.3.1: $\alpha_j \in \mathrm{Dehn}((\mathcal{G}_x)_j)$.

Note that one may verify easily that there exists a k-valued point $y_j \in \widetilde{\mathcal{H}}_{g,3,0}(k)$ (respectively, $y_j \in \widetilde{\mathcal{H}}_{g,3,1}(k)$) such that $\mathcal{G}_{y_j}$ may be identified with $(\mathcal{G}_x)_j$.

By gluing together simple coverings at unramified marked points as in the proof of Claim 4.6.D.1, we obtain a clutching morphism [cf. [10], Definition 3.6]

$$\mathcal{H}_{3,3,t-1} \times \mathcal{H}_{1,3,1} \times \cdots \times \mathcal{H}_{1,3,1} \longrightarrow \overline{\mathcal{H}}_{g,3,0}$$
$$(\text{respectively, } \mathcal{H}_{3,3,t} \times \mathcal{H}_{1,3,1} \times \cdots \times \mathcal{H}_{1,3,1} \longrightarrow \overline{\mathcal{H}}_{g,3,1}),$$

where the number of factors in the above product is t; the first factor in the product corresponds to the irreducible component of $(\mathcal{G}_x)_j$ that arises from v_1 and w_j; the factors other than the first factor correspond to $w_{j'}$, for $j' \in \{1, 2, \ldots, t\} \setminus \{j\}$. Since the image of this clutching morphism is contained in the normal locus of $\overline{\mathcal{H}}_{g,3,0}$ (respectively, $\overline{\mathcal{H}}_{g,3,1}$) [cf. Theorem 1.5; Corollary 1.9; [11], §3.23], we thus obtain a clutching morphism

$$\mathcal{H}_{3,3,t-1} \times \mathcal{H}_{1,3,1} \times \cdots \times \mathcal{H}_{1,3,1} \longrightarrow \widetilde{\mathcal{H}}_{g,3,0}$$
$$(\text{respectively, } \mathcal{H}_{3,3,t} \times \mathcal{H}_{1,3,1} \times \cdots \times \mathcal{H}_{1,3,1} \longrightarrow \widetilde{\mathcal{H}}_{g,3,1}).$$

Note that $y_j \in \widetilde{\mathcal{H}}_{g,3,0}(k)$ (respectively, $y_j \in \widetilde{\mathcal{H}}_{g,3,1}(k)$) is contained in the image of the above clutching morphism, and that

$$\alpha_j \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})}(\rho_{g,3,0}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_j) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})$$
$$(\text{respectively, } \alpha_j \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})}(\rho_{g,3,1}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_j) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}}))$$

naturally determines, by considering the above clutching morphism, an element of

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{3,3(t-1)})}(\rho_{3,3,t-1}(H^{\dagger}))$$
$$(\text{respectively, } Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{3,3t})}(\rho_{3,3,t}(H^{\dagger}))),$$

where $H^{\dagger} \subseteq \Pi_{\mathcal{H}_{3,3,t-1}}$ (respectively, $H^{\dagger} \subseteq \Pi_{\mathcal{H}_{3,3,t}}$) is an open subgroup of $\Pi_{\mathcal{H}_{3,3,t-1}}$ (respectively, $\Pi_{\mathcal{H}_{3,3,t}}$). Since $t \geq 2$, this element is trivial by Claim 4.6.B and Claim 4.6.E.1. In particular, $\alpha_j \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{y_j}) \subseteq \mathrm{Aut}(\mathcal{G}_{y_j}) = \mathrm{Aut}((\mathcal{G}_x)_j)$. On the other hand,

$$\alpha_j \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})}(\rho_{g,3,0}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_j) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})$$
$$(\text{respectively, } \alpha_j \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})}(\rho_{g,3,1}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_j) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}}))$$

naturally determines, by considering the above clutching morphism, an element of

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{1,3})}(\rho_{1,3,1}(H^{\ddagger}))$$

for each component of the above product $\mathcal{H}_{1,3,1} \times \cdots \times \mathcal{H}_{1,3,1}$, where $H^{\ddagger} \subseteq \Pi_{\mathcal{H}_{1,3,1}}$ is an open subgroup of $\Pi_{\mathcal{H}_{1,3,1}}$. These elements are trivial by Claim 4.6.C. This completes the proof of Claim 4.6.E.3.1.

