Combinatorial mutations on representation-theoretic polytopes

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Mirror symmetry for $\mathbb{P}^2(\mathbb{C})$

$$\mathbb{P}^2(\mathbb{C}) \quad \xleftrightarrow{\text{mirror}} \quad f := x + y + \tfrac{1}{xy} \colon (\mathbb{C}^\times)^2 \to \mathbb{C}$$

The **period** $\pi_f(t)$ of f is computed as follows:

$$\begin{aligned}
\pi_f(t) &:= \left(\frac{1}{2\pi\sqrt{-1}}\right)^2 \int_{|x|=|y|=1} \frac{1}{1-tf}\frac{dx}{x}\frac{dy}{y} \qquad (t \in \mathbb{C},\ |t| \ll 1) \\
&= \sum_{k \in \mathbb{Z}_{\geq 0}} \mathrm{Const}(f^k) \cdot t^k \qquad \text{(the Taylor expansion)} \\
&= 1 + 6t^3 + 90t^6 + 1680t^9 + 34650t^{12} + 756756t^{15} + \cdots \\
&= \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{(3k)!}{(k!)^3} t^{3k},
\end{aligned}$$

gen. func. of genus zero
GW invariants

which coincides with the **regularized quantum period** $\widehat{G}_{\mathbb{P}^2(\mathbb{C})}$ of $\mathbb{P}^2(\mathbb{C})$.

Mirror symmetry for $\mathbb{P}^2(\mathbb{C})$

$$\mathbb{P}^2(\mathbb{C}) \quad \xleftrightarrow{\text{another mirror}} \quad g := y + \frac{1}{xy} + \frac{2}{y^2} + \frac{x}{y^3}$$

The period $\pi_g(t)$ of g is also equal to $\pi_f(t) = \widehat{G}_{\mathbb{P}^2(\mathbb{C})}$.

Problem

to give relations among Laurent polynomials having the same period.

Combinatorial mutations rephrase mutations for Laurent polynomials (which preserve the period) in terms of Newton polytopes.

$$\begin{tikzcd}
f \arrow[r, mapsto, "\text{Newton polytope}"] \arrow[d, mapsto, "{\text{mutation} \atop x \mapsto \frac{xy}{x+y},\quad y \mapsto \frac{y^2}{x+y}}"'] & \mathrm{Newt}(f) \arrow[d, mapsto, "{\textbf{combinatorial mutation} \atop (\textbf{today})}"] \\
g \arrow[r, mapsto, "\text{Newton polytope}"] & \mathrm{Newt}(g)
\end{tikzcd}$$

Combinatorial mutations for lattice polytopes

- $N \simeq \mathbb{Z}^m$: a $\mathbb{Z}$-lattice of rank m,
- $M := \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$,
- $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$,
- $H_{w,h} := \{v \in N_{\mathbb{R}} \mid \langle w, v \rangle = h\}$ for $w \in M$ and $h \in \mathbb{Z}$,
- $P \subseteq N_{\mathbb{R}}$: an integral convex polytope with the vertex set $V(P) \subseteq N$,
- $w \in M$: a primitive vector,
- $F \subseteq H_{w,0}$: an integral convex polytope.

Assumption

For every $h \in \mathbb{Z}_{\leq -1}$, there exists a possibly-empty integral convex polytope $G_h \subseteq N_{\mathbb{R}}$ such that

$G_h \longleftrightarrow$ "$(P \cap H_{w,h}) - |h|F$"

$$V(P) \cap H_{w,h} \subseteq G_h + |h|F \subseteq P \cap H_{w,h}.$$

If this assumption holds, then we say that the combinatorial mutation $\mathrm{mut}_w(P, F)$ of P is well-defined.

