# Parabolic Positive Representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$

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Recent advances in combinatorial representation theory
RIMS, Kyoto University

# Positive Representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$

# Definition of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$

## Definition

$\mathcal{U}_q(\mathfrak{sl}_2)$ = *Hopf-algebra* $\langle E, F, K^{\pm 1}\rangle$ *over* $\mathbb{C}(q)$ *such that*

$$KE = q^2 EK, \qquad KF = q^{-2} FK, \qquad [E,F] = \frac{K-K^{-1}}{q-q^{-1}}$$

*Coproduct:*

$$\Delta(E) = 1\otimes E + E\otimes K, \qquad \Delta(F) = F\otimes 1 + K^{-1}\otimes F$$
$$\Delta(K) = K\otimes K$$

*(Also counit* $\epsilon$*, antipode* $S$*)*

$\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$: $\qquad (|q|=1)$

$$E^* = E, \qquad F^* = F, \qquad K^* = K$$

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$$E^* = E, \qquad F^* = F, \qquad K^* = K$$

# Definition of $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$

## Definition

*$\mathcal{U}_q(\mathfrak{g})$= Hopf-algebra $\langle E_i, F_i, K_i^{\pm 1}\rangle_{i\in I}$ over $\mathbb{C}(q)$ such that*

$$K_iE_j = q^{a_{ij}}E_jK_i, \quad K_iF_j = q^{-a_{ij}}F_jK_i, \quad [E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q-q^{-1}}$$

*+ Serre relations.*
*Coproduct:*

$$\Delta(E_i) = 1\otimes E_i + E_i \otimes K_i, \qquad \Delta(F_i) = F_i\otimes 1 + K_i^{-1}\otimes F_i$$
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$\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$: $\qquad (|q| = 1)$

$$E_i^* = E_i, \qquad F_i^* = F_i, \qquad K_i^* = K_i$$

# Definition of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$

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$\mathcal{D}_q(\mathfrak{g})$= *Drinfeld's Double:* $\langle E_i, F_i, K_i^{\pm 1}, K_i'^{\pm 1}\rangle_{i\in I}$

$$K_iE_j = q^{a_{ij}}E_jK_i, \qquad K_iF_j = q^{-a_{ij}}F_jK_i, \qquad [E_i, F_j] = \delta_{ij}\frac{K_i - K_i'}{q-q^{-1}}$$

*+ Serre relations + Similar for* $K_i'$

*Coproduct:*

$$\Delta(E_i) = 1\otimes E_i + E_i\otimes K_i, \qquad \Delta(F_i) = F_i\otimes 1 + K_i'\otimes F_i$$
$$\Delta(K_i) = K_i\otimes K_i, \qquad \Delta(K_i') = K_i\otimes K_i'$$

*(Also counit* $\epsilon$*, antipode* $S$*)*

$$\mathcal{U}_q(\mathfrak{g}) = \mathcal{D}_q(\mathfrak{g})/\langle K_iK_i' = 1\rangle_{i\in I}$$

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Research program started in [Frenkel-I. (2012)]

- Representations by positive operators on Hilbert space.
- Generalization of Teschner's representations of $\mathcal{U}_q(\mathfrak{sl}(2,\mathbb{R}))$
  - Closure under taking tensor product $A_n$: [Schrader-Shapiro 2018]
  - Braiding structure [I. 2012]
  - Peter-Weyl Theorem $A_n$: [I.-Schrader-Shapiro 2020]
- = "Quantization of principal series representations"
- Constructed for all semisimple Lie types.

Construction:

- Lusztig's total positive space $L^2((G/B)_{>0}) \simeq L^2(\mathbb{R}_{>0}^{N=\ell(w_0)})$
- Mellin transformation: $L^2(\mathbb{R}^N_{>0}) \simeq L^2(\mathbb{R}^N)$
- $\mathcal{U}(\mathfrak{g})$ differential operator $\rightsquigarrow$ finite difference operator
- Quantization $\rightsquigarrow$ positive operators $\mathbf{e}_i, \mathbf{f}_i, K_i \in \mathcal{U}_q(\mathfrak{g}_\mathbb{R})$

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# Positive Representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$

Rescale generators by ($q = e^{\pi i b^2}$, $b \in (0,1)$)

$$\mathbf{e}_k = -i(q-q^{-1})E_k, \qquad \mathbf{f}_k = -i(q-q^{-1})F_k$$

**Theorem** (*I. (2012)*)

*There exists a family of irreducible representations $\mathcal{P}_\lambda$ of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$:*

- *Parametrized by $\lambda \in \mathbb{R}_{\geq 0}P^+ \simeq \mathbb{R}_{\geq 0}^{n=rank\mathfrak{g}}$*
- *Positivity: $\{\mathbf{e}_i, \mathbf{f}_i, K_i\}$ are represented by positive, essentially self-adjoint (unbounded) operators on $L^2(\mathbb{R}^N)$*
- *$\mathbf{e}_i, \mathbf{f}_i, K_i$ are expressed in terms of Laurent polynomials of $\{e^{\pi b x_k}, e^{2\pi b p_k}\}_{k=1}^N$*
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# Example: $\mathcal{U}_q(\mathfrak{sl}_3)$

Coordinates on $(G/B)_{>0}$:

$$\begin{pmatrix}1 & a & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & b\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & c & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\cdot\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & t & 1\end{pmatrix} \qquad a,b,c>0$$

$$=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & \\ 0 & \frac{t}{1+bt} & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1+bt & 0\\ 0 & 0 & (1+bt)^{-1}\end{pmatrix}\begin{pmatrix}1 & a+abt & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & \frac{b}{1+bt}\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & c & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$e^{tF_2}\cdot f(a,b,c)=(1+bt)^{2\lambda}f(a+abt,\frac{b}{1+bt},c), \qquad \lambda\in\mathbb{R}_{\geq 0}$$

$$F_2:=\frac{d}{dt}e^{tF_2}\Big|_{t=0}=ab\frac{\partial}{\partial a}-b^2\frac{\partial}{\partial b}+b\lambda$$

