Toward $A_2^{(2)}$ Andrews–Gordon identities

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(arXiv:1910.12461 and 2006.02630, joint with Shunsuke Tsuchioka)

Rogers–Ramanujan identities (Rogers 1894, Ramanujan 1913)

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty}, \qquad \sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}$$

where $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$, $(x,y,\cdots;q)_n = (x;q)_n(y;q)_n\cdots$.

Rogers–Ramanujan partition theorem (Schur 1917, MacMahon 1918)

$$\mathcal{R} := \{\lambda \in \mathsf{Par} \mid 1 \leq \forall i < l(\lambda),\ \lambda_i - \lambda_{i+1} \geq 2\},$$
$$\mathcal{R}' := \{\lambda \in \mathcal{R} \mid m_1(\lambda) = 0\}.$$

Then

$$\mathcal{R} \stackrel{\mathsf{PT}}{\sim} \{\lambda \in \mathsf{Par} \mid \text{parts} \equiv 1,4 \pmod 5\},$$
$$\mathcal{R}' \stackrel{\mathsf{PT}}{\sim} \{\lambda \in \mathsf{Par} \mid \text{parts} \equiv 2,3 \pmod 5\}.$$

- $\mathsf{Par} := \{\lambda = (\lambda_1,\cdots,\lambda_\ell) \mid \lambda_1 \geq \cdots \geq \lambda_\ell \geq 1\}$
  - $|\lambda| := \sum_i \lambda_i$: size
  - $\ell(\lambda) := \ell$: length
  - $m_i(\lambda) := |\{j \mid \lambda_j = i\}|$.
- $\mathsf{Par}(n) := \{\lambda \in \mathsf{Par} \mid |\lambda| = n\}$
- For $\mathcal{C},\mathcal{D} \subseteq \mathsf{Par}$, $\mathcal{C} \stackrel{\mathsf{PT}}{\sim} \mathcal{D} :\iff \forall n \geq 0,\ |\mathcal{C}\cap \mathsf{Par}(n)| = |\mathcal{D}\cap\mathsf{Par}(n)|$.

Example

$$\{\lambda \mid \forall i,\ \lambda_i - \lambda_{i+1} \geq 2\} \overset{\text{PT}}{\sim} \{\lambda \mid \forall i,\ \lambda_i \equiv 1, 4 \pmod 5\}$$

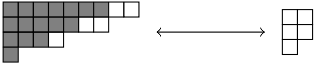
$$\begin{aligned} 9 &= 9 \\ &= 8+1 \\ &= 7+2 \\ &= 6+3 \\ &= 5+3+1 \end{aligned} \qquad \begin{aligned} 9 &= 9 \\ &= 6+1+1+1 \\ &= 4+4+1 \\ &= 4+1+1+1+1+1 \\ &= 1+\cdots+1 \end{aligned}$$

$$\{\lambda \mid \forall i,\ \lambda_i - \lambda_{i+1} \geq 2 \text{ and } m_1(\lambda) = 0\} \overset{\text{PT}}{\sim} \{\lambda \mid \forall i,\ \lambda_i \equiv 2, 3 \pmod 5\}$$

$$\begin{aligned} 9 &= 9 \\ &= 7+2 \\ &= 6+3 \end{aligned} \qquad \begin{aligned} 9 &= 7+2 \\ &= 3+3+3 \\ &= 3+2+2+2 \end{aligned}$$

Equivalence of "sum sides"

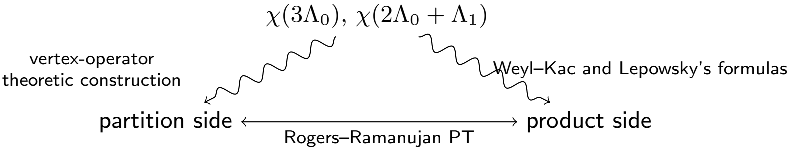
$$\sum_{\lambda \in \mathcal{R}} q^{|\lambda|} = \sum_{n \geq 0} \sum_{\substack{\ell(\lambda)=n \\ \lambda_i - \lambda_{i+1} \geq 2\ (\forall i)}} q^{|\lambda|} = \sum_{n \geq 0} q^{n^2} \sum_{\ell(\lambda) \leq n} q^{|\lambda|} = \sum_{n \geq 0} \frac{q^{n^2}}{(1-q)\cdots(1-q^n)}$$



$(9, 7, 4, 1)$ $\qquad$ $(2, 2, 1)$

Lepowsky–Wilson (1970-80s): Lie-theoretic proof for Rogers–Ramanujan identities

- Based on Lepowsky–Milne's observation (1978)
- By computing principally specialized character of level 3 standard modules of the affine Lie algebra $A_1^{(1)}$ in two ways:



(In fact, the RRPT was one of the motivations for inventing the vertex operators)

