The inequalities defining polyhedral realizations and monomial realizations of crystal bases

Yuki Kanakubo, University of Tsukuba
(Partially joint work with Toshiki Nakashima)

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Introduction

Two main objects of this talk:

- Polyhedral realizations of crystal bases $B(\infty)$.
- Monomial realizations of crystal bases $B(\lambda)$ (λ : dominant integral weight).

Polyhedral realizations of crystal bases $B(\infty)$

$\iota := (\cdots, i_3, i_2, i_1)$: infinite sequence of indices in I of a symmetrizable K.M alg. $\mathfrak{g}$ s.t.

- $i_k \neq i_{k+1}$ $(k \in \mathbb{Z}_{>0})$,
- $\sharp\{k \in \mathbb{Z}_{>0} | i_k = j\} = \infty$ (for any $j \in I$).

ex) $\mathfrak{g}$: rank 2, $\iota = (\cdots, 2, 1, 2, 1, 2, 1)$.

ex) $\mathfrak{g}$: rank 3, $\iota = (\cdots, 3, 1, 2, 3, 2, 1, 3, 1, 2, 3, 2, 1)$.

Introduction : Polyhedral realizations of $B(\infty)$

One can define a crystal structure on
$\mathbb{Z}^\infty = \{(\cdots, a_3, a_2, a_1) | a_l \in \mathbb{Z},\ a_k = 0\ (k \gg 0)\}$ associated with ι
and denote it by $\mathbb{Z}^\infty_\iota$.

Fact(Kashiwara, Nakashima-Zelevinsky)

For each ι, there exists an embedding of crystals Ψ_ι:

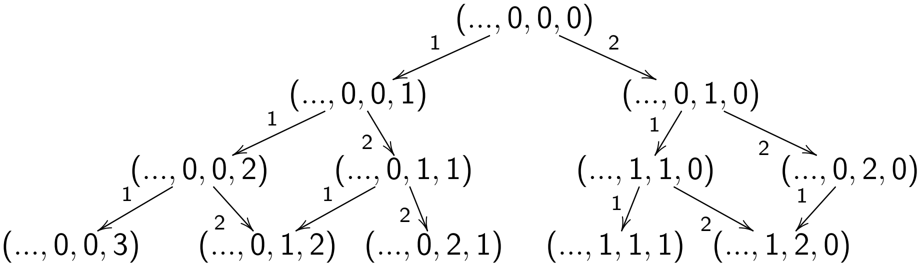
- $\Psi_\iota : B(\infty) \hookrightarrow \mathbb{Z}^\infty_\iota$.

$\mathrm{Im}(\Psi_\iota)$ ($\cong B(\infty)$) : **Polyhedral realization** of $B(\infty)$

Example : Polyhedral realizations of $B(\infty)$

$\mathfrak{g}$: type A_2, $\iota = (\cdots, 2, 1, 2, 1, 2, 1)$.

The crystal graph of $(B(\infty) \cong)\mathrm{Im}(\Psi_\iota) \subset \mathbb{Z}^\infty$ is as follows :



The set $\mathrm{Im}(\Psi_\iota)$ coincides with the set of $(\cdots, a_3, a_2, a_1) \in \mathbb{Z}^\infty$ s.t.

$$a_1 \geq 0, \;\; a_2 \geq a_3 \geq 0, \;\; a_k = 0 \; (k > 3).$$

Introduction : Polyhedral realizations of $B(\infty)$

Problem

Find an explicit form of inequalities defining $\mathrm{Im}(\Psi_\iota)$.

Nakashima-Zelevinsky(1997)

- Invention of polyhedral realizations and algorithm calculating inequalities defining $\mathrm{Im}(\Psi_\iota)$ (under a condition).
- In the case $\mathfrak{g}$: A_n or $\mathrm{A}_n^{(1)}$-type or rank2 K.M alg, and $\iota = (\cdots, n, \cdots, 2, 1, n, \cdots, 2, 1)$, an explicit form of inequalities defining $\mathrm{Im}(\Psi_\iota)$ is given.

Hoshino(2005), Kim-Shin(2008)

- $\mathfrak{g}$:simple, $\iota = (\cdots, n, \cdots, 1, n, \cdots, 1) \Rightarrow$ an explicit form of the inequalities of $\mathrm{Im}(\Psi_\iota)$

Introduction : Polyhedral realizations of $B(\infty)$

Littlemann(1998)

- We suppose that $\mathfrak{g}$ is a fin. dim. simple. Lie alg.
- $\mathbf{i}=(i_N,\cdots,i_1)$: a reduced word of the long. ele. $w_0\in W$.
- $\iota=(\cdots,i_N,\cdots,i_1)$,

$\Rightarrow \mathrm{Im}(\Psi_\iota)\subset\{(\cdots,a_N,\cdots,a_2,a_1)\in\mathbb{Z}^\infty | a_{N+1}=a_{N+2}=\cdots=0\}$.

$\therefore$ We can regard as $\mathrm{Im}(\Psi_\iota)\subset\mathbb{Z}^N$.

$\mathrm{Im}(\Psi_\iota)\subset\mathbb{Z}^N$ coincides with the set of integer points of the Littelmann's **String cone** $\mathcal{S}_{(i_1,\cdots,i_N)}\subset\mathbb{R}^N$: $\mathrm{Im}(\Psi_\iota)=\mathcal{S}_{(i_1,\cdots,i_N)}\cap\mathbb{Z}^N$.

- $\mathbf{i}$: 'nice decomposition' $\Rightarrow$ an explicit form of **string cone** $\mathcal{S}_{(\mathbf{i})}$:
  type A : $\mathbf{i}=(1,2,1,3,2,1,\cdots,n,\cdots,1)$,
  type B,C : $\mathbf{i}=$
  $(1,(2,1,2),(3,2,1,2,3),\cdots,(n,n-1\cdots,2,1,2,\cdots,n-1,n))$.

Introduction : Monomial realizations of $B(\lambda)$

Monomial realizations of crystal bases $B(\lambda)$ (Nakajima, Kashiwara)

Each element of crystal base $B(\lambda)$ is realized as a Laurent monomial of doubly indexed variables $\{X_{s,i} | s \in \mathbb{Z},\ i \in I\}$.

ex) $B(\Lambda_1)$ of type A_3

$$X_{1,1} \overset{\tilde{f}_1}{\to} \frac{X_{1,2}}{X_{2,1}} \overset{\tilde{f}_2}{\to} \frac{X_{1,3}}{X_{2,2}} \overset{\tilde{f}_3}{\to} \frac{1}{X_{2,3}}.$$

Main results

- Explicit forms of inequalities defining polyhedral realizations of $B(\infty)$ in terms of rectangular tableaux for 'adapted' sequence ι and classical Lie algebra $\mathfrak{g}$.
- A conjecture that inequalities for $B(\infty)$ are obtained from monomial realizations of $B(\lambda)$ via tropicalization.