Combinatorial mutations for lattice polytopes

Definition (Akhtar–Coates–Galkin–Kasprzyk 2012)

The **combinatorial mutation** $\mathrm{mut}_w(P, F)$ of P is defined as follows:

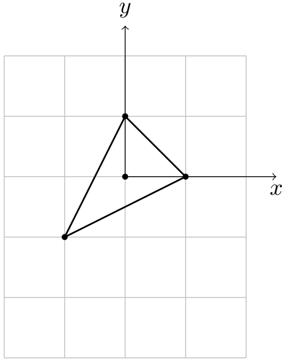
$$\mathrm{mut}_w(P, F) := \mathrm{conv}\left(\bigcup_{h \leq -1} G_h \cup \bigcup_{h \geq 0} ((P \cap H_{w,h}) + hF)\right) \subseteq N_{\mathbb{R}}.$$

Properties

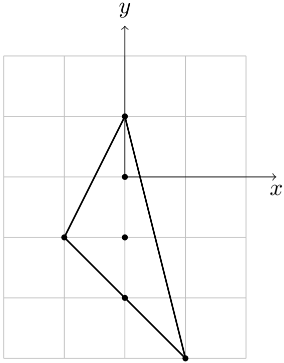
- $\mathrm{mut}_w(P, F)$ *is an integral convex polytope.*
- $\mathrm{mut}_w(P, F)$ *is independent of the choice of* $\{G_h\}_{h \leq -1}$.
- *If* $Q = \mathrm{mut}_w(P, F)$, *then we have* $P = \mathrm{mut}_{-w}(Q, F)$.

Example

For $w = (-1, -1) \in M$ and $F = \mathrm{conv}\{(0,0), (1,-1)\}$, we have



$\xmapsto{\mathrm{mut}_w(-,F)}$



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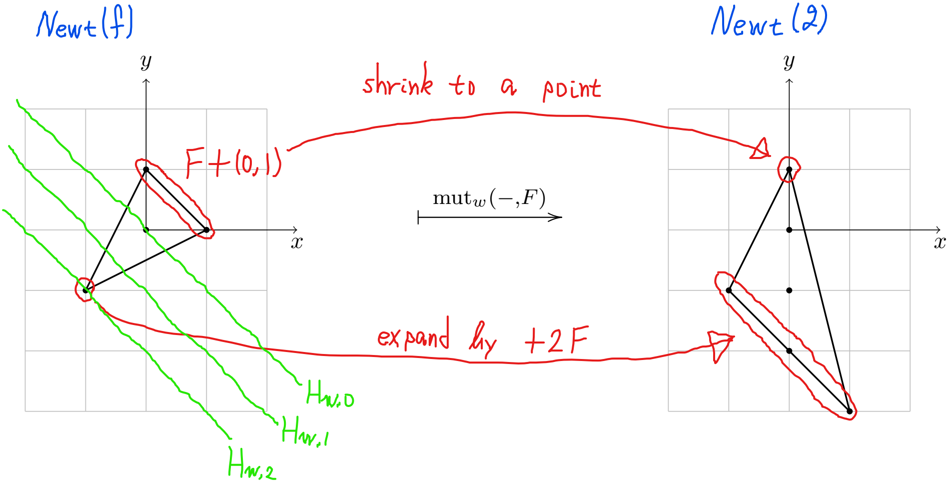


Main result

Theorem (F.–Higashitani preprint 2020)

For the anti-canonically polarized flag variety $(G/B, \mathcal{L}_{2\rho})$, the dual polytopes of the following are all related by sequences of combinatorial mutations up to unimodular equivalence: $G \curvearrowright V(2\rho)$

- *Berenstein–Littelmann–Zelevinsky's string polytopes,* $\left(\rho := \frac{1}{2} \sum_{\alpha\text{: pos. roots}} \alpha\right)$
- *Nakashima–Zelevinsky polytopes,*
- *Feigin–Fourier–Littelmann–Vinberg polytope (for type A_n or C_n).*

$$
\begin{tikzcd}
\{\text{BLZ's string polytopes}\} \arrow[rr, leftrightarrow, "\text{Kashiwara involution}"] \arrow[dr, hook, "\text{[F.–Oya preprint 2020]}"'] \arrow[dd, phantom, "\cup"] & & \{\text{NZ polytopes}\} \arrow[dl, hook', "\text{[F.–Oya preprint 2020]}"] \\
& \{g\text{-vector polytopes}\} \arrow[loop right, "\text{trop. cluster mutation}"] & \\
\text{Gelfand–Tsetlin polytope} \arrow[rr, leftrightarrow, "\text{Ardila–Bliem–Salazar's transfer map}"] & & \text{FFLV polytope}
\end{tikzcd}
$$

Flag varieties

- G: a connected, simply-connected semisimple algebraic group over $\mathbb{C}$,
- $B \subseteq G$: a Borel subgroup,
- G/B: the full flag variety,
- P_+: the set of dominant integral weights.

