Introduction to Inter-universal Teichmüller Theory I

— An Approximate Statement of the Main Theorem —

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Notation and Terminology

$$\begin{array}{cccccccc} & \mathcal{O}^{\boldsymbol{\mu}} & \stackrel{\mathrm{def}}{=} & (\mathcal{O}^{\times})_{\mathrm{tor}} & \subseteq & \mathcal{O}^{\times} & \stackrel{\mathrm{def}}{=} & \{|z|=1\} \\ \subseteq & \mathcal{O}^{\rhd} & \stackrel{\mathrm{def}}{=} & \{0<|z|\leq 1\} & \subseteq & \mathcal{O} & \stackrel{\mathrm{def}}{=} & \{|z|\leq 1\} \end{array}$$

$\mathcal{O}^{\times\boldsymbol{\mu}} \stackrel{\mathrm{def}}{=} \mathcal{O}^{\times}/\mathcal{O}^{\boldsymbol{\mu}}$

an *isomorph* of $A \stackrel{\mathrm{def}}{\Leftrightarrow}$ an object which is isomorphic to A

R_{+}: the underlying additive module of a ring R

F: a number field, i.e., $[F : \mathbb{Q}] < \infty$, s.t. $\sqrt{-1} \in F$

$\mathbb{V}(-)$: the set of primes of $(-)$

E: an elliptic curve over F which has

either good or split multiplicative reduction at $\forall v \in \mathbb{V}(F)$

$q_v \in \mathcal{O}_{F_v}^{\rhd}$: the q-parameter of E at $v \in \mathbb{V}(F)$

$q_E \stackrel{\text{def}}{=} (q_v)_{v \in \mathbb{V}(F)} \in \prod_{v \in \mathbb{V}(F)} \mathcal{O}_{F_v}^{\rhd}$

$\Rightarrow \deg q_E$ $(= [F : \mathbb{Q}]^{-1} \log(\prod \sharp(\mathcal{O}_{F_v}/q_v\mathcal{O}_{F_v})))$ $\quad (\approx 6 \cdot \mathrm{ht}_E)$

The Szpiro Conjecture for Elliptic Curves over Number Fields

A certain upper bound of ht_E, i.e., $\deg q_E$

Suppose that the following $(*)$ holds:

$(*)$: $\exists N \geq 2, \exists C \geq 0$ s.t. $\deg q_E^N \leq \deg q_E + C$

Then since $\deg q_E^N = N \cdot \deg q_E$, one may conclude that

$$\deg q_E \leq \frac{C}{N-1}.$$

In order to establish $(*)$, let us

- take two isomorphs ${}^\dagger\mathfrak{S}$, ${}^\ddagger\mathfrak{S}$ of (a part of) scheme theory,
- consider a "link" between these two isomorphs

$$\Theta_{\text{naive}}\colon\ {}^\dagger\mathfrak{S} \ni {}^\dagger q_E^N \mapsto {}^\ddagger q_E \in {}^\ddagger\mathfrak{S}, \quad \text{and}$$

- compare, via Θ_{naive}, the computation of $\deg$ of ${}^\dagger q_E^N$ (in ${}^\dagger\mathfrak{S}$) with the computation of $\deg$ of ${}^\ddagger q_E$ (in ${}^\ddagger\mathfrak{S}$).

$\Theta_{\text{naive}} \colon {}^{\dagger}\mathfrak{S} \to {}^{\ddagger}\mathfrak{S}$: ${}^{\dagger}q_E^N \mapsto {}^{\ddagger}q_E$

Very roughly speaking, the main theorem of IUT asserts that:

Relative to such a link, the computation of $\deg {}^{\dagger}q_E^N$ is, up to mild indeterminacies, *compatible* with the computation of $\deg {}^{\ddagger}q_E$.