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$$=\begin{pmatrix}1&0&0\\0&1&\\0&\frac{t}{1+bt}&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1+bt&0\\0&0&(1+bt)^{-1}\end{pmatrix}\begin{pmatrix}1&a+abt&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&\frac{b}{1+bt}\\0&0&1\end{pmatrix}\begin{pmatrix}1&c&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$e^{tF_2}\cdot f(a,b,c)=(1+bt)^{2\lambda}f(a+abt,\frac{b}{1+bt},c),\qquad \lambda\in\mathbb{R}_{\geq 0}$$

$$F_2:=\frac{d}{dt}e^{tF_2}\Big|_{t=0}=ab\frac{\partial}{\partial a}-b^2\frac{\partial}{\partial b}+b\lambda$$

# Example: $\mathcal{U}_q(\mathfrak{sl}_3)$

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$$F_2 = ab\frac{\partial}{\partial a} - b^2\frac{\partial}{\partial b} + b\lambda$$

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$$F_2 : \mathcal{F}(u,v,w) \mapsto (2\lambda + u - v + 1)\mathcal{F}(u, v-1, w)$$

Quantum Twist ($n \mapsto [n]_q$ + "Wick's rotation")

$$F_2 := \left(\frac{i}{q-q^{-1}}\right)\left(e^{\pi b(2\lambda+u-v+2p_v)} + e^{\pi b(-2\lambda-u+v+2p_v)}\right)$$

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# The goal of this talk

**Definition**

*Parabolic positive representations is a new family of positive representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ based on quantizing the parabolic induction representations on $L^2((G/P)_{>0})$, where $P \subset G$ is a parabolic subgroup.*

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# Quantum Cluster Variety

# Quantum Torus Algebra

"Quantization of cluster $\mathcal{X}$ variety" [Fock-Goncharov]

**Definition**

*Seed* $\mathbf{Q} = (Q, Q_0, B)$*:*

- $Q$ *= nodes (finite set)*
- $Q_0 \subset Q$ *= frozen nodes*
- $B = (b_{ij})$ *exchange matrix* $(|Q| \times |Q|$*, skew-symmetric,* $\frac{1}{2}\mathbb{Z}$*-valued)*

*Quantum torus algebra* $\mathcal{X}_q^{\mathbf{Q}}$ *= algebra generated by* $\{X_i\}_{i\in Q}$ *over* $\mathbb{C}[q]$ *such that*

$$X_iX_j = q^{-2b_{ij}}X_jX_i$$

$X_i$ *= quantum cluster variables*

Exchange Matrix $B \rightsquigarrow$ Quiver.

$$i \longrightarrow j \qquad\qquad X_iX_j = q^{-2}X_jX_i,$$

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*Seed* $\mathbf{Q} = (Q, Q_0, B)$*:*

- $\Lambda_{\mathbf{Q}} = \mathbb{Z}$*-Lattice with basis* $\{e_i\}_{i\in Q}$
- $(-,-)$ *skew-symmetric form,* $(e_i, e_j) := b_{ij}$.

*Quantum torus algebra* $\mathcal{X}_q^{\mathbf{Q}}$ *=algebra generated by* $\{X_\lambda\}_{\lambda\in\Lambda_{\mathbf{Q}}}$ *over* $\mathbb{C}[q^{\frac{1}{2}}]$ *such that*

$$X_{\lambda+\mu} = q^{(\lambda,\mu)} X_\lambda X_\mu$$

$X_i := X_{e_i}, \qquad X_{i_1,i_2,\ldots,i_k} := X_{e_{i_1}+e_{i_2}+\cdots+e_{i_k}}$

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# Quantum Cluster Mutations

$\mathbf{T}_q^{\mathbf{Q}} :=$ (non-commutative) field of fractions of $\mathcal{X}_q^{\mathbf{Q}}$.
Cluster mutation $\mu_k$ induces $\mu_k^q : \mathbf{T}_q^{\mathbf{Q}'} \longrightarrow \mathbf{T}_q^{\mathbf{Q}}$:

$$\mu_k^q(\widehat{X}_i) := \begin{cases} X_k^{-1} & i = k \\ X_i \prod_{r=1}^{|b_{ki}|}(1+q_i^{2r-1}X_k) & i \neq k, b_{ki} < 0 \\ X_i \prod_{r=1}^{b_{ki}}(1+q_i^{2r-1}X_k^{-1})^{-1} & i \neq k, b_{ki} > 0 \end{cases}$$

Can be rewritten as

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$\Psi_q$ = quantum dilogarithm

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Recall $q = e^{\pi i b^2}$ such that $|q| = 1$.

## Definition

*A* polarization *of $\mathcal{X}_q^{\mathbf{Q}}$ is a choice of representation of the cluster variables $X_k \in \mathcal{X}_q^{\mathbf{Q}}$ of the form $X_k = e^{2\pi b x_k}$ such that*

- *$x_j$ is self-adjoint*
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$$[x_j, x_k] = \frac{1}{2\pi i} b_{jk},$$

*acting on some Hilbert space $\mathcal{H}_{\mathbf{Q}} \simeq L^2(\mathbb{R}^N)$.*

## Remark

*Modular double $\widehat{X}_k$ acts by $X_k^{2\pi b^{-1} x_k}$ on $\mathcal{H}_{\mathbf{Q}}$.*

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## Proposition

- *Different polarizations (with the same central characters) are unitary equivalent (via* $Sp(2N)$*-action)*
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# Quantum cluster variety

$S$=Riemann surface with marked points on $\partial S$ and punctures.

Fock-Goncharov's $\mathcal{X}_{G,S}$-space= "(framed) local $G$-system"

- $\mathcal{X}_{G,S}$ has Poisson cluster $\mathcal{X}$ variety structure $\rightsquigarrow$ quantization $\mathcal{X}_{G,S}^q$
- To each triangle of ideal triangulation of $S$, assign a basic quiver.
- $G = PGL_{n+1}$: "$n$-triangulation"



$Q_{\mathfrak{sl}_4}$

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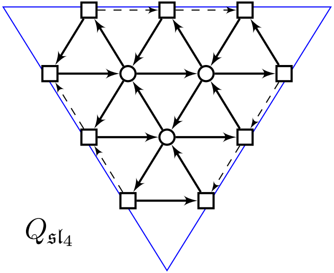

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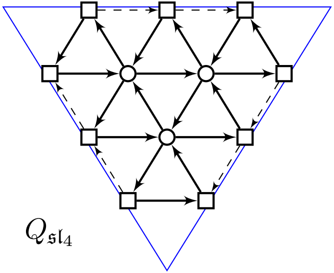


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# Basic Quiver

[I. (2016), Goncharov-Shen (2019)]

**Definition**

*Elementary quiver*

- $\overline{\mathbf{J}}_k(i)$, $i, k \in I$
- $Q = Q_0 = (I \setminus \{i\}) \cup \{i_l\} \cup \{i_r\} \cup \{k_e\}$

$$c_{i_l,j} = c_{j,i_r} = \frac{a_{ij}}{2}, \qquad c_{i,i_r} = c_{i_r,k_e} = c_{k_e,i_l} = 1$$

- $\overline{\mathbf{J}}(i)$: *without* $\{k_e\}$.