1. Polyhedral realizations of $B(\infty)$

A morphism of crystals

Definition

A **morphism** $\psi : \mathcal{B}_1 \to \mathcal{B}_2$ of crystals $\mathcal{B}_1$, $\mathcal{B}_2$ is a map $\mathcal{B}_1 \bigsqcup \{0\} \to \mathcal{B}_2 \bigsqcup \{0\}$ s.t. $\psi(0) = 0$ and for $b \in \mathcal{B}_1$,

(1) $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$ if $\psi(b) \neq 0$,

(2) $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$ if $\psi(b) \neq 0$,

(3) $\varphi_i(\psi(b)) = \varphi_i(b)$ if $\psi(b) \neq 0$,

(4) $\psi(\tilde{e}_i(b)) = \tilde{e}_i \psi(b)$ if $\psi(b) \neq 0$ and $\psi(\tilde{e}_i(b)) \neq 0$,

(5) $\psi(\tilde{f}_i(b)) = \tilde{f}_i \psi(b)$ if $\psi(b) \neq 0$ and $\psi(\tilde{f}_i(b)) \neq 0$,

for $i \in I$.

A strict morphism of crystals

Definition

A **strict morphism** $\psi : \mathcal{B}_1 \to \mathcal{B}_2$ of crystals $\mathcal{B}_1$, $\mathcal{B}_2$ is a map $\mathcal{B}_1 \bigsqcup \{0\} \to \mathcal{B}_2 \bigsqcup \{0\}$ s.t. $\psi(0) = 0$ and for $b \in \mathcal{B}_1$,

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(4) $\psi(\tilde{e}_i(b)) = \tilde{e}_i\psi(b)$,

(5) $\psi(\tilde{f}_i(b)) = \tilde{f}_i\psi(b)$,

for $i \in I$.

An injective strict morphism is said to be **strict embedding**.

A crystal structure of $\mathbb{Z}^{\infty}_{\iota}$

$$\mathbb{Z}^{\infty} := \{\mathbf{x} = (\cdots, x_4, x_3, x_2, x_1) | x_k \in \mathbb{Z},\ x_l = 0 (l \gg 0)\}.$$

$\iota := (\cdots, i_3, i_2, i_1)$: infinite sequence of I
s.t. $i_k \neq i_{k+1}$ $(k \in \mathbb{Z}_{>0})$ and $\sharp\{k \in \mathbb{Z}_{>0} | i_k = j\} = \infty$ (for any $j \in I$).

We define a crystal str. on $\mathbb{Z}^{\infty}$ ass. to ι as follows:

$$\mathrm{wt}(\mathbf{x}) := -\sum_{j \in \mathbb{Z}_{\geq 1}} x_j \alpha_{i_j},$$

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We define a crystal str. on $\mathbb{Z}^{\infty}$ ass. to ι as follows:

$$\mathrm{wt}(\mathbf{x}) := -\sum_{j \in \mathbb{Z}_{\geq 1}} x_j \alpha_{i_j},$$

$$\sigma_k(\mathbf{x}) := x_k + \sum_{j>k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j \qquad (k \in \mathbb{Z}_{\geq 1},\ \mathbf{x} \in \mathbb{Z}^{\infty}),$$

$$\varepsilon_i(\mathbf{x}) := \max_{k \in \mathbb{Z}_{\geq 1};\ i_k = i} \sigma_k(\mathbf{x}), \qquad \varphi_i(\mathbf{x}) := \langle wt(\mathbf{x}), h_i \rangle + \varepsilon_i(\mathbf{x})\ (i \in I).$$

A crystal structure of $\mathbb{Z}^{\infty}_{\iota}$

For $\mathbf{x} = (x_k)_{k \in \mathbb{Z}_{\geq 1}} \in \mathbb{Z}^{\infty}$ and $i \in I$,

$$(\tilde{f}_i(\mathbf{x}))_k := x_k + \delta_{k,m_i},$$

$$(\tilde{e}_i(\mathbf{x}))_k := x_k - \delta_{k,m'_i} \ \text{ if } \varepsilon_i(\mathbf{x}) > 0, \quad (\tilde{e}_i(\mathbf{x})) = 0 \ \text{ if } \varepsilon_i(\mathbf{x}) = 0,$$

where m_i, m'_i are combinatorially determined from $\{\sigma_k(\mathbf{x})\}_{k \in \mathbb{Z}_{\geq 1}}$.

Theorem (Nakashima-Zelevinsky)

$(\mathbb{Z}^{\infty}, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \mathrm{wt})$ is a crystal. We denote it by $\mathbb{Z}^{\infty}_{\iota}$.

Polyhedral realizations

Theorem (Nakashima-Zelevinsky)

There is a unique strict embedding of crystals $\Psi_\iota : B(\infty) \hookrightarrow \mathbb{Z}_\iota^\infty$ s.t. $\Psi_\iota(u_\infty) = (\cdots, 0, 0, 0)$. Here u_∞ is the highest weight vector of $B(\infty)$.

Definition

$\mathrm{Im}\Psi_\iota (\cong B(\infty))$ is called a **polyhedral realization** of $B(\infty)$.

Problem

Find explicit forms inequalities defining $\mathrm{Im}(\Psi_\iota)$.

2. Calculations of Polyhedral realizations

Polyhedral realizations of $B(\infty)$

- Nakashima and Zelevinsky found a way calculating the inequalities defining $\mathrm{Im}(\Psi_\iota)$ if ι satisfies a positivity condition.
- Defining a set $\Xi_\iota \subset \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^\infty, \mathbb{Z})$, they described $\mathrm{Im}(\Psi_\iota)$ as

$$\mathrm{Im}(\Psi_\iota) = \{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \geq 0, \quad \forall \varphi \in \Xi_\iota\}.$$

Let us see the **positivity condition** and a **construction of** Ξ_ι.

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Let us see the **positivity condition** and a **construction of** Ξ_ι.

For $\iota = (\cdots, i_3, i_2, i_1)$ and $k \in \mathbb{Z}_{\geq 1}$,

$$k^- := \max(\{l \in \mathbb{Z}_{\geq 1} | l < k, \ i_k = i_l\} \cup \{0\}),$$
$$k^+ := \min\{l \in \mathbb{Z}_{\geq 1} | l > k, \ i_k = i_l\}.$$

ex) $\iota = (\cdots, 2, 1, 2, 1, 2, 1) \Rightarrow 1^- = 0$, $2^- = 0$, $3^- = 1$, $4^- = 2$, $5^- = 3$, $6^- = 4$, $1^+ = 3$, $2^+ = 4$, $3^+ = 5$, $4^+ = 6$.

