Mono-anabelian Reconstruction of Number Fields

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Contents

§1 Main Result

Question: Can one reconstruct a number field from the associated absolute Galois group?

Definition

F: an NF $\overset{\text{def}}{\Leftrightarrow}$ $[F:\mathbb{Q}] < \infty$

k: an MLF $\overset{\text{def}}{\Leftrightarrow}$ $[k:\mathbb{Q}_p] < \infty$ for some p

For a topological group G,

G: of NF-type (resp. of MLF-type)

$\overset{\text{def}}{\Leftrightarrow}$ $G \cong$ the abs. Gal. gp of an NF (resp. MLF)

Theorem [Neukirch-Uchida]

$\square \in \{\circ, \bullet\}$

$F_\square$: a global field [i.e., a fin. ext. of $\mathbb{Q}$ or $\mathbb{F}_p(t)$]

$\overline{F}_\square$: a separable closure of $F_\square$

$G_{F_\square} \stackrel{\mathrm{def}}{=} \mathrm{Gal}(\overline{F}_\square / F_\square)$

$\Longrightarrow$ The natural map

$$\mathrm{Isom}(\overline{F}_\circ / F_\circ, \overline{F}_\bullet / F_\bullet) \longrightarrow \mathrm{Isom}(G_{F_\bullet}, G_{F_\circ})$$

is bijective.

In particular: $F_\circ \cong F_\bullet \Longleftrightarrow G_{F_\circ} \cong G_{F_\bullet}$.

Mochizuki's Mono-anabelian Philosophy

Give a(n) [functorial "group-theoretic"] algorithm

$$G_F \rightsquigarrow \overline{F}/F.$$

A "reconstruction" as in Theorem [N-U] is called "bi-anabelian reconstruction".

In the case where

- $\mathrm{char}(F_\square) > 0$, the proof $\Rightarrow$ mono-anab'n rec'n,
- $\mathrm{char}(F_\square) = 0$, the proof $\not\Rightarrow$ mono-anab'n rec'n.

Rough Statement of Main Theorem (1)

$\exists$A functorial "group-theoretic" algorithm

G : of NF-type

$\rightsquigarrow$ $\overline{F}(G)$: an algebraically closed field $\curvearrowleft$ G

which satisfies some conditions.

For instance:

- An isomorphism $G \xrightarrow{\sim} \mathrm{Gal}(\overline{F}/F)$ determines

$$(\overline{F}(G) \curvearrowleft G) \xrightarrow{\sim} (\overline{F} \curvearrowleft \mathrm{Gal}(\overline{F}/F)).$$

Rough Statement of Main Theorem (2)

- The Log-Frobenius compatibility

G: of NF-type

$D \subseteq G$: a dec'n gp ass'd to a nonarch'n prime

$G \hookleftarrow D$

$\overset{\text{Th'm}}{\rightsquigarrow} G \hookleftarrow D \curvearrowright \overline{k}(D)$

$\overset{\text{forget}}{\rightsquigarrow} G \hookleftarrow D \curvearrowright \text{``}\overline{k}\text{''}$ [a telecore]

$\rightsquigarrow G \hookleftarrow D \curvearrowright \text{``}\overline{k}\text{''} \curvearrowleft \mathfrak{log}$ [$\mathfrak{log}$: $\overline{k} \rightsquigarrow \log_{\overline{k}}(\mathcal{O}^{\times}_{\overline{k}})^{\mathrm{pf}}$]

$\overset{\text{forget}}{\rightsquigarrow} G \hookleftarrow D$ [a core]

$\rightsquigarrow$ ...

Remark

- One may replace "$\overline{F}$" by an absolutely Galois [i.e., Galois over $\mathbb{Q}$] solvably closed [i.e., not admitting a nontrivial abelian extension] extension of F.

- The Neukirch-Uchida theorem plays a crucial role in the proof of the main result. In particular, the [proof of the] main result does not give an alternative proof of the Neukirch-Uchida theorem.

§2 Two Keywords Related to IUT

- Mono-anabelian Reconstruction Algorithm

- Cyclotomic Synchronization Isomorphism
[sometimes “Cyclotomic Rigidity Isomorphism”]

Mono-anabelian Reconstruction Algorithm (1)

What is an MRA? For instance:

Bi-anabelian Geometry

$$\mathrm{Isom}(\overline{F}_\circ/F_\circ, \overline{F}_\bullet/F_\bullet) \xrightarrow{\sim} \mathrm{Isom}(G_{F_\bullet}, G_{F_\circ})$$

or $F_\circ \cong F_\bullet \Longleftrightarrow G_{F_\circ} \cong G_{F_\bullet}$.

