# Algebraic Effects and Handlers for Arrows

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## Goal

To derive the notions of algebraic effects and effect handlers corresponding to arrows in terms of category theory.

## Categorical Framework

To interpret arrow categorically, we use a strong monad $\mathcal{A}$ in the category Prof instead of a strong monad $\mathcal{T}$ in the category Cat.

## Profunctor

A profunctor is a categorical analogue of a relation.

| | | |
|---|---|---|
| set-theoretic | a function $f\colon X \to Y$ | a relation $r \subseteq Y \times X$ is $r\colon Y \times X \to 2$ |
| categorical | a functor $F\colon \mathbb{C} \to \mathbb{D}$ | a profunctor $R\colon \mathbb{C} \nrightarrow \mathbb{D}$ is $R\colon \mathbb{D}^{\mathrm{op}} \times \mathbb{C} \to \mathrm{Set}$ |

Let $q \subseteq Y \times X$ and $r\colon Z \times Y$ be relations. We define the composition $q \circ r \subseteq Z \times X$ as

$$r \circ q = \{(z,x) \mid \exists y \in Y.\, (z,y) \in r \wedge (y,x) \in q\}.$$

Similarly, we can define composition $R \circ Q\colon \mathbb{C} \nrightarrow \mathbb{E}$ of two profunctors $Q\colon \mathbb{C} \nrightarrow \mathbb{D}$ and $R\colon \mathbb{D} \nrightarrow \mathbb{E}$ using coend:

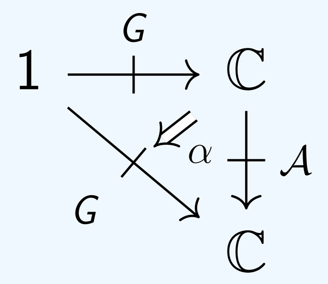
$$(R \circ Q)(Z,X) = \int^{Y \in \mathbb{D}} R(Z,Y) \times Q(Y,X).$$

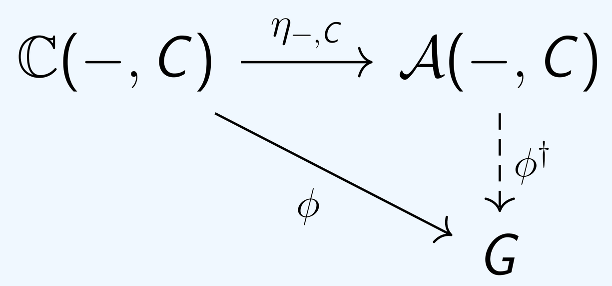
We obtain the bicategory Prof of categories and profunctors.

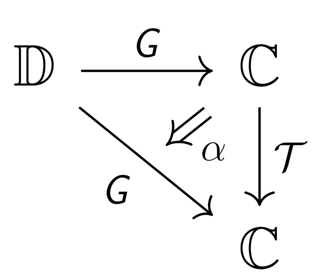
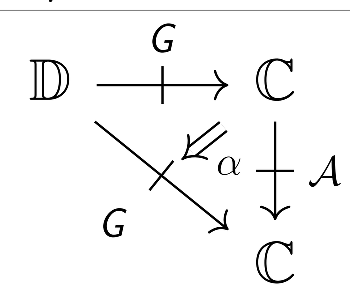
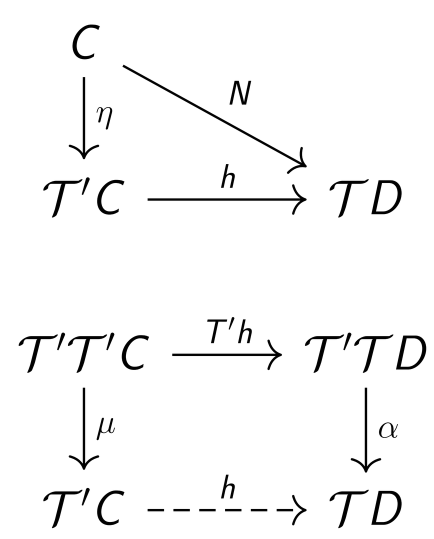
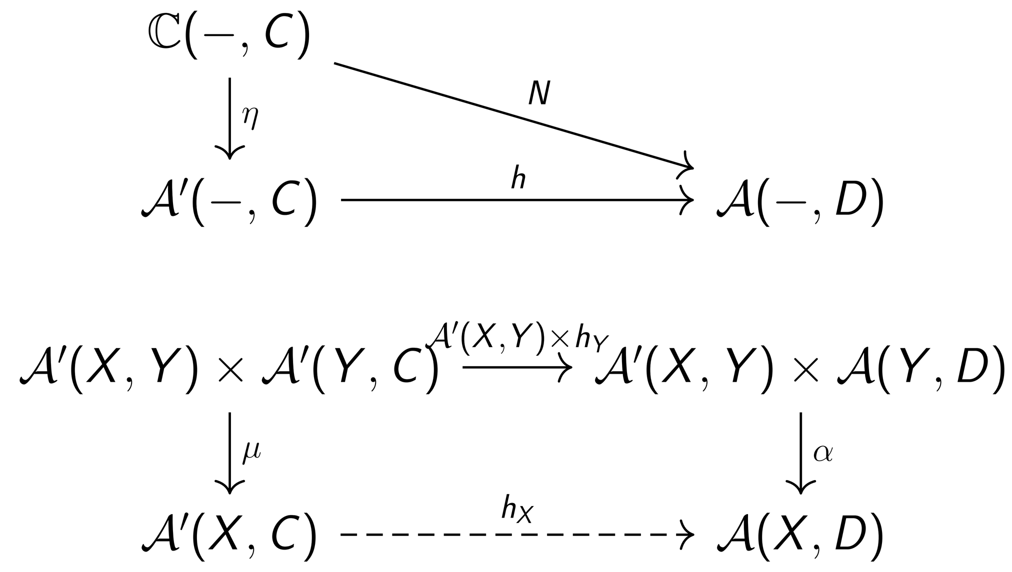
| | | |
|---|---|---|
| set-theoretic | The category Set | The category Rel |
| categorical | The 2-category Cat | The bicategory Prof |