X: affine Dynkin diagram, $\mathfrak{g}(X)$: affine Lie algebra

For $n_i \in \mathbb{Z}_{\geq 0}$ $(i \in \{\text{vertices of } X\})$,

- $V = V(\sum n_i \Lambda_i)$: irreducible integrable highest weight module (a.k.a. standard module)
- $\Omega(V)$: vacuum space w.r.t. the principal Heisenberg subalgebra
- $\chi(\sum n_i \Lambda_i)$: principally specialized character of $\Omega(V)$

Andrews–Gordon identities

Theorem (Gordon (1961))

For $k \geq 2$ and $1 \leq i \leq k$,

$$\{\lambda \mid \forall j,\ \lambda_j - \lambda_{j+k-1} \geq 2 \text{ and } m_1(\lambda) \leq i-1\} \overset{PT}{\sim} \{\lambda \mid \text{parts} \not\equiv 0, \pm i \pmod{2k+1}\}.$$

Theorem (Andrews (1974))

For $k \geq 2$ and $1 \leq i \leq k$,

$$\sum_{n_1,\cdots,n_{k-1}\geq 0} \frac{q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}} = \prod_{\substack{j>0 \\ j\not\equiv 0,\pm i \pmod{2k+1}}} \frac{1}{1-q^j}$$

where $N_i = n_i + \cdots + n_{k-1}$.

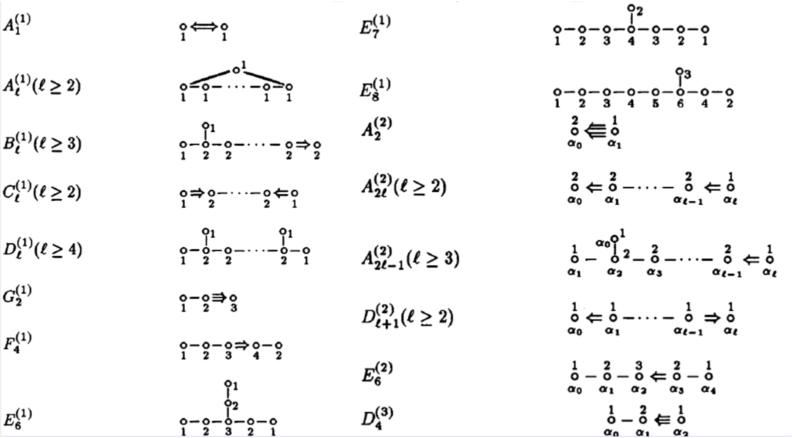
- $k = 2 \rightsquigarrow$ Rogers–Ramanujan
- There is a similar generalization for even moduli (Andrews–Bressoud identities (1979, 1980)).

Furthermore,

type	level	identities
$A_1^{(1)}$	3	Rogers–Ramanujan (mod 5)
$A_1^{(1)}$	$2k+1$	Andrews–Gordon (mod $2k+3$)
$A_1^{(1)}$	$2k$	Andrews–Bressoud (mod $2k+2$)

Big Picture

Choose $\mathfrak{g}(X)$ and $\sum_i m_i \Lambda_i$ $(m_i \geq 0)$ $\rightsquigarrow$ "Rogers–Ramanujan type" $\begin{cases} \text{partition theorem} \\ q\text{-series identity} \end{cases}$?



Big Picture

Choose $\mathfrak{g}(X)$ and $\sum_i m_i\Lambda_i$ $(m_i \geq 0)$ $\rightsquigarrow$ "Rogers–Ramanujan type" $\begin{cases} \text{partition theorem} \\ q\text{-series identity} \end{cases}$?

RR-type PT: typically

$$\{\lambda \in \text{Par} \mid \text{difference condition} \ + \ \text{forbidden patterns}\} \ \text{“}\overset{\text{PT}}{\sim}\text{”} \ \chi\Big(\sum_i m_i\Lambda_i\Big)$$

RR-type identity (one possible formulation):

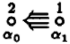
$$\sum_{n_1,\cdots,n_k\geq 0} \frac{(-1)^{\sum_i s_i n_i} q^{\sum_i A_i\binom{n_i}{2}+\sum_{i<j} B_{ij}n_in_j+\sum_i C_in_i}}{(q^{D_1};q^{D_1})_{n_1}\cdots(q^{D_k};q^{D_k})_{n_k}} = \chi\Big(\sum_i m_i\Lambda_i\Big)$$

("Andrews–Gordon type" sum)

- product side: computed via Weyl–Kac and Lepowsky's formulas
- partition and multisum sides: vague

$A_2^{(2)}$ case

level of $m_0\Lambda_0 + m_1\Lambda_1 = m_0 + 2m_1$



From vertex operator theoretic construction

level 2 Rogers–Ramanujan$|_{q\mapsto q^2}$ (mod 10)

level 3 Conjecture: Capparelli (1988)

Proof: Andrews(1994), Tamba–Xie(1995), Capparelli(1996) etc.

level 4 Conjecture: Nandi (2014)

Proof: T.–Tsuchioka (arXiv:1910.12461)