Mono-anabelian Geometry

$G_F \rightsquigarrow \overline{F}/F$: functorial, "group-theoretic"

MRA = the algorithm "$\rightsquigarrow$" discussed in mono-anabelian geometry

Mono-anabelian Reconstruction Algorithm (2)

(MRA_1) What is an example of a mono-anabelian reconstruction algorithm?

(MRA_2) Why should one consider [not only a "fully faithfulness result" in bi-AG but also] an algorithm in mono-AG?

An answer to (MRA_1):
[Of course, our result gives an exa. of an MRA...]
the local reconstruction algorithm reviewed in §3

Mono-anabelian Reconstruction Algorithm (3)

A(n) [tautological] answer to
(MRA_2) Why should one consider an algorithm in mono-AG?

The issue of "what can one do by a given reconstruction result" depends on the content of the given algorithm in the reconstruction result.

See some examples which appear in this talk.

Cyclotomic Synchronization Isomorphism (1)

[sometimes "Cyclotomic Rigidity Isomorphism"]

What is a CSI?

A CSI is a suitable isom. between cyclotomes.

(CSI_1) What is a cyclotome?

(CSI_2) How does one use a cyclotomic synchronization isomorphism?

An answer to (CSI_2):
an example in the final portion of §3

Cyclotomic Synchronization Isomorphism (2)

An answer to (CSI_1) What is a cyclotome?:

A cyclotome is an isomorph of "$\widehat{\mathbb{Z}}(1)$".

For instance:

- $\Lambda(K) \stackrel{\mathrm{def}}{=} \varprojlim_n \mu_n(K)$
- $\pi_1^{\text{ét}}(\mathrm{Spec}(K((t))))$
- $\mathrm{Hom}_{\widehat{\mathbb{Z}}}(H^2_{\text{ét}}(C, \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}})$, where

K: an algebraically closed field of characteristic 0

C: a projective smooth curve over K

Cyclotomic Synchronization Isomorphism (3)

[Recall: A CSI is a suitable isom. of cyclotomes.]

In the ring-theoretic framework of scheme theory, we have suitable isom. of various cyclotomes.
For instance:

- The inclusion $\overline{\mathbb{Q}} \hookrightarrow K$ determines
  $$\Lambda(\overline{\mathbb{Q}}) \xrightarrow{\sim} \Lambda(K).$$
- The map $c_1 \colon \mathrm{Pic}(C) \to H^2_{\text{ét}}(C, \Lambda(K))$
  determines $\Lambda(K) \xrightarrow{\sim} \mathrm{Hom}_{\widehat{\mathbb{Z}}}(H^2_{\text{ét}}(C, \widehat{\mathbb{Z}}), \widehat{\mathbb{Z}})$.

Cyclotomic Synchronization Isomorphism (4)

[Recall: A CSI is a suitable isom. of cyclotomes.]

On the other hand, in the group-theoretic framework of anabelian geometry, at least a priori, we do not have such an isomorphism.

$$\begin{array}{ccccc} \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) & \xrightarrow{\sim} & \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) & & \\ \curvearrowright & & \curvearrowright & \overset{???}{\Rightarrow} & \exists \text{a suitable} \\ \Lambda(\overline{\mathbb{Q}}_p) & & \Lambda(\overline{\mathbb{Q}}_p) & & \Lambda(\overline{\mathbb{Q}}_p) \xrightarrow{\sim} \Lambda(\overline{\mathbb{Q}}_p) \end{array}$$

[No such a "suitable" isom... — cf. $\widehat{\mathbb{Z}}^\times \curvearrowright \Lambda$.]