Theorem (Borel–Weil theory)

There exists a natural bijective map

$$P_+ \xrightarrow{\sim} \{\text{globally generated line bundles on } G/B\}, \quad \lambda \mapsto \mathcal{L}_\lambda,$$

such that $H^0(G/B, \mathcal{L}_\lambda)^*$ is the irreducible highest weight G-module with highest weight λ.

In this talk, we mainly focus on $\mathcal{L}_{2\rho}$ which is isomorphic to the anti-canonical bundle of G/B.

Newton–Okounkov bodies

- Fix a birational map from $(\mathbb{C}^\times)^m$ to G/B, which induces an isomorphism $\mathbb{C}(G/B) \simeq \mathbb{C}(t_1, \ldots, t_m)$.
- $\leq$: a total order on $\mathbb{Z}^m$, respecting the addition.
- $\tau \in H^0(G/B, \mathcal{L}_\lambda)$: a nonzero section.

The **lowest term valuation** $v_{\leq}^{\mathrm{low}} \colon \mathbb{C}(G/B) \setminus \{0\} \to \mathbb{Z}^m$ is defined by

$$v_{\leq}^{\mathrm{low}}(f/g) \coloneqq v_{\leq}^{\mathrm{low}}(f) - v_{\leq}^{\mathrm{low}}(g), \text{ and}$$
$$v_{\leq}^{\mathrm{low}}(f) \coloneqq (a_1, \ldots, a_m) \Leftrightarrow f = c t_1^{a_1} \cdots t_m^{a_m} + (\text{higher terms w.r.t. } \leq)$$

for $f, g \in \mathbb{C}[t_1, \ldots, t_m] \setminus \{0\}$, where $c \in \mathbb{C}^\times$.
We define a semigroup $S = S(G/B, \mathcal{L}_\lambda, v_{\leq}^{\mathrm{low}}, \tau) \subseteq \mathbb{Z}_{>0} \times \mathbb{Z}^m$ and a real closed convex cone $C = C(G/B, \mathcal{L}_\lambda, v_{\leq}^{\mathrm{low}}, \tau) \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}^m$ by

$$S \coloneqq \{(k, v_{\leq}^{\mathrm{low}}(\sigma/\tau^k)) \mid k \in \mathbb{Z}_{>0},\ \sigma \in H^0(G/B, \mathcal{L}_\lambda^{\otimes k}) \setminus \{0\}\},$$

C: the smallest real closed cone containing S.

Newton–Okounkov bodies

Definition (Lazarsfeld–Mustata 2009, Kaveh–Khovanskii 2012)

The **Newton–Okounkov body** $\Delta(G/B, \mathcal{L}_\lambda, v^{\mathrm{low}}_{\leq}, \tau) \subseteq \mathbb{R}^m$ of $(G/B, \mathcal{L}_\lambda)$ associated with $v^{\mathrm{low}}_{\leq}$ and τ is defined by

$$\Delta(G/B, \mathcal{L}_\lambda, v^{\mathrm{low}}_{\leq}, \tau) \coloneqq \{\boldsymbol{a} \in \mathbb{R}^m \mid (1, \boldsymbol{a}) \in C(G/B, \mathcal{L}_\lambda, v^{\mathrm{low}}_{\leq}, \tau)\}.$$

Example

The Newton–Okounkov bodies $\Delta(G/B, \mathcal{L}_\lambda, v^{\mathrm{low}}_{\leq}, \tau)$ of $(G/B, \mathcal{L}_\lambda)$ realize the following representation-theoretic polytopes:

- string polytopes (Kaveh 2015, F.–Oya 2017),
- Nakashima–Zelevinsky polytopes (F.–Naito 2017, F.–Oya 2017),
- FFLV polytopes (Feigin–Fourier–Littelmann 2017, Kiritchenko 2017).

Newton–Okounkov bodies

Assumption

The semigroup $S(G/B, \mathcal{L}_\lambda, v^{\mathrm{low}}_{\leq}, \tau)$ is finitely generated and saturated.

