Eichler integrals of Eisenstein series as q-brackets of various types of modular forms

Ken Ono (University of Virginia)

(joint work with Kathrin Bringmann and Ian Wagner)

Ramanujan's "Death bed letter"

Dear Hardy, *January 1920*

"I am extremely sorry for not writing you a single letter up to now. I discovered very interesting functions recently which I call **"Mock" ϑ-functions***. ...they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some ...."*

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Example

One of Ramanujan's examples:

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}.$$

What are mock theta functions?

Some History

In his PhD thesis ('02), Zwegers combined **Lerch-type series** *and Mordell integrals to obtain non-holomorphic Jacobi forms.*

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"Theorem" (Zwegers, 2002)

The mock theta functions are (up to powers of q) holomorphic parts of the specializations of weight $1/2$ harmonic Maass forms.

Harmonic Maass forms (note. $z = x + iy \in \mathbb{H}$)

"Definition"

A **weight** k **harmonic Maass form on** Γ *is any smooth function* f *on* $\mathbb{H}$ *satisfying:*

HARMONIC MAASS FORMS (NOTE. $z = x + iy \in \mathbb{H}$)

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$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{k} \ f(z).$$

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2. *We have that* $\Delta_k f = 0$*, where*

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

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Remark

Classical modular forms represent a density 0 subset of HMFs.

Fourier expansions of HMFs ($q := e^{2\pi i z}$)

Fundamental Lemma

If $f \in H_{2-k}$ *and* $\Gamma(a, x)$ *is the incomplete* Γ*-function, then*

$$f(z) = \sum_{n \gg -\infty} c_f^+(n) q^n + \sum_{n<0} c_f^-(n)\Gamma(k-1, 4\pi|n|y) q^n.$$

$\updownarrow$ $\updownarrow$

Holomorphic part f^+ **Nonholomorphic part** f^-

q-series "Period integral of MF"

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$\updownarrow$ **Holomorphic part** f^+ — q-series

$\updownarrow$ **Nonholomorphic part** f^- — "Period integral of MF"

Remark

Ramanujan's examples are the f^+ *with* $k = 1/2$.

Ramanujan's Strange Conjecture

Conjecture (Ramanujan)

Consider the **mock theta** $q^{-\frac{1}{24}}f(q)$ *and* **modular form** $q^{-\frac{1}{24}}b(q)$, *where*

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2},$$

$$b(q) := (1-q)(1-q^3)(1-q^5)\cdots \times \left(1 - 2q + 2q^4 - 2q^9 + \cdots\right).$$

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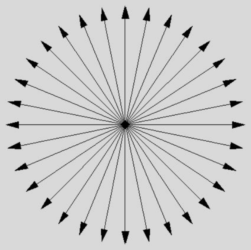
If q approaches an even order $2k$ root of unity (i.e. **pole of** f*), then*

$$f(q) - (-1)^k b(q) = O(1).$$

“q APPROACHES A ROOT OF UNITY”

Radial asymptotics, near roots of unity.



NUMERICS

Numerics

As $q \to -1$, we have

$$f(-0.994) \sim -1{\cdot}10^{31},\ f(-0.996) \sim -1{\cdot}10^{46},\ f(-0.998) \sim -6{\cdot}10^{90},$$

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$f(-0.998185) \sim -$Googol

POLES AT $q = -1$ AND $q = i$

Amazingly, Ramanujan's guess gives:

q	-0.990	-0.992	-0.994	-0.996	-0.998
$f(q)+b(q)$	$3.961\dots$	$3.969\dots$	$3.976\dots$	$3.984\dots$	$3.992\dots$

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It is true that

$$\lim_{q\to -1}(f(q)+b(q)) = 4$$

$$\lim_{q\to i}(f(q)-b(q)) = 4i.$$

Finite SUMS OF ROOTS OF UNITY.