Cyclotomic Synchronization Isomorphism (5)

In the main result of this talk:

H: a profinite group of MLF- or NF-type

$\rightsquigarrow$ a cyclotome $H \curvearrowright \Lambda(H)$ [cf. §3 and §4]

[i.e., "$\mathrm{Gal}(\overline{M}/M) \rightsquigarrow (\mathrm{Gal}(\overline{M}/M) \curvearrowright \Lambda(\overline{M}))$"]

Thus: G: of NF-type

$D \subseteq G$: a dec'n gp ass'd to a nonarch'n prime

$$\begin{array}{ccc} \rightsquigarrow D & \hookrightarrow & G \\ \curvearrowright & & \curvearrowright \\ \Lambda(D) & & \Lambda(G) \end{array} \overset{\text{our CSI}}{\Longrightarrow} \exists \text{a suitable [e.g., } D\text{-eq.]} \quad \Lambda(G) \overset{\sim}{\to} \Lambda(D) \text{ [cf. §4]}$$

Cyclotomic Synchronization Isomorphism (6)

In IUT: For instance: k: an MLF

$G \curvearrowright M$: an isomorph of $\mathrm{Gal}(\overline{k}/k) \curvearrowright \mathcal{O}^{\rhd}_{\overline{k}}$, where

$\mathcal{O}^{\rhd}_{\overline{k}}$: the monoid of nonzero integers of $\overline{k}$

[i.e., a certain "Frobenioid"]

In this situation:

"G": the étale-like portion of $G \curvearrowright M$

"M": the Frobenius-like portion of $G \curvearrowright M$

By the prev. page: $G \rightsquigarrow G \curvearrowright \Lambda(G)$: a cyclotome

Cyclotomic Synchronization Isomorphism (7)

$M^{\mathrm{gp}} \cong \overline{k}^{\times} \Rightarrow \Lambda(M) \stackrel{\mathrm{def}}{=} \varprojlim_n M^{\mathrm{gp}}[n] \cong \Lambda(\overline{k})$,
i.e., $G \curvearrowright \Lambda(M)$: a cyclotome

$G \rightsquigarrow G \curvearrowright \Lambda(G)$: the étale-like cyclotome
$M \rightsquigarrow G \curvearrowright \Lambda(M)$: the Frobenius-like cyclotome

<u>CSI [via local class field theory]</u>:

$G \curvearrowright M \rightsquigarrow$ a suitable [e.g., G-eq.] $\Lambda(G) \stackrel{\sim}{\to} \Lambda(M)$

§3 Review of the Local Theory

k: an MLF

$\mathcal{O}_k \subseteq k$: the ring of integers of k

$\mathfrak{m}_k \subseteq \mathcal{O}_k$: the maximal ideal of $\mathcal{O}_k$

$\mathcal{O}_k^{\triangleright} \stackrel{\text{def}}{=} \mathcal{O}_k \setminus \{0\} \subseteq k^{\times}$ [submonoid]

$\underline{k} \stackrel{\text{def}}{=} \mathcal{O}_k/\mathfrak{m}_k$: the residue field of $\mathcal{O}_k$

$\overline{k}$: an algebraic closure of k

$\mathcal{O}_{\overline{k}}^{\triangleright} \stackrel{\text{def}}{=} \mathcal{O}_{\overline{k}} \setminus \{0\} \subseteq \overline{k}^{\times}$ [submonoid]

$G_k \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{k}/k)$

$P_k \subseteq I_k \subseteq G_k$: the wild inertia, inertia subgps

Proposition

(i) [Local Class Field Theory]

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Im}(I_k \to G_k^{\mathrm{ab}}) & \longrightarrow & G_k^{\mathrm{ab}} & \longrightarrow & G_k/I_k & \longrightarrow & 1 \\
 & & \downarrow \wr & & \downarrow \wr & & \downarrow \wr & & \\
1 & \longrightarrow & \mathcal{O}_k^\times & \longrightarrow & (k^\times)^\wedge & \longrightarrow & \widehat{\mathbb{Z}} & \longrightarrow & 1 \\
 & & \| & & \uparrow \cup & & \uparrow \cup & & \\
1 & \longrightarrow & \mathcal{O}_k^\times & \longrightarrow & k^\times & \xrightarrow{\mathrm{ord}_k} & \mathbb{Z} & \longrightarrow & 1
\end{array}$$

— where the right-hand upper vertical arrow maps $\mathrm{Frob}_{\underline{k}} \in G_k/I_k$ to $1 \in \widehat{\mathbb{Z}}$.