Higher levels

- Hirschhorn (1979), Capprelli (2004): partition theorems regarding level 5,7
- McLaughlin–Sills (2008): single sum=product identities regarding level 6
- T.–Tsuchioka (arXiv:2006:02630): AG-type sum=product q-series identities regarding level 5, 7
- Kanade–Russell (arXiv:2010.01008): different sum=product identities for all level

In each case, it is unclear whether they are related to vertex operators

$A_2^{(2)}$ level 4 identities (conjectured by Nandi (2014))

Theorem (T.–Takigiku, arXiv:1910.12461)

(N1) $\lambda_i - \lambda_{i+1} \neq 1$,

(N2) $\lambda_i - \lambda_{i+2} \geq 3$,

(N3) $\lambda_i - \lambda_{i+2} = 3 \implies \lambda_i \neq \lambda_{i+1}$,

(N4) $\lambda_i - \lambda_{i+2} = 3$ *and* $2 \nmid \lambda_i \implies \lambda_{i+1} \neq \lambda_{i+2}$,

(N5) $\lambda_i - \lambda_{i+2} = 4$ *and* $2 \nmid \lambda_i \implies \lambda_i \neq \lambda_{i+1}$ *and* $\lambda_{i+1} \neq \lambda_{i+2}$,

(N6) *No subwords of* $(\lambda_1 - \lambda_2, \lambda_2 - \lambda_3, \cdots, \lambda_{l(\lambda)-1} - \lambda_{l(\lambda)})$ *match the pattern* $(3, 2^*, 3, 0)$.

(2^*: *zero or more repetitions of* 2)

$$\mathcal{N} := \{\lambda \in \text{Par} \mid \text{(N1)-(N6)}\},$$
$$\mathcal{N}_1 := \{\lambda \in \mathcal{N} \mid m_1(\lambda) = 0\},$$
$$\mathcal{N}_2 := \{\lambda \in \mathcal{N} \mid m_i(\lambda) \leq 1 \text{ for } i = 1, 2, 3\},$$
$$\mathcal{N}_3 := \left\{ \lambda \in \mathcal{N} \;\middle|\; \begin{array}{c} m_1(\lambda) = m_3(\lambda) = 0,\ m_2(\lambda) \leq 1 \\ \lambda \text{ does not contain } (2k+3, 2k, 2k-2, \cdots, 4, 2) \text{ for } k \geq 1 \end{array} \right\}.$$

Then

$$\mathcal{N}_1 \overset{PT}{\sim} \{\lambda \mid \text{parts} \equiv \pm 2, \pm 3, \pm 4 \pmod{14}\},$$
$$\mathcal{N}_2 \overset{PT}{\sim} \{\lambda \mid \text{parts} \equiv \pm 1, \pm 4, \pm 6 \pmod{14}\},$$
$$\mathcal{N}_3 \overset{PT}{\sim} \{\lambda \mid \text{parts} \equiv \pm 2, \pm 5, \pm 6 \pmod{14}\}.$$

A common recipe for proving a partition identity $\mathcal{C} \overset{\mathsf{PT}}{\sim} \mathcal{D}$ (of RR type)

$\mathcal{C}$: difference conditions, $\mathcal{D}$: mod conditions

Step 1 Find a q-difference equation for the (x,q)-generating series $f_{\mathcal{C}}(x,q) := \sum_{\lambda \in \mathcal{C}} x^{\ell(\lambda)} q^{|\lambda|}$.

(e.g.) Rogers–Ramanujan: $f_{\mathcal{R}}(x,q) = f_{\mathcal{R}}(xq,q) + xq f_{\mathcal{R}}(xq^2,q)$

Step 2 Solve the q-difference equation and obtain a q-series expression for $f_{\mathcal{C}}(1,q)$.

(e.g.) $f_{\mathcal{R}} =: \sum_n A_n(q) x^n$

$$\rightsquigarrow \quad A_n = q^n A_n + q^{2n-1} A_{n-1}.$$

$$\therefore A_n = \frac{q^{2n-1}}{1-q^n} A_{n-1} = \frac{q^{n^2}}{(q;q)_n}. \quad \therefore f_{\mathcal{R}}(x,q) = \sum_{n \geq 0} \frac{x^n q^{n^2}}{(q;q)_n}.$$

Step 3 Use some formulas (theorems) on q-series and prove $f_{\mathcal{C}}(1,q) = f_{\mathcal{D}}(1,q)$.

(e.g.) $$\sum_{n \geq 0} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty}$$

- in this case, it's just the Rogers–Ramanujan (q-series) identity
- Slater (1952) gave > 100 similar identities (single sum $=$ product), based on Bailey's ${}_6\psi_6$ summation formula and "Bailey's Lemma". Now almost 200 such identities are recorded in a recently published book of Sills.

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In RR case, this equation is algorithmically obtained via the theory of *linked partition ideals* by Andrews.