Calculations of Polyhedral realization of $B(\infty)$

- For $k \in \mathbb{Z}_{\geq 1}$, we define $x_k \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\infty}, \mathbb{Z})$ as $x_k(\cdots, a_3, a_2, a_1) := a_k$.
- Using x_k, each $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\infty}, \mathbb{Z})$ can be written as

$$\varphi = \sum c_k x_k, \quad c_k \in \mathbb{Z}.$$

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- Let $\beta_k \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\infty}, \mathbb{Z})$ $(k \in \mathbb{Z}_{\geq 1})$ be

$$\beta_k = x_k + \sum_{k<j<k^+} \langle h_{i_k}, \alpha_{i_j} \rangle x_j + x_{k^+}.$$

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- For $\varphi = \sum c_k x_k \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\infty}, \mathbb{Z})$ and $k \in \mathbb{Z}_{\geq 1}$, we define $S_k(\varphi) \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\infty}, \mathbb{Z})$ as

$$S_k(\varphi) := \begin{cases} \varphi - c_k \beta_k & \text{if } c_k \geq 0, \\ \varphi - c_k \beta_{k^-} & \text{if } c_k < 0. \end{cases}$$

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where we set $\beta_0 := 0$. We also define

$$\Xi_{\iota} := \{S_{j_l} \cdots S_{j_1} x_{j_0} | l \geq 0, j_0, j_1, \cdots, j_l \geq 1\}.$$

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Positivity condition

For any $\varphi = \sum c_k x_k \in \Xi_\iota$, if $k^- = 0$ ($k \in \mathbb{Z}_{\geq 1}$) then $c_k \geq 0$.

Theorem (Nakashima-Zelevinsky)

If ι satisfies the positivity condition then

$$\mathrm{Im}(\Psi_\iota)(\cong B(\infty)) = \{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \geq 0, \quad \forall \varphi \in \Xi_\iota\}.$$

An example of the polyhedral realization of $B(\infty)$

Example) $\mathfrak{g}$: type A_2, $\iota = (\cdots, 2, 1, 2, 1, 2, 1)$.

$$1^- = 2^- = 0,\ k^- > 0\ (k > 2).$$

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We rewrite a vector $(\cdots, x_6, x_5, x_4, x_3, x_2, x_1)$ as

$$(\cdots, x_{3,2}, x_{3,1}, x_{2,2}, x_{2,1}, x_{1,2}, x_{1,1}). \qquad (x_{l,1} = x_{2l-1}, \ x_{l,2} = x_{2l})$$

Similarly, we rewrite $S_{l,1} = S_{2l-1}, \ S_{l,2} = S_{2l}$.

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Similarly, we rewrite $S_{l,1} = S_{2l-1},\ S_{l,2} = S_{2l}$.

Recall) positivity condition $\Leftrightarrow$ the coefficients of $x_1 = x_{1,1}$ and $x_2 = x_{1,2}$ in each $\varphi \in \Xi_\iota$ are non-negative.

An example of the polyhedral realization of $B(\infty)$

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$$x_{k,1} \underset{S_{k+1,1}}{\overset{S_{k,1}}{\rightleftarrows}} x_{k,2} - x_{k+1,1} \underset{S_{k+1,2}}{\overset{S_{k,2}}{\rightleftarrows}} -x_{k+1,2},$$

$$x_{k,2} \underset{S_{k+1,2}}{\overset{S_{k,2}}{\rightleftarrows}} x_{k+1,1} - x_{k+1,2} \underset{S_{k+2,1}}{\overset{S_{k+1,1}}{\rightleftarrows}} -x_{k+2,1},$$

for $k \in \mathbb{Z}_{\geq 1}$ and other actions are trivial.

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for $k \in \mathbb{Z}_{\geq 1}$ and other actions are trivial. Thus,

$\Xi_\iota = \{x_{k,1},\ x_{k,2}-x_{k+1,1},\ -x_{k+1,2},\ x_{k,2},\ x_{k+1,1}-x_{k+1,2}, -x_{k+2,1} | k \geq 1\}$.

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The coefficients of $x_{1,1}$ and $x_{1,2}$ in each $\varphi \in \Xi_\iota$ are non-negative.
$\therefore$ ι **satisfies the positivity condition** and

$$\mathrm{Im}(\Psi_\iota) = \{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \geq 0, \quad \forall \varphi \in \Xi_\iota\}$$

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The coefficients of $x_{1,1}$ and $x_{1,2}$ in each $\varphi \in \Xi_\iota$ are non-negative.

$\therefore$ ι **satisfies the positivity condition** and

$$\begin{aligned} \mathrm{Im}(\Psi_\iota) &= \{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \geq 0, \quad \forall \varphi \in \Xi_\iota\} \\ &= \{\mathbf{x} \in \mathbb{Z}^\infty | x_{k+1,2} = x_{k+2,1} = 0\ (k \geq 1),\ x_{1,2} \geq x_{2,1} \geq 0,\ x_{1,1} \geq 0\}. \end{aligned}$$

An example which does not satisfy the positivity condition

Example) $\mathfrak{g}$: type A_3, $\iota = (\cdots, 2, 1, 2, 3, 2, 1)$.

$$x_1 \overset{S_1}{\rightarrow} -x_5 + x_4 + x_2 \overset{S_2}{\rightarrow} -x_5 + x_3 \overset{S_5}{\rightarrow} -x_4 + x_3 - x_2 + x_1.$$

An example which does not satisfy the positivity condition

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Thus, $-x_4 + x_3 - x_2 + x_1 \in \Xi_\iota$ and $2^- = 0$.
$\therefore$ ι does **not** satisfy the positivity condition.

3. An explicit form of the polyhedral realization for $B(\infty)$

Infinite sequences adapted to A

$A = (a_{i,j})_{i,j\in I}$: The Cartan matrix of $\mathfrak{g}$

Definition

If ι satisfies the following condition, we say ι is **adapted to** A :
For $i, j \in I$ with $i \neq j$ and $a_{i,j} \neq 0$, the subsequence of ι consisting of i, j is

$$(\cdots, i, j, i, j, i, j, i, j) \quad \text{or} \quad (\cdots, j, i, j, i, j, i, j, i).$$

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Example) $\mathfrak{g}$: type A_3, $\iota=(\cdots,2,1,3,2,1,3,2,1,3)$

- subsequence consisting of 1, 2 : $(\cdots,2,1,2,1,2,1)$
- subsequence consisting of 2, 3 : $(\cdots,2,3,2,3,2,3)$
- Since $a_{1,3}=0$ we do not need consider the pair 1, 3.

Thus, ι is adapted to A.

