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An Algorithm for $(n-3)$-Connectivity Augmentation Problem: Jump System Approach

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Abstract

We consider the problem of making a given $(k-1)$-connected graph k-connected by adding a minimum number of new edges, which we call the k-connectivity augmentation problem. This problem is polynomially solvable when k is a fixed number or $k = n-1, n-2$, and it is open for general k. Here n is the number of vertices of the input graph. In this paper, we deal with the problem when $k = n-3$. By considering the complement graph, the $(n-3)$-connectivity augmentation problem can be reduced to the problem of finding a maximum square-free 2-matching in a simple subcubic graph.

We give a polynomial-time algorithm to find a maximum square-free 2-matching in a simple subcubic graph, which yields a polynomial-time algorithm for the $(n-3)$-connectivity augmentation problem. Our algorithm is based on the fact that the square-free 2-matchings are endowed with a matroid structure called a jump system. We also show that the weighted $(n-3)$-connectivity augmentation problem can be solved in polynomial time if the weights are induced by a function on the vertex set, whereas the problem is NP-hard for general weights.

1 Introduction

For an integer k, a graph (digraph) is *k-connected* if it contains more than k vertices and it remains connected (respectively, strongly connected) when we delete at most $k-1$ vertices from the graph (respectively, digraph). In this paper, we deal with the problem of making a given graph or digraph k-connected by adding a minimum number of new edges. Concerning the directed case, Frank and Jordán gave a min-max formula and also an algorithm relying on the ellipsoid method for finding the minimum [11]. In [12] they also provided a combinatorial algorithm to make a $(k-1)$-connected digraph k-connected. However, their algorithm is polynomial-time only for a fixed k, that is, the running time is polynomial in the size of the digraph but exponential in k. Végh and Benczúr gave a combinatorial algorithm for the general case whose running time is polynomial also in k [42].

There are only partial results in the undirected case. The solution is trivial when $k = 1$. Eswaran and Tarjan solved the problem for $k = 2$ in [9], while Watanabe and Nakamura found a characterization for the case of $k = 3$ [43]. Later, Hsu and Ramachandran [22, 23] gave a linear-time algorithm for both of these problems. For $k = 4$, a polynomial algorithm was developed by Hsu [21]. It is also known that near-optimal solutions can be found in polynomial time for every

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k, see [24, 25]. In [26], Jackson and Jordán gave an algorithm which provides an optimal solution in polynomial time for every fixed k. If the size of an optimal solution is large compared to k, their algorithm is polynomial for all k. They also obtained a min-max formula for this special case, and completely solved the problem for a new family of graphs called *k-independence free graphs*. However, the complexity of the vertex-connectivity augmentation problem is still open, and it is certainly one of the most interesting unsolved questions in this area.

An interesting special case consists of increasing the connectivity by one, that is, when the starting graph is already $(k-1)$-connected. In this paper we call this problem the *k-connectivity augmentation problem*. Hsu gave an almost linear-time algorithm to increase the connectivity from three to four in [20]. Hence a linear-time algorithm for $k = 1, 2, 3$, an almost linear-time algorithm for $k = 4$ and a polynomial time algorithm provided by [26] for fixed k are at hand. For general k, the question is open again.

On the other hand, values of k close to n are also of interest. If $k = n - 1$, then the graph should be simply extended to a complete graph and the answer is trivial since every augmenting set consists of the edges of $\overline{G}$ where $\overline{G}$ denotes the complement of G. Jim Geelen observed that a graph G is $(n-2)$-connected if and only if each vertex has degree at most one in $\overline{G}$. This implies that for $k = n-2$ the problem is equivalent to finding a maximum matching in the complement of the graph. It can be verified easily that a graph G is $(n-3)$-connected if and only if the edge set of $\overline{G}$ is a square-free 2-matching, that is, each vertex in $\overline{G}$ has degree at most two and $\overline{G}$ contains no cycle of length four. Moreover, an obvious but important observation is that if G is $(n-4)$-connected then its complement $\overline{G}$ is a subcubic graph (i.e. each vertex has degree at most three). Therefore, the $(n-3)$-connectivity augmentation problem can be reduced to the problem of finding a square-free 2-matching of maximum size, called the *square-free 2-matching problem*, in a simple subcubic graph.