## Theorem (Main theorem)

*Let $\mathcal{A}\colon \mathbb{C} \nrightarrow \mathbb{C}$ be a promonad on $\mathbb{C}$, $G$ be a presheaf on $\mathbb{C}$ and $\langle G, \alpha\colon \mathcal{A} \circ G \Rightarrow G\rangle$ be an $\mathcal{A}$-module. If we have a morphism $\phi\colon \mathbb{C}(-,C) \to G$ between the presheaves, then there is a unique homomorphism $\phi^\dagger\colon \mu^{\mathcal{A}}_{-,C} \to \alpha$ between the $\mathcal{A}$-modules that makes the following diagrams commute*




| | Monad | Arrow (without strength) |
|---|---|---|
| Data | a functor $\mathcal{T}\colon \mathbb{C} \to \mathbb{C}$ with $\eta\colon \mathrm{Id}_{\mathbb{C}} \to \mathcal{T}$ and $\mu\colon \mathcal{T} \circ \mathcal{T} \to \mathcal{T}$ | a profunctor $\mathcal{A}\colon \mathbb{C} \nrightarrow \mathbb{C}$ with $\eta\colon 1_{\mathbb{C}} \nrightarrow \mathcal{A}$ and $\mu\colon \mathcal{A} \circ \mathcal{A} \nrightarrow \mathcal{A}$ |
| Algebra |  |  |
| Algebra when $\mathbb{D} = 1$ | $C \in \mathbb{C}$ and $\alpha\colon \mathcal{T}C \to C$ satisfying proper axioms | Presheaf $G\colon \mathbb{C}^{\mathrm{op}} \to \mathrm{Set}$ and $\alpha_{C,D}\colon \mathcal{A}(C,D) \times GD \to GC$ natural in $C$, extra natural in $D$ satisfying proper axioms |
| Interpretation of an operation $\mathrm{op}\colon \gamma \twoheadrightarrow \delta$ | $[\![\mathrm{op}]\!]_I\colon (\mathcal{T}I)^{[\![\delta]\!]} \to (\mathcal{T}I)^{[\![\gamma]\!]}$ | $\langle\!\vert \mathrm{op} \vert\!\rangle_I\colon \mathcal{A}([\![\delta]\!], I) \to \mathcal{A}([\![\gamma]\!], I)$ |
| Diagram for handler |  |  |

## An Arrow Calculus with Operations and Handlers

Syntax:

| | |
|---|---|
| Types | $A, B, C, D ::= \beta \mid A \times B \mid A \to B \mid A \rightsquigarrow B$ |
| Terms | $M, N, L ::= x \mid (M, N) \mid \mathsf{fst}\, M \mid \mathsf{snd}\, M \mid \lambda x.M \mid MN \mid \lambda^\bullet x.P$ |
| Commands | $P, Q, R ::= \lfloor M \rfloor \mid \mathsf{let}\, x \Leftarrow P \,\mathsf{in}\, Q \mid L \bullet M \mid \mathsf{op}(M)$ |
| Values | $V, W ::= x \mid (V, W) \mid \lambda x.M \mid \lambda^\bullet x.P$ |
| Handlers | $H ::= \{{}_{;} x \mapsto P\} \cup \{\mathsf{op}, k \,;\, c \mapsto Q_{\mathsf{op}}\}_{\mathsf{op} \in \Sigma'}$ |
| Environment | $\Gamma, \Delta ::= \diamond \mid x : A, \Gamma$ |