Infinite sequences adapted to A

Example) $\mathfrak{g}$: type A_3, $\iota = (\cdots, 3, 2, 1, 3, 2, 3, 1, 2, 3, 1, 2, 1)$

- subsequence consisting of 1, 2 : $(\cdots, 2, 1, 2, 1, 2, 1)$
- subsequence consisting of 2, 3 : $(\cdots, 3, 2, 3, 2, 3, 2)$

ι is adapted to A.

Example) $\mathfrak{g}$: type A_3, $\iota = (\cdots, 2, 1, 2, 3, 2, 1)$

- subsequence consisting of 1, 2 : $(\cdots, 2, 1, 2, 2, 1)$

ι is **not** adapted to A.

Tableaux description of Ξ_ι

In this section, let $\mathfrak{g}$ be classical type (A_n, B_n, C_n or D_n).

In what follows, we suppose that ι is adapted to the Cartan matrix of $\mathfrak{g}$ and consider the following problem:

Problem

Find an explicit form of inequalities defining $\mathrm{Im}(\Psi_\iota)$.

Let us construct **an explicit form** of Ξ_ι by using rectangular tableaux.

Recall If ι satisfies the positivity condition then

$$\mathrm{Im}(\Psi_\iota) = \{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \geq 0, \quad \forall \varphi \in \Xi_\iota\} \; (\cong B(\infty)).$$

Tableaux description of Ξ_ι

Recall

ι is **adapted to** $A = (a_{ij}) \overset{def}{\Leftrightarrow}$
For $i, j \in I$ ($i \neq j$, $a_{i,j} \neq 0$), the subsequence of ι consisting of i, j is

$$(\cdots, i, j, i, j, i, j, i, j) \quad \text{or} \quad (\cdots, j, i, j, i, j, i, j, i).$$

Let $(p_{i,j})_{i \neq j,\ a_{i,j} \neq 0}$ be the set of integers s.t.

$$p_{i,j} = \begin{cases} 1 & \text{if } (\cdots, j, i, j, i, j, i), \\ 0 & \text{if } (\cdots, i, j, i, j, i, j). \end{cases}$$

Tableaux description of Ξ_ι

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For k $(2 \leq k \leq n)$, we set

$$P(k) := \begin{cases} p_{2,1} + p_{3,2} + \cdots + p_{n-2,n-3} + p_{n,n-2} & \text{if } k = n,\ \mathfrak{g} : \text{type } \mathrm{D}_n, \\ p_{2,1} + p_{3,2} + p_{4,3} + \cdots + p_{k,k-1} & \text{if o.w.} \end{cases}$$

Tableaux descriptions

$$\iota = (\cdots, i_k, \cdots, i_3, i_2, i_1).$$

For $k \in \mathbb{Z}_{\geq 1}$, we write

$$x_k = x_{s,j}, \quad S_k = S_{s,j},$$

if $i_k = j$ and j is appearing s times in i_k, i_{k-1}, $\cdots$, i_1.

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Example) $\iota = (\cdots, 2, 1, 3, 2, 1, 3, 2, 1, 3)$
$(\cdots, x_7, x_6, x_5, x_4, x_3, x_2, x_1) = (\cdots, x_{3,3}, x_{2,2}, x_{2,1}, x_{2,3}, x_{1,2}, x_{1,1}, x_{1,3})$

Tableaux descriptions

$\mathfrak{g} = \mathrm{A}_n$ case

For $1 \leq j \leq n+1$ and $s \in \mathbb{Z}$, we set

$$\boxed{j}^{\mathrm{A}}_{s} := x_{s+P(j),j} - x_{s+P(j-1)+1,j-1} \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\infty}, \mathbb{Z}).$$

$(x_{m,0} = x_{m,n+1} = 0$ for $m \in \mathbb{Z}$, and $x_{m,i} = 0$ $(m \leq 0, i \in I))$.

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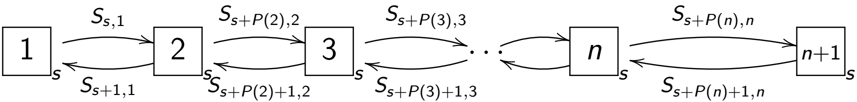
$\boxed{j}_{s} = \boxed{j}^{\mathrm{A}}_{s}$ are obtained from $\boxed{1}_{s} = x_{s,1}$ by operators $S_{m,j}$ $(1 \leq s)$:



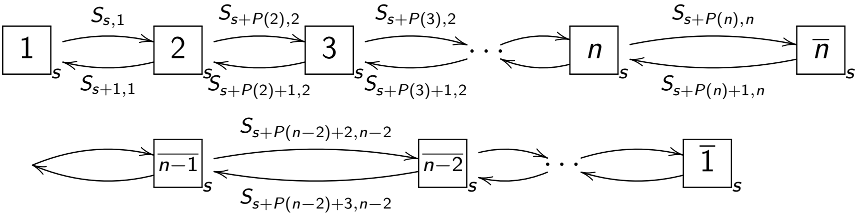
Tableaux descriptions

$\mathfrak{g} = \mathrm{B}_n$ case

For $1 \leq j \leq n$ and $s \in \mathbb{Z}$, we set

$$\boxed{j}^{\mathrm{B}}_s := x_{s+P(j),j} - x_{s+P(j-1)+1,j-1},$$

$$\boxed{\bar{j}}^{\mathrm{B}}_s := x_{s+P(j-1)+n-j+1,j-1} - x_{s+P(j)+n-j+1,j}.$$



Tableaux descriptions

$\mathfrak{g} = \mathrm{C}_n$ case

For $1 \leq j \leq n-1$ and $s \in \mathbb{Z}$, we set

$$\boxed{j}^{\mathrm{C}}_s := x_{s+P(j),j} - x_{s+P(j-1)+1,j-1},$$

$$\boxed{n}^{\mathrm{C}}_s := 2x_{s+P(n),n} - x_{s+P(n-1)+1,n-1},$$

$$\boxed{\overline{n}}^{\mathrm{C}}_s := x_{s+P(n-1)+1,n-1} - 2x_{s+P(n)+1,n},$$

$$\boxed{\overline{j}}^{\mathrm{C}}_s := x_{s+P(j-1)+n-j+1,j-1} - x_{s+P(j)+n-j+1,j}.$$

Tableaux descriptions

$\mathfrak{g} = \mathrm{D}_n$ case

For $s \in \mathbb{Z}$, we set

$$\boxed{j}^{\mathrm{D}}_s := x_{s+P(j),j} - x_{s+P(j-1)+1,j-1}, \ (1 \le j \le n-2, \ j = n),$$

$$\boxed{n-1}^{\mathrm{D}}_s := x_{s+P(n-1),n-1} + x_{s+P(n),n} - x_{s+P(n-2)+1,n-2},$$