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Theorem (F-O-R (2013))

If ζ is an even $2k$ order root of unity, then

$$\lim_{q\to\zeta}(f(q)-(-1)^k b(q)) = -4\sum_{n=0}^{k-1}(1+\zeta)^2(1+\zeta^2)^2\cdots(1+\zeta^n)^2\zeta^{n+1}.$$

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Remark

This behavior "near roots of unity" is a glimpse of **quantum modularity**.

What is going on?

Question

Ramanujan essentially discovered that

$$\lim_{q \to \zeta} (\text{Mock } \vartheta - \epsilon_\zeta \text{MF}) = \text{Quantum MF}$$

$$\uparrow$$

$$O(1) \text{ numbers}$$

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$$h_\gamma(x) := f(x) - \epsilon(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right)$$

satisfies a "suitable" property of continuity or analyticity.

Applications of HMFs and QMFs

- Integer partitions and q-series
- Eichler-Shimura theory
  (e.g. modularity of elliptic curves via **Eichler integrals**)
- Arithmetic Geometry (i.e. BSD Conjecture)
- Moonshine
- Knot invariants.
- ....

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Question

Eichler integrals of MFs are prominent in the theory of HMFs. What about for general "Eisenstein-type" series?

- *q-series identities?*
- *Harmonic Maass forms?*
- *Quantum Modular forms?*

"Eisenstein-type series"

Definition

For $a \in \mathbb{Z}$, we define the **divisor function series**

$$\mathcal{E}_{2-a}(z) := \sum_{n=1}^{\infty} \sigma_{1-a}(n) q^n = \sum_{n=1}^{\infty} \sum_{d|n} d^{1-a} q^n.$$

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Remarks

1. *For $k \geq 2$, the Eichler integral of the modular $E_{2k}(z)$ satisfies*

$$\mathcal{E}_{2-2k}(z) = -\frac{B_{2k}}{4k} \cdot \mathit{Eichler}_{E_{2k}}(z).$$

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2. *Do the $\mathcal{E}_{2-a}(z)$ give modular objects for other a?*

Executive Summary of New Results

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- Relations involving zeta-values and Bernoulli numbers

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- *(Bloch and Okounkov)* $\mathrm{SL}_2(\mathbb{Z})$ *quasimodular forms are generated by q-brackets of shifted symmetric polynomials.*
- *Do q-brackets give other types of modular forms?*

Functions on t-hooks of partitions

Notation

$$\mathcal{H}(\lambda) := \{\text{hook numbers of } \lambda\}$$
$$\mathcal{H}_t(\lambda) := \{\text{hook numbers of } \lambda \text{ that are multiples of } t\}.$$

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Definition

If $t \in \mathbb{Z}^+$ and $a \in \mathbb{C}$, then define $f_{a,t} : \mathcal{P} \to \mathbb{C}$ by

$$f_{a,t}(\lambda) := t^{a-1} \sum_{h \in \mathcal{H}_t(\lambda)} \frac{1}{h^a}.$$

EXAMPLES

Consider the partition $\lambda = 4 + 3 + 1$:

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$$\begin{array}{llll} \bullet_6 & \bullet_4 & \bullet_3 & \bullet_1 \\ \bullet_4 & \bullet_2 & \bullet_1 & \\ \bullet_1 & & & \end{array} \qquad \longleftarrow \qquad \text{Subscripts} = \text{Hook numbers}$$

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We find that $\mathcal{H}(\lambda) = \{1, 1, 1, 2, 3, 4, 4, 6\}$ and

$$\mathcal{H}_2(\lambda) = \{2, 4, 4, 6\} \quad \text{and} \quad \mathcal{H}_3(\lambda) = \{3, 6\}.$$

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Therefore, we have

$$f_{3,1}(\lambda) = 1 + 1 + 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{64} + \frac{1}{216} = \frac{307}{96},$$
$$f_{3,2}(\lambda) = 2^2 \left(\frac{1}{8} + \frac{1}{64} + \frac{1}{64} + \frac{1}{216} \right) = \frac{139}{216},$$
$$f_{3,3}(\lambda) = 3^2 \left(\frac{1}{27} + \frac{1}{216} \right) = \frac{3}{8}.$$

q-IDENTITIES

THEOREM (B-O-W)