# Basic Quiver

[I. (2016), Goncharov-Shen (2019)]

**Definition**

*Elementary quiver*

- $\mathbf{H}(\mathbf{i})$, $\quad \mathbf{i} = (i_1, ..., i_m)$ *reduced words*
- $Q = I$

$$c_{ij} := \begin{cases} \mathrm{sgn}(r-s)\frac{a_{ij}}{2} & \beta_s = \alpha_i \text{ and } \beta_r = \alpha_j \\ 0 & \text{otherwise} \end{cases}$$

- $\beta_j := s_{i_m} s_{i_{m-1}} \cdots s_{i_{j+1}}(\alpha_{i_j}), \quad \alpha_i \in \Delta_+$
- *(If $\mathbf{i} = \mathbf{i}_0$, orientation of Dynkin diagram)*

# Basic Quiver

[I. (2016), Goncharov-Shen (2019)]

**Definition**

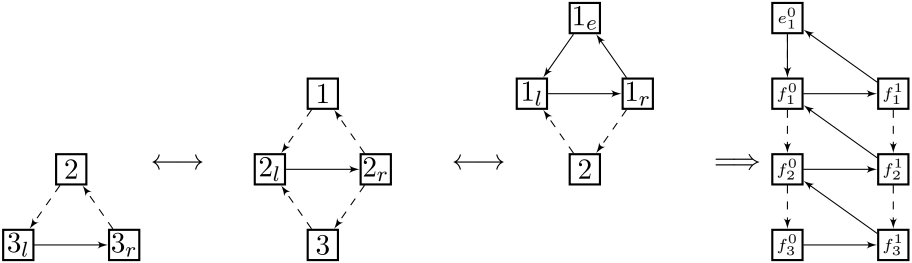
*Basic Quiver*

- $\mathbf{Q}(\mathbf{i}), \qquad \mathbf{i} = (i_1, ..., i_m)$ *reduced words*
- $\mathbf{Q} = \mathbf{J}_{\mathbf{i}}^{\#}(i_1) * \mathbf{J}_{\mathbf{i}}^{\#}(i_2) * \cdots * \mathbf{J}_{\mathbf{i}}^{\#}(i_m) * \mathbf{H}(\mathbf{i})$
- $\mathbf{J}_{\mathbf{i}}^{\#}(i_j) = \begin{cases} \overline{\mathbf{J}}_k(i_j) & \textit{if } \beta_j = \alpha_k \\ \mathbf{J}(i_j) & \textit{otherwise} \end{cases}$

# Basic Quiver

## Example

$\mathfrak{g} = \mathfrak{sl}_4$, $\mathbf{i} = (3, 2, 1)$.

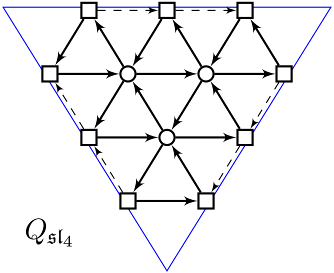



$\mathbf{J}(3)$ $\quad\longleftrightarrow\quad$ $\mathbf{J}(2)$ $\quad\longleftrightarrow\quad$ $\overline{\mathbf{J}}_1(1)$ $\quad\Longrightarrow\quad$ $\mathbf{Q}(\mathbf{i})$

# Basic Quiver

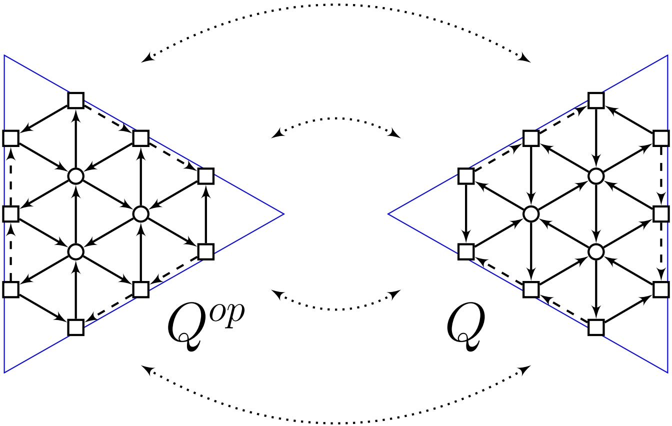
## Example

$\mathfrak{g} = \mathfrak{sl}_4, \qquad \mathbf{i}_0 = (3,2,1,3,2,3).$



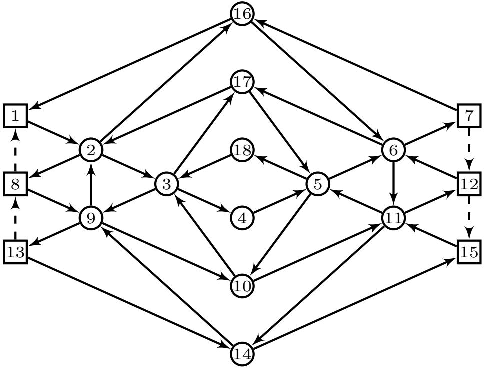
$Q_{\mathfrak{sl}_4}$

# Example: Type $A_n$ Case




Amalgamation of 2 quivers

# Example: Type $A_n$ Case




$\mathcal{D}_{\mathfrak{sl}_{n+1}}$-quiver $\rightsquigarrow \mathcal{X}_\odot := \mathcal{X}_{\mathfrak{sl}_{n+1}}$ [Schrader-Shapiro]

$$\iota : \mathcal{D}_q(\mathfrak{sl}_{n+1}) \hookrightarrow \mathcal{X}_\odot$$
$$\mathcal{U}_q(\mathfrak{sl}_{n+1}) \hookrightarrow \mathcal{X}_\odot / \langle \iota(K_i)\iota(K_i') = 1 \rangle$$