Square free 2-matchings appear also in the context of 2-matchings without short cycles. We say that a 2-matching M is *C_k-free* if M contains no cycle of length k or less. The *C_k-free 2-matching problem* is to find a C_k-free 2-matching of maximum size in a given graph. This problem has been studied as a relaxation of the Hamiltonian cycle problem. The case $k \leq 2$ is exactly the classical simple 2-matching problem, which can be solved efficiently. Papadimitriou showed that the problem is NP-hard when $k \geq 5$ (see [4]), and Hartvigsen [18] gave an augmenting path algorithm for the case $k = 3$. The C_4-free 2-matching problem is left open. In bipartite graphs, "C_4-free" and "square-free" mean the same condition. For the square-free 2-matching problem in bipartite graphs, a min-max formula [27] and polynomial-time algorithms [19, 37] are proposed.

The main result of this paper is a polynomial-time algorithm for the square-free 2-matching problem in simple subcubic graphs (Theorem 3.1), which leads to a polynomial-time algorithm for the $(n-3)$-connectivity augmentation problem (Theorem 3.2). Our algorithm is based on the theorem that square-free 2-matchings in a simple subcubic graph have a matroid structure called a jump system (Theorem 3.3). With the aid of known results on jump systems, we show that some optimization problems are also solvable in polynomial-time. We also give a faster algorithm for the square-free 2-matching problem in simple subcubic graphs, that runs in $\mathrm{O}(n^{\frac{3}{2}})$ time (Theorem 3.9).

We also discuss the weighted versions of the problems. Given a $(k-1)$-connected graph $G = (V, E)$ and a weight function $w : \overline{E} \to \mathbf{R}_+$, where $\overline{E}$ is the complement of E, the *weighted k-connectivity augmentation problem* is the problem of finding a set of edges of minimum total weight that should be added to the original graph to obtain a simple k-connected graph. Of course the weighted $(n-3)$-connectivity augmentation problem can be reduced to the problem of finding a square-free 2-matching maximizing the total weight of its edges, which we call the *weighted square-free 2-matching problem*.

Z. Király proved that the weighted square-free 2-matching problem in bipartite graphs is NP-hard (see [10]). This problem is, however, polynomially solvable in bipartite graphs if the weight function is vertex-induced on every square [32, 40]. For a subgraph $H = (V(H), E(H))$ of G, we say that w is *vertex-induced on* H if there exists a function $\pi_H : V(H) \to \mathbf{R}$ such that $w(e) = \pi_H(u) + \pi_H(v)$ for every edge $e = (u, v) \in E(H)$.

We show that the weighted square-free 2-matching problem in simple subcubic graphs can be solved in polynomial time if the weight function is vertex-induced on every square (Theorem 6.1), whereas the problem is NP-hard for general weights (Theorem 5.1). In our algorithm for the weighted problem, we use the theory of M-concave (M-convex) functions on constant-parity jump systems introduced by Murota [35].

It may be noted that jump systems and M-concave (M-convex) functions are understood as a natural framework of efficiently solvable problems. Besides studies of these structures themselves [28, 31, 35, 38], their relation to efficiently solvable combinatorial optimization problems has been revealed (see [1, 6, 29, 30, 35, 39]). This paper presents another such example and enforces the importance of these structures.

This paper is organized as follows. In Section 2, we give definitions and previous works on connectivity, square-free 2-matchings, and jump systems. In Section 3, which is the main part of this paper, we give polynomial-time algorithms for the square-free 2-matching problem in simple subcubic graphs and the $(n-3)$-connectivity augmentation problem. In our algorithms, we use the relation between square-free 2-matchings and jump systems, which is shown in Section 4. In Sections 5 and 6, we show the NP-hardness of the weighted version of the problem and give a polynomial-time algorithm for the case when the weight function is vertex-induced on every square, respectively. In Section 7, we give a min-max theorem characterizing the maximum size of a square-free 2-matching in a subcubic graph. We also prove the corresponding special case of Jordán's conjecture about the size of minimum augmenting sets.

2 Preliminaries

2.1 Connectivity and square-free 2-matchings

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E, and n and m denote the number of vertices and the number of edges, respectively. An edge connecting $u, v \in V$ is denoted by (u, v). A cycle C, which is denoted by $C = (v_1, v_2, \ldots, v_l)$, is a subgraph consisting of distinct vertices $v_1, \ldots, v_l$ and edges $(v_1, v_2), \ldots, (v_{l-1}, v_l), (v_l, v_1)$. For a subgraph H of G, the vertex set and the edge set of H are denoted by $V(H)$ and $E(H)$, respectively. Let δv denote the set of edges incident to $v \in V$.

For an integer k, we say that a graph $G = (V, E)$ is *k-connected* if $|V| \geq k + 1$ and $G - X$ is connected for every $X \subseteq V$ with $|X| \leq k - 1$. The *complement graph* of $G = (V, E)$ is the simple graph $\overline{G} = (V, \overline{E})$ such that $(u, v) \in \overline{E}$ if and only if $(u, v) \notin E$ for distinct $u, v \in V$.