($\Rightarrow$ $\deg q_E^N \overset{\text{ind.}\curvearrowright}{=} \deg q_E$ $\Rightarrow$ $(*)$ $\Rightarrow$ the Szpiro Conjecture)

Terminology

- a(n) (*arithmetic*) *holomorphic structure*

  $\overset{\text{def}}{\Leftrightarrow}$ a (structure which determines a) ring structure

- a *mono-analytic structure*

  $\overset{\text{def}}{\Leftrightarrow}$ an "underlying" ("non-holomorphic") structure of a hol. str.

(e.g.: $\mathbb{Q}_p$, $\pi_1^{\text{ét}}(\mathbb{P}^1_{\mathbb{Q}_p} \setminus \{0, 1, \infty\})$: hol.; $(\mathbb{Q}_p)_+$, $\mathbb{Q}_p^{\times}$, $G_{\mathbb{Q}_p}$: mono-an.)

$F_{\mathrm{mod}} \subseteq F$: the field of moduli of E

l: a prime number

$K \stackrel{\mathrm{def}}{=} F(E[l](\overline{F}))$

$\underline{\mathbb{V}} \subseteq \mathbb{V}(K)$: the image of a splitting of $\mathbb{V}(K) \twoheadrightarrow \mathbb{V}_{\mathrm{mod}} \stackrel{\mathrm{def}}{=} \mathbb{V}(F_{\mathrm{mod}})$

Suppose that F/F_{mod} and K/F_{mod} are *Galois*.

$S \stackrel{\mathrm{def}}{=} \left[\mathrm{Spec}\,\mathcal{O}_K/\mathrm{Gal}(K/F_{\mathrm{mod}})\right]$ (the stack-theoretic quotient)

$\Rightarrow$ The arith. div. on $\mathcal{O}_F$ determined by q_E can be descended to an arith. div. on S, i.e., by considering the arith. div. on S det'd by

$$\mathfrak{q} \stackrel{\mathrm{def}}{=} (q_{\underline{v}} \stackrel{\mathrm{def}}{=} q_{\underline{v}|_F} \in \mathcal{O}^{\triangleright}_{F_{\underline{v}|_F}} \subseteq \mathcal{O}^{\triangleright}_{K_{\underline{v}}})_{\underline{v}\in\underline{\mathbb{V}}} \in \prod_{\underline{v}\in\underline{\mathbb{V}}} \mathcal{O}^{\triangleright}_{K_{\underline{v}}}.$$

Note that $\deg \mathfrak{q} \stackrel{\mathrm{def}}{=} [F_{\mathrm{mod}} : \mathbb{Q}]^{-1} \log(\prod_{\underline{v}\in\underline{\mathbb{V}}} \sharp(\mathcal{O}_{K_{\underline{v}}}/q_{\underline{v}}\mathcal{O}_{K_{\underline{v}}})) = \deg q_E$.

Recall An arithmetic line bundle on $\mathcal{O}_K$

$=$ a certain pair of a l.b. $\mathcal{L}$ on $\mathcal{O}_K$ and a metric on $\mathcal{L} \times_{\mathbb{Z}} \mathbb{C}$

$$1 \to \boldsymbol{\mu}(K) \to K^{\times} \stackrel{\mathrm{ADiv}}{\to} \bigoplus_{w \in \mathbb{V}(K)} (K_w^{\times}/\mathcal{O}_{K_w}^{\times}) \to \mathrm{APic}\,\mathcal{O}_K \to 1$$

Categories of Arithmetic Line Bundles on S

$\mathcal{F}^{\circledast}_{\mathrm{mod}}$: the Frobenioid of arithmetic line bundles on S

Module-theoretic Description

$\mathcal{F}^{\circledast}_{\mathfrak{mod}}$: the Frobenioid of collections $\{a_{\underline{v}}\mathcal{O}_{K_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}}$ s.t.

$a_{\underline{v}} \in K_{\underline{v}}^{\times}$, $\quad a_{\underline{v}} \in \mathcal{O}_{K_{\underline{v}}}^{\times}$ for almost $\underline{v} \in \underline{\mathbb{V}}$

Multiplicative Description

$\mathcal{F}^{\circledast}_{\mathrm{MOD}}$: the Frobenioid of pairs $(T, \ \{t_{\underline{v}}\}_{\underline{v}\in\underline{\mathbb{V}}})$ s.t.