Thus, by varying j, we conclude from Claim 4.6.E.3.1 and [5], Theorem 4.8, (ii), (iv), that α is trivial. Hence it remains to verify the following:

> Claim 4.6.E.3.2: α determines an element of $\mathrm{Aut}(\mathcal{G}_x)$ that preserves w_j for each $j = 1, \ldots, t$.

The proof of Claim 4.6.E.3.2 is similar to the proof of Claim 4.6.D.2. This completes the proof of Claim 4.6.E.3.

> Claim 4.6.E.4: When $g \equiv 1 \pmod 3$, Claim 4.6.E holds.

Let us first observe that by Claim 4.6.E.1, we may assume that $g \geq 5$. By considering the covering obtained by applying Lemma 4.2, (ii), in the case where

$$t \overset{\mathrm{def}}{=} \tfrac{g+2}{3} \geq 3,$$

one may verify easily that there exists a k-valued point $x \in \widetilde{\mathcal{H}}_{g,3,0}(k)$ [so $\mathcal{G}_x$ has no cusps!] (respectively, $x \in \widetilde{\mathcal{H}}_{g,3,1}(k)$ [so $\mathcal{G}_x$ has 3 cusps]) satisfying the following conditions:

- $\mathrm{Vert}(\mathcal{G}_x) = \{v_1, v_2, v_3, w_1, w_2, \ldots, w_t\}$;
- $\mathrm{Node}(\mathcal{G}_x) = \{e_1, e_2, \ldots, e_{3t}\}$;
- $\mathcal{N}(v_1) = \{e_1, e_4, \ldots, e_{3t-2}\}$;
- $\mathcal{N}(v_2) = \{e_2, e_5, \ldots, e_{3t-1}\}$;
- $\mathcal{N}(v_3) = \{e_3, e_6, \ldots, e_{3t}\}$;
- $\mathcal{N}(w_j) = \{e_{3j-2}, e_{3j-1}, e_{3j}\}$ for $j = 1, \ldots, t$;

- for $i = 1, 2, 3$, v_i is of type $(0, t)$ (respectively, $(0, t+1)$);
- for $j = 1, \ldots, t$, w_j is of type $(1, 3)$.

Thus, let us fix $x \in \widetilde{\mathcal{H}}_{g,3,0}(k)$ (respectively, $x \in \widetilde{\mathcal{H}}_{g,3,1}(k)$) satisfying the above conditions.

Let $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,3,0}(H))$ (respectively, $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,3})}(\rho_{g,3,1}(H))$) be an outomorphism of $\Pi_{g,0}$ (respectively, $\Pi_{g,3}$). Suppose, moreover, that, relative to the isomorphism $\Pi_{g,0} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$ (respectively, $\Pi_{g,3} \xrightarrow{\sim} \Pi_{\mathcal{G}_x}$) fixed above [cf. the discussion immediately preceding Claim 4.6.B], $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,3,0}(H))$ (respectively, $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,3})}(\rho_{g,3,1}(H))$) determines an element of $\mathrm{Aut}(\mathcal{G}_x)$ that preserves w_j for each $j = 1, \ldots, t$ [cf. Claim 4.6.E.4.2]. For $j = 1, \ldots, t$, write

$$(\mathcal{G}_x)_j \stackrel{\mathrm{def}}{=} (\mathcal{G}_x)_{\rightsquigarrow \{e_{3j-2}, e_{3j-1}, \ldots, e_{3j}\}}$$

[cf. [5], Definition 2.8]; α_j for the image of α via the natural inclusion $\mathrm{Aut}^{|W|}(\mathcal{G}_x) \hookrightarrow \mathrm{Aut}((\mathcal{G}_x)_j)$ [cf. [5], Definition 2.6, (i); [5], Proposition 2.9, (ii)], where we write $W \stackrel{\mathrm{def}}{=} \{w_1, w_2, \ldots, w_t\}$.

Next, we claim the following:

Claim 4.6.E.4.1: $\alpha_j \in \mathrm{Dehn}((\mathcal{G}_x)_j)$.