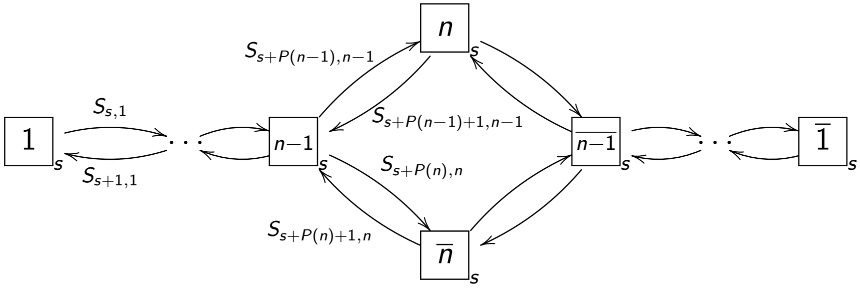
$$\boxed{\overline{n}}^{\mathrm{D}}_s := x_{s+P(n-1),n-1} - x_{s+P(n)+1,n},$$

$$\boxed{\overline{n-1}}^{\mathrm{D}}_s := x_{s+P(n-2)+1,n-2} - x_{s+P(n-1)+1,n-1} - x_{s+P(n)+1,n},$$

$$\boxed{\overline{j}}^{\mathrm{D}}_s := x_{s+P(j-1)+n-j,j-1} - x_{s+P(j)+n-j,j}, \ (1 \le j \le n-2).$$

Tableaux descriptions

$\mathfrak{g} = \mathrm{D}_n$ case



Tableaux descriptions

For $\mathrm{X} = \mathrm{A}$, B, C or D,

$$\boxed{\begin{array}{c} j_1 \\ \hline j_2 \\ \hline \vdots \\ \hline j_{k-1} \\ \hline j_k \end{array}}^{\mathrm{X}}_{s} := \boxed{j_k}^{\mathrm{X}}_{s} + \boxed{j_{k-1}}^{\mathrm{X}}_{s+1} + \cdots + \boxed{j_2}^{\mathrm{X}}_{s+k-2} + \boxed{j_1}^{\mathrm{X}}_{s+k-1}$$

Partial order set

Let us define the following posets:

- $J_A := \{1, 2, \cdots, n, n+1\}, \quad 1 < 2 < \cdots < n < n+1.$
- $J_B = J_C := \{1, 2, \cdots, n, \overline{n}, \cdots, \overline{2}, \overline{1}\},$

$$1 < 2 < \cdots < n < \overline{n} < \cdots < \overline{2} < \overline{1}.$$

- $J_D := \{1, 2, \cdots, n, \overline{n}, \cdots, \overline{2}, \overline{1}\},$

$$1 < 2 < \cdots < n-1 < \begin{smallmatrix} n \\ \overline{n} \end{smallmatrix} < \overline{n-1} < \cdots < \overline{2} < \overline{1}.$$

For $j \in \{1, 2, \cdots, n\}$, we set $|j| = |\overline{j}| = j$.

Tableaux descriptions

For $\mathrm{X} = \mathrm{A}, \mathrm{B}$,

$$\mathrm{Tab}_{\mathrm{X},\iota} := \left\{ \begin{array}{|c|}\hline j_1 \\ \hline j_2 \\ \hline \vdots \\ \hline j_k \\ \hline \end{array}^{\mathrm{X}}_{s} \middle| k \in I,\ j_i \in J_X,\ 1 - P(k) \le s,\ (*)^{\mathrm{X}}_k \right\}$$

$(*)^{\mathrm{A}}_k : 1 \le j_1 < j_2 < \cdots < j_k \le n+1,$

$$(*)^{\mathrm{B}}_k : \begin{cases} 1 \le j_1 < j_2 < \cdots < j_k \le \overline{1} & \text{for } k < n, \\ 1 \le j_1 < j_2 < \cdots < j_n \le \overline{1}, \quad |j_l| \ne |j_m|\ (l \ne m) & \text{for } k = n. \end{cases}$$

Tableaux descriptions

$$\mathrm{Tab}_{\mathrm{C},\iota} := \left\{ \boxed{\begin{array}{c} j_1 \\ \hline j_2 \\ \hline \vdots \\ \hline j_k \end{array}}^{\mathrm{C}}_{s} \middle| k \in \{1, \cdots, n-1\},\ j_i \in J_C,\ 1 - P(k) \le s,\ (*)^{\mathrm{C}}_k \right\}$$

$$\cup \left\{ \boxed{\begin{array}{c} \overline{n+1} \\ \hline j_1 \\ \hline \vdots \\ \hline j_t \end{array}}^{\mathrm{C}}_{s} \middle| 0 \le t \le n,\ j_i \in J_C,\ 1 - P(n) \le s,\ (*)^{\mathrm{C}}_n \right\},$$

where $\boxed{\overline{n+1}}^{\mathrm{C}}_{s} := x_{s+P(n),n}$, $(*)^{\mathrm{C}}_k : \begin{cases} 1 \le j_1 < \cdots < j_k \le \overline{1} & \text{for } k < n, \\ \overline{n} \le j_1 < \cdots < j_t \le \overline{1} & \text{for } k = n. \end{cases}$

Tableaux descriptions

$$\mathrm{Tab}_{\mathrm{D},\iota} := \left\{ \boxed{\begin{array}{c} j_1 \\ \vdots \\ j_k \end{array}}^{\mathrm{D}}_{s} \middle| k \in \{1, \cdots, n-2\},\ j_i \in J_D,\ 1 - P(k) \le s,\ (*)^{\mathrm{D}}_{k} \right\}$$

$$\cup \left\{ \boxed{\begin{array}{c} \overline{n+1} \\ j_1 \\ \vdots \\ j_t \end{array}}^{\mathrm{D}}_{s} \middle| \begin{array}{l} 0 \le t \le n,\ j_i \in J_D, \\ 1 - P(n-1) \le s, \\ (*)^{\mathrm{D}}_{n-1} \end{array} \right\} \cup \left\{ \boxed{\begin{array}{c} \overline{n+1} \\ j_1 \\ \vdots \\ j_t \end{array}}^{\mathrm{D}}_{s} \middle| \begin{array}{l} 0 \le t \le n, \\ j_i \in J_D, \\ 1 - P(n) \le s, \\ (*)^{\mathrm{D}}_{n} \end{array} \right\}.$$

$$(*)^{\mathrm{D}}_{k} : \begin{cases} j_1 \not\ge j_2 \not\ge \cdots \not\ge j_k & \text{if } k < n-1, \\ \overline{n} \le j_1 < \cdots < j_t \le \overline{1},\ t : \text{odd} & \text{if } k = n-1, \\ \overline{n} \le j_1 < \cdots < j_t \le \overline{1},\ t : \text{even} & \text{if } k = n. \end{cases} \qquad \boxed{\overline{n+1}}^{\mathrm{D}}_{s} := x_{s+P(n),n}$$

Explicit forms of Ξ_ι via tableaux descriptions

Theorem (K-Nakashima)

$\mathfrak{g}$: type X (=A, B, C, or D), ι : adapted to the Cartan matrix of $\mathfrak{g}$.
Then ι satisfies the positivity condition and

$$\Xi_\iota = \mathrm{Tab}_{\mathrm{X},\iota}.$$

$\Rightarrow$An explicit form of Ξ_ι via rectangular tableaux.