The *degree* of a vertex $v \in V$ in G is the number of edges incident with v. The *degree sequence* of an edge set $F \subseteq E$ is the vector $d_F \in \mathbf{Z}^V$ such that $d_F(v)$ is the number of edges in F incident with v. Note that if a self-loop e is incident with v, e is counted twice. We say that a graph $G = (V, E)$ is *subcubic* (respectively, *cubic*) if $d_E(v) \leq 3$ (respectively, $d_E(v) = 3$) for every $v \in V$. An edge set $M \subseteq E$ is said to be a *2-matching* (respectively *2-factor*) if $d_M(v) \leq 2$ (respectively $d_M(v) = 2$) for every $v \in V$. In other words, a 2-matching is a vertex-disjoint collection of paths and cycles. For a simple undirected graph $G = (V, E)$, an edge set $M \subseteq E$ is a *square-free 2-matching* if M is a 2-matching that contains no cycle of length four as a subgraph.

By the definition of k-connectivity, for an integer t, a simple graph $G = (V, E)$ is $(n - t)$-connected if and only if $\overline{G}$ contains no complete bipartite graph with $t + 1$ vertices. Thus, we can observe the following:

- G is $(n - 2)$-connected if and only if $\overline{G}$ contains no $K_{1,2}$, that is, $\overline{E}$ is a matching.
- G is $(n-3)$-connected if and only if $\overline{G}$ contains no $K_{1,3}$ and no $K_{2,2}$, that is, $\overline{E}$ is a square-free 2-matching.
- G is $(n - 4)$-connected if and only if $\overline{G}$ contains no $K_{1,4}$ and no $K_{2,3}$, in particular $\overline{G}$ is subcubic.

In what follows, we deal with simple graphs when we consider the $(n-3)$-connectivity augmentation problem and the square-free 2-matching problem, and so we often omit to declare that the graph is simple. Non-simple graphs appear only when we shrink graphs.

Definition 2.1. Let $C = (v_1, v_2, v_3, v_4)$ be a cycle of length four in $G = (V, E)$. *Shrinking* of C in G consists of the following operations:

- identify v_1 with v_3, and denote the corresponding vertex by u_1,
- identify v_2 with v_4, and denote the corresponding vertex by u_2, and
- identify all edges between u_1 and u_2.

In the obtained graph, the edge between u_1 and u_2 corresponding to $E(C)$ is called a *square-edge*.

Let $C_1, C_2, \ldots, C_q$ be edge-disjoint cycles of length four, and let $G^\circ = (V^\circ, E^\circ)$ be the graph obtained from $G = (V, E)$ by shrinking $C_1, C_2, \ldots, C_q$. Note that G° might have self-loops and parallel edges, whereas G does not. We also note that if G is subcubic, $C_1, C_2, \ldots, C_q$ are vertex-disjoint and G° is also subcubic. In a shrunk graph G°, a *square* is a cycle of length four whose vertices are not incident to a square-edge. In other words, a cycle in G° is a square if its corresponding edges in G form a cycle of length four. We say that an edge set in a shrunk graph G° is *square-free* if it contains no square.

2.2 Jump systems

Let V be a finite set. For $u \in V$, we denote by χ_u the *characteristic vector* of u, with $\chi_u(u) = 1$ and $\chi_u(v) = 0$ for $v \in V \setminus \{u\}$. For $x, y \in \mathbf{Z}^V$, a vector $s \in \mathbf{Z}^V$ is called an (x, y)-*increment* if $x(u) < y(u)$ and $s = \chi_u$ for some $u \in V$, or $x(u) > y(u)$ and $s = -\chi_u$ for some $u \in V$.

A *jump system*, introduced by Bouchet and Cunningham [2], is defined as follows.

Definition 2.2 (Jump system [2])**.** A nonempty set $J \subseteq \mathbf{Z}^V$ is said to be a *jump system* if it satisfies an exchange axiom, called the 2-*step axiom*:

> For any $x, y \in J$ and for any (x, y)-increment s with $x + s \notin J$, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$.

A set $J \subseteq \mathbf{Z}^V$ is a *constant-parity system* if $x(V) - y(V)$ is even for any $x, y \in J$. Here $x(S) = \sum_{v \in S} x(v)$ for $x \in \mathbf{Z}^V$ and $S \subseteq V$. For constant-parity jump systems, J. F. Geelen observed a stronger exchange property:

(EXC) For any $x, y \in J$ and for any (x,y)-increment s, there exists an $(x+s,y)$-increment t such that $x+s+t \in J$ and $y-s-t \in J$.