Note that one may verify easily that there exists a k-valued point $y_j \in \widetilde{\mathcal{H}}_{g,3,0}(k)$ (respectively, $y_j \in \widetilde{\mathcal{H}}_{g,3,1}(k)$) such that $\mathcal{G}_{y_j}$ may be identified with $(\mathcal{G}_x)_j$.

By gluing together simple coverings at unramified marked points as in the proof of Claim 4.6.D.1, we obtain a clutching morphism [cf. [10], Definition 3.6]

$$\mathcal{H}_{1,3,t-1} \times \mathcal{H}_{1,3,1} \times \cdots \times \mathcal{H}_{1,3,1} \longrightarrow \overline{\mathcal{H}}_{g,3,0}$$
$$(\text{respectively, } \mathcal{H}_{1,3,t} \times \mathcal{H}_{1,3,1} \times \cdots \times \mathcal{H}_{1,3,1} \longrightarrow \overline{\mathcal{H}}_{g,3,1}),$$

where the number of factors in the above product is t; the first factor in the product corresponds to the irreducible component of $(\mathcal{G}_x)_j$ that arises from v_1, v_2, v_3, and w_j; the factors other than the first factor correspond to $w_{j'}$, for $j' \in \{1, 2, \ldots, t\} \setminus \{j\}$. Since the image of this clutching morphism is contained in the normal locus of $\overline{\mathcal{H}}_{g,3,0}$ (respectively, $\overline{\mathcal{H}}_{g,3,1}$) [cf. Theorem 1.5; Corollary 1.9; [11], §3.23], we thus obtain a clutching morphism

$$\mathcal{H}_{1,3,t-1} \times \mathcal{H}_{1,3,1} \times \cdots \times \mathcal{H}_{1,3,1} \longrightarrow \widetilde{\mathcal{H}}_{g,3,0}$$
$$(\text{respectively, } \mathcal{H}_{1,3,t} \times \mathcal{H}_{1,3,1} \times \cdots \times \mathcal{H}_{1,3,1} \longrightarrow \widetilde{\mathcal{H}}_{g,3,1}).$$

Note that $y_j \in \widetilde{\mathcal{H}}_{g,3,0}(k)$ (respectively, $y_j \in \widetilde{\mathcal{H}}_{g,3,1}(k)$) is contained in the image of the above clutching morphism, and that

$$\alpha_j \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})}(\rho_{g,3,0}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_j) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})$$
$$(\text{respectively, } \alpha_j \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})}(\rho_{g,3,1}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_j) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}}))$$

naturally determines, by considering the above clutching morphism, an element of

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{1,3(t-1)})}(\rho_{1,3,t-1}(H^\dagger))$$
$$(\text{respectively, } Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{1,3t})}(\rho_{1,3,t}(H^\dagger))),$$

where $H^\dagger \subseteq \Pi_{\mathcal{H}_{1,3,t-1}}$ (respectively, $H^\dagger \subseteq \Pi_{\mathcal{H}_{1,3,t}}$) is an open subgroup of $\Pi_{\mathcal{H}_{1,3,t-1}}$ (respectively, $\Pi_{\mathcal{H}_{1,3,t}}$). Since $t \geq 3$, this element is trivial by Claim 4.6.B. In particular, $\alpha_j \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{y_j}) \subseteq \mathrm{Aut}(\mathcal{G}_{y_j}) = \mathrm{Aut}((\mathcal{G}_x)_j)$. On the other hand,

$$\alpha_j \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})}(\rho_{g,3,0}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_j) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})$$
$$(\text{respectively, } \alpha_j \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})}(\rho_{g,3,1}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_j) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}}))$$

naturally determines, by considering the above clutching morphism, an element of

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{1,3})}(\rho_{1,3,1}(H^{\ddagger}))$$

for each component of the above product $\mathcal{H}_{1,3,1} \times \cdots \times \mathcal{H}_{1,3,1}$, where $H^{\ddagger} \subseteq \Pi_{\mathcal{H}_{1,3,1}}$ is an open subgroup of $\Pi_{\mathcal{H}_{1,3,1}}$. These elements are trivial by Claim 4.6.C. This completes the proof of Claim 4.6.E.4.1.

