Intermediate Symplectic Characters and Applications

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Plan:

- Intermediate symplectic characters
- Determinant formulas
- Application 1 : Brent–Krattenthaler–Warnaar's identity
- Application 2 : shifted plane partition enumeration

This talk is based on

- A bialternant formula for odd symplectic characters and its application, Josai Math. Monographs **12** (2020), 99–116. arXiv:1905.12964.
- Intermediate symplectic characters and shifted plane partitions of shifted double staircase shape, arXiv:2009.14037.

Intermediate Symplectic Characters

Schur functions (irreducible characters of GL_n)

Let n be a positive integer, and λ a partition of length $\leq n$. An n-semistandard tableau of shape λ is a filling of the boxes in the Young diagram $D(\lambda)$ with entries from $1, 2, \ldots, n$ such that

- the entries in each row are weakly increasing;
- the entries in each column are strictly increasing;

Then the Schur function $s_\lambda(x_1, \ldots, x_n)$ is defined by

$$s_\lambda(x_1, \ldots, x_n) = \sum_T \boldsymbol{x}^T, \quad \boldsymbol{x}^T = \prod_{i=1}^{n} x_i^{\#(i\text{'s in } T)}.$$

where T runs over all n-semistandard tableaux of shape λ.

$$T = \begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 2 & 2 & 4 \\ \hline 2 & 3 & 3 \\ \cline{1-3} 4 & 4 \\ \cline{1-2}\end{array}, \quad \boldsymbol{x}^T = x_1^2 x_2^3 x_3^2 x_4^3$$

Symplectic characters (irreducible characters of Sp_{2n})

An n-symplectic tableau of shape λ is a filling of the boxes in the Young diagram $D(\lambda)$ with entries from $1 < \overline{1} < 2 < \overline{2} < \cdots < n < \overline{n}$ such that

- the entries in each row are weakly increasing;
- the entries in each column are strictly increasing;
- the entries in the ith row are greater than or equal to i.

Then the symplectic character $\mathrm{sp}_\lambda(x_1, \dots, x_n)$ is defined by

$$\mathrm{sp}_\lambda(x_1, \dots, x_n) = \sum_T \boldsymbol{x}^T, \quad \boldsymbol{x}^T = \prod_{i=1}^n x_i^{\#(i\text{'s in } T) - \#(\overline{i}\text{'s in } T)}.$$

where T runs over all n-symplectic tableaux of shape λ.

$T =$

1	1	$\overline{1}$	$\overline{2}$	4
2	$\overline{2}$	3		
$\overline{4}$	5			

$, \quad \boldsymbol{x}^T = x_1 x_2^{-1} x_3 x_5$

Intermediate symplectic tableaux

Let n and k be two integers such that $n > 0$ and $0 \leq k \leq n$, and λ a partition of length $\leq n$. A $(k, n-k)$-symplectic tableau of shape λ is a filling of the boxes in the Young diagram $D(\lambda)$ with entries from

$$\{1 < \overline{1} < 2 < \overline{2} < \cdots < k < \overline{k} < k+1 < k+2 < \cdots < n\}$$

satisfying the following three conditions:

- the entries in each row are weakly increasing;
- the entries in each column are strictly increasing;
- the entries in the ith row are greater than or equal to i.

For example,

$$T = \begin{array}{|c|c|c|c|c|} \hline \overline{1} & 2 & 2 & 3 & 4 \\ \hline 2 & \overline{2} & 5 & \multicolumn{2}{c}{} \\ \cline{1-3} 4 & 4 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array}$$

is a $(2, 3)$-symplectic tableau of shape $(5, 3, 2)$.

Intermediate symplectic tableaux

Let n and k be two integers such that $n > 0$ and $0 \leq k \leq n$. A $(k, n-k)$-symplectic tableau of shape λ is a filling of the boxes in the Young diagram $D(\lambda)$ with entries from

$$\{1 < \overline{1} < 2 < \overline{2} < \cdots < k < \overline{k} < k+1 < k+2 < \cdots < n\}$$

satisfying the following three conditions:

- The entries in each row are weakly increasing.
- The entries in each column are strictly increasing.
- the entries in the ith row are greater than or equal to i.

