Tame elements of the Grothendieck groups of special biserial algebras

joint work with Laurent Demonet, Osamu Iyama

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Setting

- K: an algebraically closed field.
- A: a finite dimensional K-algebra.
- $\operatorname{mod} A$: the cat. of fin. dim. A-modules.
- $\operatorname{proj} A$: the cat. of fin. gen. proj. A-modules.
- $|\bullet|$: the number of iso. of indec. direct summands.
- $n := |A|$.
- $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$: the homotopy category of $\operatorname{proj} A$.
- $K_0(C)$: the Grothendieck group of a category C.
- $K_0(\operatorname{proj} A)$ has a $\mathbb{Z}$-basis $([P(i)])_{i=1}^n$ of all the indec. proj.

Presentation spaces

Definition

Let $\theta \in K_0(\operatorname{proj} A)$. We set

(1) $P_0^\theta, P_1^\theta \in \operatorname{proj} A$ so that
$\theta = [P_0^\theta] - [P_1^\theta]$ and $\operatorname{add} P_0^\theta \cap \operatorname{add} P_1^\theta = \{0\}$.

(2) $\operatorname{PHom}_A(\theta) := \operatorname{Hom}_A(P_1^\theta, P_0^\theta)$: the presentation space of θ.

(3) For $f \in \operatorname{PHom}_A(\theta)$, $P_f := (P_1^\theta \xrightarrow{f} P_0^\theta) \in \mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$.

- We consider $\operatorname{PHom}_A(\theta)$ as an irreducible algebraic variety.
- "For each general $f \in \operatorname{PHom}_A(\theta)$ ..." means
"$\exists O \subset \operatorname{PHom}_A(\theta)$: open & dense, $\forall f \in O$ ...".

Direct sums in $K_0(\operatorname{proj} A)$

Definition [Derksen–Fei]

Let $\theta, \theta_1, \theta_2 \in K_0(\operatorname{proj} A)$.

(1) $\theta_1 \oplus \theta_2 :\Longleftrightarrow$ each general $f \in \mathrm{PHom}_A(\theta_1 + \theta_2)$ admits $f_i \in \mathrm{PHom}_A(\theta_i)$ such that $P_f \cong P_{f_1} \oplus P_{f_2}$ in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$.

- We sometimes write $\theta_1 + \theta_2 = \theta_1 \oplus \theta_2$.

(2) θ: indecomposable

$:\Longleftrightarrow \ \theta \neq 0$ and $(\theta = \theta_1 \oplus \theta_2 \Rightarrow \theta_1 = 0$ or $\theta_2 = 0)$

$\Longleftrightarrow \ P_f$ is indec. in $\mathsf{K}^{\mathrm{b}}(\operatorname{proj} A)$ for each general $f \in \mathrm{PHom}_A(\theta)$.

Canonical decompositions

Theorem [DF, Plamondon]

Let $\theta, \theta_1, \theta_2 \in K_0(\operatorname{proj} A)$.

(1) $\theta_1 \oplus \theta_2 \iff \exists f_i, f_i' \in \mathrm{PHom}_A(\theta_i)$ $(i = 1, 2)$,

$$\mathrm{Hom}_{\mathsf{K^b}(\operatorname{proj} A)}(P_{f_1}, P_{f_2}[1]) = 0, \quad \mathrm{Hom}_{\mathsf{K^b}(\operatorname{proj} A)}(P_{f_2'}, P_{f_1'}[1]) = 0.$$

In this case, both conditions hold for each general pair (f_1, f_2).

(2) There exists a canonical decomposition of θ: i.e.

$$\theta = \theta_1 \oplus \theta_2 \oplus \cdots \oplus \theta_m$$

with each θ_i indecomposable.
It is unique up to reordering.

Rigid and tame elements

Definition [DF]

Let $\theta \in K_0(\mathrm{proj}\, A)$: indecomposable.

(1) θ: rigid $:\Longleftrightarrow$ $\exists f \in \mathrm{PHom}_A(\theta)$, P_f is a 2-term presilting; i.e.

$$\mathrm{Hom}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A)}(P_f, P_f[1]) = 0.$$

- Then, $\forall$general $f' \in \mathrm{PHom}_A(\theta)$, $P_{f'} \cong P_f$ in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A)$ [P].
- $\{$rigid elements$\} \leftrightarrow \{$indec. 2-term presilting obj. in $\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\, A)\}$.

(2) θ: tame $:\Longleftrightarrow$ θ is not rigid, but $\theta \oplus \theta$ holds.