Thus, by varying j, we conclude from Claim 4.6.E.4.1 and [5], Theorem 4.8, (ii), (iv), that α is trivial. Hence it remains to verify the following:

Claim 4.6.E.4.2: α determines an element of $\mathrm{Aut}(\mathcal{G}_x)$ that preserves w_j for each $j = 1, \ldots, t$.

The proof of Claim 4.6.E.4.2 is similar to the proof of Claim 4.6.D.2. This completes the proof of Claim 4.6.E.4.

Claim 4.6.E.5: When $g \equiv 2 \pmod 3$, Claim 4.6.E holds.

Let us first observe that by Claim 4.6.E.2, we may assume that $g \geq 8$. By considering the covering obtained by applying Lemma 4.2, (iii), in the case where

$$t \stackrel{\mathrm{def}}{=} \tfrac{g+1}{3} \geq 3,$$

one may verify easily that there exists a k-valued point $x \in \widetilde{\mathcal{H}}_{g,3,0}(k)$ [so $\mathcal{G}_x$ has no cusps!] (respectively, $x \in \widetilde{\mathcal{H}}_{g,3,1}(k)$ [so $\mathcal{G}_x$ has 3 cusps]) satisfying the following conditions:

- $\mathrm{Vert}(\mathcal{G}_x) = \{v_1, v_2, w_1, w_2, \ldots, w_t\}$;
- $\mathrm{Node}(\mathcal{G}_x) = \{e_1, e_2, \ldots, e_{3t}\}$;
- $\mathcal{N}(v_1) = \{e_1, e_2, e_4, e_5, \ldots, e_{3t-2}, e_{3t-1}\}$;
- $\mathcal{N}(v_2) = \{e_3, e_6, \ldots, e_{3t}\}$;
- $\mathcal{N}(w_j) = \{e_{3j-2}, e_{3j-1}, e_{3j}\}$ for $j = 1, \ldots, t$;
- v_1 is of type $(0, 2t)$ (respectively, $(0, 2t+2)$);
- v_2 is of type $(0, t)$ (respectively, $(0, t+1)$);
- for $j = 1, \ldots, t$, w_j is of type $(1, 3)$.

Thus, let us fix $x \in \widetilde{\mathcal{H}}_{g,3,0}(k)$ (respectively, $x \in \widetilde{\mathcal{H}}_{g,3,1}(k)$) satisfying the above conditions.

Let $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,3,0}(H))$ (respectively, $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,3})}(\rho_{g,3,1}(H))$) be an outomorphism of $\Pi_{g,0}$ (respectively, $\Pi_{g,3}$). Suppose, moreover, that, relative to the isomorphism $\Pi_{g,0} \stackrel{\sim}{\to} \Pi_{\mathcal{G}_x}$ (respectively, $\Pi_{g,3} \stackrel{\sim}{\to} \Pi_{\mathcal{G}_x}$) fixed above [cf. the discussion immediately preceding Claim 4.6.B], $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,3,0}(H))$ (respectively, $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,3})}(\rho_{g,3,1}(H))$) determines an element of $\mathrm{Aut}(\mathcal{G}_x)$ that preserves w_j for each $j = 1, \ldots, t$ [cf. Claim 4.6.E.5.2]. For $j = 1, \ldots, t$, write

$$(\mathcal{G}_x)_j \stackrel{\mathrm{def}}{=} (\mathcal{G}_x)_{\rightsquigarrow\{e_{3j-2}, e_{3j-1}, \ldots, e_{3j}\}}$$

[cf. [5], Definition 2.8]; α_j for the image of α via the natural inclusion $\mathrm{Aut}^{|W|}(\mathcal{G}_x) \hookrightarrow \mathrm{Aut}((\mathcal{G}_x)_j)$ [cf. [5], Definition 2.6, (i); [5], Proposition 2.9, (ii)], where we write $W \stackrel{\mathrm{def}}{=} \{w_1, w_2, \ldots, w_t\}$.

Next, we claim the following:

Claim 4.6.E.5.1: $\alpha_j \in \mathrm{Dehn}((\mathcal{G}_x)_j)$.