- Par $\ni \lambda \leftrightarrow (f_1, f_2, \dots)$ where $f_i = m_i(\lambda)$
- $\pi_A := \emptyset$, $\pi_B := (1)$
- $\mathcal{R} = \{\lambda \mid \lambda_i - \lambda_{i+1} \geq 2\} \leftrightarrow \{(f_1, f_2, \dots) \mid \forall j,\ f_j + f_{j+1} \leq 1\}$

  $\leftrightarrow$ infinite sequences in $\{A, B\}$ with "BB" forbidden

  (e.g.) $\lambda = (6,3,1) \leftrightarrow (1,0,1,0,0,1,0,0,\dots) \leftrightarrow BABAABAA\cdots$

  $$\rightsquigarrow \begin{pmatrix} F_A(x) \\ F_B(x) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ xq & 0 \end{pmatrix} \begin{pmatrix} F_A(xq) \\ F_B(xq) \end{pmatrix} \text{ where } F_X(x) := \sum_{\substack{\lambda\in\mathcal{R} \\ \text{begin with X}}} x^{\ell(\lambda)} q^{|\lambda|}$$

  $\rightsquigarrow$ qdiff. eq. for $f_{\mathcal{R}}(x,q) = F_A(xq^{-1})$ can be obtained

Linked partition ideal $=: \mathcal{C}$ ($\subseteq$ Par) which can be encoded to infinite sequences with *forbidden patterns of finite length*

- Applicable to most of known RR-type partition theorems
- Not applicable to Nandi's partitions $\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3$ (due to "$*$" in (N6))

Theorem (T.–Tsuchioka)

If $\mathcal{C} \subseteq \mathrm{Par}$ *can be encoded to infinite sequences with forbidden patterns given by a regular expression, then a* q*-difference equation for* $f_{\mathcal{C}}(x,q)$ *can be algorithmically obtained.*

Regular expression (on a finite set Σ):

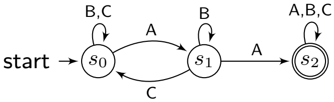
- $\emptyset$, $\{a\}$ $(a \in \Sigma)$, Σ are regular expr.
- Closed under $\cap$, $\cup$, concatenation, $*$, ^{c}

Point: for a regular expression X on Σ,

$$\text{a sequence in } \Sigma \text{ not containing a pattern in } X \longleftrightarrow \text{a “walk” on } \exists \textit{ finite automata}$$

(e.g.) $\Sigma = \{A, B, C\}$, $X = AB^*A$. For a sequence in Σ,

AB^*A is forbidden $\Longleftrightarrow$ never falling into s_2 in



- $\mathcal{N} \ni \lambda \leftrightarrow (\underbrace{f_1, f_2}, \underbrace{f_3, f_4}, \underbrace{f_5, f_6}, \cdots) \leftrightarrow$ infinite sequence in $\{0,1,2,3,4\}$

  $\forall k \geq 1,\ (f_{2k+1}, f_{2k+2}) \in \{\pi_0 = \emptyset, \pi_1 = (2), \pi_2 = (2,2), \pi_3 = (1), \pi_4 = (1,1)\}$

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$\longleftrightarrow$ infinite sequence in $\{0,1,2,3,4\}$ with forbidden patterns:

$$\begin{array}{ll} (1,\{2,3,4\}), & (2,\{1,2,3,4\}), \\ (3,\{2,4\}), & (4,\{2,3,4\}), \\ (1,0,4), & (2,0,\{3,4\}), \\ (3,0,4), & (4,1^*,0,\{3,4\}). \end{array}$$

Here,
$\{a, b, \cdots\}$ is exactly one of $a, b, \cdots$,
a^* is zero or more reps of a.

$+$

Forbidden prefixes:

$\mathcal{N}_1$: (3), (4)

$\mathcal{N}_2$: (2), (4), $(0,4)$

$\mathcal{N}_3$: (2), (3), (4), $(0,4)$, $(1^*,0,3)$

- $\mathcal{N} \ni \lambda \leftrightarrow (\underbrace{f_1, f_2}, \underbrace{f_3, f_4}, \underbrace{f_5, f_6}, \cdots) \leftrightarrow$ infinite sequence in $\{0,1,2,3,4\}$

  $\forall k \geq 1,\ (f_{2k+1}, f_{2k+2}) \in \{\pi_0 = \emptyset, \pi_1 = (2), \pi_2 = (2,2), \pi_3 = (1), \pi_4 = (1,1)\}$

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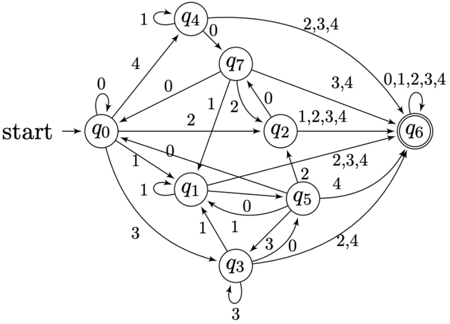
input sequence for the below automaton s.t. it never falls into q_6 $\longleftrightarrow$ infinite sequence in $\{0,1,2,3,4\}$ with forbidden patterns:



$$(1,\{2,3,4\}),\ (2,\{1,2,3,4\}),$$
$$(3,\{2,4\}),\ (4,\{2,3,4\}),$$
$$(1,0,4),\ (2,0,\{3,4\}),$$
$$(3,0,4),\ (4,1^*,0,\{3,4\}).$$