T: an $F^{\times}_{\mathrm{mod}}$-torsor, $\quad t_{\underline{v}} \in T \times^{F^{\times}_{\mathrm{mod}}} K_{\underline{v}}^{\times}/\mathcal{O}_{K_{\underline{v}}}^{\times}$

$\Rightarrow$ The holomorphic structure of F_{mod} determines

$$\begin{array}{ccccc} \mathcal{F}^{\circledast}_{\mathfrak{mod}} & \xrightarrow{\sim} & \mathcal{F}^{\circledast}_{\mathrm{mod}} & \xrightarrow{\sim} & \mathcal{F}^{\circledast}_{\mathrm{MOD}} \\ \{a_{\underline{v}}\mathcal{O}_{K_{\underline{v}}}\} & & \mapsto & & (F^{\times}_{\mathrm{mod}},\ \{\mathrm{ord}_{\underline{v}}(a_{\underline{v}})\}) \end{array}$$

$\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}}$, $\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}}$, $\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}}$: the resp. realifications of $\mathcal{F}^{\circledast}_{\mathrm{mod}}$, $\mathcal{F}^{\circledast}_{\mathfrak{mod}}$, $\mathcal{F}^{\circledast}_{\mathrm{MOD}}$,

i.e., obtained by replacing $\bigoplus_{\underline{v}}(K^{\times}_{\underline{v}}/\mathcal{O}^{\times}_{K_{\underline{v}}})$ by $\bigoplus_{\underline{v}}((K^{\times}_{\underline{v}}/\mathcal{O}^{\times}_{K_{\underline{v}}}) \otimes \mathbb{R})$

($\Rightarrow$ The hol. str. of F_{mod} determines $\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}} \xrightarrow{\sim} \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{mod}} \xrightarrow{\sim} \mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}}$)

The multiplication by $1/N$ on $\bigoplus_{\underline{v}}((K_{\underline{v}}^{\times}/\mathcal{O}_{K_{\underline{v}}}^{\times}) \otimes \mathbb{R})$ determines

$$\Theta_{\mathrm{naive}}\colon\ {}^{\dagger}\mathcal{F}_{\mathrm{MOD}}^{\circledast\mathbb{R}} \xrightarrow{\ \sim\ } {}^{\ddagger}\mathcal{F}_{\mathrm{MOD}}^{\circledast\mathbb{R}} \quad \text{which maps} \quad {}^{\dagger}\mathfrak{q}^{N} \mapsto {}^{\ddagger}\mathfrak{q}.$$

Remark

- Θ_{naive} may be regarded as a "deformation of value groups".
- The link "Θ_{naive}" will be eventually established by means of nonarchimedean theta functions (cf. p.21).

$\mathcal{F}_{\mathfrak{mod}}^{\circledast}$	depending on hol. str.	suited to deg. estimates
$\mathcal{F}_{\mathrm{MOD}}^{\circledast}$	only multiplicative str.	not suited to deg. estimates

Goal

Relative to a link such as Θ_{naive}, the computation of $\deg {}^{\dagger}\mathfrak{q}^N$ is, up to mild indeterminacies, *compatible* with the comp. of $\deg {}^{\ddagger}\mathfrak{q}$.

Note that $\nexists$a ring automorphism of $K_{\underline{v}}$ s.t. $q_{\underline{v}}^N \mapsto q_{\underline{v}}$ (if $q_{\underline{v}} \neq 1$).

Thus, Θ_{naive} *cannot be compatible* with the holomorphic structures, i.e., Θ_{naive} may be compatible with only certain mono-analytic str.