(ii)

$\{\mathrm{char}(\underline{k})\} = \{\, l : \text{prime} \mid \dim_{\mathbb{Q}_l}(G_k^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l) \geq 2 \,\}$

Write $p \stackrel{\mathrm{def}}{=} \mathrm{char}(\underline{k})$.

(iii) $d_k \stackrel{\mathrm{def}}{=} [k : \mathbb{Q}_p] = \dim_{\mathbb{Q}_p}(G_k^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_p) - 1$

(iv) $f_k \stackrel{\mathrm{def}}{=} [\underline{k} : \mathbb{F}_p] = \log_p(\sharp(G_k^{\mathrm{ab}})_{\mathrm{tor}}^{(p')} + 1)$

(v) $I_k = \bigcap_{K/k\text{: fin. s.t. } d_K/f_K = d_k/f_k} G_K$

(vi) $P_k \subseteq I_k$: the unique pro-p-Sylow subgroup

(vii) $\{\, \mathrm{Frob}_{\underline{k}} \in G_k/I_k \,\}$

$= \{\, \gamma \in G_k/I_k \mid \gamma \text{ acts on } I_k/P_k \text{ by } p^{f_k} \,\}$

(viii) $U_k^{(1)} \stackrel{\text{def}}{=} 1 + \mathfrak{m}_k \subseteq \mathcal{O}_k^\times$: unique pro-$p$-Sylow

(ix) $\overline{k}^\times = \varinjlim_{K/k\text{: fin.}} K^\times$

$\mathcal{O}_{\overline{k}}^\rhd = \varinjlim_{K/k\text{: fin.}} \mathcal{O}_K^\rhd$

(x) $\Lambda(\overline{k}) \stackrel{\text{def}}{=} \text{“}\widehat{\mathbb{Z}}(1)\text{”} = \varprojlim_n \overline{k}^\times[n]$

(xi) $1 \to \overline{k}^\times[n] \to \overline{k}^\times \xrightarrow{n} \overline{k}^\times \to 1 \quad \curvearrowleft \quad G_k$

induces an <u>injection</u>

$$\mathrm{Kmm}_k\colon\ k^\times \hookrightarrow H^1(G_k, \Lambda(\overline{k})).$$

Local Mono-anabelian Reconstruction (1)

G: of MLF-type

(1) $p(G)$: [unique] prime l

s.t. $\dim_{\mathbb{Q}_l}(G^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_l) \geq 2$

(2) $d(G) \stackrel{\mathrm{def}}{=} \dim_{\mathbb{Q}_{p(G)}}(G^{\mathrm{ab}} \otimes_{\widehat{\mathbb{Z}}} \mathbb{Q}_{p(G)}) - 1$

(3) $f(G) \stackrel{\mathrm{def}}{=} \log_{p(G)}(\sharp(G^{\mathrm{ab}})^{(p(G)')}_{\mathrm{tor}} + 1)$

(4) $I(G) \stackrel{\mathrm{def}}{=} \bigcap_{G^\dagger \subseteq G\text{: open s.t. } \frac{d(G^\dagger)}{f(G^\dagger)} = \frac{d(G)}{f(G)}} G^\dagger$

(5) $P(G) \subseteq I(G)$: [unique] pro-$p(G)$-Sylow

Local Mono-anabelian Reconstruction (2)

(6) $\mathrm{Frob}(G) \in G/I(G)$: [unique] elem't $\in G/I(G)$

which acts on $I(G)/P(G)$ by $p(G)^{f(G)}$

(7) $k^{\times}(G) \stackrel{\mathrm{def}}{=} G^{\mathrm{ab}} \times_{G/I(G)} \mathrm{Frob}(G)^{\mathbb{Z}} \subseteq G^{\mathrm{ab}}$

(8) $\mathcal{O}^{\triangleright}(G) \stackrel{\mathrm{def}}{=} G^{\mathrm{ab}} \times_{G/I(G)} \mathrm{Frob}(G)^{\mathbb{N}} \subseteq k^{\times}(G)$

(9) $\mathcal{O}^{\times}(G) \stackrel{\mathrm{def}}{=} \mathrm{Im}(I(G) \to G^{\mathrm{ab}}) \subseteq \mathcal{O}^{\triangleright}(G)$

(10) $U^{(1)}(G) \subseteq \mathcal{O}^{\times}(G)$: [unique] pro-$p(G)$-Sylow

Local Mono-anabelian Reconstruction (3)