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For $\mathcal{N}_1$ (resp. $\mathcal{N}_2$, $\mathcal{N}_3$), the start state is changed to q_7 (resp. q_3, q_4)

$$\rightsquigarrow \begin{pmatrix} F_0(x) \\ F_1(x) \\ F_2(x) \\ F_3(x) \\ F_4(x) \\ F_5(x) \\ F_7(x) \end{pmatrix} = \begin{pmatrix} 1 & xq^2 & x^2q^4 & xq & x^2q^2 & 0 & 0 \\ 0 & xq^2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & xq^2 & 0 & xq & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & xq^2 & 0 & 1 \\ 1 & xq^2 & x^2q^4 & xq & 0 & 0 & 0 \\ 1 & xq^2 & x^2q^4 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} F_0(xq^2) \\ F_1(xq^2) \\ F_2(xq^2) \\ F_3(xq^2) \\ F_4(xq^2) \\ F_5(xq^2) \\ F_7(xq^2) \end{pmatrix}$$

where
$$f_{\mathcal{N}_1}(x,q) = F_7(x),$$
$$f_{\mathcal{N}_2}(x,q) = F_3(x),$$
$$f_{\mathcal{N}_3}(x,q) = F_4(x).$$

Now Step 1 for Nandi's partitions is done:

q-diff. eq. for $\mathcal{N}_1$: ($\mathcal{N}_2$, $\mathcal{N}_3$: similar)

$$0 = \sum_{i=0}^{5} p_{2i}(x,q) f_{\mathcal{N}_1}(xq^{2i}, q), \qquad \text{where} \quad \begin{cases} p_0(x,q) = 1, \\ p_2(x,q) = -1 - x\left(q^2+q^3+q^4\right), \\ p_4(x,q) = xq^4\left(1 - x + xq^3 + xq^4 + xq^5\right), \\ p_6(x,q) = x^2q^6\left(-1 + xq^4(1+q+q^2-q^5)\right), \\ p_8(x,q) = x^3q^{13}(1+q+q^2)(1-xq^6), \\ p_{10}(x,q) = x^3q^{17}(1-xq^6)(1-xq^8). \end{cases}$$

(Step 2) By standard arguments (and some miracles) we can solve them:

$$f_{\mathcal{N}_1}(x,q) = (-q;q)_\infty \sum_{n\geq 0} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}(q^2;q^2)_n} \qquad \text{etc.}$$

(Step 3) Use three formulas of Slater (1952): $\displaystyle\sum_{n\geq 0} \frac{(-1)^n q^{n^2}}{(-q;q)_{2n}(q^2;q^2)_n} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \frac{(q^6,q^8,q^{14};q^{14})_\infty}{(q^3,q^{11};q^{14})_\infty}$ etc.

Simultaneously we get

$$\sum_{i,j\geq 0} \frac{(-1)^j q^{\binom{i}{2}+2\binom{j}{2}+2ij+A_a(i,j)}}{(q;q)_i(q^2;q^2)_j} = \begin{cases} 1/(q^2,q^3,q^4,q^{10},q^{11},q^{12};q^{14})_\infty & (a=1), \\ 1/(q^1,q^4,q^6,q^8,q^{10},q^{13};q^{14})_\infty & (a=2), \\ 1/(q^2,q^5,q^6,q^8,q^9,q^{12};q^{14})_\infty & (a=3) \end{cases}$$

where $A_1(i,j) = i+j$, $A_2(i,j) = i+3j$, $A_3(i,j) = 2i+3j$.

$\rightsquigarrow$ matches the "Big Picture" (multisum = product). What about higher levels?

Theorem (T.–Tsuchioka, arXiv:2006:02630)

$[x;q]_\infty := (x, q/x; q)_\infty, \quad [x, y, \dots; q]_\infty := [x;q]_\infty [y;q]_\infty \cdots.$

Level 5:

$$\sum_{i,j,k\geq 0} (-1)^k \frac{q^{\binom{i}{2}+8\binom{j}{2}+2\binom{k}{2}+2ij+2ik+4jk+i+5j+k}}{(q;q)_i(q^2;q^2)_j(q^2;q^2)_k} = \frac{1}{[q^2,q^3,q^4,q^5;q^{16}]_\infty} \left(= \chi_{A_2^{(2)}}(5\Lambda_0)\right)$$

$$\sum_{i,j,k\geq 0} (-1)^k \frac{q^{\binom{i}{2}+8\binom{j}{2}+2\binom{k}{2}+2ij+2ik+4jk+i+7j+3k}}{(q;q)_i(q^2;q^2)_j(q^2;q^2)_k} = \frac{1}{[q,q^4,q^6,q^7;q^{16}]_\infty} \left(= \chi_{A_2^{(2)}}(\Lambda_0+2\Lambda_1)\right)$$