This property characterizes a constant-parity jump system (see [35] for details).

Theorem 2.3. *A nonempty set J is a constant-parity jump system if and only if it satisfies* (EXC).

A constant-parity jump system is a generalization of the base family of a matroid, an even delta-matroid [44, 45], and a base polyhedron of an integral polymatroid (or a submodular system) [13].

The degree sequences of all subgraphs in an undirected graph is a typical example of a constant-parity jump system [2, 31]. Cunningham [3] showed that the set of degree sequences of all C_k-free 2-matchings is a jump system for $k \leq 3$, but not a jump system for $k \geq 5$. Szabó [39] showed that it is also a jump system when $k = 4$.

Efficient algorithms for optimization problems on jump systems are studied in [36, 38]. For a set $S \subseteq \mathbf{Z}^V$, we define $\Phi(S) = \max_{v \in V}\{\max_{y \in S} y(v) - \min_{y \in S} y(v)\}$.

Theorem 2.4 (Shioura and Tanaka [38])**.** *Let $J \subseteq \mathbf{Z}^V$ be a finite jump system, and $c \in \mathbf{R}^V$ be a vector. Suppose that a vector $x_0 \in J$ is given, and we can check whether $x \in J$ or not in γ time. Then, we can find a vector $x \in J$ maximizing cx in* $\mathrm{O}(n^3 \log \Phi(J)\gamma)$ *time.*

We can also find a vector maximizing the sum of univariate concave functions efficiently. A univariate function $\phi : \mathbf{Z} \to \mathbf{R}$ is *concave* if it satisfies

$$2\phi(x) \geq \phi(x-1) + \phi(x+1)$$

for any $x \in \mathbf{Z}$. A univariate function ϕ is *convex* if $-\phi$ is *concave*.

Theorem 2.5 (Murota and Tanaka [36])**.** *Let $J \subseteq \mathbf{Z}^V$ be a finite jump system, and $\phi_v : \mathbf{Z} \to \mathbf{R}$ be a univariate concave function for each $v \in V$. Suppose that a vector $x_0 \in J$ is given, and we can check whether $x \in J$ or not in γ time. Then, we can find a vector $x \in J$ maximizing $\sum_{v \in V} \phi_v(x)$ in* $\mathrm{O}(n^3 \Phi(J)\gamma)$ *time.*

Note that Shioura and Tanaka [38] gave an algorithm for the problem that runs in $\mathrm{O}(n^4(\log \Phi(J))^2\gamma)$ time. However, if $\Phi(J)$ is fixed, it is slower than the algorithm in Theorem 2.5.

3 Polynomial-time algorithms for the problems

3.1 Main results

Let γ_1 denote the time to solve the b-factor problem when $b(v) \leq 2$, that is, for a not necessarily simple graph $G = (V, E)$ with $|V| = n$ and a vector $b \in \{0,1,2\}^V$, we can determine whether there exists an edge set $F \subseteq E$ such that $d_F = b$ in γ_1 time. It is the same as the running time of finding a maximum cardinality matching, and γ_1 is bounded by $\mathrm{O}(\sqrt{n}m \log_n \frac{n^2}{m})$ [17]. In subcubic graphs, since $m = \mathrm{O}(n)$, we have $\gamma_1 = \mathrm{O}(n^{\frac{3}{2}})$.

Our first results are the following.

Theorem 3.1. *In subcubic graphs, the square-free 2-matching problem can be solved in* $\mathrm{O}(n^3\gamma_1)$ *time.*

Theorem 3.2. *The $(n-3)$-connectivity augmentation problem is solvable in* $\mathrm{O}(n^3\gamma_1)$ *time.*

Theorem 3.2 obviously follows from Theorem 3.1. Note that we can construct the complement graph in $\mathrm{O}(n^2)$ time, which is shorter than $\mathrm{O}(n^3\gamma_1)$ time. Our proof for Theorem 3.1 is based on the fact that the degree sequences of all square-free 2-matchings in a subcubic graph form a jump system. Let $J_{\mathrm{sq}}(G) \subseteq \mathbf{Z}^V$ denote the set of all degree sequences of square-free 2-matchings in G, that is,

$$J_{\mathrm{sq}}(G) = \{d_M \mid M \text{ is a simple square-free 2-matching in } G\}.$$

Then the following theorem holds.

Theorem 3.3 (Szabó [39])**.** *For any subcubic graph G, $J_{\mathrm{sq}}(G)$ is a constant-parity jump system.*

Although a stronger result is given in [39], we give a new proof for this theorem in Section 4 which can be extended to the weighted version.