If t is a positive integer and $a \in \mathbb{C}$, then we have

$$\langle f_{a,t} \rangle_q = \mathcal{E}_{2-a}(tz).$$

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1. *Proof follows easily from recent work of Han and Ji.*

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Remarks

1. *Proof follows easily from recent work of Han and Ji.*
2. *Think "log-derivative" of the Nekrasov-Okounkov & Westbury formula*

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = \prod_{n=1}^{\infty} (1-q^n)^{z-1}.$$

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A weight k **sesquiharmonic Maass form** is a real analytic modular form that is annihilated by $\Delta_{k,2} := -\xi_k \circ \xi_{2-k} \circ \xi_k$, where $\xi_k := 2iy^k \overline{\frac{\partial}{\partial \overline{z}}}$.

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Theorem (B-O-W)

$\mathbb{E}_0(tz)$ is a wgt zero sesquiharmonic Maass form on $\Gamma_0(t)$, where

$$\mathbb{E}_0(tz) := ty + \frac{6}{\pi}\left(\gamma - \log(2) - \frac{\log(ty)}{2} - \frac{6\zeta'(2)}{\pi^2} + \langle f_{2,t}\rangle_q + \sum_{n=1}^{\infty} \sigma_{-1}(n)\overline{q}^{tn}\right).$$

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$$:= (ty)^{2k-1} + \frac{2 \cdot (2k)!}{B_{2k}(4\pi)^{2k-1}} \left(\zeta(2k-1) + \langle f_{2k,t} \rangle_q + \sum_{n=1}^{\infty} \sigma_{1-2k}(n) \Gamma^*(2k-1, 4\pi tny) q^{-tn} \right).$$

Harmonic Maass forms ($a \geq 4$ even)

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Proof.

- Eichler integrals of holomorphic modular forms are "mock modular".

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Proof.

- Eichler integrals of holomorphic modular forms are "mock modular".
- The nonholomorphic part is the "period integral" of $E_{2k}(z)$.

□

Modularity of $\langle f_{2k,t} \rangle_q$ (Case $k \geq 1$)

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Notation

For $k \in \mathbb{N}$, *we define the* **Bernoulli number polynomial**

$$P_{-2k}(z) := -\frac{1}{2}(2\pi i)^{2k+1} \sum_{m=0}^{k+1} \frac{B_{2m}}{(2m)!} \frac{B_{2k+2-2m}}{(2k+2-2m)!} \cdot z^{2m-1}.$$

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Corollary (B-O-W)

If k and t are positive integers and

$$M_{-2k,t}(z) := \langle f_{2k+2,t} \rangle_q - \frac{1}{2} P_{-2k}(tz) + \frac{1}{2}\zeta(2k+1),$$

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then for $z \in \mathbb{H}$ we have

$$M_{-2k,t}(z) = (tz)^{2k} M_{-2k,t}\left(-\frac{1}{t^2 z}\right).$$

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NOTATION

We require functions

$$P_t(z) := -t\left(t + \frac{\pi i}{12}\right)z + \frac{1}{z} \qquad \text{and} \qquad L_t(z) := -\frac{1}{4} \cdot \log(tz).$$

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Notation

We require functions

$$P_t(z) := -t\left(t + \frac{\pi i}{12}\right) z + \frac{1}{z} \qquad \text{and} \qquad L_t(z) := -\frac{1}{4}\cdot \log(tz).$$

Corollary (B-O-W)

If t is a positive integer and

$$M_t(z) := \langle f_t\rangle_q + P_t(z) + L_t(z),$$

then for all $z \in \mathbb{H}$ we have

$$M_t(z) = M_t\left(-\frac{1}{t^2 z}\right).$$

Algebraic Parts of Dedekind's eta values

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Definition (Dedekind)

The **Dedekind eta-function** is defined by

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Theorem (Chowla and Selberg (1967))

Suppose that $D<0$ is a fundamental discriminant and let

$$\Omega_D := \frac{1}{\sqrt{2\pi|D|}} \left(\prod_{j=1}^{|D|} \Gamma\left(\frac{j}{|D|}\right)^{\chi_D(j)} \right)^{\frac{1}{2h'(D)}}.$$