(*note:* $\chi_{A_2^{(2)}}(3\Lambda_0+\Lambda_1) = \frac{1}{[q,q^3,q^5,q^7;q^{16}]_\infty} = \frac{1}{(q;q^2)_\infty}$)

Level 7:

$$\sum_{i,j,k\geq 0} \frac{q^{\binom{i}{2}+8\binom{j}{2}+10\binom{k}{2}+2ij+2ik+8jk+i+4j+5k}}{(q;q)_i(q^2;q^2)_j(q^2;q^2)_k} = \frac{1}{[q,q^3,q^4,q^5,q^7,q^9;q^{20}]_\infty} \left(= \chi_{A_2^{(2)}}(5\Lambda_0+\Lambda_1)\right)$$

$$\sum_{i,j,k\geq 0} \frac{q^{\binom{i}{2}+8\binom{j}{2}+10\binom{k}{2}+2ij+2ik+8jk+i+8j+9k}}{(q;q)_i(q^2;q^2)_j(q^2;q^2)_k} = \frac{1}{[q,q^3,q^5,q^7,q^8,q^9;q^{20}]_\infty} \left(= \chi_{A_2^{(2)}}(\Lambda_0+3\Lambda_1)\right)$$

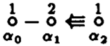
$$\sum_{i,j,k,\ell\geq 0} (-1)^k \frac{q^{\binom{i}{2}+2\binom{j}{2}+2\binom{k}{2}+8\binom{\ell}{2}+ij+ik+2i\ell+4jk+4j\ell+4k\ell+i+3j+k+6\ell}}{(q;q)_i(q^2;q^2)_j(q^2;q^2)_k(q^4;q^4)_\ell} = \frac{1}{[q^2,q^3,q^4,q^5,q^6,q^7;q^{20}]_\infty} \left(= \chi_{A_2^{(2)}}(7\Lambda_0)\right)$$

$$\sum_{i,j,k,\ell\geq 0} (-1)^k \frac{q^{\binom{i}{2}+2\binom{j}{2}+2\binom{k}{2}+8\binom{\ell}{2}+ij+ik+2i\ell+4jk+4j\ell+4k\ell+i+2j+4k+8\ell}}{(q;q)_i(q^2;q^2)_j(q^2;q^2)_k(q^4;q^4)_\ell} = \frac{1}{[q,q^2,q^5,q^6,q^8,q^9;q^{20}]_\infty} \left(= \chi_{A_2^{(2)}}(3\Lambda_0+2\Lambda_1)\right)$$

Final step of the proof: use Slater's formulas

$$\sum_{n\geq 0} \frac{q^{2n^2}}{(q;q)_{2n}} = \chi_{A_2^{(2)}}(5\Lambda_0), \quad \sum_{n\geq 0} \frac{q^{2n^2+2n}}{(q;q)_{2n+1}} = \chi_{A_2^{(2)}}(\Lambda_0+2\Lambda_1), \quad \sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_{2n}} = \chi_{A_2^{(2)}}(7\Lambda_0), \quad \sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_{2n+1}} = \chi_{A_2^{(2)}}(3\Lambda_0+2\Lambda_1).$$

Kanade–Russell's conjectures

Supposedly related to $D_4^{(3)}$ level 3 modules (level of $a\Lambda_0 + b\Lambda_1 + c\Lambda_2 = a + 2b + 3c$) 

$$\chi_{D_4^{(3)}}(3\Lambda_0) = \frac{1}{(q^2, q^3, q^6, q^7; q^9)_\infty}, \chi_{D_4^{(3)}}(\Lambda_0 + \Lambda_1) = \frac{1}{(q, q^3, q^6, q^8; q^9)_\infty}, \chi_{D_4^{(3)}}(\Lambda_2) = \frac{1}{(q^3, q^4, q^5, q^6; q^9)_\infty}.$$

Conjecture (Kanade–Russell, 2014)

Let

$$\mathcal{K} = \left\{ \lambda \in \mathsf{Par} \;\middle|\; \begin{array}{c} \lambda_i - \lambda_{i+2} \geq 3, \\ \lambda_i - \lambda_{i+1} \leq 1 \implies 3 | \lambda_i + \lambda_{i+1} \end{array} \right\}.$$

Then

$$\begin{aligned} \mathcal{K} &\overset{PT}{\sim} \{\lambda \mid \textit{parts} \equiv \pm 1, \pm 3 \pmod 9\}, \\ \{\lambda \in \mathcal{K} \mid m_1(\lambda) = 0\} &\overset{PT}{\sim} \{\lambda \mid \textit{parts} \equiv \pm 2, \pm 3 \pmod 9\}, \\ \{\lambda \in \mathcal{K} \mid m_1(\lambda) = m_2(\lambda) = 0\} &\overset{PT}{\sim} \{\lambda \mid \textit{parts} \equiv \pm 3, \pm 4 \pmod 9\}. \end{aligned}$$