Example

If $A = K(1 \rightrightarrows 2)$, $\theta := [P(1)] - [P(2)]$ is tame.

Tame algebras are E-tame

Definition

A: E-tame $:\Longleftrightarrow$ any indec. $\theta \in K_0(\mathrm{proj}\, A)$ is rigid or tame.

Theorem [Geiss–Labardini-Fragoso–Schröer] (cf. [P–Yurikusa])

Let A be a representation-tame algebra, then A is E-tame,
and if $\theta \in K_0(\mathrm{proj}\, A)$ is tame, then
for each general $f \in \mathrm{PHom}_A(\theta)$, they are isomorphic bricks:

- $H^0(P_f) = \mathrm{Coker}\, f$,
- $H^{-1}(\nu P_f) = \mathrm{Ker}\, \nu f \cong \tau\, \mathrm{Coker}\, f$.
  - X: brick $:\Longleftrightarrow$ $\mathrm{End}_A(X) \cong K$,
  - ν is the Nakayama functor,
  - τ is the Auslander–Reiten translation.

They use properties of tame algebras by [Crawley-Boevey].

Today's topic and strategy

In the rest, we assume that A be a representation-tame algebra.

Problem

We want to know

- whether $K_0(\mathrm{proj}\,A)$ has some tame element;
- where tame elements lie in $K_0(\mathrm{proj}\,A)$.

In this talk, I will give the answer for special biserial algebras.
The strategy is:

- A tame element $\theta \in K_0(\mathrm{proj}\,A)$ cannot have any rigid direct summand.
- How can we know whether a rigid element $\sigma \in K_0(\mathrm{proj}\,A)$ is a direct summand of a given $\theta \in K_0(\mathrm{proj}\,A)$ or not?
- We can use the Euler form and numerical torsion pairs.

Euler form

- The Euler form $\langle \bullet, \bullet \rangle \colon K_0(\operatorname{proj} A) \times K_0(\operatorname{mod} A) \to \mathbb{Z}$:

$$\langle P(i), S(j) \rangle = \delta_{i,j}.$$

  - $K_0(\operatorname{proj} A)$ has a $\mathbb{Z}$-basis $(P(i))_{i=1}^n$ of all the indec. proj.
  - $K_0(\operatorname{mod} A)$ has a $\mathbb{Z}$-basis $(S(i))_{i=1}^n$ of all the simple with $P(i) \twoheadrightarrow S(i)$.
- Each $\theta \in K_0(\operatorname{proj} A)$ gives a linear map

$$\theta := \langle \theta, \bullet \rangle \colon K_0(\operatorname{mod} A) \to \mathbb{Z}.$$

- We can extend them to $K_0(C)_{\mathbb{R}} := K_0(C) \otimes_{\mathbb{Z}} \mathbb{R}$.

Numerical torsion pairs

Definition [Baumann–Kamnitzer–Tingley]

For $\theta \in K_0(\mathrm{proj}\, A)_{\mathbb{R}}$, we define torsion pairs $(\overline{\mathcal{T}}_\theta, \mathcal{F}_\theta), (\mathcal{T}_\theta, \overline{\mathcal{F}}_\theta)$ by

$$\begin{aligned}
\overline{\mathcal{T}}_\theta &:= \{X \in \mathrm{mod}\, A \mid \forall Y \twoheadleftarrow X,\ \theta(Y) \geq 0\}, \\
\mathcal{T}_\theta &:= \{X \in \mathrm{mod}\, A \mid 0 \neq \forall Y \twoheadleftarrow X,\ \theta(Y) > 0\}, \\
\overline{\mathcal{F}}_\theta &:= \{X \in \mathrm{mod}\, A \mid \forall Y \subset X,\ \theta(Y) \leq 0\}, \\
\mathcal{F}_\theta &:= \{X \in \mathrm{mod}\, A \mid 0 \neq \forall Y \subset X,\ \theta(Y) < 0\}.
\end{aligned}$$

Definition [King]

For $\theta \in K_0(\mathrm{proj}\, A)_{\mathbb{R}}$, we set the θ-semistable subcat. by

$$\mathcal{W}_\theta := \overline{\mathcal{T}}_\theta \cap \overline{\mathcal{F}}_\theta \quad \text{(a wide subcategory of } \mathrm{mod}\, A\text{)}.$$

Rigid/tame elements and torsion pairs

- For $\theta \in K_0(\mathsf{proj}\, A)$ and $f \in \mathsf{PHom}_A(\theta)$, set $C_f := \mathsf{Coker}\, f$ and $K_f := \mathsf{Ker}\, \nu f$.
  - $\mathsf{Hom}_{\mathsf{K}^{\mathsf{b}}(\mathsf{proj}\, A)}(P_{f_1}, P_{f_2}[1]) \cong D\, \mathsf{Hom}_A(C_{f_2}, K_{f_1})$.
  - $\theta(X) = \dim_K \mathsf{Hom}_A(C_f, X) - \dim_K \mathsf{Hom}_A(X, K_f)$.
- If $\sigma = [U] \in K_0(\mathsf{proj}\, A)$ with an indec. 2-term presilting obj. U, then set $C_\sigma := H^0(U)$ and $K_\sigma := H^{-1}(\nu U)$.