If $k = 0$ or $k = n$, then

$(0, n)$-symplectic tableaux $=$ n-semistandard tableaux,

$(n, 0)$-symplectic tableaux $=$ n-symplectic tableaux.

Intermediate symplectic characters

Given a partition λ of length $\leq n$, we define the $(k, n-k)$-symplectic character $\mathrm{sp}_\lambda^{(k,n-k)}(x_1, \dots, x_n)$ by

$$\mathrm{sp}_\lambda^{(k,n-k)}(x_1, \dots, x_k | x_{k+1}, \dots, x_n) = \sum_T \boldsymbol{x}^T,$$

where T runs over all $(k, n-k)$-symplectic tableaux of shape λ, and

$$\boldsymbol{x}^T = \prod_{i=1}^{k} x_i^{\#(i\text{'s in } T) - \#(\bar{i}\text{'s in } T)} \prod_{i=k+1}^{n} x_i^{\#(i\text{'s in } T)}.$$

If $k = 0$ or $k = n$, then

$$\mathrm{sp}_\lambda^{(0,n)}(\boldsymbol{x}) = s_\lambda(\boldsymbol{x}) : \text{Schur function},$$
$$\mathrm{sp}_\lambda^{(n,0)}(\boldsymbol{x}) = \mathrm{sp}_\lambda(\boldsymbol{x}) : \text{symplectic character}.$$

Intermediate symplectic groups

Let $V = \mathbb{C}^{n+k}$ be the $(n+k)$-dimensional complex vector space with basis $e_1, e_{\bar{1}}, \ldots, e_k, e_{\bar{k}}, e_{k+1}, \ldots, e_n$. Let $\langle\ ,\ \rangle$ be the skew-symmetric bilinear form (not necessarily non-degenerate) on V defined by

$$\langle e_\alpha, e_\beta \rangle = \begin{cases} 1 & \text{if } \alpha = i \text{ and } \beta = \bar{i} \text{ for } 1 \le i \le k, \\ -1 & \text{if } \alpha = \bar{i} \text{ and } \beta = i \text{ for } 1 \le i \le k, \\ 0 & \text{otherwise.} \end{cases}$$

Then the intermediate symplectic group $\mathbf{Sp}_{2k,n-k}$ is defined by

$$\mathbf{Sp}_{2k,n-k} = \{g \in \mathbf{GL}(V) : \langle gv, gw \rangle = \langle v, w \rangle \ (v, w \in V)\}.$$

If $k = 0$ or $k = n$, then

$$\mathbf{Sp}_{0,n} \cong \mathbf{GL}_n, \quad \mathbf{Sp}_{2n,0} \cong \mathbf{Sp}_{2n}.$$

If $k = n-1$, then the group $\mathbf{Sp}_{2k,1}$ is called the odd symplectic group.

Representations of intermediate symplectic groups

Recall that $\mathbf{Sp}_{2k,n-k} \subset \mathbf{GL}(V)$ with $V = \mathbb{C}^{n+k}$.

Theorem (Proctor) Let λ be a partition of d with length $\leq n$. Let

$V^\lambda =$ an irreducible $\mathbf{GL}_{n+k}$-submodule of $V^{\otimes d}$ corresp. to λ,

$V_0^\lambda =$ the trace-free subspace of V^λ.

Then

- V_0^λ is an indecomposable $\mathbf{Sp}_{2k,n-k}$-module.
- V_0^λ has a weight basis indexed by $(k, n-k)$-symplectic tableaux of shape λ.