Multisum expression (Kurşungöz, 2018)

$$\sum_{\lambda \in \mathcal{K}} q^{|\lambda|} = \sum_{n,m \geq 0} \frac{q^{n^2+3m^2+3nm}}{(q;q)_n (q^3;q^3)_m} \qquad \text{etc.}$$

$A_{\text{odd}}^{(2)}$ level 2 identities

Another seemingly interesting family of partition identities:

- $A_3^{(2)}$ Alladi–Schur's identities
- $A_5^{(2)}$ Göllnitz–Gordon identities
- $A_7^{(2)}$ Rogers–Ramanujan identities
- $A_9^{(2)}$ Kanade–Russell's conjecture (2018), proved by Bringmann–Jennings-Shaffer–Mahlburg (2019)

(In general, $\chi_{A_{2n-1}^{(2)}}((\delta_{i0}+\delta_{i1})\Lambda_0+\Lambda_i) = \frac{(q^{2n+2};q^{2n+2})_\infty}{(q^2;q^2)_\infty}\frac{[q^{2i};q^{2n+2}]_\infty}{[q^i;q^{2n+2}]_\infty}$)

Interestingly, the product sides of ($A_2^{(2)}$, level 4) are (some of) those of ($A_{11}^{(2)}$, level 2).

$$\sum_{i,k\geq 0}(-1)^k\frac{q^{\binom{i}{2}+2\binom{k}{2}+2ik+i+k}}{(q;q)_i(q^2;q^2)_k} = \frac{1}{[q^2,q^3,q^4;q^{14}]_\infty}(=\chi_{A_2^{(2)}}(4\Lambda_0)=\chi_{A_{11}^{(2)}}(\Lambda_3)),$$

$$\sum_{i,k\geq 0}(-1)^k\frac{q^{\binom{i}{2}+2\binom{k}{2}+2ik+i+3k}}{(q;q)_i(q^2;q^2)_k} = \frac{1}{[q,q^4,q^6;q^{14}]_\infty}(=\chi_{A_2^{(2)}}(2\Lambda_0+\Lambda_1)=\chi_{A_{11}^{(2)}}(\Lambda_0+\Lambda_1)),$$

$$\sum_{i,k\geq 0}(-1)^k\frac{q^{\binom{i}{2}+2\binom{k}{2}+2ik+2i+3k}}{(q;q)_i(q^2;q^2)_k} = \frac{1}{[q^2,q^5,q^6;q^{14}]_\infty}(=\chi_{A_2^{(2)}}(2\Lambda_1)=\chi_{A_{11}^{(2)}}(\Lambda_5)).$$

Conjectures for $A_{13}^{(2)}$ level 2 (T.–Tsuchioka, arXiv:2006:02630)

(1) $F_1(2,2,2) = \chi_{A_{13}^{(2)}}(\Lambda_3)$, $F_1(4,2,6) = \chi_{A_{13}^{(2)}}(\Lambda_5)$ and $F_1(6,4,6) = \chi_{A_{13}^{(2)}}(\Lambda_7)$, where

$$F_1(a,b,c) := \sum_{i,j,k\geq 0} (-1)^k \frac{q^{4\binom{i}{2}+2\binom{j}{2}+4\binom{k}{2}+2ij+4ik+4jk+ai+bj+ck}}{(q;q)_i(q^2;q^2)_j(q^4;q^4)_k}.$$

(2) $F_2(1,3,12) = \chi_{A_{13}^{(2)}}(\Lambda_0+\Lambda_1)$, $F_2(1,1,8) = \chi_{A_{13}^{(2)}}(\Lambda_3)$ and $F_2(3,3,16) = \chi_{A_{13}^{(2)}}(\Lambda_7)$, where

$$F_2(a,b,c) := \sum_{i,j,k\geq 0} (-1)^j \frac{q^{\binom{i}{2}+2\binom{j}{2}+16\binom{k}{2}+2ij+4ik+4jk+ai+bj+ck}}{(q;q)_i(q^2;q^2)_j(q^4;q^4)_k}.$$

(3) $F_3(1,5,1,12) = \chi_{A_{13}^{(2)}}(\Lambda_5)$, where

$$F_3(a,b,c,d) := \sum_{i,j,k,\ell\geq 0} (-1)^k \frac{q^{2\binom{i}{2}+4\binom{j}{2}+2\binom{k}{2}+16\binom{\ell}{2}+2ij+2ik+4i\ell+4jk+8j\ell+4k\ell+ai+bj+ck+d\ell}}{(q;q)_i(q^2;q^2)_j(q^2;q^2)_k(q^4;q^4)_\ell}.$$

Conjectures for $A_{13}^{(2)}$ level 2 (T.–Tsuchioka, arXiv:2006:02630)

(2) $F_2(1,3,12)=\chi_{A_{13}^{(2)}}(\Lambda_0+\Lambda_1)$, $F_2(1,1,8)=\chi_{A_{13}^{(2)}}(\Lambda_3)$ and $F_2(3,3,16)=\chi_{A_{13}^{(2)}}(\Lambda_7)$, where

$$F_2(a,b,c):=\sum_{i,j,k\geq 0}(-1)^j\frac{q^{\binom{i}{2}+2\binom{j}{2}+16\binom{k}{2}+2ij+4ik+4jk+ai+bj+ck}}{(q;q)_i(q^2;q^2)_j(q^4;q^4)_k}.$$