Since A is a representation-tame algebra, we have the following.

Proposition / (2) by [Y, Brüstle–Smith–Treffinger]

(1) For any $\theta \in K_0(\mathsf{proj}\, A)$, $C_f \in \overline{\mathcal{T}}_\theta$ and $K_f \in \overline{\mathcal{F}}_\theta$ hold for each general $f \in \mathsf{PHom}_A(\theta)$.

(2) For any rigid $\sigma \in K_0(\mathsf{proj}\, A)$, $C_\sigma \in \mathcal{T}_\sigma$ and $K_\sigma \in \mathcal{F}_\sigma$.

(3) For any tame $\eta \in K_0(\mathsf{proj}\, A)$, $C_f \cong K_f$ is a simple object in $\mathcal{W}_\eta$ for each general $f \in \mathsf{PHom}_A(\theta)$.

Direct sums and numerical torsion pairs

Proposition [ADI]

Let $\sigma \in K_0(\operatorname{proj} A)$ be rigid and $\theta, \theta_1, \theta_2 \in K_0(\operatorname{proj} A)$.

(1) $\theta_1 \oplus \theta_2 \iff \mathcal{T}_{\theta_1} \subset \overline{\mathcal{T}}_{\theta_2}, \mathcal{F}_{\theta_1} \subset \overline{\mathcal{F}}_{\theta_2}$
$\iff \exists f_1 \in \operatorname{PHom}_A(\theta_1), C_{f_1} \in \overline{\mathcal{T}}_{\theta_2}, K_{f_1} \in \overline{\mathcal{F}}_{\theta_2}$.

(2) $\theta = \sigma \oplus (\theta - \sigma) \iff C_\sigma \in \mathcal{T}_\theta, K_\sigma \in \mathcal{F}_\theta$.

Definition

For each rigid $\sigma \in K_0(\operatorname{proj} A)$, we set

$$N_\sigma := \{\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}} \mid C_\sigma \in \mathcal{T}_\theta, K_\sigma \in \mathcal{F}_\theta\}.$$

Lemma

For each rigid $\sigma \in K_0(\operatorname{proj} A)$, N_σ is an open neighborhood of $\mathbb{R}_{>0}\sigma$.

“Purely non-rigid” area

Definition

We set a closed subset

$$R_0 = R_0(A) := K_0(\text{proj}\, A)_{\mathbb{R}} \setminus \bigcup_{\sigma \in K_0(\text{proj}\, A)\text{: rigid}} N_\sigma .$$

Lemma

(1) For $\theta \in K_0(\text{proj}\, A)$,
$\theta \in R_0 \iff$ all indec. direct summands of θ are tame.

(2) $R_0 \cap K_0(\text{proj}\, A) \neq \{0\} \iff \exists \theta \in K_0(\text{proj}\, A)$: tame.

Remark [Zimmermann–Zvonareva, A]

$R_0 \neq \{0\} \iff A$ is τ-tilting infinite (infin. many rigid elements).

- We do not know whether $R_0 \neq \{0\} \Longrightarrow R_0 \cap K_0(\text{proj}\, A) \neq \{0\}$.

Our result

Result

We determined R_0 for all special biserial algebras $A = KQ/I$.

- The main tool is maximal nonzero paths p in A.
  - We focus on whether the string module $M(p)$ for such p belongs to $\overline{\mathcal{T}}_\theta$ or $\overline{\mathcal{F}}_\theta$.
- R_0 is a union of finitely many rational polyhedral cones for special biserial algebras.
  - Thus, $R_0 \neq \{0\} \Longrightarrow R_0 \cap K_0(\operatorname{proj} A) \neq \{0\}$ holds.