Theorem (Steinert preprint 2019)

If the assumption holds for $\lambda = 2\rho$, then $\Delta := \Delta(G/B, \mathcal{L}_{2\rho}, v^{\mathrm{low}}_{\leq}, \tau)$ contains exactly one lattice point $\boldsymbol{a}$ in its interior, and the dual polytope

$$\Delta^\vee := \{\boldsymbol{u} \in N_\mathbb{R} \mid \langle \boldsymbol{u}' - \boldsymbol{a}, \boldsymbol{u} \rangle \geq -1 \text{ for all } \boldsymbol{u}' \in \Delta\}$$

is an integral convex polytope, where $\Delta \subseteq \mathbb{R}^m =: M_\mathbb{R}$.

Aim

to relate dual polytopes $\Delta(G/B, \mathcal{L}_{2\rho}, v^{\mathrm{low}}_{\leq}, \tau)^\vee$ associated with different kinds of valuations $v^{\mathrm{low}}_{\leq}$ using combinatorial mutations.

Cluster varieties

Let us consider an $\mathcal{A}$-cluster variety

$$\mathcal{A} = \bigcup_{\mathbf{s}} \mathcal{A}_{\mathbf{s}} = \bigcup_{\mathbf{s}} \mathrm{Spec}(\mathbb{C}[A_{j;\mathbf{s}}^{\pm 1} \mid j \in \{1, \ldots, m\} = J_{\mathrm{uf}} \sqcup J_{\mathrm{fr}}]),$$

where $\mathbf{s}$ runs over the set of seeds which are mutually mutation equivalent, and the tori are glued via the following birational cluster mutations:

$$\mu_k^*(A_{i;\mathbf{s}'}) = \begin{cases} A_{i;\mathbf{s}} & (i \neq k), \\ A_{k;\mathbf{s}}^{-1}\left(\prod_{\varepsilon_{k,j}>0} A_{j;\mathbf{s}}^{\varepsilon_{k,j}} + \prod_{\varepsilon_{k,j}<0} A_{j;\mathbf{s}}^{-\varepsilon_{k,j}}\right) & (i = k) \end{cases}$$

if $\mathbf{s}' = \mu_k(\mathbf{s})$, where $\varepsilon = (\varepsilon_{i,j})_{i,j}$ is the exchange matrix of $\mathbf{s}$.

Definition (Berenstein–Fomin–Zelevinsky 2005)

The ring $\mathbb{C}[\mathcal{A}]$ of regular functions is called an **upper cluster algebra**.

Cluster structures on unipotent cells

Let U^- be the unipotent radical of the opposite Borel subgroup B^-, and

$$G/B = \bigsqcup_{w \in W} BwB/B$$

the Bruhat decomposition of G/B, where W is the Weyl group.

Definition

For $w \in W$, the **unipotent cell** U_w^- is defined by

$$U_w^- := BwB \cap U^- \subseteq G.$$

Theorem (Berenstein–Fomin–Zelevinsky 2005)

The coordinate ring $\mathbb{C}[U_w^-]$ admits an upper cluster algebra structure.

Cluster structures on unipotent cells

Each reduced word $\boldsymbol{i} = (i_1, \ldots, i_m)$ for w induces a seed $\mathbf{s}_{\boldsymbol{i}} = ((A_{j;\mathbf{s}_{\boldsymbol{i}}})_j, \varepsilon^{\boldsymbol{i}})$ for U_w^- given by

- $A_{j;\mathbf{s}_{\boldsymbol{i}}} \in \mathbb{C}[U_w^-]$ is the restriction of the generalized minor $\Delta_{s_{i_1} \cdots s_{i_j} \varpi_{i_j}, \varpi_{i_j}} \in \mathbb{C}[G]$ for $1 \leq j \leq m$;
- if we write $\varepsilon^{\boldsymbol{i}} = (\varepsilon_{s,t})_{s,t}$, then

$$\varepsilon_{s,t} = \begin{cases} 1 & \text{if } s^+ = t, \\ -1 & \text{if } s = t^+, \\ \langle \alpha_{i_s}, h_{i_t} \rangle & \text{if } s < t < s^+ < t^+, \\ -\langle \alpha_{i_s}, h_{i_t} \rangle & \text{if } t < s < t^+ < s^+, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$k^+ \coloneqq \min(\{k+1 \leq j \leq m \mid i_j = i_k\} \cup \{m+1\}).$$

Cluster structures on unipotent cells

Theorem (Kang–Kashiwara–Kim–Oh 2018, Qin preprint 2020)

For $w \in W$, the upper global basis $\mathbf{B}_w^{\mathrm{up}} \subseteq \mathbb{C}[U_w^-]$ is (the specialization at $q = 1$ of) a common triangular basis. In particular, the following hold.