Explicit forms of $\mathrm{Im}(\Psi_\iota)$ via tableaux descriptions

Corollay

$$\mathrm{Im}(\Psi_\iota) = \{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \geq 0, \ \forall \varphi \in \mathrm{Tab}_{\mathrm{X},\iota}\}$$

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$$\mathrm{Im}(\Psi_\iota) = \{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \geq 0,\ \forall \varphi \in \mathrm{Tab}_{\mathrm{X},\iota}\}$$
$$= \{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \geq 0,\ \forall \varphi \in \mathrm{Tab}^n_{\mathrm{X},\iota}, \quad x_{m,i} = 0\ (\forall i \in I,\ m > n).\}$$

$$\mathrm{Tab}^n_{\mathrm{X},\iota} := \{ \begin{array}{|c|} \hline j_1 \\ \hline j_2 \\ \hline \vdots \\ \hline j_k \\ \hline \end{array}{}^{\mathrm{X}}_{s} \in \mathrm{Tab}_{\mathrm{X},\iota} | s \leq n\} \ :\ \text{a finite set}$$

Thus, $\mathrm{Im}(\Psi_\iota)$ can be written via finitely many inequalities and finitely many variables $x_{m,i}$.

Explicit forms of Ξ_ι via tableaux descriptions

Example) $\mathfrak{g}$: type A_2, $\iota = (\cdots, 2, 1, 2, 1, 2, 1)$. We have $p_{2,1} = 0$.

$$\begin{aligned} &\mathrm{Tab}^2_{\mathrm{A},\iota} \\ &= \left\{ \boxed{i}^{\mathrm{A}}_s \,\middle|\, 1 \le i \le 3,\ 1 \le s \le 2 \right\} \cup \left\{ \begin{array}{|c|}\hline i \\ \hline j \\ \hline \end{array}^{\mathrm{A}}_s \,\middle|\, 1 \le i < j \le 3,\ 1 \le s \le 2 \right\} \\ &= \{x_{s,i} - x_{s+1,i-1} | 1 \le i \le 3,\ 1 \le s \le 2\} \\ &\quad \cup \{x_{s+1,i} - x_{s+2,i-1} + x_{s,j} - x_{s+1,j-1} | 1 \le i < j \le 3,\ 1 \le s \le 2\}. \end{aligned}$$

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&= \ \{x_{s,i} - x_{s+1,i-1} | 1 \le i \le 3,\ 1 \le s \le 2\} \\
&\quad \cup \{x_{s+1,i} - x_{s+2,i-1} + x_{s,j} - x_{s+1,j-1} | 1 \le i < j \le 3,\ 1 \le s \le 2\}.
\end{aligned}$$

$$\mathrm{Im}(\Psi_\iota) = \{\mathbf{x} \in \mathbb{Z}^\infty | \varphi(\mathbf{x}) \ge 0,\ \forall \varphi \in \mathrm{Tab}^2_{\mathrm{A},\iota},\ x_{m,1} = x_{m,2} = 0\ (3 \le m)\}$$

Explicit forms of Ξ_ι via tableaux descriptions

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$$
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&= \quad \{ x_{s,i} - x_{s+1,i-1} | 1 \le i \le 3,\ 1 \le s \le 2 \} \\
&\qquad \cup \{ x_{s+1,i} - x_{s+2,i-1} + x_{s,j} - x_{s+1,j-1} | 1 \le i < j \le 3,\ 1 \le s \le 2 \}.
\end{aligned}
$$

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$$

$$
= \{ \mathbf{x} \in \mathbb{Z}^\infty | x_{1,2} \ge x_{2,1} \ge 0,\ x_{1,1} \ge 0,\ x_{m+1,2} = x_{m+2,1} = 0,\ \forall m \in \mathbb{Z}_{\ge 1} \}.
$$

4. Polyhedral realizations and monomial realizations

Mono. real. and polyh. real. of $B(\infty)$

In this section, we will compare **inequalities defining polyhedral realizations** and **monomial realizations**.

Mono. real. and polyh. real. of $B(\infty)$

In this section, we will compare **inequalities defining polyhedral realizations** and **monomial realizations**.

Recall For $j \in \mathbb{Z}_{\geq 1}$, $\exists\ S_j : \mathrm{Hom}(\mathbb{Z}^\infty, \mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}^\infty, \mathbb{Z})$. We put

$$\Xi_{\iota,k} := \{S_{j_l} \cdots S_{j_1} x_{s,k} | l \in \mathbb{Z}_{\geq 0}, j_1, \cdots, j_l,\ s \in \mathbb{Z}_{\geq 1}\} \quad (k \in I),$$

Putting

$$\Xi_\iota = \bigcup_{k \in I} \Xi_{\iota,k},$$

if ι satisfies the 'positivity condition' then

$$\mathrm{Im}(\Psi_\iota) = \{\mathbf{a} \in \mathbb{Z}^\infty | \varphi(\mathbf{a}) \geq 0,\ \forall \varphi \in \Xi_\iota\}.$$

We compare $\Xi_{\iota,k}$ and a monomial realization of $B(\Lambda_k)$.

Monomial realization of crystal base

- $\mathfrak{g}$: symmetrizable K.M Lie algebra with generalized Cartan matrix A,
- ι : adapted to $A = (a_{i,j})$.