On the other hand, the membership problem of $J_{\mathrm{sq}}(G)$ can be solved in polynomial time, whose proof is given in Section 3.2.

Lemma 3.4. *Given a subcubic graph $G = (V, E)$ and a vector $x \in \mathbf{Z}^V$, we can determine whether $x \in J_{\mathrm{sq}}(G)$ or not in $\mathrm{O}(\gamma_1)$ time.*

By combining Theorems 2.4 and 3.3 and Lemma 3.4, we obtain Theorem 3.1. Note that $(0, 0, \ldots, 0) \in \mathbf{Z}^V$ is a vector contained in $J_{\mathrm{sq}}(G)$.

We give a faster algorithm for the square-free 2-matching problem in Section 3.3, which does not use jump systems. However, the advantage of using a jump system is that we can immediately extend the result to optimization problems with the aid of some results on jump systems.

When the weight function is vertex-induced on V, the weighted square-free 2-matching problem is the problem of finding a square-free 2-matching M maximizing a linear function of d_M. Therefore, by Theorems 2.4 and 3.3 and Lemma 3.4, we obtain the following corollaries.

Corollary 3.5. *The weighted square-free 2-matching problem in subcubic graphs is solvable in $\mathrm{O}(n^3\gamma_1)$ time if the weight function is vertex-induced on V.*

Corollary 3.6. *The weighted $(n-3)$-connectivity augmentation problem is solvable in $\mathrm{O}(n^3\gamma_1)$ time if the weight function is vertex-induced on V.*

In the same way as these corollaries, we obtain the following by Theorem 2.5.

Corollary 3.7. *Let $\phi_v : \mathbf{Z} \to \mathbf{R}$ be a univariate concave function for each $v \in V$. For a subcubic graph $G = (V, E)$, we can find a square-free 2-matching M maximizing*

$$\sum_{v \in V} \phi_v(d_M(v))$$

in $\mathrm{O}(n^3\gamma_1)$ time.

Corollary 3.8. *Let $\phi_v : \mathbf{Z} \to \mathbf{R}$ be a univariate convex function for each $v \in V$. For an $(n-4)$-connected graph $G = (V, E)$, we can find in $\mathrm{O}(n^3\gamma_1)$ time an edge set $E' \subseteq \overline{E}$ minimizing*

$$\sum_{v \in V} \phi_v(d_{E \cup E'}(v))$$

such that $G' = (V, E \cup E')$ is a simple $(n-3)$-connected graph.

3.2 Proof for Lemma 3.4

In this subsection, we give a proof for Lemma 3.4.

Take edge-disjoint cycles $C_1, C_2, \ldots, C_q$ of length four maximally such that $x(v) = 2$ for each $v \in \bigcup V(C_i)$. Note that these cycles are also vertex-disjoint since the graph is subcubic. Obviously, if there is a cycle C_i such that $V(C_i)$ spans a K_4 then $x \notin J_{\mathrm{sq}}(G)$. Thus, we may assume that $V(C_i)$ does not span a K_4.

Let $G^\circ = (V^\circ, E^\circ)$ denote the graph obtained from $G = (V, E)$ by shrinking $C_1, C_2, ..., C_q$ as in Definition 2.1. Define $E_1 \subseteq E$ as the set of all shrunk edges, that is, $E_1 = E(C_1) \cup \cdots \cup E(C_q)$, and let $E_0 = E \setminus E_1$. Similarly, define $V_1 \subseteq V$ as the set of all shrunk vertices, that is, $V_1 = V(C_1) \cup \cdots \cup V(C_q)$, and let $V_0 = V \setminus V_1$. Therefore E_0 and V_0 are also subsets of E° and V°, respectively. Note that E° may contain self-loops and also parallel edges.

Let $x^\circ \in \mathbf{Z}^{V^\circ}$ be the vector obtained from x by setting

$$x^\circ(v) = \begin{cases} x(v) & \text{if } v \in V_0, \\ 2 & \text{if } v \in V^\circ \setminus V_0. \end{cases}$$

We will show that $x \in J_{\mathrm{sq}}(G)$ if and only if x° is the degree sequence of some 2-matching in G°.

Let $x \in J_{\mathrm{sq}}(G)$ and let M be a square-free 2-matching in $G = (V, E)$ with $d_M = x$. Note that $|E(C_i) \cap M| = 2$ or $|E(C_i) \cap M| = 3$ for $i = 1, 2, \ldots, p$, because G is subcubic. Let u_1^i and u_2^i denote the vertices arising when shrinking $C_i = (v_1^i, v_2^i, v_3^i, v_4^i)$. Let I denote the set of indices for which $|E(C_i) \cap M| = 3$. Then define M° as

$$M^\circ = (M \cap E_0) \cup \left(\bigcup_{i \in I} \{(u_1^i, u_2^i)\} \right).$$

One can see easily that M° is a 2-matching in G° with $d_{M^\circ} = x^\circ$.