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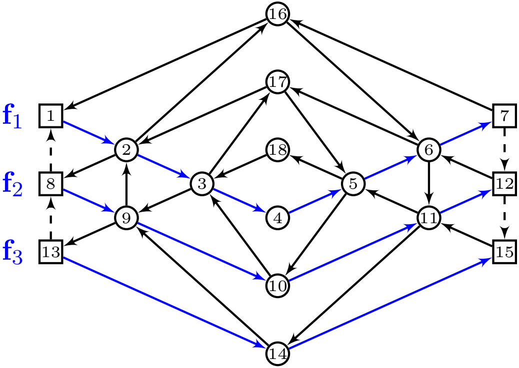



Embedding of $F_i \in \mathfrak{D}_{\mathfrak{sl}_4} \hookrightarrow \mathcal{X}_\odot$

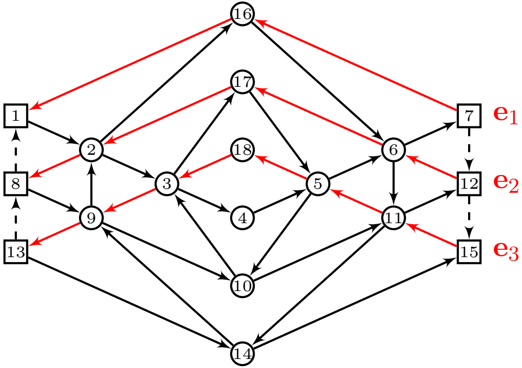
$$\mathbf{f}_1 = X_1 + X_{1,2} + X_{1,2,3} + X_{1,2,3,4} + X_{1,2,3,4,5} + X_{1,2,3,4,5,6}$$

$$\mathbf{f}_2 = X_8 + X_{8,9} + X_{8,9,10} + X_{8,9,10,11}$$

$$\mathbf{f}_3 = X_{13} + X_{13,14}$$

$$K_1' = X_{1,2,3,4,5,6,7} \qquad K_2' = X_{8,9,10,11,12} \qquad K_3' = X_{13,14,15}$$

# Example: Type $A_n$ Case




Embedding of $E_i \in \mathfrak{D}_{\mathfrak{sl}_4} \hookrightarrow \mathcal{X}_\odot$

$$\mathbf{e}_1 = X_7 + X_{7,16}$$

$$\mathbf{e}_2 = X_{12} + X_{12,6} + X_{12,6,17} + X_{12,6,17,2}$$

$$\mathbf{e}_3 = X_{15} + X_{15,11} + X_{15,11,5} + X_{15,11,5,18} + X_{15,11,5,18,3} + X_{15,11,5,18,3,9}$$

$$K_1 = X_{7,16,1} \qquad K_2 = X_{12,6,17,2,8} \qquad K_3 = X_{15,11,5,18,3,9,13}$$

# Positive Representations of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$

**Theorem** (*Schrader-Shapiro, I. (2016)*)

- *There exists an embedding*

$$\mathcal{D}_q(\mathfrak{g}) \hookrightarrow \mathcal{X}_\odot$$

*corresponding to the quiver $\mathcal{D}_\mathfrak{g}$ associated to* 

- *We recover the positive representations $\mathcal{P}_\lambda \simeq \mathcal{H}_\odot$ through a polarization of $\mathcal{X}_\odot$.*

**Theorem** (*I. (2016)*)

- *The generators $\mathbf{e}_i, \mathbf{f}_i, K_i$ are represented by positive polynomials (i.e. over $\mathbb{N}[q,q^{-1}]$) in the cluster variables $X_i \in \mathcal{X}_\odot$.*
- *The generators $\mathbf{e}_i, \mathbf{f}_i, K_i$ are labeled by paths on the $\mathcal{D}_\mathfrak{g}$ quiver.*
- *$\mathbf{f}_i$ paths are simple - coincide with Feigin's homomorphism.*

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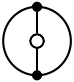
- *There exists an embedding*

$$\mathcal{D}_q(\mathfrak{g}) \hookrightarrow \mathcal{X}_\odot$$

*corresponding to the quiver $\mathcal{D}_\mathfrak{g}$ associated to* 

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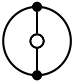
*corresponding to the quiver $\mathcal{D}_\mathfrak{g}$ associated to* 

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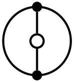
*corresponding to the quiver $\mathcal{D}_\mathfrak{g}$ associated to* 

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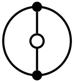
*corresponding to the quiver* $\mathcal{D}_\mathfrak{g}$ *associated to* 

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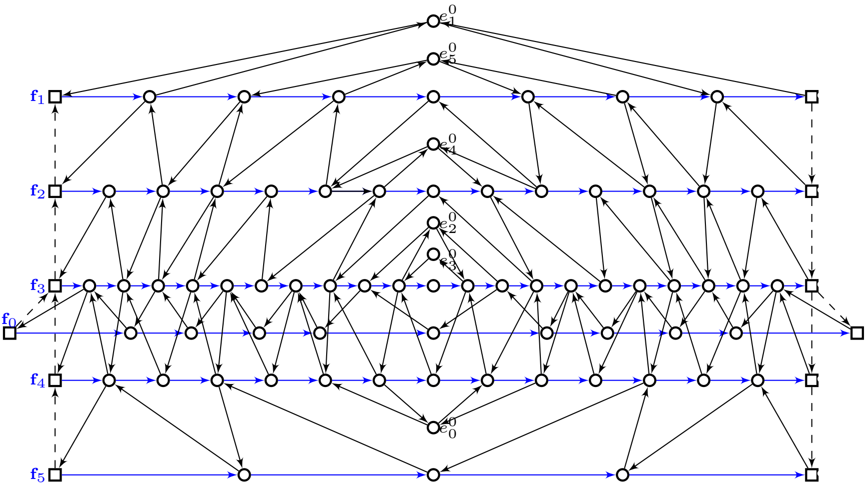
# $E_6$ embedding