Hence

$$\mathrm{sp}_\lambda^{(k,n-k)}(x_1, \ldots, x_k | x_{k+1}, \ldots, x_n) = \text{the character of } V_0^\lambda.$$

Determinant Formulas

Jacobi–Trudi formulas for s_λ and sp_λ

For a partition of length $\leq n$, we have

$$s_\lambda(x_1, \dots, x_n) = \det \Big(h_{\lambda_i - i + j}(x_1, \dots, x_n) \Big)_{1 \leq i, j \leq n},$$

and

$$\mathrm{sp}_\lambda(x_1, \dots, x_n) = \frac{1}{2} \det \begin{pmatrix} h_{\lambda_i - i + j}(x_1^{\pm 1}, \dots, x_n^{\pm 1}) \\ \qquad + h_{\lambda_i - i - j + 2}(x_1^{\pm 1}, \dots, x_n^{\pm 1}) \end{pmatrix}_{1 \leq i, j \leq n},$$

where $h_r(x_1, \dots, x_n)$ is the rth complete symmetric polynomial in x_1, $\dots, x_n$ and

$$h_r(x_1^{\pm 1}, \dots, x_n^{\pm 1}) = h_r(x_1, x_1^{-1}, \dots, x_n, x_n^{-1}).$$

Jacobi–Trudi formulas for $\mathrm{sp}^{(k,n-k)}$ (1/2)

By using a lattice path interpretation of $(k, n-k)$-symplectic tableaux and Lindström–Gessel–Viennot lemma, we have

Proposition For a partition λ of length $\leq n$, we have

$$\mathrm{sp}_\lambda^{(k,n-k)}(\boldsymbol{x}) = \det K_\lambda^{(k,n-k)},$$

where $K_\lambda^{(k,n-k)}$ is the $n \times n$ matrix with (i,j) entry given by

$$\begin{cases} h_{\lambda_i - i + j}(x_j, x_j^{-1}, \dots, x_k, x_k^{-1}, x_{k+1}, \dots, x_n) & \text{if } 1 \leq j \leq k, \\ h_{\lambda_i - i + j}(x_j, \dots, x_n) & \text{if } k+1 \leq j \leq n. \end{cases}$$

Jacobi–Trudi formulas for $\mathrm{sp}^{(k,n-k)}$ (2/2)

By performing column operations on the matrix $K_\lambda^{(k,n-k)}$, we obtain

Proposition For a partition λ of length $\le n$, we have

$$\mathrm{sp}_\lambda^{(k,n-k)}(\boldsymbol{x}) = \det H_\lambda^{(k,n-k)},$$

where $H_\lambda^{(k,n-k)}$ is the $n \times n$ matrix with (i,j) entry given by

$$\begin{cases} h_{(\lambda_i-i+1)}(x_1^{\pm 1}, \dots, x_k^{\pm 1}, x_{k+1}, \dots, x_n) & \text{if } j = 1, \\ h_{(\lambda_i-i+j)}(x_1^{\pm 1}, \dots, x_k^{\pm 1}, x_{k+1}, \dots, x_n) & \\ \quad + h_{(\lambda_i-i-j+2)}(x_1^{\pm 1}, \dots, x_k^{\pm 1}, x_{k+1}, \dots, x_n) & \text{if } 2 \le j \le k, \\ h_{\lambda_i-i+j}(x_{k+1}, \dots, x_n) & \text{if } k+1 \le j \le n. \end{cases}$$

This formula reduces to the Jacobi–Trudi formulas for Schur functions $(k=0)$ and symplectic Schur functions $(k=n)$.

Bialternant formulas for s_λ and sp_λ

The Schur functions and the symplectic Schur functions are expressed as ratios of two determinants:

$$s_\lambda(x_1, \dots, x_n) = \frac{\det\left(x_j^{\lambda_i+n-i}\right)_{1 \le i, j \le n}}{\det\left(x_j^{n-i}\right)_{1 \le i, j \le n}},$$

$$\mathrm{sp}_\lambda(x_1, \dots, x_n) = \frac{\det\left(x_j^{\lambda_i+n-i+1} - x_j^{-(\lambda_i+n-i+1)}\right)_{1 \le i, j \le n}}{\det\left(x_j^{n-i+1} - x_j^{-(n-i+1)}\right)_{1 \le i, j \le n}},$$

and

$$\det\left(x_j^{n-i}\right)_{1 \le i, j \le n} = \prod_{1 \le i < j \le n} (x_i - x_j),$$

$$\det\left(x_j^{n-i+1} - x_j^{-(n-i+1)}\right)_{1 \le i, j \le n}$$
$$= \prod_{i=1}^{n} (x_i - x_i^{-1}) \prod_{1 \le i < j \le n} \left(x_i^{1/2} x_j^{1/2} - x_i^{-1/2} x_j^{-1/2}\right)\left(x_i^{1/2} x_j^{-1/2} - x_i^{-1/2} x_j^{1/2}\right).$$