(1) *For all $b \in \mathbf{B}_w^{\mathrm{up}}$ and $\mathbf{s}$, there uniquely exists $g_{\mathbf{s}}(b) = (g_1, \ldots, g_m) \in \mathbb{Z}^m$ (the* **extended g-vector** *of b) such that*

$$b \in A_{1;\mathbf{s}}^{g_1} \cdots A_{m;\mathbf{s}}^{g_m}\Big(1 + \sum_{0 \neq (a_j)_{j \in J_{\mathrm{uf}}} \in \mathbb{Z}_{\geq 0}^{J_{\mathrm{uf}}}} \mathbb{Z} \prod_{j \in J_{\mathrm{uf}}} \widehat{X}_j^{a_j}\Big),$$

where $\widehat{X}_j := A_{1;\mathbf{s}}^{\varepsilon_{j,1}} \cdots A_{m;\mathbf{s}}^{\varepsilon_{j,m}}$.

(2) *If $\mathbf{s}' = \mu_k(\mathbf{s})$, then $g_{\mathbf{s}'}(b) = \mu_k^T(g_{\mathbf{s}}(b))$ for all $b \in \mathbf{B}_w^{\mathrm{up}}$, where $\mu_k^T \colon \mathbb{R}^m \to \mathbb{R}^m,\ (g_1, \ldots, g_m) \mapsto (g_1', \ldots, g_m')$, is the* **tropicalized cluster mutation** *given by*

$$g_i' := \begin{cases} g_i + \max\{\varepsilon_{k,i}, 0\} g_k - \varepsilon_{k,i} \min\{g_k, 0\} & (i \neq k), \\ -g_k & (i = k). \end{cases}$$

g-vectors as higher rank valuations

Definition (Qin 2017)

For each seed $\mathbf{s} = ((A_{j;\mathbf{s}})_j, \varepsilon)$, define a partial order $\preceq_{\mathbf{s}}$ on $\mathbb{Z}^m$ by

$$g' \preceq_{\mathbf{s}} g \Leftrightarrow g' - g \in \sum_{j \in J_{\mathrm{uf}}} \mathbb{Z}_{\geq 0}(\varepsilon_{j,1}, \ldots, \varepsilon_{j,m}).$$

This $\preceq_{\mathbf{s}}$ is called the **dominance order** associated with $\mathbf{s}$.

Fix a total order $\leq_{\mathbf{s}}$ on $\mathbb{Z}^m$ refining the opposite dominance order $\preceq_{\mathbf{s}}^{\mathrm{op}}$.

Definition (F.–Oya preprint 2020)

For each seed $\mathbf{s}$, define a valuation $v_{\mathbf{s}}$ on $\mathbb{C}(U_w^-) = \mathbb{C}(\mathcal{A}) = \mathbb{C}(A_{1;\mathbf{s}}, \ldots, A_{m;\mathbf{s}})$ to be the lowest term valuation $v_{\leq_{\mathbf{s}}}^{\mathrm{low}}$.

Corollary

For all $b \in \mathbf{B}_w^{\mathrm{up}}$ and $\mathbf{s}$, the equality $v_{\mathbf{s}}(b) = g_{\mathbf{s}}(b)$ holds.

g-vector polytopes

Let $\tau_\lambda \in H^0(G/B, \mathcal{L}_\lambda)$ be a lowest weight vector. For the longest element $w_0 \in W$, the unipotent cell $U^-_{w_0}$ is an open subvariety of G/B.

Theorem (F.–Oya preprint 2020)

Let $\mathbf{s}$ *be a seed for* $U^-_{w_0}$, $\lambda \in P_+$, *and* $\boldsymbol{i}$ *a reduced word for* w_0.

