

K-theory and G-theory of derived algebraic stacks

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Abstract. These are some notes on the basic properties of algebraic K-theory and G-theory of derived algebraic spaces and stacks, and the theory of fundamental classes in this setting.

Keywords and phrases: algebraic K-theory, virtual fundamental classes, derived algebraic geometry, stacks

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