Bialternant formulas for $\mathrm{sp}^{(k,n-k)}$ (1/2)

Theorem Given a partition λ of length $\leq n$, we define $A_\lambda^{(k,n-k)}$ to be the $n \times n$ matrix with (i,j) entry given by

$$\begin{cases} h_{\lambda_i+k-i+1}(x_j, x_{k+1}, \dots, x_n) - h_{\lambda_i+k-i+1}(x_j^{-1}, x_{k+1}, \dots, x_n) & \text{if } 1 \leq j \leq k, \\ x_j^{\lambda_i+n-i} & \text{if } k+1 \leq j \leq n, \end{cases}$$

Then we have

$$\mathrm{sp}_\lambda^{(k,n-k)}(x_1, \dots, x_n) = \frac{\det A_\lambda^{(k,n-k)}}{\det A_\emptyset^{(k,n-k)}}.$$

This formula reduces to the bialternant formulas for Schur functions ($k=0$) and symplectic Schur functions ($k=n$).

Bialternant formulas for $\mathrm{sp}^{(k,n-k)}$ (1/2)

Theorem Given a partition λ of length $\leq n$, we define $A_\lambda^{(k,n-k)}$ to be the $n \times n$ matrix with (i,j) entry given by

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Then we have

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Remark The denominator factors as

$$\det A_\emptyset^{(k,n-k)} = \prod_{i=1}^{k} (x_i - x_i^{-1}) \prod_{1 \leq i < j \leq k} (x_i^{1/2} x_j^{1/2} - x_i^{-1/2} x_j^{-1/2})(x_i^{1/2} x_j^{-1/2} - x_i^{-1/2} x_j^{1/2})$$
$$\times \prod_{k+1 \leq i < j \leq n} (x_i - x_j).$$

Sketch of proof

We can find an $n \times n$ matrix M such that

$$A_\lambda^{(k,n-k)} = H_\lambda^{(k,n-k)} M$$

for any partitions λ of length $\leq n$. Since $H_\emptyset^{(k,n-k)}$ is a upper-triangular matrix with diagonal entries 1, we have

$$\det A_\emptyset^{(k,n-k)} = \det H_\emptyset^{(k,n-k)} \cdot \det M = \det M.$$

Hence we obtain

$$\det A_\lambda^{(k,n-k)} = \det H_\lambda^{(k,n-k)} \cdot \det M = \det H_\lambda^{(k,n-k)} \cdot \det A_\emptyset^{(k,n-k)},$$

and

$$\mathrm{sp}_\lambda^{(k,n-k)}(\boldsymbol{x}) = \det H_\lambda^{(k,n-k)} = \frac{\det A_\lambda^{(k,n-k)}}{\det A_\emptyset^{(k,n-k)}}.$$

Bialternant formulas for $\mathrm{sp}^{(k,n-k)}$ (2/2)

Corollary Given a partition λ of length $\leq n$, we define $\overline{A}_\lambda^{(k,n-k)}$ to be the $n \times n$ matrix with (i,j) entry given by

$$\overline{a}_{i,j} = \begin{cases} \dfrac{x_j^{\lambda_i+k-i+1}}{\prod_{l=k+1}^{n}(1-x_j^{-1}x_l)} - \dfrac{x_j^{-(\lambda_i+k-i+1)}}{\prod_{l=k+1}^{n}(1-x_j x_l)} & \text{if } 1 \leq j \leq k, \\ x_j^{\lambda_i+n-i} & \text{if } k+1 \leq j \leq n. \end{cases}$$

Then we have

$$\mathrm{sp}_\lambda^{(k,n-k)}(x_1,\dots,x_n) = \frac{\det \overline{A}_\lambda^{(k,n-k)}}{\det \overline{A}_\emptyset^{(k,n-k)}}.$$

This formula also reduces to the bialternant formulas for Schur functions ($k=0$) and symplectic Schur functions ($k=n$).