(For instance, Θ_{naive} is *compatible* with the local Galois group $G_{\underline{v}} \stackrel{\text{def}}{=} \text{Gal}(\overline{F}_{\underline{v}}/K_{\underline{v}})$ for each finite $\underline{v} \in \underline{\mathbb{V}}$

— cf. Θ_{naive} "$=$" a deformation of value groups.)

On the other hand:

Remark

The "degree computation" is, at least a priori, performed by means of the holomorphic structure under consideration.

Thus, in order to obtain a certain compatibility of the degree computations, we have to establish a "*multiradial representation*" of the degree computations whose coric data consist of suitable mono-analytic structures.

Multiradial Algorithm

Suppose that we are given

- a mathematical object R, i.e., a *radial data*,
- an "underlying" object C of R, i.e., a *coric data*, and
- a func'l algorithm Φ whose input data is (an isomorph of) R.

Example

- R: the one-dimensional complex linear space $\mathbb{C}$

  C: the underlying two-dimensional real linear space $\mathbb{R}^{\oplus 2}$
- R: the field $\mathbb{Q}_p$ C: the underlying additive module $(\mathbb{Q}_p)_+$
- R: the étale fundamental gp $\pi_1^{\text{ét}}(V)$ of a hyperbolic curve $V/\mathbb{Q}_p$

  C: the absolute Galois group $G_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$

Roughly speaking, we shall say that the algorithm Φ is:

- *coric* if Φ depends only on C
- *multiradial* if Φ (is rel'd to R but) may be described in terms of C
- *uniradial* if Φ is not multiradial, i.e., essentially depends on R

If one starts with a coric data "C" and applies the alg'm Φ, then:

- *uniradial* $\Rightarrow$ the output *depends* on the choice of a "spoke"
- *multiradial* $\Rightarrow$ the output is *unaffected* by alterations in a "spoke"

$$\begin{array}{ccccc} \cdots & & R_1 & & \cdots \\ & & \downarrow & & \\ R_2 & \longrightarrow & \boxed{C} & \longleftarrow & R_3 \end{array}$$

(Tautological) Example

$(R, C) \cong (\mathbb{C}, \mathbb{R}^{\oplus 2})$

- $\Phi(R)$ = the holomorphic structure on $R \Rightarrow$ uniradial
- $\Phi(R)$ = the real analytic structure on $R \Rightarrow$ coric
- $\Phi(R)$ = the $\mathrm{GL}_2(\mathbb{R})$-orbit of the hol. str. on $R \Rightarrow$ multiradial

incompatible

H.S. on ${}^{\dagger}\mathbb{C}$

$\downarrow$

H.S. of ${}^{\ddagger}\mathbb{C} \longrightarrow \boxed{\mathbb{R}^{\oplus 2}}$

uniradial

compatible

H.S. of ${}^{\dagger}\mathbb{C}$

$\mathrm{GL}_2(\mathbb{R}) \downarrow \curvearrowright$

H.S. of ${}^{\ddagger}\mathbb{C} \xrightarrow[\curvearrowright]{\mathrm{GL}_2(\mathbb{R})} \boxed{\mathbb{R}^{\oplus 2}}$

multiradial

(cf. $\mathrm{GL}_2(\mathbb{R})/\mathbb{C}^{\times} = \mathbb{H}_+ \sqcup \mathbb{H}_-$)

Summary

- We want to obtain a certain compatibility of the degree computations relative to a link such as Θ_{naive}.
- Θ_{naive} *cannot be compatible* w/ the holomorphic str., i.e., Θ_{naive} is compatible w/ only certain mono-an. str., e.g., $G_{\underline{v}}$.
- On the other hand, the degree computation is, at least a priori, performed by means of the holomorphic structure.
- Thus, we have to establish a *multiradial representation* of the degree computations whose coric data are suitable mono-an. str.