Special biserial algebras

Definition

We say that $A = KQ/I$ is a special biserial algebra
with Q a finite quiver and $I \subset KQ$ an admissible ideal if

(a) I is generated by a set $X \subset KQ$ such that
each $x \in X$ is a path in Q or $p - q$ with $p \neq q$ paths in Q,

(b) $\forall i \in Q_0$, $\exists_{\leq 2}$ arrows from i in Q,

(c) $\forall i \in Q_0$, $\exists_{\leq 2}$ arrows to i in Q,

(d) $\alpha \in Q_1$ to $i \in Q_0$, $\beta \neq \gamma \in Q_1$ from $i \Rightarrow \alpha\beta \in I$ or $\alpha\gamma \in I$,

(e) $\alpha \in Q_1$ from $i \in Q_0$, $\beta \neq \gamma \in Q_1$ to $i \Rightarrow \beta\alpha \in I$ or $\gamma\alpha \in I$.

Remark following from [Eisele–Janssens–Raedschelders, A]

On (a), if paths $p \neq q$ in Q satisfy $p = q \neq 0$ in A,
then $R_0(A/\langle p \rangle) = R_0(A)$, since $p \in Z(A) \cap \operatorname{rad} A$.

String modules and band modules

In the rest, let $A = KQ/I$ be a string algebra
(i.e. a special biserial algebra with I generated by paths).

Definition

Let s, b be a walk in Q.

- s: string in A $:\Longleftrightarrow$ s does not contain $\alpha^{\pm 1}\alpha^{\mp 1}$ or $(\text{paths in } I)^{\pm 1}$.
- b: band in A $:\Longleftrightarrow$ b^2 is a string in A.

We can construct a string module $M(s)$ for each string s and
a band module $M(b, \lambda)$ for each band b and $\lambda \in K^{\times}$.

Proposition [Crawley-Boevey, Krause]

Every indec. A-module is a string module or a band module.

E-tameness of special biserial algebras

Lemma

(a) If $\sigma \in K_0(\operatorname{proj} A)$ is rigid, then C_σ, K_σ are string modules or 0, and

$$C_\sigma \in \mathcal{T}_\sigma, \quad K_\sigma \in \mathcal{F}_\sigma.$$

(b) If $\eta \in K_0(\operatorname{proj} A)$ is tame, then a band b_η exists such that

$$C_f \cong K_f \cong M(b_\eta, \lambda_f)\text{: a simple object in } \mathcal{W}_\eta$$

for each general $f \in \operatorname{PHom}_A(\eta)$.
Here, $\lambda_f \in K^\times$ depends on f, but b_η does not.

Symbols for maximal paths

Definition

(1) $\mathrm{MP}_*(A) := \{p\text{: length} \geq 1 \mid p \neq 0, p\alpha = 0 \text{ in } A\ (\forall \alpha \in Q_1)\}$.

(2) $\mathrm{MP}^*(A) := \{p\text{: length} \geq 1 \mid p \neq 0, \alpha p = 0 \text{ in } A\ (\forall \alpha \in Q_1)\}$.

(3) $\mathrm{MP}(A) := \mathrm{MP}_*(A) \cap \mathrm{MP}^*(A)$.

We write ε_i for the path of length 0 for each $i \in Q_0$.

Definition

(1) $\overline{\mathrm{MP}}_*(A) := \mathrm{MP}_*(A) \cup \{\varepsilon_i \mid \exists_{\leq 1} \text{ arrow from } i\}$.

(2) $\overline{\mathrm{MP}}^*(A) := \mathrm{MP}^*(A) \cup \{\varepsilon_i \mid \exists_{\leq 1} \text{ arrow to } i\}$.

(3) $\overline{\mathrm{MP}}(A) := \overline{\mathrm{MP}}_*(A) \cap \overline{\mathrm{MP}}^*(A)$.

We introduce a subclass of special biserial algebras
for which $\overline{\mathrm{MP}}(A)$ satisfies nice properties.

Truncated gentle algebras

Definition

A special biserial algebra $A = KQ/I$ is a truncated gentle algebra if $\exists I_1, I_2 \subset KQ$: ideals satisfying $I = I_1 + I_2$ and

(i) I_1 is generated by a set J_1 of paths of length 2 such that

- $\alpha \in Q_1$ to $i \in Q_0$, $\beta \neq \gamma \in Q_1$ from $i \Rightarrow \alpha\beta \notin J_1$ or $\alpha\gamma \notin J_1$.
- $\alpha \in Q_1$ from $i \in Q_0$, $\beta \neq \gamma \in Q_1$ to $i \Rightarrow \beta\alpha \notin J_1$ or $\gamma\alpha \notin J_1$.