Application 1
Brent–Krattenthaler–Warnaar's identity

Brent–Krattenthaler–Warnaar's identity

Brent, Krattenthaler and Warnaar found the following identity in their study of discrete analogues of Macdonald–Mehta integrals.

Theorem (Brent–Krattenthaler–Warnaar) For a nonnegative integer r, we have

$$\sum_{\lambda \subset (r^{n+1})} z^{-r} \mathrm{sp}_\lambda^{(n,1)}(x_1, \dots, x_n | z) \, \mathrm{sp}_\lambda^{(n,1)}(y_1, \dots, y_n | z) = \mathrm{sp}_{(r^{2n+1})}(x_1, \dots, x_n, y_1, \dots, y_n, z),$$

where λ runs over all partitions with $l(\lambda) \le n+1$ and $\lambda_1 \le r$.

We can use the bialternant formula for $\mathrm{sp}_\lambda^{(n,1)}$ together with the Cauchy–Binet formula to give an alternate proof.

Sketch of proof (1/2)

Let $T(\boldsymbol{x}|z) = \left(t_{i,p}\right)_{1 \le i \le n+1,\, 0 \le p \le r+n}$ be the $(n+1) \times (r+n+1)$ matrix with (i,p) entry

$$t_{i,p} = \begin{cases} x_i^p/(1 - x_i^{-1}z) - x_i^{-p}/(1 - x_i z) & \text{if } 1 \le i \le n, \\ z^p & \text{if } i = n+1. \end{cases}$$

Then we have

$$\mathrm{sp}^{(n,1)}(x_1, \dots, x_n|z) = \frac{\det T(\lambda_1 + n, \lambda_2 + n - 1, \dots, \lambda_n + 1, \lambda_{n+1})}{\det T(n, n-1, \dots, 1, 0)},$$

where $T(I)$ is the submatrix of T obtained by picking up columns indexed by I. By applying the Cauchy–Binet formula

$$\sum_I \det X(I) \det Y(I) = \det \left(X {}^t Y \right),$$

we can express the summation of BKW formula in terms of the determinant of $T(\boldsymbol{x}|z) {}^t T(\boldsymbol{y}|z)$.

Sketch of proof (2/2)

We can evaluate $\det\left(T(\boldsymbol{x}|z)\,{}^{t}T(\boldsymbol{y}|z)\right)$ by using the following Lemma.

Lemma Let $C = \left(C_{i,j}\right)_{1\le i,\,j\le n+1}$ and $V = \left(V_{i,j}\right)_{1\le i,\,j\le 2n+1}$ be the matrices given by

$$C_{i,j} = \begin{cases} \dfrac{(1-x_i z)(1-y_j z)}{1-x_i y_j} - \dfrac{a_i(x_i - z)(1-y_j z)}{x_i - y_j} \\ \quad + \dfrac{b_j(1-x_i z)(y_j - z)}{x_i - y_j} - \dfrac{a_i b_j (x_i - z)(y_j - z)}{1 - x_i y_j} & \text{if } 1 \le i,\, j \le n, \\ 1 - a_i & \text{if } i = n+1 \text{ and } 1 \le j \le n, \\ 1 - b_j & \text{if } 1 \le i \le n \text{ and } j = n+1, \\ \dfrac{1-c}{1-z^2} & \text{if } i = j = n+1, \end{cases}$$

$$V_{i,j} = x_i^{j-1} - a_i x_i^{2n+1-j}, \quad V_{n+i,j} = y_i^{j-1} - b_i y_i^{2n+1-j}, \quad V_{2n+1,j} = z^{j-1} - c z^{2n+1-j}.$$