Conversely, let M° be a 2-matching in $G^\circ = (V^\circ, E^\circ)$ with $d_{M^\circ} = x^\circ$. Let $C = (v_1, v_2, v_3, v_4)$ be one of the shrunk cycles and let u_1, u_2 be the corresponding vertices in G°. If $(u_1, u_2) \notin M^\circ$ then either $\{(v_1, v_2), (v_3, v_4)\}$ or $\{(v_1, v_4), (v_2, v_3)\}$ can be added to $M^\circ \cap E_0$ without forming a square since G is subcubic (we use here the assumption that $V(C_i)$ does not span a K_4). One can also see that if $(u_1, u_2) \in M^\circ$ then three properly chosen edges of C can be added to $M^\circ \cap E_0$ without forming a square (see Figure 1). What we do exactly is that we blow up the cycles one by one. In each step we extend the actual 2-matching to a new one in the extended graph using one of the two extensions described above in such a way that the arising 2-matching has no square. Recall that a square is defined as a cycle of length four whose all four vertices are contained in V_0. In this way $M^\circ \cap E_0$ can be extended to a square-free 2-matching M of $G = (V, E)$ with $d_M = x$.

The above reduction can be done in linear time and we can determine whether x° is a degree sequence of a 2-matching or not in $\mathrm{O}(\gamma_1)$ time which proves the lemma.

3.3 Faster algorithm

In this subsection, we give another algorithm for the square-free 2-matching problem, that runs in $\mathrm{O}(\gamma_1)$ time. A faster algorithm for the $(n-3)$-connectivity augmentation problem follows from the algorithm. However, in this case, we have to consider the time to construct the complement graph, which is denoted by γ_0. Obviously, γ_0 is bounded by $\mathrm{O}(n^2)$, but it depends on how the input graph is represented.

Theorem 3.9. *The square-free 2-matching problem in subcubic graphs can be solved in $\mathrm{O}(\gamma_1)$ time. The $(n-3)$-connectivity augmentation problem is solvable in $\mathrm{O}(\gamma_0 + \gamma_1)$ time, where γ_0 is the time to construct the complement graph.*

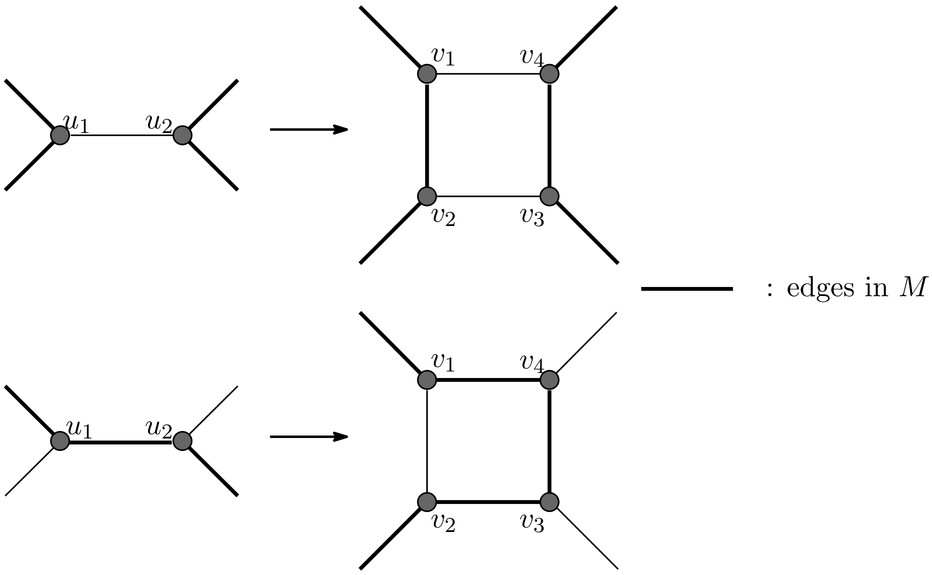


Figure 1: Constructing M from M°

Proof. Let $G = (V, E)$ be a subcubic graph. If G contains a complete graph on four nodes then this K_4 forms a component of G since the graph is subcubic. Clearly, a maximum square-free 2-matching contains exactly three edges of such a component. By handling these components separately, we may assume that G contains no K_4.