(11) $\overline{k}^{\times}(G) \stackrel{\text{def}}{=} \varinjlim_{G^{\dagger}\subseteq G:\ \text{open}} k^{\times}(G^{\dagger})$

$\overline{\mathcal{O}}^{\triangleright}(G) \stackrel{\text{def}}{=} \varinjlim_{G^{\dagger}\subseteq G:\ \text{open}} \mathcal{O}^{\triangleright}(G^{\dagger})$

$\Lambda(G) \stackrel{\text{def}}{=} \varprojlim_{n} \overline{k}^{\times}(G)[n] \qquad \stackrel{\text{conj.}}{\curvearrowleft} G$

(12) $\mathrm{Kmm}(G)\colon\ k^{\times}(G) \hookrightarrow H^1(G, \Lambda(G))$:

the injection induced by

$1 \to \overline{k}^{\times}(G)[n] \to \overline{k}^{\times}(G) \xrightarrow{n} \overline{k}^{\times}(G) \to 1 \quad \stackrel{\text{conj.}}{\curvearrowleft} \quad G$

Local Mono-anabelian Reconstruction (4)

Let $\alpha \colon G_k \xrightarrow{\sim} G$ be an isomorphism. Then:

(i) $\mathrm{char}(\underline{k}) = p(G)$, $d_k = d(G)$, $f_k = f(G)$.

(ii) α determines a commutative diagram

$$\begin{array}{ccccc} P_k & \xrightarrow{\subset} & I_k & \xrightarrow{\subset} & G_k \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow\alpha \\ P(G) & \xrightarrow{\subset} & I(G) & \xrightarrow{\subset} & G; \end{array}$$

moreover, $G_k/I_k \xrightarrow{\sim} G/I(G)$ maps $\mathrm{Frob}_{\underline{k}}$ to $\mathrm{Frob}(G)$.

Local Mono-anabelian Reconstruction (5)

(iii) α [and the fld str. on k] det. a comm. dia'm

$$\begin{array}{ccccccc} U_k^{(1)} & \stackrel{\subset}{\longrightarrow} & \mathcal{O}_k^{\times} & \stackrel{\subset}{\longrightarrow} & \mathcal{O}_k^{\rhd} & \stackrel{\subset}{\longrightarrow} & k^{\times} \\ \wr\downarrow & & \wr\downarrow & & \wr\downarrow & & \wr\downarrow \\ U^{(1)}(G) & \stackrel{\subset}{\longrightarrow} & \mathcal{O}^{\times}(G) & \stackrel{\subset}{\longrightarrow} & \mathcal{O}^{\rhd}(G) & \stackrel{\subset}{\longrightarrow} & k^{\times}(G). \end{array}$$

(iv) The dia'm of (iii) det. (G_k, G)-equiv't isom.

$$\overline{k}^{\times} \stackrel{\sim}{\longrightarrow} \overline{k}^{\times}(G), \quad \mathcal{O}_{\overline{k}}^{\rhd} \stackrel{\sim}{\longrightarrow} \overline{\mathcal{O}}^{\rhd}(G),$$

$$\Lambda(\overline{k}) \stackrel{\sim}{\longrightarrow} \Lambda(G).$$

Local Mono-anabelian Reconstruction (6)

(v)
$k^\times \xrightarrow{\sim} k^\times(G)$ of (iii) and $\Lambda(\overline{k}) \xrightarrow{\sim} \Lambda(G)$ of (iv) fit into a commutative diagram

$$\begin{array}{ccc} k^\times & \xrightarrow{\mathrm{Kmm}_k} & H^1(G_k, \Lambda(\overline{k})) \\ \wr\downarrow & & \wr\downarrow \\ k^\times(G) & \xrightarrow{\mathrm{Kmm}(G)} & H^1(G, \Lambda(G)). \end{array}$$

Remark

In general:

$$G_k \not\rightsquigarrow \text{ the field } k.$$

Indeed: $\exists$a pair of MLF $(k_\circ, k_\bullet)$ s.t.

$$G_{k_\circ} \simeq G_{k_\bullet} \quad \text{but} \quad k_\circ \not\simeq k_\bullet.$$

On the other hand:

$$G_k + \text{ram'n fil'n} \rightsquigarrow \text{ the field } k \quad \text{[Mochizuki]}$$

$$G_k + \text{Hodge-Tate rep.} \rightsquigarrow \text{ the field } k \quad \text{[H]}$$

Cyclotomic Synchronization Isomorphism in IUT

An answer to (CSI_2) How does one use a CSI?