The set $p = (p_{j,k})_{j,k \in I,\ j \neq k,\ a_{j,k} \neq 0}$ is as before:

$$p_{j,k} = \begin{cases} 1 & \text{if} \quad (\cdots, k, j, k, j, k, j), \\ 0 & \text{if} \quad (\cdots, j, k, j, k, j, k). \end{cases}$$

For doubly-indexed variables $\{X_{s,i} \mid i \in I,\ s \in \mathbb{Z}\}$,

$$\mathcal{Y} := \left\{ X = \prod_{s \in \mathbb{Z},\ i \in I} X_{s,i}^{\zeta_{s,i}} \middle| \zeta_{s,i} \in \mathbb{Z}, \ \ \text{only finitely many } \zeta_{s,i} \neq 0 \right\}.$$

Monomial realization of crystal base

For $X = \prod_{s\in\mathbb{Z},\ i\in I} X_{s,i}^{\zeta_{s,i}} \in \mathcal{Y}$, $\quad \mathrm{wt}(X) := \sum_{i,s} \zeta_{s,i}\Lambda_i$,

$$\varphi_i(X) := \max\left\{\sum_{k\le s} \zeta_{k,i} \,\middle|\, s\in\mathbb{Z}\right\}, \quad \varepsilon_i(X) := \varphi_i(X) - \mathrm{wt}(X)(h_i).$$

Setting

$$A_{s,k} := X_{s,k}X_{s+1,k} \prod_{j\neq k,\ a_{j,k}\neq 0} X_{s+p_{j,k},j}^{a_{j,k}}.$$

Define the Kashiwara operators as

$$\tilde{f}_i X = \begin{cases} A_{n_{f_i},i}^{-1}X & \text{if} \quad \varphi_i(X) > 0, \\ 0 & \text{if} \quad \varphi_i(X) = 0, \end{cases} \qquad \tilde{e}_i X = \begin{cases} A_{n_{e_i},i}X & \text{if} \quad \varepsilon_i(X) > 0, \\ 0 & \text{if} \quad \varepsilon_i(X) = 0, \end{cases}$$

where the integers n_{f_i} and n_{e_i} are combinatorially determined from exponents of X.

Monomial realization of crystal base

Theorem (Nakajima, Kashiwara)

(i) The 6-tuple $\mathcal{Y}(p) = (\mathcal{Y}, \mathrm{wt}, \varphi_i, \varepsilon_i, \tilde{f}_i, \tilde{e}_i)_{i \in I}$ is a crystal.

(ii) If a monomial $X \in \mathcal{Y}$ satisfies $\varepsilon_i(X) = 0$ for all $i \in I$, then

$$B(\mathrm{wt}(X)) \cong \{\tilde{f}_{j_s} \cdots \tilde{f}_{j_1} X | s \geq 0,\ j_1, \cdots, j_s \in I\} \setminus \{0\}.$$

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1. We denote the embedding of crystals by $\mu : B(\mathrm{wt}(X)) \hookrightarrow \mathcal{Y}$.
2. If $X \in \{\prod_{s\in\mathbb{Z},\ i\in I} X_{s,i}^{\zeta_{s,i}} \in \mathcal{Y} \mid \zeta_{s,i} \geq 0\}$, then $\varepsilon_i(X) = 0$ $(i \in I)$.

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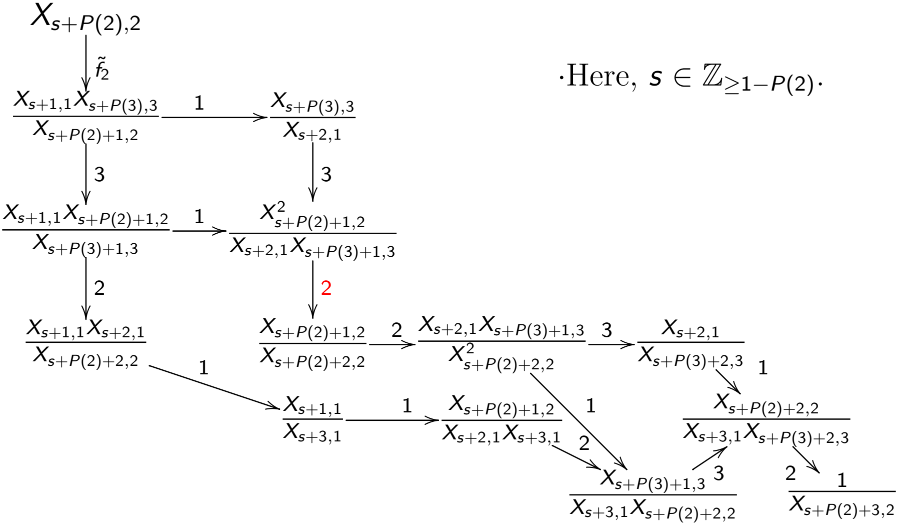
Rem) For the construction of monomial realizations, one can take the set of integers $p = (p_{j,k})_{j,k \in I,\ j \neq k,\ a_{j,k} \neq 0}$ more generally.

Ex) A monomial realization for $B(\Lambda_2)$ of type C_3

$\cdot$Here, $s \in \mathbb{Z}_{\geq 1-P(2)}$.