(ii) I_2 is generated by a set J_2 of cycles of length ≥ 3 such that

- $\forall c \in J_2$, $c^2 \notin I_1$;
- let $\alpha \in Q_1$ appear two cycles c, c' in Q and $c \in J_2$, then $c' \in J_2 \iff c'$ is a cyclic permutation of c.

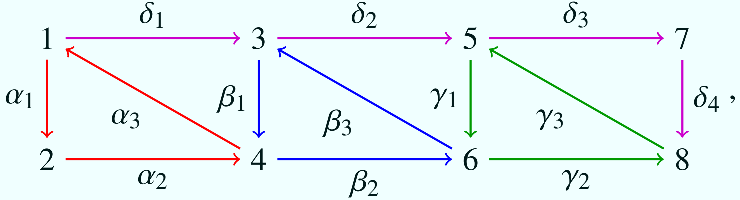
Example

- Finite dimensional gentle algebras ($I_2 = 0$).
- $A/\operatorname{soc} A$ for Brauer graph algebras A with enough multiplicities.

Example of truncated gentle algebras

$A = KQ/(I_1 + I_2)$ is a truncated gentle algebra, where

$Q :=$



$$I_1 := \langle \alpha_3\delta_1, \delta_1\beta_1, \beta_3\delta_2, \alpha_2\beta_2, \beta_1\alpha_3, \delta_2\gamma_1, \gamma_3\delta_3, \beta_2\gamma_2, \gamma_1\beta_3, \delta_4\gamma_3 \rangle,$$
$$I_2 := \langle \alpha_i\alpha_{i+1}\alpha_{i+2}, \beta_i\beta_{i+1}\beta_{i+2}, \gamma_i\gamma_{i+1}\gamma_{i+2} \mid i \in \{1,2,3\} \rangle.$$

We have

$$\mathsf{MP}(A) = \{\alpha_i\alpha_{i+1}, \beta_i\beta_{i+1}, \gamma_i\gamma_{i+1}, \delta_1\delta_2\delta_3\delta_4 \mid i \in \{1,2,3\}\},$$
$$\overline{\mathsf{MP}}(A) = \mathsf{MP}(A) \cup \{\varepsilon_2, \varepsilon_7\}.$$

Classification of maximal paths

Proposition

Let A be a truncated gentle algebra.
$\mathsf{MP}(A)$ consists of (i), (ii), and $\overline{\mathsf{MP}}(A)$ consists of (i), (ii), (iii).
In (i) and (ii), α and β denote the first and the last arrows of p.

(i) The paths p of length ≥ 1 satisfying
$p \neq 0$ in A and $\forall \gamma \in Q_1$, $\beta\gamma = \gamma\alpha = 0$ in A.

(ii) The paths p of length ≥ 2 satisfying
$p \neq 0$ in A and $\exists \gamma \in Q_1$ s.t

- $c_p := p\gamma$ and $c^p := \gamma p$ are cycles in Q,
- $\beta\gamma \neq 0$, $\gamma\alpha \neq 0$, $p\gamma = 0$ and $\gamma p = 0$ in A.

(iii) The paths ε_i for $i \in Q_0$ satisfying
$\exists_{\leq 1}$ arrow from i and $\exists_{\leq 1}$ arrow to i.

Main Theorem

Theorem [ADI]

Let A be a truncated gentle algebra. For $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$, TFAE.

(a) $\theta \in R_0$.

(b) $\forall p \in \overline{\mathsf{MP}}_*(A)$, $M(p) \notin \mathcal{T}_\theta$, and $\forall p \in \overline{\mathsf{MP}}^*(A)$, $M(p) \notin \mathcal{F}_\theta$.

(c) $\forall p \in \overline{\mathsf{MP}}(A)$,
if p is of type (ii), then $\theta(M(p)) = 0$; otherwise, $M(p) \in \mathcal{W}_\theta$.

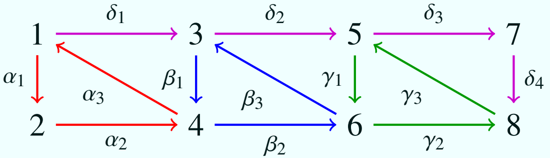
In particular, R_0 is a rational polyhedral cone.

Remark

(1) If $\theta \in K_0(\operatorname{proj} A)$ is indivisible and on a ray (1-dim. face) of R_0, then θ is tame.

(2) In general, R_0 itself does not give all information on tame elements.

Example of Theorem

$Q :=$ 

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$\overline{\mathsf{MP}}(A) = \{\alpha_i\alpha_{i+1}, \beta_i\beta_{i+1}, \gamma_i\gamma_{i+1}, \delta_1\delta_2\delta_3\delta_4, \varepsilon_2, \varepsilon_7 \mid i \in \{1,2,3\}\}$.