Then we have

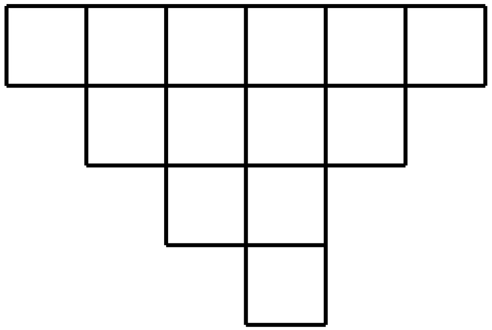
$$\det C = \frac{(-1)^n}{(1-z^2)\prod_{i=1}^{n}\prod_{j=1}^{n}(x_i - y_j)(1 - x_i y_j)} \det V.$$

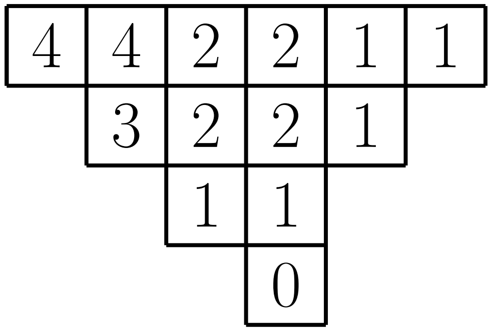
Application 2
Shifted Plane Partition Enumeration

Shifted plane partitions

Given a strict partition μ, a shifted plane partition of shape μ is a filling of the shifted Young diagram $S(\mu)$ with nonnegative integers where the entries are weakly decreasing along rows and down columns.

Example





are the shifted diagram of $(6,4,2,1)$ and a shifted plane partition of shape $(6,4,2,1)$ respectively.

We put

$$\mathcal{A}^m(S(\mu)) = \{\text{shifted plane partitions of shape } \mu \text{ with entries} \le m\}.$$

Shifted plane partitions of shifted double staircase shape

We put $\delta_r = (r, r-1, \ldots, 2, 1)$. Hopkins and Lai prove the following theorem by counting lozenge tilings of a certain region in the triangular lattice.

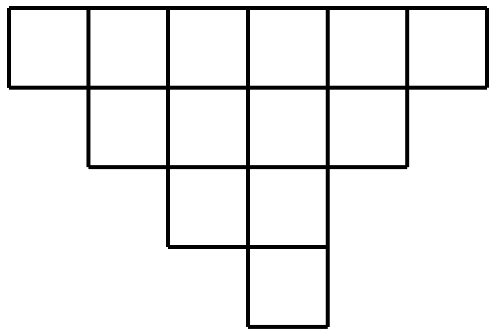
Theorem (Hopkins–Lai) If $0 \le k \le n$, then the number of shifted plane partitions of shape

$$\delta_n + \delta_k = (n+k, n+k-2, \ldots, n-k+2, n-k, n-k-1, \ldots, 2, 1)$$

with entries bounded by m is equal to

$$\#\mathcal{A}^m(S(\delta_n + \delta_k)) = \prod_{1 \le i \le j \le n} \frac{m+i+j-1}{i+j-1} \prod_{1 \le i \le j \le k} \frac{m+i+j}{i+j}.$$