An Approximate Statement of the Main Theorem of IUT (tentative)

$\exists$A suitable multiradial algorithm whose output data consist of

the following three objects $\curvearrowleft$ mild indeterminacies

- $\{(\mathcal{O}_{K_{\underline{v}}})_*\}_{\underline{v}\in\underline{\mathbb{V}}}$ $(* = +$ if $\underline{v}$ is finite, $* = \emptyset$ if $\underline{v}$ is infinite$)$
- $\mathfrak{q}^N \curvearrowright \prod_{\underline{v}\in\underline{\mathbb{V}}} (\mathcal{O}_{K_{\underline{v}}})_*$
- $F_{\mathrm{mod}} \curvearrowright \prod_{\underline{v}\in\underline{\mathbb{V}}} \left((K_{\underline{v}})_*\text{``via } (\mathcal{O}_{K_{\underline{v}}})_*\text{''}\right)$

Moreover, this algorithm is *compatible* with

$$\Theta_{\mathrm{naive}} \colon {}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}};\ {}^{\dagger}\mathfrak{q}^N \mapsto {}^{\ddagger}\mathfrak{q}.$$

Main Theorem of IUT ⇒ Szpiro Conjecture

$\Theta_{\mathrm{naive}} : {}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}}; \quad {}^{\dagger}\mathfrak{q}^{N} \mapsto {}^{\ddagger}\mathfrak{q}$

$\overset{\text{Th'm}}{\Rightarrow}$

$$\begin{pmatrix} \{({}^{\dagger}\mathcal{O}_{K_{\underline{v}}})_{*}\}_{\underline{v}\in\underline{\mathbb{V}}} \\ {}^{\dagger}\mathfrak{q}^{N} \curvearrowright \prod ({}^{\dagger}\mathcal{O}_{K_{\underline{v}}})_{*} \\ {}^{\dagger}F_{\mathrm{mod}} \curvearrowright \prod ({}^{\dagger}K_{\underline{v}})_{*} \end{pmatrix} \overset{\text{ind.}\curvearrowright}{\xrightarrow{\sim}} \begin{pmatrix} \{({}^{\ddagger}\mathcal{O}_{K_{\underline{v}}})_{*}\}_{\underline{v}\in\underline{\mathbb{V}}} \\ {}^{\ddagger}\mathfrak{q}^{N} \curvearrowright \prod ({}^{\ddagger}\mathcal{O}_{K_{\underline{v}}})_{*} \\ {}^{\ddagger}F_{\mathrm{mod}} \curvearrowright \prod ({}^{\ddagger}K_{\underline{v}})_{*} \end{pmatrix}$$

$\Rightarrow$

- $c_{\mathfrak{mod}} : {}^{\dagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}} \overset{\text{ind.}\curvearrowright}{\xrightarrow{\sim}} {}^{\ddagger}\mathcal{F}^{\circledast}_{\mathfrak{mod}}$ which maps $\{{}^{\dagger}q_{\underline{v}}^{N}\,{}^{\dagger}\mathcal{O}_{K_{\underline{v}}}\} \mapsto \{{}^{\ddagger}q_{\underline{v}}^{N}\,{}^{\ddagger}\mathcal{O}_{K_{\underline{v}}}\}$
- $c_{\square} : {}^{\square}\mathcal{F}^{\circledast}_{\mathfrak{mod}} \xrightarrow{\sim} {}^{\square}\mathcal{F}^{\circledast}_{\mathrm{MOD}}$ which maps $\{{}^{\square}q_{\underline{v}}^{(N)}\,{}^{\square}\mathcal{O}_{K_{\underline{v}}}\} \mapsto {}^{\square}\mathfrak{q}^{(N)}$

which are *compatible* with Θ_{naive}, i.e., fit into a diagram that is *commutative*, up to *mild indeterminacies*

$$\begin{array}{ccc} {}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}} & \xrightarrow{c_{\mathfrak{mod}}} & {}^{\ddagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathfrak{mod}} \\ c_{\dagger}\downarrow & & \downarrow c_{\ddagger} \\ {}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}} & \xrightarrow{\Theta_{\mathrm{naive}}} & {}^{\ddagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}} \end{array}$$