$\mathbf{i}_0 = (3\ 43\ 034\ 230432\ 12340321\ 5432103243054321)$



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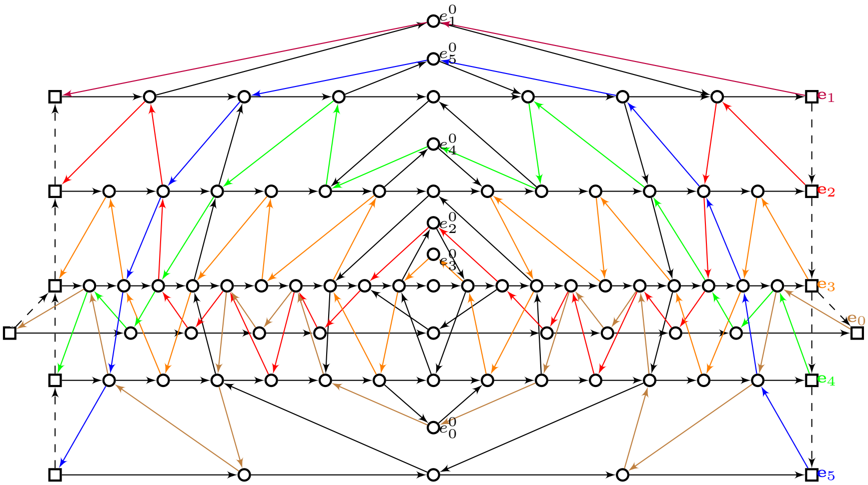

# Minimal Positive Representation for $\mathcal{U}_q(\mathfrak{sl}(n+1,\mathbb{R}))$

# Minimal Positive Representation

- Parabolic subgroups $\longleftrightarrow J \subset I$
- $P_J := B_- L_J$, Levi subgroup $L_J = \langle T, U_j^+, U_j^- \rangle_{j \in J}$
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**Example**

*For* $G = SL_4$, $J = \{1, 2\} \subset I = \{1, 2, 3\}$

$$P_J = \begin{pmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix}$$

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Previous recipe produces a representation $\mathcal{P}_\lambda^J$ for $\mathcal{U}_q(\mathfrak{sl}(4,\mathbb{R}))$, $(\lambda \in \mathbb{R})$

$$\begin{aligned}
\pi_\lambda^J(\mathbf{e}_1) &= e^{\pi b(u-2p_u)} + e^{\pi b(-u-2p_u)}\\
\pi_\lambda^J(\mathbf{e}_2) &= e^{\pi b(-u+v-2p_v)} + e^{\pi b(u-v-2p_v)}\\
\pi_\lambda^J(\mathbf{e}_3) &= e^{\pi b(-v+w-2p_w)} + e^{\pi b(v-w-2p_w)}\\
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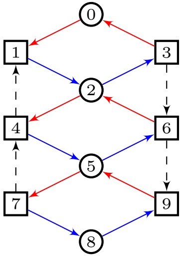
\pi_\lambda^J(\mathbf{K}_1) &= e^{\pi b(-2u+v)}\\
\pi_\lambda^J(\mathbf{K}_2) &= e^{\pi b(u-2v+w)}\\
\pi_\lambda^J(\mathbf{K}_3) &= e^{\pi b(v-2w+2\lambda)}
\end{aligned}$$

acting on $L^2(\mathbb{R}^3)$ as positive self-adjoint operators.

# Minimal Positive Representation




$$\mathbf{D}(\mathbf{i}) := \mathbf{Q}(\mathbf{i}^{op}) * \mathbf{Q}(\mathbf{i}), \qquad \mathbf{i} = (3,2,1)$$

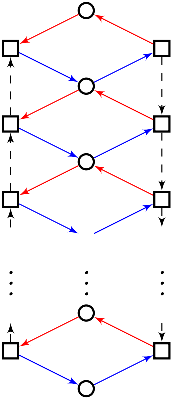
$$\begin{aligned} e_1 &= X_3 + X_{3,0} & K_1 &= X_{3,0,1} \\ e_2 &= X_6 + X_{6,2} & K_2 &= X_{6,2,4} \\ e_3 &= X_9 + X_{9,5} & K_3 &= X_{9,5,7} \\ f_1 &= X_1 + X_{1,2} & K_1' &= X_{1,2,3} \\ f_2 &= X_4 + X_{4,5} & K_2' &= X_{4,5,6} \\ f_3 &= X_7 + X_{7,8} & K_3' &= X_{7,8,9} \end{aligned}$$

Central character: $\pi(X_{0,2,5,8}) = e^{-4\pi b\lambda}$

# Minimal Positive Representation

**Theorem** (*I. (2020)*)

*The polarization of the quiver* $\mathbf{D}(\mathbf{i})$ *for* $\mathbf{i} = (n, ..., 3, 2, 1)$ *gives a representation* $\mathcal{P}_\lambda^J$ *of* $\mathcal{U}_q(\mathfrak{sl}(n+1, \mathbb{R}))$ *acting on* $L^2(\mathbb{R}^n)$ *as positive self-adjoint operators.*



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**Theorem** (*I. (2020)*)

- *The non-simple generators*

$$\mathbf{e}_\alpha := T_{i_1} \cdots T_{i_{k-1}}(\mathbf{e}_k)$$

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*is non-zero, where $T_i$ = Lusztig's braid group action.*

- *The universal $\mathcal{R}$ operator is well-defined*

$$\mathcal{R} = \mathcal{K} \prod_{\alpha \in \Phi_+} g_b(\mathbf{e}_\alpha \otimes \mathbf{f}_\alpha)$$

- *The Casimirs $\mathbf{C}_k$ acts by real-valued scalar, and lie outside the positive spectrum of the usual positive representations.*

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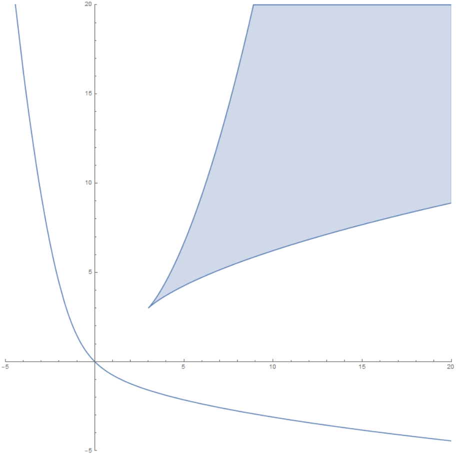
# Casimirs