Observation: Letting $k=0$ in (2), two of $A_{11}^{(2)}$ level 2 sums appear:

$$\sum_{i,j\geq 0}(-1)^j\frac{q^{\binom{i}{2}+2\binom{j}{2}+2ij+i+j}}{(q;q)_i(q^2;q^2)_j}=\frac{1}{[q^2,q^3,q^4;q^{14}]_\infty}(=\chi_{A_2^{(2)}}(4\Lambda_0)=\chi_{A_{11}^{(2)}}(\Lambda_3)),$$

$$\sum_{i,j\geq 0}(-1)^j\frac{q^{\binom{i}{2}+2\binom{j}{2}+2ij+i+3j}}{(q;q)_i(q^2;q^2)_j}=\frac{1}{[q,q^4,q^6;q^{14}]_\infty}(=\chi_{A_2^{(2)}}(2\Lambda_0+\Lambda_1)=\chi_{A_{11}^{(2)}}(\Lambda_0+\Lambda_1)),$$

$$\sum_{i,j\geq 0}(-1)^j\frac{q^{\binom{i}{2}+2\binom{j}{2}+2ij+2i+3j}}{(q;q)_i(q^2;q^2)_j}=\frac{1}{[q^2,q^5,q^6;q^{14}]_\infty}(=\chi_{A_2^{(2)}}(2\Lambda_1)=\chi_{A_{11}^{(2)}}(\Lambda_5)).$$

(cf. Andrews–Gordon identities)

$$\sum_{n_1,\cdots,n_{k-1}\geq 0}\frac{q^{N_1^2+\cdots+N_{k-1}^2+N_i+\cdots+N_{k-1}}}{(q;q)_{n_1}(q;q)_{n_2}\cdots(q;q)_{n_{k-1}}}=\frac{(q^i,q^{2k+1-i},q^{2k+1};q^{2k+1})_\infty}{(q;q)_\infty}(=\chi_{A_1^{(1)}}((2k-i)\Lambda_0+(i-1)\Lambda_1)$$

where $N_i=n_i+\cdots+n_{k-1}$.

- "$n_{k-1}=0$ part" $\rightsquigarrow \chi_{A_1^{(1)}}((2k-2-i)\Lambda_0+(i-1)\Lambda_1)$

Similar phenomenon between $A_2^{(2)}$ level 4 and 5:

Level 4:

$$\sum_{i,j\geq 0}(-1)^j \frac{q^{\binom{i}{2}+2\binom{j}{2}+2ij+i+j}}{(q;q)_i(q^2;q^2)_j} = \frac{1}{[q^2,q^3,q^4;q^{14}]_\infty}(=\chi_{A_2^{(2)}}(4\Lambda_0)=\chi_{A_{11}^{(2)}}(\Lambda_3)),$$

$$\sum_{i,j\geq 0}(-1)^j \frac{q^{\binom{i}{2}+2\binom{j}{2}+2ij+i+3j}}{(q;q)_i(q^2;q^2)_j} = \frac{1}{[q,q^4,q^6;q^{14}]_\infty}(=\chi_{A_2^{(2)}}(2\Lambda_0+\Lambda_1)=\chi_{A_{11}^{(2)}}(\Lambda_0+\Lambda_1)),$$

$$\sum_{i,j\geq 0}(-1)^j \frac{q^{\binom{i}{2}+2\binom{j}{2}+2ij+2i+3j}}{(q;q)_i(q^2;q^2)_j} = \frac{1}{[q^2,q^5,q^6;q^{14}]_\infty}(=\chi_{A_2^{(2)}}(2\Lambda_1)=\chi_{A_{11}^{(2)}}(\Lambda_5)).$$

Level 5:

$$\sum_{i,j,k\geq 0}(-1)^k \frac{q^{\binom{i}{2}+8\binom{j}{2}+2\binom{k}{2}+2ij+2ik+4jk+i+5j+k}}{(q;q)_i(q^2;q^2)_j(q^2;q^2)_k} = \frac{1}{[q^2,q^3,q^4,q^5;q^{16}]_\infty}(=\chi_{A_2^{(2)}}(5\Lambda_0))$$

$$\sum_{i,j,k\geq 0}(-1)^k \frac{q^{\binom{i}{2}+8\binom{j}{2}+2\binom{k}{2}+2ij+2ik+4jk+i+7j+3k}}{(q;q)_i(q^2;q^2)_j(q^2;q^2)_k} = \frac{1}{[q,q^4,q^6,q^7;q^{16}]_\infty}(=\chi_{A_2^{(2)}}(\Lambda_0+2\Lambda_1))$$

- $j=0$ part $\rightsquigarrow$ double sums in level 4

Q. Does it generalize to the "Big Picture"?

Thank you!