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If $\tau \in \mathbb{Q}(\sqrt{D}) \cap \mathbb{H}$, then we have

$$\eta\left(-\frac{1}{\tau}\right) \in \overline{\mathbb{Q}} \cdot \sqrt{\Omega_D}.$$

Ramanujan's Examples

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Ramanujan discovered that

$$\eta(i/2) = 2^{\frac{1}{8}} \cdot \Omega_{-4}^{\frac{1}{2}}, \quad \eta(i) = \Omega_{-4}^{\frac{1}{2}}, \quad \eta(2i) = \frac{1}{2^{\frac{3}{8}}} \cdot \Omega_{-4}^{\frac{1}{2}}, \quad \eta(4i) = \frac{\left(\sqrt{2}-1\right)^{\frac{1}{4}}}{2^{\frac{13}{16}}} \cdot \Omega_{-4}^{\frac{1}{2}}.$$

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where

$$\Omega_{-4} = \frac{1}{2\sqrt{2\pi}} \cdot \frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)},$$

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Modularity for Gen Fcn of $f_{a,1}$

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Notation

For $a \in \mathbb{C}$ *and* $k \in \mathbb{N}$ *define*

$$H_a(z) := q^{-\frac{1}{24}} \sum_{\lambda \in \mathcal{P}} f_{a,1}(\lambda) q^{|\lambda|}.$$

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COROLLARY (B-O-W)

If $z \in \mathbb{H}$ and $k \in \mathbb{N}$, then

$$H_{2k+2}\left(-\frac{1}{z}\right) - \frac{1}{z^{2k}\sqrt{-iz}} H_{2k+2}(z) = \frac{\Psi_{-2k}(z)}{\eta\left(-\frac{1}{z}\right)}.$$

Chowla-Selberg for $H_a(z)$

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Corollary (B-O-W)

If $k \in \mathbb{N}$ and $\tau \in \mathbb{Q}(\sqrt{D}) \cap \mathbb{H}$, where $D < 0$ is a fundamental discriminant, then

$$H_{2k+2}\left(-\frac{1}{\tau}\right) - \frac{1}{\tau^{2k}\sqrt{-i\tau}} H_{2k+2}(\tau) \in \overline{\mathbb{Q}} \cdot \frac{\Psi_{-2k}(\tau)}{\sqrt{\Omega_D}}.$$

Numerical Examples

Numerical calculation gives

$$H_4(2i) \approx 5.887 \cdot 10^{-6}, \qquad H_4\left(\frac{i}{2}\right) \approx 0.05420,$$

$$H_6(2i) \approx 5.887 \cdot 10^{-6}, \qquad H_6\left(\frac{i}{2}\right) \approx 0.05398.$$

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We have proven that

$$H_4\left(\frac{i}{2}\right) + \frac{1}{2^{\frac{5}{2}}} H_4(2i) = \frac{1}{2^{\frac{1}{8}}} \cdot \frac{\Psi_{-2}(2i)}{\sqrt{\Omega_{-4}}}$$

and

$$H_6\left(\frac{i}{2}\right) - \frac{1}{2^{\frac{9}{2}}} H_6(2i) = \frac{1}{2^{\frac{1}{8}}} \cdot \frac{\Psi_{-4}(2i)}{\sqrt{\Omega_{-4}}}$$

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What can be said if $a \leq -1$ *is odd?*

Example

For instance, if $a = -1$ then we have

$$\langle f_{-1,1} \rangle_q = \sum_{n=1}^{\infty} \sigma_2(n) q^n.$$

Holomorphic Quantum modular forms

Definition (Zagier)

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$$h_\gamma(x) := f(x) - \epsilon(\gamma)(cx+d)^{-k} f\left(\frac{ax+b}{cx+d}\right)$$

is holomorphic on a "larger domain" than $\mathbb{H}$.