## Example

*$\mathcal{U}_q(\mathfrak{sl}(3,\mathbb{R}))$, the possible action of $(\mathbf{C}_1,\mathbf{C}_2)$ (by scalars) on $\mathcal{P}_\lambda$ and $\mathcal{P}_\lambda^J$:*



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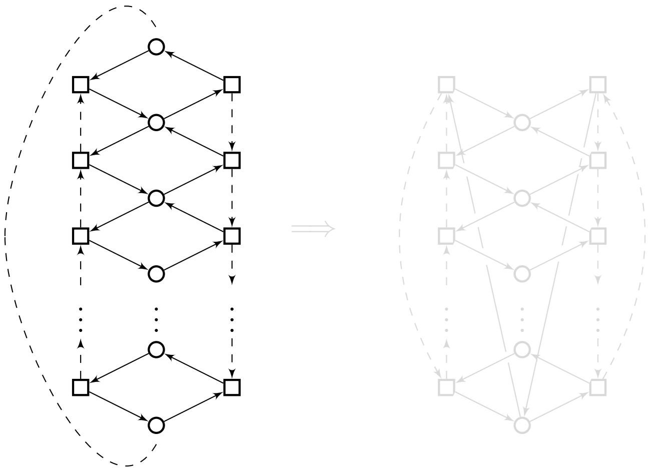

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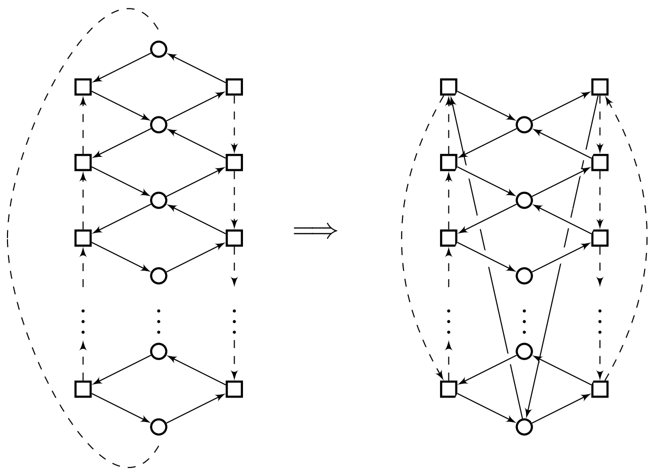

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**Theorem** (*I. (2020)*)

*The positive representation of* $\mathcal{U}_q(\widehat{\mathfrak{sl}}_{n+1})$ *defined by the polarization of the previous quiver is unitarily equivalent to Jimbo's evaluation module* $\mathcal{P}_\lambda^\mu$, $\mu \in \mathbb{R}$

$$\mathcal{U}_q(\widehat{\mathfrak{sl}}_{n+1}) \longrightarrow \mathcal{U}_q(\mathfrak{sl}_{n+1})$$

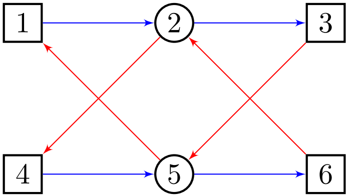
*of the minimal positive representations* $\mathcal{P}_\lambda^J$ *of* $\mathcal{U}_q(\mathfrak{sl}_{n+1})$, *where*

$$e^{\pi b\mu} := \pi(D_0^{\frac{1}{n+1}} D_1)$$

*(*$D_0$*=product of all middle vertices,* $D_1$*= product of all right vertices.)*

# Positive representation of $\mathcal{U}_q(\widehat{\mathfrak{sl}}_2)$

## Example




$$\mathbf{f}_0 = X_1 + X_{1,2} \qquad \mathbf{e}_0 = X_3 + X_{3,5}$$

$$\mathbf{f}_1 = X_4 + X_{4,5} \qquad \mathbf{e}_1 = X_6 + X_{6,2}$$

*Serre relation ($a_{01} = a_{10} = -2$):*

$$X_i^3 X_j - [3]_q X_i^2 X_j X_i + [3]_q X_i X_j X_i^2 - X_j X_i^3 = 0, \qquad i \neq j$$

# General Construction

# Main Theorem

Parabolic induction $\longleftrightarrow$ truncating $\mathbf{i}_J \subset \mathbf{i}_0$ where $\mathbf{i}_J, \mathbf{i}_0$ are the longest word of the Weyl groups $W_J \subset W$.

$$w_0 = w_J \overline{w}$$
$$\overline{w} \longleftrightarrow \bar{\mathbf{i}}$$

**Example**

$$W_{\mathfrak{sl}_4} \subset W_{\mathfrak{sl}_5}$$
$$\mathbf{i}_0 = (1,2,1,3,2,1,4,3,2,1)$$

Observe that
$$\mathbf{Q}(\mathbf{i}) = \mathbf{Q}(\mathbf{i}_J) * \mathbf{Q}(\bar{\mathbf{i}})$$

In general, we have realization of $\mathcal{U}_q(\mathfrak{g}_{\mathbb{R}})$ on the quantum torus algebra associated to the symplectic double $\mathbf{D}(\bar{\mathbf{i}})$.