Recall: k: an MLF

$G \curvearrowright M$: an isomorph of $\mathrm{Gal}(\overline{k}/k) \curvearrowright \mathcal{O}^{\rhd}_{\overline{k}}$

$\Rightarrow M^{\mathrm{gp}} \cong \overline{k}^{\times} \Rightarrow \Lambda(M) \stackrel{\mathrm{def}}{=} \varprojlim_n M^{\mathrm{gp}}[n] \cong \Lambda(\overline{k})$,

i.e., $G \curvearrowright \Lambda(M)$: a cyclotome

On the other hand: $G \rightsquigarrow G \curvearrowright \Lambda(G)$: a cyclotome

<u>CSI [via local class field theory]:</u>

$G \curvearrowright M \rightsquigarrow$ a suitable [e.g., G-eq.] $\Lambda(G) \xrightarrow{\sim} \Lambda(M)$

Cyclotomic Synchronization Isomorphism in IUT

- The $\mathrm{Kmm}(G^\dagger)$'s $\Rightarrow$
$\overline{\mathcal{O}}^{\triangleright}(G) \hookrightarrow \varinjlim_{G^\dagger \subseteq G} H^1(G^\dagger, \Lambda(G^\dagger)\,(= \Lambda(G)))$

- The $(1 \to M^{\mathrm{gp}}[n] \to M^{\mathrm{gp}} \stackrel{n}{\to} M^{\mathrm{gp}} \to 1 \curvearrowleft G^\dagger)$'s
$\Rightarrow \ M \hookrightarrow \varinjlim_{G^\dagger \subseteq G} H^1(G^\dagger, \Lambda(M))$

In fact: Our $\Lambda(G) \stackrel{\sim}{\to} \Lambda(M)$ is a unique isom. s.t.
$\varinjlim H^1(\Lambda(G)) \stackrel{\sim}{\to} \varinjlim H^1(\Lambda(M)) \stackrel{!!!}{\Rightarrow} \overline{\mathcal{O}}^{\triangleright}(G) \stackrel{\sim}{\to} M$.

Thus, we obtain a "Kummer isomorphism", i.e.,
"étale-like monoid $\stackrel{\sim}{\to}$ Frobenius-like portion".

Cyclotomic Synchronization Isomorphism in IUT

$G_i \curvearrowright M_i$: an isomorph of $G_{\overline{k}_i} \curvearrowright \mathcal{O}^{\triangleright}_{\overline{k}_i}$ $[i = 1, 2]$

Given an isom. $G_1 \xrightarrow{\sim} G_2$

[i.e., two "math. worlds" $G_1 \curvearrowright M_1$, $G_2 \curvearrowright M_2$ are glued by an "étale bridge" $G_1 \xrightarrow{\sim} G_2$]

Then:

$$M_1 \overset{\text{Kmm via CSI}}{\xleftarrow{\sim}} \overline{\mathcal{O}}^{\triangleright}(G_1) \overset{\text{given}}{\xrightarrow{\sim}} \overline{\mathcal{O}}^{\triangleright}(G_2) \overset{\text{Kmm via CSI}}{\xrightarrow{\sim}} M_2$$

Thus: $G_1 \xrightarrow{\sim} G_2 \rightsquigarrow (G_1 \curvearrowright M_1) \xrightarrow{\sim} (G_2 \curvearrowright M_2)$

Cyclotomic Synchronization Isomorphism in IUT

Recall: the above discussion

$\Leftarrow$ the $\exists$ of $\mathrm{Kmm}(G)$: $k^\times(G) \hookrightarrow H^1(G, \Lambda(G))$

$\Leftarrow$ $k^\times(G) \subseteq \overline{k}^\times(G) \supseteq \overline{k}^\times(G)_{\mathrm{tor}} \rightsquigarrow \Lambda(G)$,

i.e., the relationship between the algorithms for constructing $k^\times(G)$ and $\Lambda(G)$ [cf. (MRA_2)]

Remark: In IUT, $\exists$other various CSI, e.g., a CSI via a mono-Θ environment.