$$\Rightarrow \left(\begin{array}{ccc} \{{}^{\dagger}q_{v}^{N}\,{}^{\dagger}\mathcal{O}_{K_{\underline{v}}}\} & \Rightarrow & \{{}^{\ddagger}q_{\underline{v}}^{N}\,{}^{\ddagger}\mathcal{O}_{K_{\underline{v}}}\}, \quad \{{}^{\ddagger}q_{\underline{v}}\,{}^{\ddagger}\mathcal{O}_{K_{\underline{v}}}\} \\ \Uparrow & & \Downarrow \\ {}^{\dagger}\mathfrak{q}^{N} & \Leftarrow & {}^{\ddagger}\mathfrak{q} \end{array} \right)$$

$$\Rightarrow \quad \{{}^{\ddagger}q_{\underline{v}}\,{}^{\ddagger}\mathcal{O}_{K_{\underline{v}}}\} \text{ “} \overset{\text{log-vol.}}{\subseteq} \text{”} \bigcup_{\text{indeterminacies}} \{{}^{\ddagger}q_{\underline{v}}^{N}\,{}^{\ddagger}\mathcal{O}_{K_{\underline{v}}}\}$$

$\Rightarrow \quad -\deg \mathfrak{q} \leq -\deg \mathfrak{q}^{N} + C$, i.e., $(*)$ in p.4

$\Rightarrow$ the Szpiro Conjecture

An Approximate Statement of the Main Theorem of IUT (tentative)

$\exists$A suitable multiradial algorithm whose output data consist of the following three objects $\curvearrowleft$ mild indeterminacies

- $\{(\mathcal{O}_{K_{\underline{v}}})_*\}_{\underline{v}\in\underline{\mathbb{V}}}$ $\quad$ ($* = +$ if $\underline{v}$ is finite, $\quad * = \emptyset$ if $\underline{v}$ is infinite)
- $\mathfrak{q}^N \curvearrowright \prod_{\underline{v}\in\underline{\mathbb{V}}} (\mathcal{O}_{K_{\underline{v}}})_*$
- $F_{\mathrm{mod}} \curvearrowright \prod_{\underline{v}\in\underline{\mathbb{V}}} \left((K_{\underline{v}})_*\text{“via } (\mathcal{O}_{K_{\underline{v}}})_*\text{''}\right)$

Moreover, this algorithm is *compatible* with

$$\Theta_{\mathrm{naive}} \colon {}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}} \xrightarrow{\sim} {}^{\ddagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}};\ {}^{\dagger}\mathfrak{q}^N \mapsto {}^{\ddagger}\mathfrak{q}.$$

Recall ($\underline{v} \in \underline{\mathbb{V}}$: fin.; $p \stackrel{\text{def}}{=} p_{\underline{v}}$) $\quad (\mathcal{O}_{K_{\underline{v}}})_+ \in$ output, $G_{\underline{v}}$: coric

Unfortunately, it is known that:

$\nexists$*a func'l* (w.r.t. open injections) *alg'm* for rec. "$(\mathcal{O}_{K_{\underline{v}}})_+$" from "$G_{\underline{v}}$".

$\mathcal{I}_{K_{\underline{v}}} \stackrel{\text{def}}{=} \frac{1}{2p}\mathrm{Im}(\mathcal{O}^\times_{K_{\underline{v}}} \to \mathcal{O}^\times_{K_{\underline{v}}} \otimes_{\mathbb{Z}} \mathbb{Q} \stackrel{\log_p}{\stackrel{\sim}{\to}} (K_{\underline{v}})_+)$: the *log-shell* of $K_{\underline{v}}$