Take edge-disjoint cycles $C_1, C_2, \ldots, C_q$ of length four maximally. Our first observation is that for any maximum square-free 2-matching M in G either $|M \cap C_i| = 2$ or $|M \cap C_i| = 3$ for every $C_i = (v_1^i, v_2^i, v_3^i, v_4^i)$. Moreover, we may assume the following:

(A) If $|M \cap C_i| = 2$ then $M \cap C_i = \{(v_1^i, v_2^i), (v_3^i, v_4^i)\}$ or $\{(v_1^i, v_4^i), (v_2^i, v_3^i)\}$.

Let $G^\circ = (V^\circ, E^\circ)$ denote the graph obtained from $G = (V, E)$ by shrinking $C_1, C_2, \ldots, C_q$. Define E_0, E_1 and V_0, V_1 on the same lines with Lemma 3.4.

We will show that for any maximum square-free 2-matching M in G satisfying the condition (A) we can find a 2-matching in G° with $|M^\circ| = |M| - 2q$. Conversely, for any maximum 2-matching M° in G° we can define a square-free 2-matching M in G so that $|M| = |M^\circ| + 2q$. Since a 2-matching in G° with maximum cardinality can be found in $\mathrm{O}(\gamma_1)$ time that would prove the theorem.

The correspondence described in Lemma 3.4 works again. Namely, let M be a maximum square-free 2-matching in G satisfying the condition (A) and let I denote the set of indices for which $|E(C_i) \cap M| = 3$. Then define M° as

$$M^\circ = (M \cap E_0) \cup \left(\bigcup_{i \in I} \{(u_1^i, u_2^i)\} \right).$$

One can see easily that M° is a 2-matching in G° and the observation above implies $|M^\circ| = |M| - 2q$.

Conversely, let M° be a maximum 2-matching in G°. Let $C = (v_1, v_2, v_3, v_4)$ be one of the shrunk cycles and let u_1, u_2 be the corresponding vertices in G°. If $(u_1, u_2) \notin M^\circ$ then either $\{(v_1, v_2), (v_3, v_4)\}$ or $\{(v_1, v_4), (v_2, v_3)\}$ can be added to $M^\circ \cap E_0$ without forming a square since G is subcubic (again, we use here the assumption that G contains no K_4). One can also see that if $(u_1, u_2) \in M^\circ$ then three properly chosen edges of C can be added to $M^\circ \cap E_0$ without forming a square. In both cases, the size of the 2-matching increases by two. Hence $M^\circ \cap E_0$ can be extended to a square-free 2-matching M of $G = (V, E)$ with $|M| = |M^\circ| + 2q$.

Now it is understandable why K_4's are handled differently. If we let G contain a K_4 then after shrinking the cycles the K_4 corresponds to an edge with two self-loops at the end-vertices in G°. However, a maximum 2-matching in G° contains the two self-loops and a maximum square-free 2-matching in G contains three edges from the K_4 so in this case the size of the 2-matching increases only by one when blowing back the corresponding cycle.

As above, the square-free 2-matching problem can be reduced to the ordinary maximum 2-matching problem, which can be solved in $\mathrm{O}(\gamma_1)$ time.

The latter half of the theorem is immediately derived from the first half. □

4 Proof for Theorem 3.3

This section is devoted to the proof for Theorem 3.3, that is, we show that $J_{\mathrm{sq}}(G)$ is a constant-parity jump system for any subcubic graph G. Recall that G is simple. In this section, we give an algorithm for finding an $(x+s, y)$-increment t such that $x + s + t \in J_{\mathrm{sq}}(G)$ and $y - s - t \in J_{\mathrm{sq}}(G)$. Without loss of generality, we assume that $s = -\chi_u$ for some $u \in V$.

4.1 Preliminaries

In this subsection, we give some preliminaries for the proof.

Let M and N be edge sets in an undirected (not necessarily simple) graph. We say that a path $P = (v_0, v_1, v_2, \ldots, v_l)$ is an (M, N)-*alternating path* if

- $(v_i, v_{i+1}) \in M \setminus N$ if i is even,
- $(v_i, v_{i+1}) \in N \setminus M$ if i is odd, and
- $(v_i, v_{i+1}) \neq (v_j, v_{j+1})$ for $i \neq j$.

Obviously, $d_{M \Delta E(P)} = d_M - \chi_{v_0} + (-1)^l \chi_{v_l}$ and $d_{N \Delta E(P)} = d_N + \chi_{v_0} - (-1)^l \chi_{v_l}$. By taking the longest (M, N)-alternating path, we can see the following.