§4 Reconstruction of Global Cyclotomes

Set of Nonarchimedean Primes

G: of NF-type

- $\widetilde{\mathcal{V}}(G)$

$\overset{\text{def}}{=}$ {maximal subgps of G of MLF-type} $\overset{\text{conj.}}{\curvearrowleft}$ G

- $\mathcal{V}(G) \overset{\text{def}}{=} \widetilde{\mathcal{V}}(G)/G$ by Neukirch's work

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F: an NF $G_F \overset{\text{def}}{=} \mathrm{Gal}(\overline{F}/F)$

- $\mathcal{V}_{\overline{F}} \overset{\text{def}}{=}$ { nonarch'n primes of $\overline{F}$ } $\curvearrowleft$ G_F
- $\mathcal{V}_F \overset{\text{def}}{=}$ { nonarch'n primes of F } $\cong \mathcal{V}_{\overline{F}}/G_F$

Local Modules/Group of Finite Idèles

For $v \in \mathcal{V}(G)$, by considering the "diagonal",
$\mathcal{O}^{\times}(v) \subseteq k^{\times}(v) \subseteq \prod_{D \in v} k^{\times}(D) \subseteq \prod_{D \in v} D^{\mathrm{ab}}$
by §3

$\Rightarrow \mathbb{I}^{\Sigma}(G) \stackrel{\mathrm{def}}{=} (\prod_{v \in \Sigma} k^{\times}(v)) \times (\prod_{v \notin \Sigma} \mathcal{O}^{\times}(v))$

$\mathbb{I}^{\mathrm{fin}}(G) \stackrel{\mathrm{def}}{=} \varinjlim_{\Sigma \subseteq \mathcal{V}(G) \colon \mathrm{finite}} \mathbb{I}^{\Sigma}(G)$

For $v \in \mathcal{V}_F$,
$\mathcal{O}_{F_v}^{\times} \subseteq F_v^{\times}$, where F_v: the completion of F at v
$\mathbb{I}_F^{\mathrm{fin}}$: the group of finite idèles of F

Homomorphism via Global Class Field Theory

The $(k^\times(v) \hookrightarrow D^{\mathrm{ab}} \to G^{\mathrm{ab}})$'s $\Rightarrow \mathbb{I}^{\mathrm{fin}}(G) \to G^{\mathrm{ab}}$

$\mathbb{I}_F$: the group of idèles of F

By global class field theory:

$$(\mathbb{I}_F^{\mathrm{fin}} \hookrightarrow)\ \mathbb{I}_F \twoheadrightarrow (\mathbb{I}_F/F^\times \twoheadrightarrow)\ G_F^{\mathrm{ab}}$$

Moreover: the $(F^\times \hookrightarrow F_v^\times)$'s $\Rightarrow F^\times \hookrightarrow \mathbb{I}_F^{\mathrm{fin}}$.

Remark: $F^\times \hookrightarrow \mathbb{I}_F^{\mathrm{fin}} \to G_F^{\mathrm{ab}}$ is nontriv. in general!

[i.e., "$F^\times$" of "$F^\times \hookrightarrow \mathbb{I}_F^{\mathrm{fin}}$" $\neq$ "$F^\times$" of "$\mathbb{I}_F/F^\times$"]

Proposition

It holds that:

$$\mathrm{Ker}(\mathbb{I}_F^{\mathrm{fin}} \to G_F^{\mathrm{ab}})_{\mathrm{tor}} \subseteq (F^\times)_{\mathrm{tor}} \; (= \mu(F))$$

If, moreover, F is totally imaginary, then:

$$\mathrm{Ker}(\mathbb{I}_F^{\mathrm{fin}} \to G_F^{\mathrm{ab}})_{\mathrm{tor}} = (F^\times)_{\mathrm{tor}} \; (= \mu(F))$$

Global Cyclotome

$$\mu(G) \stackrel{\text{def}}{=} \varinjlim_{G^\dagger \subseteq G:\,\text{open}} \mathrm{Ker}(\mathbb{I}^{\text{fin}}(G^\dagger) \to (G^\dagger)^{\text{ab}})_{\text{tor}}$$

$$\Lambda(G) \stackrel{\text{def}}{=} \varprojlim_n \mu(G)[n] \qquad \overset{\text{conj.}}{\curvearrowleft} \; G$$