(1) $\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{s}}, \tau_\lambda)$ *does not depend on the choice of a refinement* $\leq_{\mathbf{s}}$ *of the opposite dominance order* $\preceq^{\mathrm{op}}_{\mathbf{s}}$.

(2) $\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{s}}, \tau_\lambda)$ *is a rational convex polytope.*

(3) $S(G/B, \mathcal{L}_\lambda, v_{\mathbf{s}}, \tau_\lambda)$ *is finitely generated and saturated.*

(4) *If* $\mathbf{s}' = \mu_k(\mathbf{s})$, *then* $\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{s}'}, \tau_\lambda) = \mu_k^T(\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{s}}, \tau_\lambda))$.

(5) $\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{s}_{\boldsymbol{i}}}, \tau_\lambda)$ *is unimodularly equivalent to the string polytope* $\Delta_{\boldsymbol{i}}(\lambda)$ *by an explicit unimodular transformation.*

(6) *There is a seed* $\mathbf{s}^{\mathrm{mut}}_{\boldsymbol{i}}$ *such that* $\Delta(G/B, \mathcal{L}_\lambda, v_{\mathbf{s}^{\mathrm{mut}}_{\boldsymbol{i}}}, \tau_\lambda)$ *is unimodularly equivalent to the Nakashima–Zelevinsky polytope* $\widetilde{\Delta}_{\boldsymbol{i}}(\lambda)$.

g-vector polytopes

Corollary (F.–Higashitani preprint 2020)

The unique interior lattice point $\boldsymbol{a}_{\mathbf{s}} = (a_j)_{1 \leq j \leq m}$ *of* $\Delta(G/B, \mathcal{L}_{2\rho}, v_{\mathbf{s}}, \tau_{2\rho})$ *is given by*

$$a_j = \begin{cases} 0 & (\text{if } j \in J_{\mathrm{uf}}), \\ 1 & (\text{if } j \in J_{\mathrm{fr}}). \end{cases}$$

Corollary

Let $\boldsymbol{i} = (i_1, \ldots, i_m)$ *be a reduced word for* w_0. *Then the unique interior lattice point* $\boldsymbol{a}_{\boldsymbol{i}} = (a_j)_j$ *of the string polytope* $\Delta_{\boldsymbol{i}}(2\rho)$ *is given by*

$$a_j = \sum_{k \in J_{\mathrm{fr}};\ j \leq k} \langle s_{i_{j+1}} \cdots s_{i_k} \varpi_{i_k}, h_{i_j} \rangle.$$

Relation with combinatorial mutations

Theorem (F.–Higashitani preprint 2020)

(1) *The dual polytopes* $\Delta(G/B, \mathcal{L}_{2\rho}, v_{\mathbf{s}}, \tau_{2\rho})^\vee$ *for seeds* $\mathbf{s}$ *are all related by sequences of combinatorial mutations up to unimodular equivalence.*

(2) *In particular, the dual polytopes* $\Delta_{\boldsymbol{i}}(2\rho)^\vee$ *and* $\widetilde{\Delta}_{\boldsymbol{i}}(2\rho)^\vee$ *of string polytopes and Nakashima–Zelevinsky polytopes for reduced words* $\boldsymbol{i}$ *are all related by sequences of combinatorial mutations up to unimodular equivalence.*

Remark

More strongly, we can realize the tropicalized cluster mutation μ_k^T as a dual operation of a combinatorial mutation up to unimodular equivalence. This dual operation was introduced by Akhtar–Coates–Galkin–Kasprzyk (2012).

Marked poset polytopes

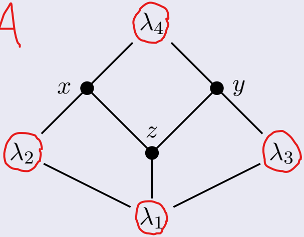
- $\widetilde{\Pi}$: a poset equipped with a partial order $\preceq$,
- $A \subseteq \widetilde{\Pi}$: a subset of $\widetilde{\Pi}$ containing all minimal elements and maximal elements in $\widetilde{\Pi}$,
- $\lambda = (\lambda_a)_{a \in A} \in \mathbb{R}^A$ such that $\lambda_a \leq \lambda_b$ whenever $a \preceq b$ in $\widetilde{\Pi}$, which is called a **marking**.