$S(\delta_4 + \delta_2) =$



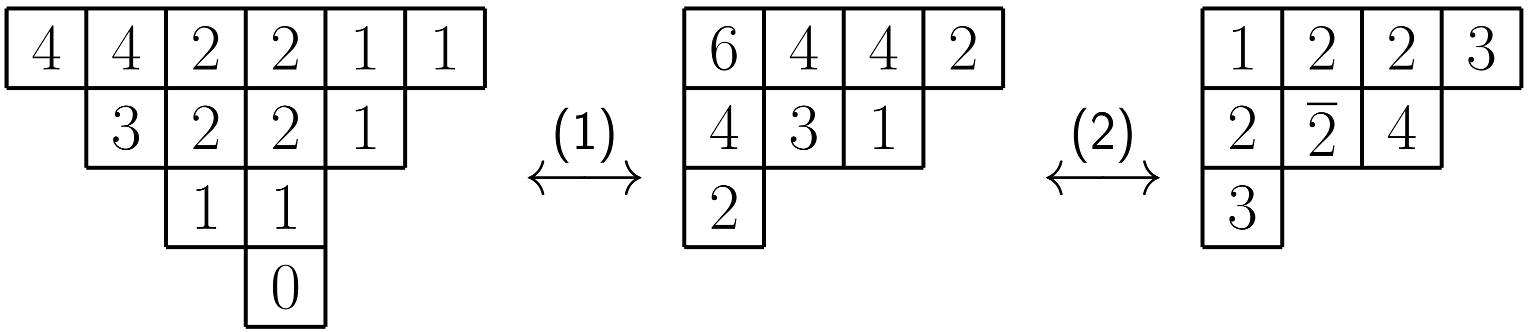
Bijection

For a shifted plane partition σ, we define the profile of σ to be the partition $(\sigma_{1,1}, \sigma_{2,2}, \dots)$.

Lemma For a partition λ, there exists a bijection between

- shifted plane partitions of shape $\delta_n + \delta_k$ with profile λ, and
- $(k, n-k)$-symplectic tableaux of shape λ.

Example If $n = 4$ and $k = 2$, then



(1) conjugate each row;
(2) replace $1, 2, 3, 4, 5, 6$ with $4, 3, \overline{2}, 2, \overline{1}, 1$ respectively.

Generating functions and intermediate symplectic characters

For $\sigma \in \mathcal{A}^m(S(\delta_n+\delta_k))$, we define

$$w(\sigma) = kt_0(\sigma) + \sum_{l=0}^{n-k-1} t_l(\sigma) - nt_{n-k}(\sigma) + \sum_{l=n-k}^{n+k-1} (-1)^{l-n+k+1}(l-n+k+1)t_l(\sigma),$$

$$v(\sigma) = \left(k - \frac{1}{2}\right) t_0(\sigma) + \sum_{l=0}^{n-k-1} t_l(\sigma) - nt_{n-k}(\sigma) + \sum_{l=n-k}^{n+k-1} (-1)^{l-n+k+1}(l-n+k)t_l(\sigma),$$

where $t_l(\sigma) = \sum_i \sigma_{i,i+l}$ is the lth trace of σ. If $k=0$, then we have

$$w(\sigma) = \sum_{1\le i\le j\le n} \sigma_{i,j}, \quad v(\sigma) = \frac{1}{2}\sum_{i=1}^{n} \sigma_{i,i} + \sum_{1\le i<j\le n} \sigma_{i,j},$$

and, if $k=n$, then we have

$$w(\sigma) = \sum_{l=0}^{2n-1}(-1)^{l+1}(l+1)t_l(\sigma), \quad v(\sigma) = -\frac{1}{2}t_0(\sigma) + \sum_{l=1}^{2n-1}(-1)^{l+1}lt_l(\sigma).$$

Generating functions and intermediate symplectic characters

For $\sigma \in \mathcal{A}^m(S(\delta_n + \delta_k))$, we define

$$w(\sigma) = kt_0(\sigma) + \sum_{l=0}^{n-k-1} t_l(\sigma) - nt_{n-k}(\sigma) + \sum_{l=n-k}^{n+k-1} (-1)^{l-n+k+1}(l-n+k+1)t_l(\sigma),$$

$$v(\sigma) = \left(k - \frac{1}{2}\right) t_0(\sigma) + \sum_{l=0}^{n-k-1} t_l(\sigma) - nt_{n-k}(\sigma) + \sum_{l=n-k}^{n+k-1} (-1)^{l-n+k+1}(l-n+k)t_l(\sigma),$$

where $t_l(\sigma) = \sum_i \sigma_{i,i+l}$ is the lth trace of σ.