Note that one may verify easily that there exists a k-valued point $y_j \in \widetilde{\mathcal{H}}_{g,3,0}(k)$ (respectively, $y_j \in \widetilde{\mathcal{H}}_{g,3,1}(k)$) such that $\mathcal{G}_{y_j}$ may be identified with $(\mathcal{G}_x)_j$.

By gluing together simple coverings at unramified marked points as in the proof of Claim 4.6.D.1, we obtain a clutching morphism [cf. [10], Definition 3.6]

$$\mathcal{H}_{2,3,t-1} \times \mathcal{H}_{1,3,1} \times \cdots \times \mathcal{H}_{1,3,1} \longrightarrow \overline{\mathcal{H}}_{g,3,0}$$
$$(\text{respectively, } \mathcal{H}_{2,3,t} \times \mathcal{H}_{1,3,1} \times \cdots \times \mathcal{H}_{1,3,1} \longrightarrow \overline{\mathcal{H}}_{g,3,1}),$$

where the number of factors in the above product is t; the first factor in the product corresponds to the irreducible component of $(\mathcal{G}_x)_j$ that arises from v_1, v_2, and w_j; the factors other than the first factor correspond to $w_{j'}$, for $j' \in \{1, 2, \ldots, t\} \setminus \{j\}$. Since the image of this clutching morphism is contained in the normal locus of $\overline{\mathcal{H}}_{g,3,0}$ (respectively, $\overline{\mathcal{H}}_{g,3,1}$) [cf. Theorem 1.5; Corollary 1.9; [11], §3.23], we thus obtain a clutching morphism

$$\mathcal{H}_{2,3,t-1} \times \mathcal{H}_{1,3,1} \times \cdots \times \mathcal{H}_{1,3,1} \longrightarrow \widetilde{\mathcal{H}}_{g,3,0}$$
$$(\text{respectively, } \mathcal{H}_{2,3,t} \times \mathcal{H}_{1,3,1} \times \cdots \times \mathcal{H}_{1,3,1} \longrightarrow \widetilde{\mathcal{H}}_{g,3,1}).$$

Note that $y_j \in \widetilde{\mathcal{H}}_{g,3,0}(k)$ (respectively, $y_j \in \widetilde{\mathcal{H}}_{g,3,1}(k)$) is contained in the image of the above clutching morphism, and that

$$\alpha_j \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})}(\rho_{g,3,0}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_j) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})$$
$$(\text{respectively, } \alpha_j \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})}(\rho_{g,3,1}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_j) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}}))$$

naturally determines, by considering the above clutching morphism, an element of

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{2,3(t-1)})}(\rho_{2,3,t-1}(H^\dagger))$$
$$(\text{respectively, } Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{2,3t})}(\rho_{2,3,t}(H^\dagger))),$$

where $H^\dagger \subseteq \Pi_{\mathcal{H}_{2,3,t-1}}$ (respectively, $H^\dagger \subseteq \Pi_{\mathcal{H}_{2,3,t}}$) is an open subgroup of $\Pi_{\mathcal{H}_{2,3,t-1}}$ (respectively, $\Pi_{\mathcal{H}_{2,3,t}}$). Since $t \geq 3$, this element is trivial by Claim 4.6.B. In particular, $\alpha_j \in \mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_{y_j}) \subseteq \mathrm{Aut}(\mathcal{G}_{y_j}) = \mathrm{Aut}((\mathcal{G}_x)_j)$. On the other hand,

$$\alpha_j \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})}(\rho_{g,3,0}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_j) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})$$
$$(\text{respectively, } \alpha_j \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})}(\rho_{g,3,1}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_j) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}}))$$

naturally determines, by considering the above clutching morphism, an element of

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{1,3})}(\rho_{1,3,1}(H^\ddagger))$$

for each component of the above product $\mathcal{H}_{1,3,1} \times \cdots \times \mathcal{H}_{1,3,1}$, where $H^\ddagger \subseteq \Pi_{\mathcal{H}_{1,3,1}}$ is an open subgroup of $\Pi_{\mathcal{H}_{1,3,1}}$. These elements are trivial by Claim 4.6.C. This completes the proof of Claim 4.6.E.5.1.