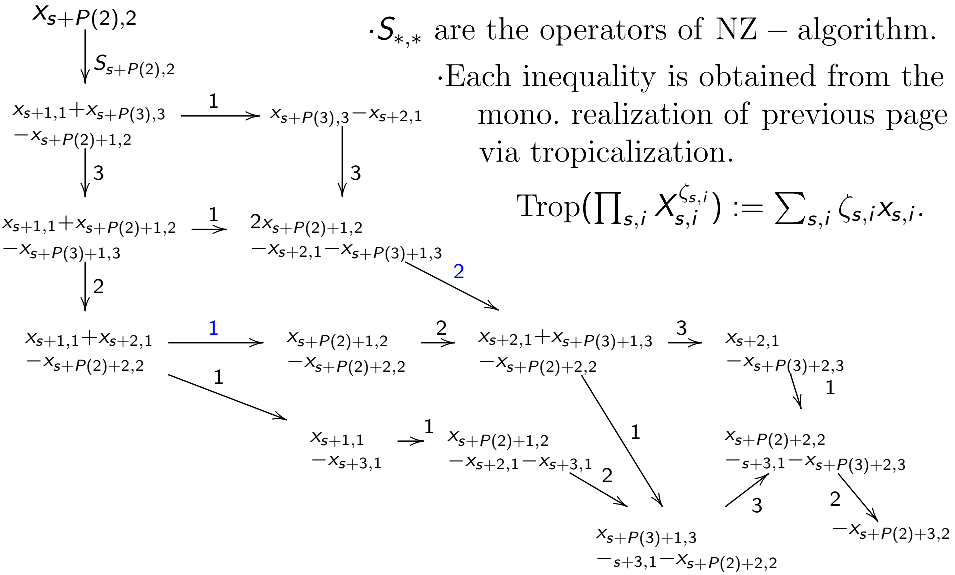


Inequalities $\Xi_{\iota,2}$ of polyh. real. of type B_3

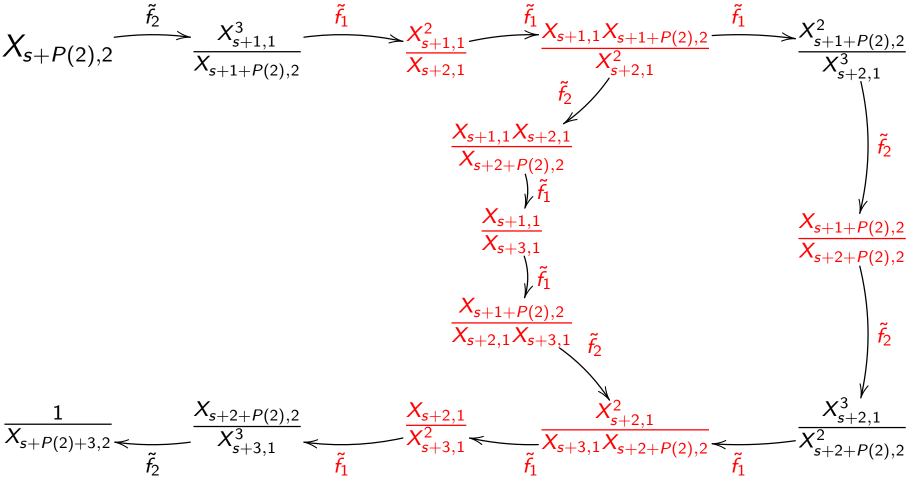
$\cdot S_{*,*}$ are the operators of NZ – algorithm.

$\cdot$Each inequality is obtained from the mono. realization of previous page via tropicalization.

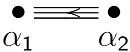
$$\mathrm{Trop}\left(\prod_{s,i} X_{s,i}^{\zeta_{s,i}}\right) := \sum_{s,i} \zeta_{s,i} x_{s,i}.$$



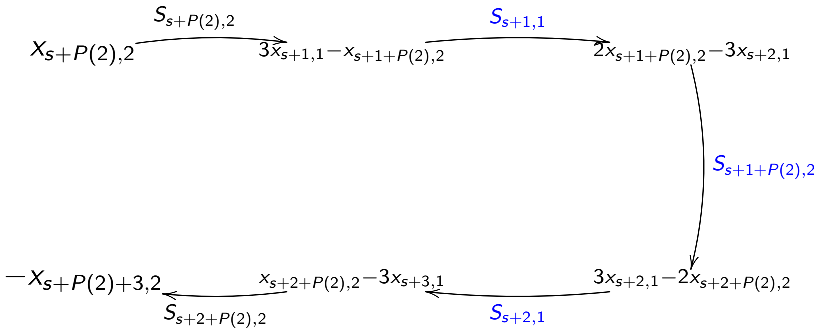
Ex) Monomial realization for $B(\Lambda_2)$ of type G_2

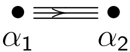


Here, $P(2) = p_{2,1}$ and $s \in \mathbb{Z}_{\geq 1-P(2)}$.



Inequalities $\Xi_{\iota,2}$ of polyh. real. of type G_2





- Each inequality is obtained from mono. realization of the previous page via tropicalization.

Inequalities of polyh. real. for $B(\infty)$ of type G_2

- $\mathrm{Trop}(\frac{X_{s+1,1}^2}{X_{s+2,1}}) = 2x_{s+1,1} - x_{s+2,1} \geq 0$ follows from

$$\mathrm{Trop}(\frac{X_{s+1,1}^3}{X_{s+1+P(2),2}}) = 3x_{s+1,1} - x_{s+1+P(2),2} \geq 0,$$

$$\mathrm{Trop}(\frac{X_{s+1+P(2),2}^2}{X_{s+2,1}^3}) = 2x_{s+1+P(2),2} - 3x_{s+2,1} \geq 0,$$

$\because 3(2x_{s+1,1} - x_{s+2,1}) = 2(3x_{s+1,1} - x_{s+1+P(2),2}) + (2x_{s+1+P(2),2} - 3x_{s+2,1})$

- Similarly, the inequalities obtained from red monomials follow by other inequalities obtained from black monomials.

Inequalities of polyh. real. for $B(\infty)$ of type G_2

- $\mathrm{Trop}(\frac{X_{s+1,1}^2}{X_{s+2,1}}) = 2x_{s+1,1} - x_{s+2,1} \geq 0$ follows from

$$\mathrm{Trop}(\frac{X_{s+1,1}^3}{X_{s+1+P(2),2}}) = 3x_{s+1,1} - x_{s+1+P(2),2} \geq 0,$$

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- Similarly, the inequalities obtained from red monomials follow by other inequalities obtained from black monomials.

$\Xi_{\iota,k} \cup \{\text{extra homomorphisms}\} \overset{1:1}{\leftrightarrow} \text{Mono. real. of } B(\Lambda_k) \text{ of } {}^L\mathfrak{g}.$

We say $\varphi \in \mathrm{Hom}(\mathbb{Z}^\infty, \mathbb{Z})$ is extra if $\varphi(x) \geq 0$ follows from $\psi(x) \geq 0$ $(\forall \psi \in \Xi_{\iota,k})$.

Mono. real. and polyh. real. for $B(\infty)$

Conjecture

If ι is adapted then the positivity condition holds and for each $k \in I$,

$$\Xi_{\iota,k} \cup \{\text{extra homomorphisms}\} = \mathrm{Trop}\left(\coprod_{s\in\mathbb{Z}_{\geq 1}} \mu^{\iota}_{s,k}(B(\Lambda_k))\right),$$

where $B(\Lambda_k)$ is the crystal base of $U_q({}^L\mathfrak{g})$ and ${}^L\mathfrak{g}$ is the Lie algebra associated with generalized Cartan matrix ${}^tA$ and $\mu^{\iota}_{s,k}$ is the monomial realization with highest weight vector $X_{s,k}$.

Theorem (K.)

In the case $\mathfrak{g}$ is type A_n, B_n, G_2 or $A_1^{(1)}$, the conjecture is true.

Remarks

If conjecture holds we obtain

$$\mathrm{Im}(\Psi_\iota) =$$

$$\{\mathbf{a} \in \mathbb{Z}^\infty | \varphi(\mathbf{a}) \geq 0,\ \forall \varphi \in \mathrm{Trop}\left(\coprod_{k\in I,\ s\in\mathbb{Z}_{\geq 1}} \mu^\iota_{s,k}(B(\Lambda_k)) \right) \}.$$

As for relations between polyh. real and mono. real.,

- Kim-Shin explicitly gave an isomorphism between a monomial realization for $B(\infty)$ and polyhedral realization $\mathrm{Im}(\Psi_\iota)$ in the case $\mathfrak{g}$ is classical type or G_2 and ι is specific one.
- Nakashima proved that inequalities defining $\mathrm{Im}(\Psi_\iota)$ are obtained from monomial realizations of $B(\Lambda_k)$ ($k = 1, 2, \cdots, n$) in the case $\mathfrak{g}$ is classical type and ι is specific one.