- $\mathcal{I}_{K_{\underline{v}}}$: a finitely generated free $\mathbb{Z}_p$-module
- $(\mathcal{O}_{K_{\underline{v}}})_+, \ \log_p(\mathcal{O}^\times_{K_{\underline{v}}}) \subseteq \mathcal{I}_{K_{\underline{v}}} \subseteq \mathcal{I}_{K_{\underline{v}}} \otimes_{\mathbb{Z}} \mathbb{Q} = (K_{\underline{v}})_+$
- $[\mathcal{I}_{K_{\underline{v}}} : (\mathcal{O}_{K_{\underline{v}}})_+]$ $(< \infty)$ can be comp'd by the top. gp str. of $G_{\underline{v}}$
- $G_{\underline{v}} \underset{\text{algorithm}}{\overset{\exists\text{func'l}}{\Rightarrow}}$ an isomorph of $\mathcal{I}_{K_{\underline{v}}}$

Thus, "$\{(\mathcal{O}_{K_{\underline{v}}})_*\}_{\underline{v}\in\underline{\mathbb{V}}}$" in Th'm should be replaced by $\{\mathcal{I}_{\underline{v}} \stackrel{\text{def}}{=} \mathcal{I}_{K_{\underline{v}}}\}_{\underline{v}\in\underline{\mathbb{V}}}$

(where the log-shell at an infinite $\underline{v} \in \underline{\mathbb{V}} \stackrel{\text{def}}{=} \pi \cdot \mathcal{O}_{K_{\underline{v}}}$).

Recall $\mathfrak{q}^N$, $F_{\mathrm{mod}} \in$ output

Unfortunately, by various indeterminacies arising from the operation of "passing from holom'c str. to mono-anal'c str.", it is difficult to obtain multiradial representations of $\mathfrak{q}^N$, F_{mod} themselves directly. To establish a mul'l alg'm of the desired type, we rep. multiradially a *suitable function* whose special value is $q_{\underline{v}}^N$ or an $\in F_{\mathrm{mod}}$.

- "$q_{\underline{v}}^N$" will be represented as a special value of a (multiradially represented) *theta function*.
- "F_{mod}" will be represented as a set of special values of (multiradially represented) κ-*coric functions*.

An Approximate Statement of the Main Theorem of IUT

For a "general E/F",

$\exists$a suitable multiradial algorithm whose output data consist of

the following three objects $\curvearrowleft$ mild indeterminacies

- the collection of log-shells $\{\mathcal{I}_{\underline{v}}\}_{\underline{v} \in \underline{\mathbb{V}}}$
- the theta values $(= \{\mathfrak{q}^{j^2/2l}\}_{1 \leq j \leq l^* \overset{\text{def}}{=} \frac{l-1}{2}}) \curvearrowright \prod_{\underline{v} \in \underline{\mathbb{V}}} \mathcal{I}_{\underline{v}}$
- F_{mod} via κ-coric functions $\curvearrowright \prod_{\underline{v} \in \underline{\mathbb{V}}} ((K_{\underline{v}})_+$ "via $\mathcal{I}_{\underline{v}}$"$)$

Moreover, this alg'm is *compatible* w/ the Θ-link (more precisely, $\Theta^{\times \boldsymbol{\mu}}_{\mathrm{LGP}}$-link) "${}^{\dagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}} \overset{\sim}{\to} {}^{\ddagger}\mathcal{F}^{\circledast\mathbb{R}}_{\mathrm{MOD}}$"; "†theta values $\mapsto {}^{\ddagger}\mathfrak{q}^{1/2l}$".

Fundamental Strategy

$\square$ is, for instance, a log-shell, a theta function, or a κ-coric function.

- Start with a usual/existing $\square$ (i.e., a *Frobenius-like* $\square$).
- Construct *links* by means of such Frobenius-like objects.
- Take an *étale-like* object closely related to $\square$

  (e.g., "$\pi_1^{\mathrm{temp}}(\underline{\underline{X}}_{\underline{v}})$" for a theta function — cf. II and III).
- Give a multiradial mono-anabelian algorithm of reconstructing $\square$

  from the étale-like object, i.e., construct a suitable *étale-like* $\square$.
- Establish "multiradial Kummer-detachment" of $\square$, i.e.,

  a suitable Kummer isomorphism "Frob.-like $\square \xrightarrow{\sim}$ étale-like $\square$".