Thus, by varying j, we conclude from Claim 4.6.E.5.1 and [5], Theorem 4.8, (ii), (iv), that α is trivial. Hence it remains to verify the following:

Claim 4.6.E.5.2: α determines an element of $\mathrm{Aut}(\mathcal{G}_x)$ that preserves w_j for each $j = 1, \ldots, t$.

The proof of Claim 4.6.E.5.2 is similar to the proof of Claim 4.6.D.2. This completes the proof of Claim 4.6.E.5.

Next, we verify the following assertion:

Claim 4.6.F: Suppose that

$$r = 0,\ d = 2 \text{ (respectively, } r = 1,\ d = 2).$$

Then the injection $\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k) \hookrightarrow Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,dr})}(\rho_{g,d,r}(H))$ is surjective. Moreover, the description of $\mathrm{Aut}_{(\mathcal{H}_{g,d,r})_k}((\mathcal{C}_{g,d,r})_k)$ in the statement of Theorem 4.6, (ii), holds.

By Claim 4.6.C, we may assume that $g \geq 3$ (respectively, $g \geq 2$). Since the hyperelliptic involution determines a nontrivial element of $\mathrm{Aut}_{(\mathcal{H}_{g,2,r})_k}((\mathcal{C}_{g,2,r})_k)$, it suffices to show that the cardinality of the centralizer $Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,2r})}(\rho_{g,2,r}(H))$ is equal to 2. Next, let us observe that by considering the covering obtained by applying Lemma 4.5 in the case where

$$t \stackrel{\mathrm{def}}{=} g+1,$$

one may verify easily that there exists a k-valued point $x \in \widetilde{\mathcal{H}}_{g,2,0}(k)$ [so $\mathcal{G}_x$ has no cusps!] (respectively, $x \in \widetilde{\mathcal{H}}_{g,2,1}(k)$ [so $\mathcal{G}_x$ has 2 cusps]) satisfying the following conditions:

- $\mathrm{Vert}(\mathcal{G}_x) = \{v_1, v_2\}$ [the hyperelliptic involution permutes v_1, v_2];
- $\mathrm{Node}(\mathcal{G}_x) = \{e_1, e_2, \ldots, e_t\}$ [the hyperelliptic involution permutes the branches of each e_j, for $j = 1, \ldots, t$];
- $\mathcal{N}(v_1) = \mathcal{N}(v_2) = \mathrm{Node}(\mathcal{G}_x)$;
- for $i = 1, 2$, v_i is of type $(0, t)$ (respectively, $(0, t+1)$).

Thus, let us fix $x \in \widetilde{\mathcal{H}}_{g,2,0}(k)$ (respectively, $x \in \widetilde{\mathcal{H}}_{g,2,1}(k)$) satisfying the above conditions.

Let $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,2,0}(H))$ (respectively, $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,2})}(\rho_{g,2,1}(H))$) be an outomorphism of $\Pi_{g,0}$ (respectively, $\Pi_{g,2}$). Suppose, moreover, that, relative to the isomorphism $\Pi_{g,0} \stackrel{\sim}{\to} \Pi_{\mathcal{G}_x}$ (respectively, $\Pi_{g,2} \stackrel{\sim}{\to} \Pi_{\mathcal{G}_x}$) fixed above [cf. the discussion immediately preceding Claim 4.6.B], $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,2,0}(H))$ (respectively, $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,2})}(\rho_{g,2,1}(H))$) determines an element of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x)$ [cf. Claim 4.6.F.2]. For $j = 1, \ldots, t$, write

$$(\mathcal{G}_x)_j \stackrel{\mathrm{def}}{=} (\mathcal{G}_x)_{\rightsquigarrow \{e_j\}}$$

[cf. [5], Definition 2.8]; α_j for the image of α via the natural inclusion $\mathrm{Aut}^{|E|}(\mathcal{G}_x) \hookrightarrow \mathrm{Aut}((\mathcal{G}_x)_j)$ [cf. [5], Definition 2.6, (i); [5], Proposition 2.9, (ii)], where we write $E \stackrel{\mathrm{def}}{=} \{e_1, e_2, \ldots, e_t\}$.

Next, we claim the following:

Claim 4.6.F.1: $\alpha_j \in \mathrm{Dehn}((\mathcal{G}_x)_j)$.