Typing rules for terms:

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma \,;\, x : A \vdash M \,!\, B}{\Gamma \vdash \lambda^\bullet x : A.M : A \rightsquigarrow B}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} \qquad \frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B} \qquad \ldots$$

Typing rules for commands:

$$\frac{\Gamma, \Delta \vdash M : A}{\Gamma \,;\, \Delta \vdash \lfloor M \rfloor \,!\, A} \qquad \frac{\Gamma \vdash L : A \rightsquigarrow B \quad \Gamma, \Delta \vdash M : A}{\Gamma \,;\, \Delta \vdash L \bullet M \,!\, B} \qquad \frac{\Gamma \,;\, \Delta \vdash P \,!\, A \quad \Gamma \,;\, x : A, \Delta \vdash Q \,!\, B}{\Gamma \,;\, \Delta \vdash \mathsf{let}\, x \Leftarrow P \,\mathsf{in}\, Q \,!\, B} \qquad \frac{\mathsf{op}\colon \gamma \twoheadrightarrow \delta \in \Sigma \quad \Gamma, \Delta \vdash M : \gamma}{\Gamma \,;\, \Delta \vdash \mathsf{op}(M) \,!\, \delta}$$

$$\frac{\Gamma \,;\, x : C \vdash P \,!\, D \quad (\Gamma, k : \delta \rightsquigarrow D \,;\, c : \gamma \vdash Q_{\mathsf{op}} \,!\, D)_{(\mathsf{op}\colon \gamma \twoheadrightarrow \delta) \in \Sigma'}}{\Gamma \vdash \{{}_{;} x \mapsto P\} \cup \{\mathsf{op}, k \,;\, c \mapsto Q_{\mathsf{op}}\}_{\mathsf{op} \in \Sigma'} : C \Rightarrow D} \qquad \frac{\Gamma \,;\, \Delta \vdash P \,!\, C \quad \Gamma \vdash H : C \Rightarrow D}{\Gamma \,;\, \Delta \vdash \mathsf{handle}\, P \,\mathsf{with}\, H \,!\, D}$$

## Operational Semantics

Evaluation context:

$$\mathcal{E} ::= [-] \mid \mathcal{E}N \mid V\mathcal{E} \mid \mathcal{E} \bullet N \mid (\lambda^\bullet x.P) \bullet \mathcal{E} \mid \lfloor \mathcal{E} \rfloor \mid \mathsf{op}(\mathcal{E}) \mid \mathsf{fst}\, \mathcal{E} \mid \mathsf{snd}\, \mathcal{E} \mid (\mathcal{E}, M) \mid (V, \mathcal{E})$$

$$\mathcal{F} ::= [-] \mid \mathsf{let}\, x \Leftarrow \mathcal{F} \,\mathsf{in}\, Q$$

Reduction:

$$(\lambda x.M)V \to M[V/x]$$
$$(\lambda^\bullet x.P) \bullet V \to P[V/x]$$
$$\mathsf{let}\, x \Leftarrow \lfloor V \rfloor \,\mathsf{in}\, Q \to Q[V/x]$$
$$\ldots$$

$$\frac{M \to M'}{\mathcal{E}[M] \to \mathcal{E}[M']} \qquad \frac{P \to P'}{\mathcal{F}[P] \to \mathcal{F}[P']}$$

$\mathsf{handle} \lfloor V \rfloor \,\mathsf{with}\, H$
$\to P[V/x]$

$\mathsf{handle}\, \mathcal{F}[\mathsf{op}(V)] \,\mathsf{with}\, H$
$\to Q_{\mathsf{op}}[V/c, (\lambda^\bullet y : \delta.\, \mathsf{handle}\, \mathcal{F}[\lfloor y \rfloor] \,\mathsf{with}\, H)/k]$
where $H = \{{}_{;} x \mapsto P\} \cup \{\mathsf{op}, k \,;\, c \mapsto Q_{\mathsf{op}}\}_{\mathsf{op} \in \Sigma}$.

## Theorem

*The arrow calculus with operations and handlers is safe, that is, preservation and progress hold.*

## Formalisation

We formalised the arrow calculus with operations and handlers using Agda, a proof assistant system based on Martin-Löf type theory, and proved the safety theorem.