By our theorem, R_0 consists of $\theta = \sum_{i=1}^{n} a_i[P(i)]$ ($a_i \in \mathbb{R}$) satisfying

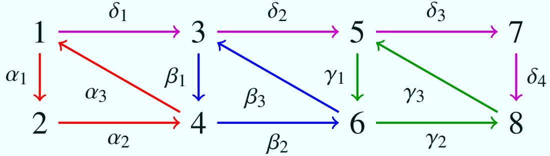
- $a_1 + a_2 + a_4 = 0$, $a_3 + a_4 + a_6 = 0$, $a_5 + a_6 + a_8 = 0$,
- $a_1 + a_3 + a_5 + a_7 + a_8 = 0$,
  $a_3 + a_5 + a_7 + a_8, a_5 + a_7 + a_8, a_7 + a_8, a_8 \leq 0$,
- $a_2 = 0$, $a_7 = 0$.

Thus, $R_0 = \{x\eta_1 + y\eta_2 \mid x, y \in \mathbb{R}_{\geq 0}\}$, where

$$\eta_1 := [P(1)] - [P(4)] - [P(5)] + [P(6)], \quad \eta_2 := [P(5)] - [P(8)].$$

Example of Theorem

$Q :=$



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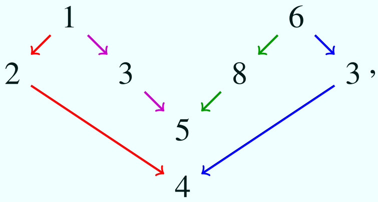
$R_0 = \{x\eta_1 + y\eta_2 \mid x, y \in \mathbb{R}_{\geq 0}\}$, where

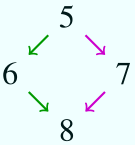
$$\eta_1 := [P(1)] - [P(4)] - [P(5)] + [P(6)], \quad \eta_2 := [P(5)] - [P(8)].$$

η_1, η_2 are tame, and the corresponding bands are

$b_{\eta_1} =$



, $b_{\eta_2} =$



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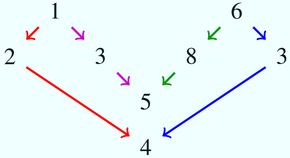
Since $S(5) \in \mathcal{F}_{\eta_1} \cap \mathcal{T}_{\eta_2}$, $\eta_1 \oplus \eta_2$ does not hold.

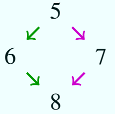
Example of Theorem

η_1, η_2 are tame, and the corresponding bands are

$b_{\eta_1} =$



, $b_{\eta_2} =$



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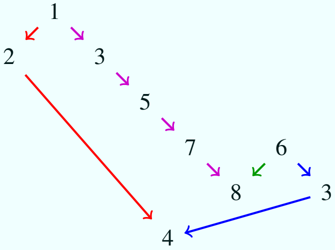
Actually, $\eta_3 := \eta_1 + \eta_2 = [P(1)] - [P(4)] + [P(6)] - [P(8)]$ is also tame:

$b_{\eta_3} =$



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Since $\eta_1 \oplus \eta_3$ and $\eta_2 \oplus \eta_3$ in this case, no other tame element exists.

Proof of (c) $\Longrightarrow$ (b)

Theorem [ADI]

Let A be a truncated gentle algebra. For $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$, TFAE.

(a) $\theta \in R_0$.

(b) $\forall p \in \overline{\mathsf{MP}}_*(A)$, $M(p) \notin \mathcal{T}_\theta$, and $\forall p \in \overline{\mathsf{MP}}^*(A)$, $M(p) \notin \mathcal{F}_\theta$.

(c) $\forall p \in \overline{\mathsf{MP}}(A)$,
if p is of type (ii), then $\theta(M(p)) = 0$; otherwise, $M(p) \in \mathcal{W}_\theta$.

Proof of (c) $\Rightarrow$ (b)

Let $p \in \overline{\mathsf{MP}}_*(A)$.

- If $p \in \overline{\mathsf{MP}}(A)$ and is of type (ii), then $\theta(M(p)) = 0$ gives $M(p) \notin \mathcal{T}_\theta$.
- Otherwise, $\exists q$: path, $qp \in \overline{\mathsf{MP}}(A)$,
so $M(qp) \in \mathcal{W}_\theta$, which implies $M(p) \notin \mathcal{T}_\theta$.