Thus, we obtain a <u>global cyclotome</u> $G \curvearrowright \Lambda(G)$!

$$\mu(\overline{F}) = \varinjlim_{E/F:\,\text{fin.}} \mathrm{Ker}(\mathbb{I}^{\text{fin}}_E \to G_E^{\text{ab}})_{\text{tor}}$$ by Prop.

$$\Lambda(\overline{F}) \stackrel{\text{def}}{=} \varprojlim_n \mu(\overline{F})[n]$$

Local-global CSI

By our construction:

$$\mu(G) \subseteq \varinjlim_{G^\dagger} \mathbb{I}^{\mathrm{fin}}(G^\dagger) \subseteq \varinjlim_{G^\dagger} \prod_{D^\dagger \in \widetilde{\mathcal{V}}(G^\dagger)} k^\times(D^\dagger)$$

Thus, for $D \in \widetilde{\mathcal{V}}(G)$, we have a homomorphism $\mu(G) \to \varinjlim_{D^\dagger \subseteq D} k^\times(D^\dagger) = \overline{k}^\times(D)$, which ind. a D-eq. isom. $\Lambda(G) \xrightarrow{\sim} \Lambda(D)$: Local-global CSI!

[cf. (MRA_2)]

$\mu(\overline{F}) \subseteq \overline{F}^\times \to \overline{F}_v^\times$ induces $\Lambda(\overline{F}) \xrightarrow{\sim} \Lambda(\overline{F}_v)$.

“Outline” of the Proof of the Main Result

$G_F \stackrel{\text{def}}{=} \mathrm{Gal}(\overline{F}/F)$

(1) By Neukirch’s work,

$$G_F \rightsquigarrow G_F \curvearrowright \mathcal{V}_{\overline{F}} \rightsquigarrow \mathcal{V}_F = \mathcal{V}_{\overline{F}}/G_F.$$

(2) By Class Field Theory + Local Rec’n,

the multiplicative groups $F^\times \subseteq \prod_{v \in \mathcal{V}_F} F_v^\times$.

$\rightsquigarrow \ \mathcal{M}_F \stackrel{\text{def}}{=} (F, \mathcal{O}_F, \mathcal{V}_F, \{U_{(v)}\}_{v \in \mathcal{V}_F})$, where

- the monoid F with respect to "$\times$",
- the submonoid $\mathcal{O}_F \subseteq F$,
- the set $\mathcal{V}_F$, and
- the subgps $U_{(v)} \stackrel{\text{def}}{=} 1 + \mathfrak{m}_{(v)} \subseteq F$ for $v \in \mathcal{V}_F$.

(3) By Uchida's Lemma for NF,

$\mathcal{M}_F \ \rightsquigarrow$ "$+$" of F,

i.e., the field structure of F. $\square$

Mono-anabelian Reconstruction Algorithm

A characterization approach for NF:

By the Neukirch-Uchida theorem, the functor from the cat. of pairs "$\overline{F}/F$" [F: an NF] to the cat. of prof. gps of NF-type given by "$\overline{F}/F \rightsquigarrow \mathrm{Gal}(\overline{F}/F)$" is an equiv. of category.

Let $\mathcal{F}$ be a quasi-inverse of "$\overline{F}/F \rightsquigarrow \mathrm{Gal}(\overline{F}/F)$".
$\Rightarrow$ A functorial assignment $G \rightsquigarrow \mathcal{F}(G)$
[similar to the assignment as in the main result].

Mono-anabelian Reconstruction Algorithm

In other words:

the reconstruction by the following algorithm:

For G of NF-type, the desired "$\overline{F}/F$" is the uniquely determined [by N-U Th'm] pair $\overline{F}/F$ s.t. $G \cong \mathrm{Gal}(\overline{F}/F)$.

Mono-anabelian Reconstruction Algorithm

Problems of this approach:

- This essentially depends on the choice of $\mathcal{F}$ [or the universes w.r.t. the above two categories — cf. fully faithful + essentially surjective $\Leftrightarrow$ equivalence of categories].

- The relationship of G and "$\overline{F}/F$" depends on the choice of an isom. $G \xrightarrow{\sim} \mathrm{Gal}(\overline{F}/F)$, i.e., any "ring-theoretic basepoint" [or "ring-theoretic label"] of G is not determined by G itself.