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Grothendieck spaces: the landscape and perspectives*

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Abstract. In 1973, Diestel published his seminal paper *Grothendieck spaces and vector measures* that drew a connection between Grothendieck spaces (Banach spaces for which weak- and weak*-sequential convergences in the dual space coincide) and vector measures. This connection was developed further in his book with J. Uhl Jr. *Vector measures*. Additionally, Diestel's paper included a section with several open problems about the structural properties of Grothendieck spaces, and only half of them have been solved to this day.

The present paper aims at synthetically presenting the state of the art at subjectively selected corners of the theory of Banach spaces with the Grothendieck property, describing the main examples of spaces with this property, recording the solutions to Diestel's problems, providing generalisations/extensions or new proofs of various results concerning Grothendieck spaces, and adding to the list further problems that we believe are of relevance and may reinvigorate a better-structured development of the theory.

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