Proof of (b) $\Rightarrow$ (c)

Theorem [ADI]

Let A be a truncated gentle algebra. For $\theta \in K_0(\mathrm{proj}\, A)_{\mathbb{R}}$, TFAE.

(a) $\theta \in R_0$.

(b) $\forall p \in \overline{\mathsf{MP}}_*(A)$, $M(p) \notin \mathcal{T}_\theta$, and $\forall p \in \overline{\mathsf{MP}}^*(A)$, $M(p) \notin \mathcal{F}_\theta$.

(c) $\forall p \in \overline{\mathsf{MP}}(A)$,
if p is of type (ii), then $\theta(M(p)) = 0$; otherwise, $M(p) \in \mathcal{W}_\theta$.

Proof of (b) $\Rightarrow$ (c)

- If $\exists p \in \overline{\mathsf{MP}}(A)$ is of type (ii) and $\theta(M(p)) > 0$,
  then $\exists q \in \overline{\mathsf{MP}}(A)$ s.t. c_q is a cyclic perm. of c_p, and $M(q) \in \mathcal{T}_\theta$.
- If $\exists p \in \overline{\mathsf{MP}}(A)$ is of type (i) or (iii) and $M(p) \notin \mathcal{W}_\theta$, then
  - $\exists X \hookrightarrow M(p)$, $X \in \mathcal{T}_\theta$ (then $\exists q \in \overline{\mathsf{MP}}_*(A)$, $X \in \mathcal{T}_\theta$); or
  - $\exists X \twoheadleftarrow M(p)$, $X \in \mathcal{F}_\theta$ (then $\exists q \in \overline{\mathsf{MP}}^*(A)$, $X \in \mathcal{F}_\theta$).

Proof of (b) $\Rightarrow$ (a)

Theorem [ADI]

Let A be a truncated gentle algebra. For $\theta \in K_0(\operatorname{proj} A)_{\mathbb{R}}$, TFAE.

- **(a)** $\theta \in R_0$.
- **(b)** $\forall p \in \overline{\mathsf{MP}}_*(A)$, $M(p) \notin \mathcal{T}_\theta$, and $\forall p \in \overline{\mathsf{MP}}^*(A)$, $M(p) \notin \mathcal{F}_\theta$.
- **(c)** $\forall p \in \overline{\mathsf{MP}}(A)$,
  if p is of type (ii), then $\theta(M(p)) = 0$; otherwise, $M(p) \in \mathcal{W}_\theta$.

Proof of (b) $\Rightarrow$ (a)

Assume $\theta \notin R_0$, then $\exists \sigma \in K_0(\operatorname{proj} A)$, $\theta = \sigma \oplus (\theta - \sigma)$,
so $C_\sigma \in \mathcal{T}_\theta$ and $K_\sigma \in \mathcal{F}_\theta$, thus (b) does not hold, since

- $\exists p \in \overline{\mathsf{MP}}_*(A)$, $M(p) \twoheadleftarrow C_\sigma$ (then $M(p) \in \mathcal{T}_\theta$); or
- $\exists p \in \overline{\mathsf{MP}}^*(A)$, $M(p) \hookrightarrow K_\sigma$ (then $M(p) \in \mathcal{F}_\theta$).

(a) $\Rightarrow$ (b) remains, and it is the most important part!

Proof of (a) $\Rightarrow$ (b)

Let A be a truncated gentle algebra.

Lemma

Let $\theta, \eta \in K_0(\operatorname{proj} A)$ with η tame and $\theta \oplus \eta$.
If $p \in \overline{\mathsf{MP}}_*(A)$ with $M(p) \in \mathcal{T}_\theta$, then $M(p) \in \mathcal{W}_\eta$.

(1) Since $\theta \oplus \eta$, $\exists f \in \mathsf{PHom}_A(\eta)$, $K_f \in \overline{\mathcal{F}}_\theta$.
Thus, $M(p) \in \mathcal{T}_\theta$ implies $\mathsf{Hom}_A(M(p), K_f) = 0$, so $M(p) \in \overline{\mathcal{T}}_\eta$.

(2) If $\mathsf{Hom}_A(C_f, M(p)) \neq 0$, then $\exists q$: a path, $C_f \twoheadrightarrow M(q) \hookrightarrow M(p)$.
Since C_f is a band module for some band b,

- $q \notin \overline{\mathsf{MP}}_*(A)$, so $p \in \overline{\mathsf{MP}}(A)$, and is of type (ii).
- The band b contains some path $q\alpha r$ ($\exists \alpha \in Q_1$, $\exists r$: a path).
- Choose the longest r, then $M(p) \twoheadrightarrow M(r) \hookrightarrow C_f = K_f$, which contradicts (1).