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- *A polarization of $\mathcal{X}_q^{\mathbf{D}(\bar{\mathbf{i}})}$ induces a family of irreducible representations $\mathcal{P}_\lambda^J$ of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ parametrized by $\lambda \in \mathbb{R}^{|I\setminus J|}$ as positive self-adjoint operators on $L^2(\mathbb{R}^{l(\overline{w})})$.*

**Corollary**

*The parabolic positive representations $\mathcal{P}_\lambda^J$ is obtained as a quantum twist of the parabolic induction, by ignoring the variables $u_i$ corresponding to the Levi subgroups $L_J$ of $P_J$ in the quotient $G/P_J$.*

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## Theorem (*I. (2020)*)

- *There is a homomorphism*

$$\mathcal{D}_q(\mathfrak{g}) \longrightarrow \mathcal{X}_q^{\mathbf{D}(\bar{\mathbf{i}})}$$

  *such that the image of universally Laurent polynomials.*

- *A polarization of $\mathcal{X}_q^{\mathbf{D}(\bar{\mathbf{i}})}$ induces a family of irreducible representations $\mathcal{P}_\lambda^J$ of $\mathcal{U}_q(\mathfrak{g}_\mathbb{R})$ parametrized by $\lambda \in \mathbb{R}^{|I \setminus J|}$ as positive self-adjoint operators on $L^2(\mathbb{R}^{l(\overline{w})})$.*

## Corollary

*The parabolic positive representations $\mathcal{P}_\lambda^J$ is obtained as a quantum twist of the parabolic induction, by ignoring the variables $u_i$ corresponding to the Levi subgroups $L_J$ of $P_J$ in the quotient $G/P_J$.*

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# Idea of Proof

## Definition

*The* *Heisenberg double* $\mathcal{H}_q^{\pm}(\mathfrak{g}) := \langle \mathbf{e}_i^{\pm}, \mathbf{f}_i^{\pm}, \mathbf{K}_i^{\pm}, \mathbf{K'}_i^{\pm} \rangle$ *satisfying*

$$\frac{[\mathbf{e}_i^+, \mathbf{f}_j^+]}{q-q^{-1}} = \delta_{ij}\mathbf{K'}_i^+, \qquad \frac{[\mathbf{e}_i^-, \mathbf{f}_j^-]}{q-q^{-1}} = \delta_{ij}\mathbf{K}_i^-$$

*and other standard quantum group relations.*

## Proposition

*The embedding* $\mathcal{D}_q(\mathfrak{g}) \hookrightarrow \mathcal{X}_q^{\mathbf{D}(\mathbf{i}_0)} \subset \mathcal{X}_q^{\mathbf{Q}(\mathbf{i}_0^{op})} \otimes \mathcal{X}_q^{\mathbf{Q}(\mathbf{i})}$ *decomposes as*

$$\mathbf{e}_i = \mathbf{e}_i^+ + \mathbf{K}_i^+\mathbf{e}_i^-, \qquad \mathbf{f}_i = \mathbf{f}_i^- + \mathbf{K'}_i^-\mathbf{f}_i^+$$

$$\mathbf{K}_i = \mathbf{K}_i^+\mathbf{K}_i^- \qquad \mathbf{K}'_i = \mathbf{K'}_i^+\mathbf{K'}_i^-$$

*where* $\mathcal{H}_q^+(\mathfrak{g}) \hookrightarrow 1 \otimes \mathcal{X}_q^{\mathbf{Q}(\mathbf{i}_0)}$, $\mathcal{H}_q^-(\mathfrak{g}) \hookrightarrow \mathcal{X}_q^{\mathbf{Q}(\mathbf{i}_0^{op})} \otimes 1$

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## Definition

*The* *generalized Heisenberg double* $\mathcal{H}^{\pm}_{q,\omega}(\mathfrak{g}) := \langle \mathbf{e}_i^{\pm}, \mathbf{f}_i^{\pm}, \mathbf{K}_i^{\pm}, \mathbf{K}_i'^{\pm} \rangle$

$$\frac{[\mathbf{e}_i^+, \mathbf{f}_j^+]}{q-q^{-1}} = \delta_{ij}\mathbf{K}_i'^+ + \omega_{ij}\mathbf{K}_i^+, \qquad \frac{[\mathbf{e}_i^-, \mathbf{f}_j^-]}{q-q^{-1}} = \delta_{ij}\mathbf{K}_i{}^- - \omega_{ij}\mathbf{K}_i'^-$$

*and other standard quantum group relations, where* $\omega_{ij} \in \mathbb{C}$.

## Proposition

*If* $\mathcal{H}^{\pm}_{q,\omega}(\mathfrak{g})$ *are commuting copies, then*

$$\mathbf{e}_i = \mathbf{e}_i^+ + \mathbf{K}_i^+\mathbf{e}_i^-, \qquad \mathbf{f}_i = \mathbf{f}_i^- + \mathbf{K}_i'^-\mathbf{f}_i^+$$
$$\mathbf{K}_i = \mathbf{K}_i^+\mathbf{K}_i^- \qquad \mathbf{K}_i' = \mathbf{K}_i'^+\mathbf{K}_i'^-$$

*gives a homomorphic image of* $\mathcal{U}_q(\mathfrak{g})$.

# Idea of Proof

## Definition

*Let $J \subset I$. The* *double Dynkin involution* *of $i \in I$ is the unique index $i^{**} \in I$ such that*

$$w_0 s_i = s_{i^*} w_0 = s_{i^*} w_J \overline{w} = w_J s_{i^{**}} \overline{w}.$$

$$\Longleftrightarrow i^{**} := (i^{*_W})^{*_{W_J}}$$

*where $i^{*_{W_J}} = i$ if $i \notin J$.*

# Decomposition Lemma

**Lemma** (*Decomposition Lemma*)

*The embedding* $\mathcal{H}_q^+(\mathfrak{g}) \hookrightarrow \mathcal{X}_q^{\mathbf{Q}(\mathbf{i}_0)} \subset \mathcal{X}_q^{\mathbf{Q}(\mathbf{i}_J)} \otimes \mathcal{X}_q^{\mathbf{Q}(\bar{\mathbf{i}})}$ *can be decomposed into the form*

$$e_i^+ = \overline{e_i} + \overline{K_i} e_{i^{**}}^J, \qquad f_i^+ = f_i^J + {K'_i}^J \overline{f_i},$$
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*where* $e_i^J = f_i^J = 0$ *and* $K_i^J = {K'_i}^J = 1$ *if* $i \notin J$, *such that*

- $X_i^J \in \mathcal{X}_q^{\mathbf{Q}(\mathbf{i}_J)} \otimes 1$ *and* $\overline{X_i} \in 1 \otimes \mathcal{X}_q^{\mathbf{Q}(\bar{\mathbf{i}})}$ *for* $X = e, f, K, K'$
- $\{e_i^J, f_i^J, K_i^J, {K'_i}^J\} \simeq \mathcal{H}_q^+(\mathfrak{g}_J)$ *in* $\mathcal{X}_q^{\mathbf{Q}(\mathbf{i}_J)}$ *where* $\mathfrak{g}_J \subset \mathfrak{g}$.
- *We have on* $\mathcal{X}_q^{\mathbf{Q}(\bar{\mathbf{i}})}$ *for some* $\omega_{ij} \in \{0,1\}$,