The triple $(\widetilde{\Pi}, A, \lambda)$ is called a **marked poset**.

Example

$\widetilde{\Pi} =$

A



$\lambda = (\lambda_1, \ldots, \lambda_4)$

Marked poset polytopes

Definition (Ardila–Bliem–Salazar 2011)

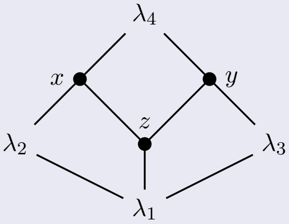
$$\mathcal{O}(\widetilde{\Pi}, A, \lambda) \coloneqq \{(x_p)_{p \in \widetilde{\Pi} \setminus A} \in \mathbb{R}^{\widetilde{\Pi} \setminus A} \mid$$
$$x_p \leq x_q \text{ if } p \prec q,\ \lambda_a \leq x_p \text{ if } a \prec p,\ x_p \leq \lambda_a \text{ if } p \prec a\},$$
$$\mathcal{C}(\widetilde{\Pi}, A, \lambda) \coloneqq \{(x_p)_{p \in \widetilde{\Pi} \setminus A} \in \mathbb{R}_{\geq 0}^{\widetilde{\Pi} \setminus A} \mid$$
$$\sum_{i=1}^{k} x_{p_i} \leq \lambda_b - \lambda_a \text{ if } a \prec p_1 \prec \cdots \prec p_k \prec b\}.$$

The polytope $\mathcal{O}(\widetilde{\Pi}, A, \lambda)$ is called the **marked order polytope**, and $\mathcal{C}(\widetilde{\Pi}, A, \lambda)$ is called the **marked chain polytope**.

If $\lambda \in \mathbb{Z}^A$, then these polytopes are integral convex polytopes.

Marked poset polytopes

Example



$$\mathcal{O}(\widetilde{\Pi}, A, \lambda) = \{(x, y, z) \in \mathbb{R}^3 \mid \lambda_2 \le x \le \lambda_4,\ \lambda_1 \le z \le \min\{x, y\},\ \lambda_3 \le y \le \lambda_4\},$$

$$\mathcal{C}(\widetilde{\Pi}, A, \lambda) = \{(x, y, z) \in \mathbb{R}^3_{\ge 0} \mid x \le \lambda_4 - \lambda_2,\ x + z \le \lambda_4 - \lambda_1,\ y + z \le \lambda_4 - \lambda_1,\ y \le \lambda_4 - \lambda_3\}.$$

Transfer maps and combinatorial mutations

Define a piecewise-affine map $\phi\colon \mathbb{R}^{\widetilde{\Pi}\setminus A} \to \mathbb{R}^{\widetilde{\Pi}\setminus A}$, $(x_p)_p \mapsto (x'_p)_p$, by

$$x'_p \coloneqq \min(\{x_p - x_{p'} \mid p' \lessdot p,\ p' \in \widetilde{\Pi} \setminus A\} \cup \{x_p - \lambda_{p'} \mid p' \lessdot p,\ p' \in A\})$$

for $p \in \widetilde{\Pi} \setminus A$, where $q \lessdot p$ means that $q \prec p$ and that there is no $q' \in \widetilde{\Pi} \setminus \{p, q\}$ with $q \prec q' \prec p$.

Theorem (Ardila–Bliem–Salazar 2011)

The map ϕ gives a piecewise-affine bijection from $\mathcal{O}(\widetilde{\Pi}, A, \lambda)$ to $\mathcal{C}(\widetilde{\Pi}, A, \lambda)$, which is called a **transfer map**.

Remark

For a usual poset Π, the **order polytope** $\mathcal{O}(\Pi)$, the **chain polytope** $\mathcal{C}(\Pi)$, and the **transfer map** ϕ between them were originally introduced by Stanley (1986).