(3) Therefore, $\mathsf{Hom}_A(C_f, M(p)) = 0$, so $M(p) \in \mathcal{W}_\eta$.

Proof of (a) $\Longrightarrow$ (b)

Let A be a truncated gentle algebra.

Lemma

Let $\theta, \eta \in K_0(\text{proj}\, A)$ with η tame and $\theta \oplus \eta$.
If $p \in \overline{\text{MP}}_*(A)$ with $M(p) \in \mathcal{T}_\theta$, then $M(p) \in \mathcal{W}_\eta$.

Proposition

Let $\theta \in R_0 \cap K_0(\text{proj}\, A)$, then $M(p) \notin \mathcal{T}_\theta$.

For an arbitrary $\theta \in R_0 \subset K_0(\text{proj}\, A)_\mathbb{R}$,
we use a sequence $(\theta_i)_{i=1}^{\infty}$ with $\theta_i \in K_0(\text{proj}\, A)_\mathbb{Q}$ converging to θ.
As of now, we cannot assume $\theta_i \in R_0 \cap K_0(\text{proj}\, A)_\mathbb{Q}$.
Thus, we also need the following.

Proof of (a) $\Rightarrow$ (b)

Let A be a truncated gentle algebra.

Lemma

Let $\theta, \eta \in K_0(\operatorname{proj} A)$ with η tame and $\theta \oplus \eta$.
If $p \in \overline{\mathsf{MP}}_*(A)$ with $M(p) \in \mathcal{T}_\theta$, then $M(p) \in \mathcal{W}_\eta$.

Proposition

Let $\theta \in R_0 \cap K_0(\operatorname{proj} A)$, then $M(p) \notin \mathcal{T}_\theta$.

Lemma

There exists $E \in \mathbb{Z}_{\geq 0}$ s.t.

- $\forall \theta, \forall \sigma \in K_0(\operatorname{proj} A)$ with σ rigid and $\theta \oplus \sigma$,
- $\forall p \in \overline{\mathsf{MP}}_*(A)$ with $M(p) \in \mathcal{T}_\theta$,

$M(p) \in \overline{\mathcal{T}}_\sigma$ and $0 \leq \sigma(M(p)) \leq E$.

General result

Let A' be a special biserial algebra.
Then, $\exists A$: a truncated gentle algebra with $A \twoheadrightarrow A'$ and $|A| = |A'|$.
For $\theta \in K_0(\operatorname{proj} A')_{\mathbb{R}}$, set $\mathcal{W}'_\theta := \operatorname{Filt}_A(\mathcal{W}_\theta \cap \operatorname{mod} A') \subset \operatorname{mod} A$.

Theorem [ADI]

For each $\theta \in K_0(\operatorname{proj} A')_{\mathbb{R}}$, TFAE.

(a) $\theta \in R_0(A')$;

(b) $\theta \in R_0(A)$, and $\forall p' \in \overline{\mathsf{MP}}_*(A')$, $M(p') \notin \mathcal{T}_\theta$;

(b′) $\theta \in R_0(A)$, and $\forall p' \in \overline{\mathsf{MP}}^*(A')$, $M(p') \notin \mathcal{F}_\theta$;

(c) $\forall p \in \overline{\mathsf{MP}}(A)$,
if p is of type (ii), then
$\exists q \in \overline{\mathsf{MP}}(A)$, c_q is a cyclic perm. of c_p and $M(q) \in \mathcal{W}'_\theta$;
otherwise, $M(p) \in \mathcal{W}'_\theta$.

Thus, $R_0(A')$ is a union of fin. many rational polyhedral cones.

Application

Corollary / (a) $\Leftrightarrow$ (b) by [Schroll–Treffinger–Valdivieso]

Let A be a special biserial algebra. TFAE.

(a) A is τ-tilting finite.

(b) There exists no band A-module which is a brick.

(c) There exists no tame element $\eta \in K_0(\operatorname{proj} A)$.

(d) $R_0 \cap K_0(\operatorname{proj} A) = \{0\}$.

(e) $R_0 = \{0\}$.

We use our theorem to show (d) $\Rightarrow$ (e).

- (a) $\Rightarrow$ (b): [STV] (cf. [Demonet–Iyama–Jasso]).
- (b) $\Rightarrow$ (c) $\Rightarrow$ (d) and (e) $\Rightarrow$ (a) follow from facts cited in this talk.
- [STV] showed (b) $\Rightarrow$ (a) in a way different from ours.

Thank you for your attention.