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Suppose that $a \leq -1$ is odd. Then the following are true:

(1) *We have that $\langle f_{a,t} \rangle_q$ is a holomorphic weight $2-a$ quantum modular form. In particular, we have the modular transformations*

$$\mathcal{E}_{2-a}(z) - z^{a-2}\mathcal{E}_{2-a}\left(-\frac{1}{z}\right) = \frac{1}{2\pi}\int_{\mathrm{Re}(s)=1-\frac{a}{2}} \frac{\Gamma(s)\zeta(s)\zeta(s+a-1)}{(2\pi)^s \sin\left(\frac{\pi s}{2}\right)} z^{-s} ds.$$

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Remark ("Larger domain")

For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the $h_{\mathcal{E}_k,\gamma}(z)$ extends to a holomorphic function on

$$\mathbb{C}_\gamma := \begin{cases} \mathbb{C} \setminus \left(-\infty, -\frac{d}{c}\right) & c > 0, \\ \mathbb{C} \setminus \left(-\frac{d}{c}, \infty\right) & c < 0. \end{cases}$$

Asymptotic Expansions

Notation

If $a \leq -1$ is odd, then we have

$$\widehat{G}_{2-a}(t) := \sum_{n=1}^{\infty} \sigma_{1-a}(n) e^{-nt} = \mathcal{E}_{2-a}\left(\frac{it}{2\pi}\right).$$

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With $k = 2 - a$, the series above agrees, as $t \to 0^{+}$, with

$$\widetilde{G}_{k}(t) := \frac{\Gamma(k)\zeta(k)}{t^{k}} + \frac{\zeta(2-k)}{t} + \sum_{n=0}^{\infty} \frac{B_{n+1}}{n+1} \frac{B_{n+k}}{n+k} \frac{(-t)^{n}}{n!}.$$

CASE WHERE $a = -1$

t	$\widehat{G}_3(t)$	$\widetilde{G}_3(t)$	$\widehat{G}_3(t)/\widetilde{G}_3(t)$
2	≈ 0.2602861623	≈ 0.2602864321	≈ 0.9999989634
1.5	≈ 0.6578359053	≈ 0.6578359052	≈ 0.9999999998
1	≈ 2.3214805734	≈ 2.3214805734	≈ 1.0000000000
0.5	≈ 19.0665916994	≈ 19.0665916994	≈ 1.0000000000
0.1	≈ 2403.2805424358	≈ 2403.2805424358	≈ 1.0000000000
$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	∞	∞	1

t-HOOK FUNCTIONS ON PARTITIONS

Definition

If $t \in \mathbb{Z}^+$ and $a \in \mathbb{C}$, then define $f_{a,t} : \mathcal{P} \to \mathbb{C}$ by

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If t is a positive integer and $a \in \mathbb{C}$, then we have

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Positive even a

Theorem (B-O-W)

$\mathbb{E}_0(tz)$ is a wgt zero sesquiharmonic Maass form on $\Gamma_0(t)$, where

$$\mathbb{E}_0(tz) := ty + \frac{6}{\pi}\left(\gamma - \log(2) - \frac{\log(ty)}{2} - \frac{6\zeta'(2)}{\pi^2} + \langle f_{2,t}\rangle_q + \sum_{n=1}^{\infty}\sigma_{-1}(n)\overline{q}^{tn}\right).$$

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THEOREM (B-O-W)

If $k \geq 2$, then $\mathbb{E}_{2-2k}(tz)$ is a weight $2-2k$ harmonic Maass form on $\Gamma_0(t)$, where

$$\mathbb{E}_{2-2k}(tz)$$
$$:= (ty)^{2k-1} + \frac{2\cdot(2k)!}{B_{2k}(4\pi)^{2k-1}}\left(\zeta(2k-1) + \langle f_{2k,t}\rangle_q + \sum_{n=1}^{\infty}\sigma_{1-2k}(n)\Gamma^*(2k-1, 4\pi tny)q^{-tn}\right).$$

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REMARK

These asymptotics are analogous to Ramanujan's $O(1)$ numbers that arise with "classical" quantum modular forms.