Transfer maps and combinatorial mutations

Assumption

The poset $\widetilde{\Pi}$ is **pure**, *that is, all maximal chains in $\widetilde{\Pi}$ have the same length. In particular, the* **rank function** $r\colon \widetilde{\Pi} \to \mathbb{Z}_{\geq 0}$ *is well-defined, where $r(p)$ is the length of a chain starting from a minimal element in $\widetilde{\Pi}$ and ending at p.*

Let λ^r denote the marking of $(\widetilde{\Pi}, A)$ given by

$$(\lambda^r)_a := r(a) \quad \text{for} \quad a \in A.$$

Then $\mathcal{O}(\widetilde{\Pi}, A, \lambda^r)$ (resp., $\mathcal{C}(\widetilde{\Pi}, A, \lambda^r)$) has a unique interior lattice point.

Theorem (F.–Higashitani preprint 2020)

The dual polytopes $\mathcal{O}(\widetilde{\Pi}, A, \lambda^r)^\vee$ and $\mathcal{C}(\widetilde{\Pi}, A, \lambda^r)^\vee$ are related by a sequence of combinatorial mutations.

Construction of the combinatorial mutation sequence

Theorem (F.–Higashitani preprint 2020)

The dual polytopes $\mathcal{O}(\widetilde{\Pi}, A, \lambda^r)^\vee$ and $\mathcal{C}(\widetilde{\Pi}, A, \lambda^r)^\vee$ are related by a sequence of combinatorial mutations.

For each $p \in \widetilde{\Pi} \setminus A$, set

- $w_p := -\boldsymbol{e}_p$,
- $F_p := \mathrm{conv}(\{-\boldsymbol{e}_{p'} \mid p' \lessdot p,\ p' \in \widetilde{\Pi} \setminus A\} \cup \{\boldsymbol{0} \mid p' \lessdot p,\ p' \in A\})$.

We write $\widetilde{\Pi} \setminus A = \{p_1, \ldots, p_N\}$, where $p_i \prec p_j$ only if $i > j$. Then

$$\mathcal{C}(\widetilde{\Pi}, A, \lambda^r)^\vee = \mathrm{mut}_{w_{p_N}}(-, F_{p_N}) \circ \cdots \circ \mathrm{mut}_{w_{p_1}}(-, F_{p_1})(\mathcal{O}(\widetilde{\Pi}, A, \lambda^r)^\vee).$$

Remark

More strongly, the transfer map ϕ coincides with the dual operation of $\mathrm{mut}_{w_{p_N}}(-, F_{p_N}) \circ \cdots \circ \mathrm{mut}_{w_{p_1}}(-, F_{p_1})$ up to translations.

Gelfand–Tsetlin polytopes and FFLV polytopes

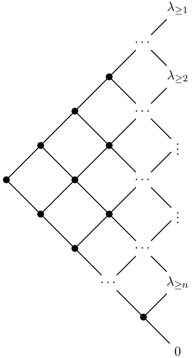


Figure: The marked poset in type A_n, where $\lambda_{\geq k} := \sum_{l \geq k} \langle \lambda, h_l \rangle$ for $\lambda \in P_+$ and $1 \leq k \leq n$.

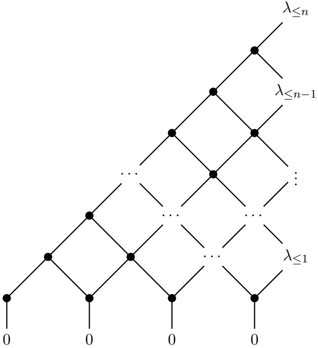


Figure: The marked poset in type C_n, where $\lambda_{\leq k} := \sum_{l \leq k} \langle \lambda, h_l \rangle$ for $\lambda \in P_+$ and $1 \leq k \leq n$.

Gelfand–Tsetlin polytopes and FFLV polytopes

Let $(\widetilde{\Pi}, A, \lambda)$ be one of the marked posets given in the previous slide.

- The marked order polytope $\mathcal{O}(\widetilde{\Pi}, A, \lambda)$ coincides with the Gelfand–Tsetlin polytope $GT(\lambda)$.
- The marked chain polytope $\mathcal{C}(\widetilde{\Pi}, A, \lambda)$ coincides with the FFLV polytope $FFLV(\lambda)$.
- The weight $\lambda = 2\rho$ corresponds to the marking λ^r given by the rank function r.

Corollary

Let G be of type A_n or C_n. Then the dual polytopes $GT(2\rho)^\vee$ and $FFLV(2\rho)^\vee$ are related by a sequence of combinatorial mutations.

Thank you for your attention!