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# Proof of Lemma

- Decomposition of $\mathbf{f}_i, K_i'$ follows from explicit calculation using Feigin's embedding.
- Decomposition of $\mathbf{e}_i, K_i$ requires combinatorics of Coxeter moves:

**Lemma** (*I. (2020)*)

*If $l(s_i w s_j) = l(w)$, then there is a sequence of Coxeter moves that brings the reduced word of $w \in W$:*

$$\mathbf{i} = (i, ...,) \mapsto \mathbf{i}' = (..., j)$$

*where the sequence of Coxeter moves splits into 2 stages, the second of which increase in indices consecutively from first to last letter.*

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# Proof of Lemma

## Example

$\mathfrak{g} = \mathfrak{sl}_5$: $\mathbf{i} = (1,2,1,3,4,3,2,3,1,2) \rightsquigarrow (.........., 4)$?

*Stage 1:*

$$(1,2,1,3,4,3,2,3,1,2) \rightsquigarrow (1,2,1,3,2,1,4,3,2,1)$$

*Stage 2:*

$$\begin{aligned} &(1,2,1,3,2,1,4,3,2,1) \\ \rightsquigarrow &(2,1,2,3,2,1,4,3,2,1) \\ \rightsquigarrow &(2,1,3,2,3,1,4,3,2,1) \\ \rightsquigarrow &(2,1,3,2,1,3,4,3,2,1) \\ \rightsquigarrow &(2,1,3,2,1,4,3,4,2,1) \\ \rightsquigarrow &(2,1,3,2,1,4,3,2,4,1) \\ \rightsquigarrow &(2,1,3,2,1,4,3,2,1,4) \end{aligned}$$

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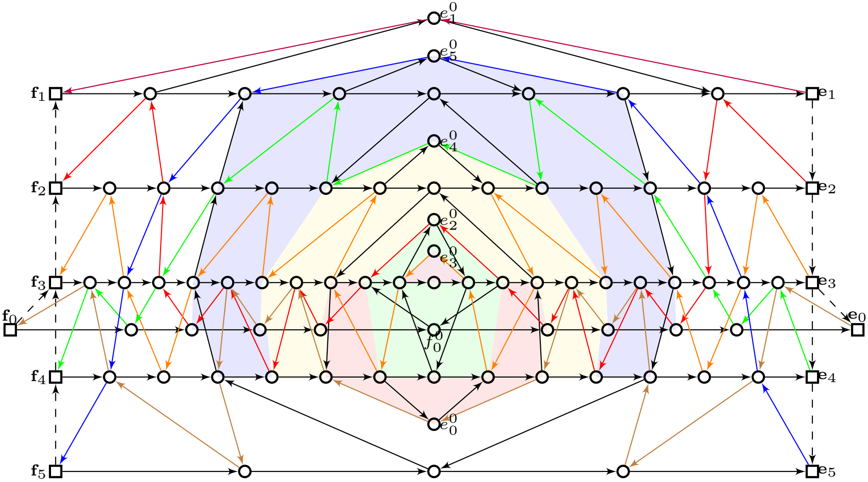
# Example: $E_6$




$$A_1 \subset A_2 \subset A_3 \subset D_4 \subset D_5 \subset E_6$$

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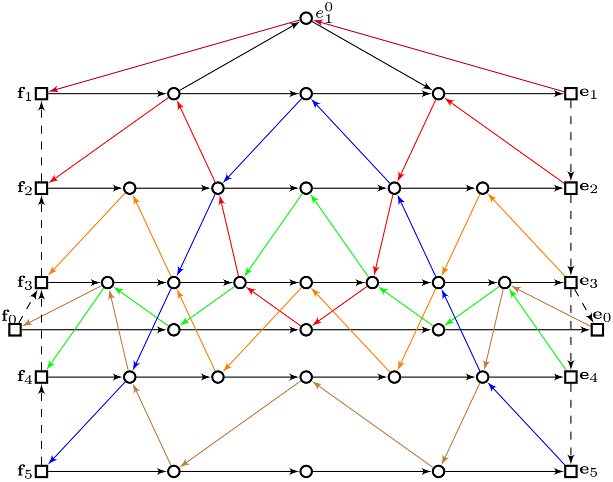
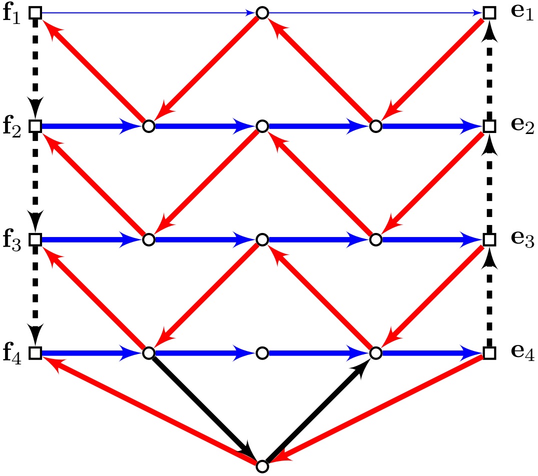



$$D_5 \subset E_6$$

# Example: $B_4$




$$J = \{1,2,3\} \subset I = \{1,2,3,4\}, \qquad 1 = short$$

# Further Discussions

- Lusztig's braid group action $T_i$ as cluster mutations on $\mathbf{Q}(\bar{\mathbf{i}})$?

$$\mathbf{e}_i \longleftrightarrow \mathbf{f}_{i^*}, \qquad \mathbf{f}_i \longleftrightarrow \mathbf{e}_{i^*}$$

- Geometric meaning of the cluster structure of $\mathbf{D}(\bar{\mathbf{i}})$
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  - $\mathcal{R}$ matrix well-defined $\Longrightarrow$ new braided tensor category?
  - Study the spectrum of Casimir operators $\mathbf{C}_k$.
  - Proved for $J = \emptyset$ and $\mathfrak{g} = \mathfrak{sl}_{n+1}$. [Schrader-Shapiro]
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Thank you for your attention!