Note that one may verify easily that there exists a k-valued point $y_j \in \widetilde{\mathcal{H}}_{g,2,0}(k)$ (respectively, $y_j \in \widetilde{\mathcal{H}}_{g,2,1}(k)$) such that $\mathcal{G}_{y_j}$ corresponds to $(\mathcal{G}_x)_j$.

By gluing together simple coverings at unramified marked points as in the proof of Claim 4.6.D.1, we obtain a clutching morphism

$$\mathcal{H}_{0,2,t-1} \times \mathcal{H}_{0,2,1} \times \cdots \times \mathcal{H}_{0,2,1} \longrightarrow \overline{\mathcal{H}}_{g,2,0}$$
$$\text{(respectively, } \mathcal{H}_{0,2,t} \times \mathcal{H}_{0,2,1} \times \cdots \times \mathcal{H}_{0,2,1} \longrightarrow \overline{\mathcal{H}}_{g,2,1}),$$

where the number of factors in the above product is t; the first factor in the product corresponds to the irreducible component of $(\mathcal{G}_x)_j$ that arises from v_1, v_2, and e_j; the factors other than the first factor correspond to $e_{j'}$, for $j' \in \{1, 2, \ldots, t\} \setminus \{j\}$. Since the image of this clutching morphism is contained in the normal locus of

$\overline{\mathcal{H}}_{g,2,0}$ (respectively, $\overline{\mathcal{H}}_{g,2,1}$) [cf. Theorem 1.5; Corollary 1.9; [11], §3.23], we thus obtain a clutching morphism

$$\mathcal{H}_{0,2,t-1} \times \mathcal{H}_{0,2,1} \times \cdots \times \mathcal{H}_{0,2,1} \longrightarrow \widetilde{\mathcal{H}}_{g,2,0}$$
$$(\text{respectively, } \mathcal{H}_{0,2,t} \times \mathcal{H}_{0,2,1} \times \cdots \times \mathcal{H}_{0,2,1} \longrightarrow \widetilde{\mathcal{H}}_{g,2,1}).$$

Note that $y_j \in \widetilde{\mathcal{H}}_{g,2,0}(k)$ (respectively, $y_j \in \widetilde{\mathcal{H}}_{g,2,1}(k)$) is contained in the image of the above clutching morphism, and that

$$\alpha_j \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})}(\rho_{g,2,0}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_j) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})$$
$$(\text{respectively, } \alpha_j \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}})}(\rho_{g,2,1}(H)) \cap \mathrm{Aut}((\mathcal{G}_x)_j) \subseteq \mathrm{Out}^{\mathrm{C}}(\Pi_{\mathcal{G}_{y_j}}))$$

naturally determines, by considering the above clutching morphism, an element of

$$Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{0,2(t-1)})}(\rho_{0,2,t-1}(H^\dagger))$$
$$(\text{respectively, } Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{0,2t})}(\rho_{0,2,t}(H^\dagger))),$$

where $H^\dagger \subseteq \Pi_{\mathcal{H}_{0,2,t-1}}$ (respectively, $H^\dagger \subseteq \Pi_{\mathcal{H}_{0,2,t}}$) is an open subgroup of $\Pi_{\mathcal{H}_{0,2,t-1}}$ (respectively, $\Pi_{\mathcal{H}_{0,2,t}}$). Since, by assumption, $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,2,0}(H))$ (respectively, $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,2})}(\rho_{g,2,1}(H))$) determines an element of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x)$, and $t \geq 4$ (respectively, $t \geq 3$), this element is trivial by Claim 4.6.B. This completes the proof of Claim 4.6.F.1.

Thus, by varying j, we conclude from Claim 4.6.F.1 and [5], Theorem 4.8, (ii), (iv), that α is trivial. Hence it remains to verify the following:

> Claim 4.6.F.2: α determines an element of $\mathrm{Aut}(\mathcal{G}_x)$ that preserves e_j for each $j = 1, \ldots, t$. In particular, by taking the composite with the hyperelliptic involution if necessary, $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,0})}(\rho_{g,2,0}(H))$ (respectively, $\alpha \in Z_{\mathrm{Out}^{\mathrm{C}}(\Pi_{g,2})}(\rho_{g,2,1}(H))$) determines an element of $\mathrm{Aut}^{|\mathrm{grph}|}(\mathcal{G}_x)$.