Proposition For a fixed partition λ, the generating functions of shifted plane partitions of shape $\delta_n + \delta_k$ with profile λ are given by

$$\sum_{\sigma} q^{w(\sigma)} = \mathrm{sp}_{\lambda}^{(k,n-k)}(q, q^2, \dots, q^n),$$

$$\sum_{\sigma} q^{v(\sigma)} = \mathrm{sp}_{\lambda}^{(k,n-k)}(q^{1/2}, q^{3/2}, \dots, q^{n-1/2}).$$

Character identity

By using the bialternant formula and the Ishikawa–Wakayama minor summation formula, we can prove

Theorem Let $0 \le k \le n$. For a nonnegative integer m, we have

$$\sum_{\lambda \subset (m^n)} \mathrm{sp}_{\lambda}^{(k,n-k)}(x_1, \dots, x_k | x_{k+1}, \dots, x_n)$$
$$= \mathrm{o}^{B}_{((m/2)^n)}(x_1, \dots, x_n) \cdot \mathrm{sp}_{((m/2)^k)}(x_1, \dots, x_k) \cdot (x_{k+1} \cdots x_n)^{m/2},$$

where o^{B}_{ν} and sp_{ν} are the odd orthogonal and symplectic character corresponding to a partition ν respectively.

If $k = 0$, then we have

$$\sum_{\lambda \subset (m^n)} s_{\lambda}(x_1, \dots, x_n) = \mathrm{o}^{B}_{((m/2)^n)}(x_1, \dots, x_n) \cdot (x_1 \cdots x_n)^{m/2},$$

which Macdonald used to prove the MacMahon and Bender–Knuth conjectures on symmetric plane partitions.

Application to shifted plane partition enumeration

By specializing $x_i = q^i$ or $x_i = q^{i-1/2}$, we obtain

Corollary

$$\sum_{\sigma \in \mathcal{A}^m(S(\delta_n+\delta_k))} q^{w(\sigma)} = \frac{1}{q^{mk(k+1)/2}} \prod_{1 \le i \le j \le n} \frac{[m+i+j-1]}{[i+j-1]} \prod_{1 \le i \le j \le k} \frac{[m+i+j]}{[i+j]}.$$

$$\sum_{\sigma \in \mathcal{A}^m(S(\delta_n+\delta_k))} q^{v(\sigma)} = \frac{1}{q^{mk^2/2}} \prod_{i=1}^{n} \frac{[m/2+i-1/2]}{[i-1/2]} \prod_{1 \le i < j \le n} \frac{[m+i+j-1]}{[i+j-1]} \times \prod_{i=1}^{k} \frac{[m/2+i]}{[i]} \prod_{1 \le i < j \le k} \frac{[m+i+j]}{[i+j]}.$$

By putting $q = 1$, we have

Corollary (Hopkins–Lai)

$$\# \mathcal{A}^m(S(\delta_n + \delta_k)) = \prod_{1 \le i \le j \le n} \frac{m+i+j-1}{i+j-1} \prod_{1 \le i \le j \le k} \frac{m+i+j}{i+j}.$$

Variations

Theorem Let $0 \le k \le n$. For a nonnegative even integer m, we have

$$\sum_{\lambda \subset (m^n):\text{even}} \mathrm{sp}_\lambda^{(k,n-k)}(x_1, \dots, x_k | x_{k+1}, \dots, x_n)$$

$$= \mathrm{sp}_{((m/2)^n)}(x_1, \dots, x_n) \cdot \mathrm{sp}_{((m/2)^k)}(x_1, \dots, x_k) \cdot (x_{k+1} \cdots x_n)^{m/2}.$$

Theorem Let $0 \le k \le n$. For a nonnegative integer a and m, we have

$$\sum_{(a^n) \subset \lambda \subset ((a+m)^n)} \mathrm{sp}_\lambda^{(k,n-k)}(x_1, \dots, x_k | x_{k+1}, \dots, x_n)$$

$$= \mathrm{o}^B_{((m/2)^n)}(x_1, \dots, x_n) \cdot \mathrm{sp}_{((m/2+a)^k)}(x_1, \dots, x_k) \cdot (x_{k+1} \cdots x_n)^{m/2+a}.$$

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