The proof of Claim 4.6.F.2 is similar to the proof of Claim 4.6.D.2. This completes the proof of Claim 4.6.F.

Thus, in summary, we have proven assertion (ii) in the following cases:

$$r \geq 2 \text{ (Claim 4.6.B, Claim 4.6.C)},$$
$$r \leq 1, d \geq 4 \text{ (Claim 4.6.D)},$$
$$r \leq 1, d = 3 \text{ (Claim 4.6.E)},$$
$$r \leq 1, d = 2 \text{ (Claim 4.6.F)}.$$

Since these cases cover all of the possibilities for r and d, this completes the proof of assertion (ii).

Assertion (iii) follows immediately from assertion (ii). This completes the proof of Theorem 4.6. □

Acknowledgements

The author would like to thank Professor Yuichiro Hoshi and Professor Shinichi Mochizuki for introducing me to the field of combinatorial anabelian geometry and for many helpful discussions, as well as for their warm encouragement. Moreover, the author also would like to thank the referee for reading carefully and giving the author valuable suggestions.

References

[1] E. Arbarello, M. Cornalba, P. Griffiths and J. Harris, *Geometry of algebraic curves I*, *Grundlehren der mathematischen Wissenschaften* **267**, Springer-Verlag, 1985.

[2] W. Fulton, Hurwitz schemes and the irreducibility of moduli of algebraic curves, *Annals of Math.* **90** (1969), 542–575.

[3] J. Harris and D. Mumford, On the Kodaira dimension of the moduli space of curves, *Inv. Math.* **67**, 23–86.

[4] Y. Hoshi, The exactness of the log homotopy sequence, *Hiroshima Math. J.* **39** (2009), 61–121.

[5] Y. Hoshi and S. Mochizuki, Topics surrounding the combinatorial anabelian geometry of hyperbolic curves I: Inertia groups and profinite Dehn twists, *Galois-Teichmüller Theory and Arithmetic Geometry*, *Adv. Stud. Pure Math.* **63**, *Math. Soc. Japan*, 2012, 659–811.

[6] Y. Hoshi and S. Mochizuki, On the combinatorial anabelian geometry of nodally nondegenerate outer representations, *Hiroshima Math. J.* **41** (2011), 275–342.

[7] F. Kato, Log smooth deformation and moduli of log smooth curves, *Internat. J. Math.* **11** (2000), 215–232.

[8] K. Kato, Logarithmic structures of Fontaine-Illusie, *Proceedings of the First JAMI Conference,* Johns-Hopkins University Press, 1990, 191–224.

[9] K. Kato, Toric singularities, *Amer. J. Math.* **116** (1994), 1073–1099.

[10] F. F. Knudsen, The projectivity of the moduli space of stable curves, II: The Stacks $M_{g,n}$, *Math. Scand.* **52** (1983), 161–199.

[11] S. Mochizuki, The geometry of the compactification of the Hurwitz scheme, *Publ. Res. Inst. Math. Sci.* **31** (1995), 355–441.

[12] S. Mochizuki, Extending families of curves over log regular schemes, *J. Reine Angew. Math.* **511** (1999), 43–71.

[13] S. Mochizuki, The local pro-p anabelian geometry of curves, *Invent. Math.* **138** (1999), 319–423.

[14] S. Mochizuki, Semi-graphs of anabelioids, *Publ. Res. Inst. Math. Sci.* **42** (2006), 221–322.

[15] S. Mochizuki, A combinatorial version of the Grothendieck conjecture, *Tohoku Math. J.* **59** (2007), 455–479.

[16] S. Mochizuki, On the combinatorial cuspidalization of hyperbolic curves, *Osaka J. Math.* **47** (2010), 651–715.

[17] S. Mochizuki and A. Tamagawa, The algebraic and anabelian geometry of configuration spaces, *Hokkaido Math. J.* **37** (2008), 75–131.

Shota Tsujimura
Address: Research Institute for Mathematical Sciences, Kyoto University Kyoto 606-8502, Japan
Email: stsuji@kurims.kyoto-u.ac.jp