Lemma 4.1. *For 2-matchings M, N in an undirected graph and for a (d_M, d_N)-increment $s = -\chi_u$, there exists an (M, N)-alternating path P beginning with $v_0 = u$ such that both $M \Delta E(P)$ and $N \Delta E(P)$ are 2-matchings (not necessarily square-free), $d_{M \Delta E(P)} = d_M + s + t$, and $d_{N \Delta E(P)} = d_N - s - t$ for some $(x+s, y)$-increment t.*

Let L be a subset of edges and let $C_1, C_2, \ldots, C_q$ be edge-disjoint cycles of length four such that $|E(C_i) \cap L| = 3$ for $i = 1, 2, \ldots, p$. If an edge set $L^\circ \subseteq E^\circ$ is obtained from $L \subseteq E$ by shrinking $C_1, C_2, \ldots, C_q$, we say that L° is the *shrunk edge set* of L, and L is an *expanded edge set* of L°. Note that the shrunk edge set L° contains all square-edges in G°.

We now define a map $\phi : \mathbf{Z}^V \to \mathbf{Z}^{V^\circ}$ by

$$(\phi(x))(u) = \sum \{x(v) \mid v \in V,\ v \text{ corresponds to } u\} \\ - 2|\{\text{square-edges incident to } u\}| \qquad (1)$$

for $x \in \mathbf{Z}^V$ and $u \in V^\circ$. One can see that for an edge set $L \subseteq E$ satisfying that $|E(C_i) \cap L| = 3$ for $i = 1, 2, \ldots, p$, $\phi(d_L)$ is the degree sequence of the shrunk edge set of L. Conversely, the following lemma holds.

Lemma 4.2 (Kobayashi and Takazawa [30]). *Let $L^\circ \subseteq E^\circ$ be a 2-matching in G° that contains all square-edges and x be a vector in $\{0,1,2\}^V$. If $\phi(x)$ is the degree sequence of L°, there exists an expanded edge set L of L° in G such that $d_L = x$. Furthermore, such L is unique.*

4.2 Finding an $(x+s, y-s)$-increment

Although we need an $(x+s, y)$-increment t to prove Theorem 3.3, in this subsection, we give a procedure to find an $(x+s, y-s)$-increment t such that $x+s+t \in J_{\rm sq}(G)$ and $y-s-t \in J_{\rm sq}(G)$. After that, we modify the procedure to obtain an $(x+s, y)$-increment t in Section 4.3.

For given degree sequences $x, y \in J_{\rm sq}(G)$, take edge sets $M, N \subseteq E$ such that $d_M = x$ and $d_N = y$. Let $s = -\chi_u$ be an (x, y)-increment for some $u \in V$. Let $C_1, C_2, \ldots, C_q$ be edge-disjoint cycles of length four in G such that $E(C_i) \subseteq M \cup N$ and $|E(C_i) \cap M| = |E(C_i) \cap N| = 3$ for $i = 1, 2, \ldots, p$. We take such $C_1, C_2, \ldots, C_q$ maximally, and shrink them. Let $G^\circ = (V^\circ; E^\circ)$ be the obtained graph, and let $M^\circ, N^\circ, x^\circ, y^\circ, u^\circ$ and s° be counterparts in G° to M, N, x, y, u and s, respectively. Recall that a square is a cycle of length four whose vertices are not incident to a square-edge. Then, G° satisfy the following condition.

(B) Both edge sets M° and N° contain all square-edges in G°, and G° has no square C such that $E(C) \subseteq M^\circ \cup N^\circ$ and $|E(C) \cap M^\circ| = |E(C) \cap N^\circ| = 3$.

In order to obtain an $(x+s, y-s)$-increment t, it suffices to find an $(x^\circ + s^\circ, y^\circ - s^\circ)$-increment t° and edge sets M^*, N^* in the shrunk graph G° such that M^* and N^* are square-free 2-matchings in G°, $d_{M^*} = x^\circ + s^\circ + t^\circ$, and $d_{N^*} = y^\circ - s^\circ - t^\circ$. This is because a unit vector t corresponding to t° is a desired $(x+s, y-s)$-increment by Lemma 4.2. Thus, in what follows, we describe a procedure that finds an $(x^\circ + s^\circ, y^\circ - s^\circ)$-increment t° and edge sets M^*, N^* in G°.

Let $P = (v_0, v_1, v_2, \ldots, v_l)$ be an (M°, N°)-alternating path beginning with $v_0 = u^\circ$ such that both $M^\circ \Delta E(P)$ and $N^\circ \Delta E(P)$ are 2-matchings, $d_{M^\circ \Delta E(P)} = d_{M^\circ} + s^\circ + t^\circ$, and $d_{N^\circ \Delta E(P)} = d_{N^\circ} - s^\circ - t^\circ$ for some $(x^\circ + s^\circ, y^\circ)$-increment t°. The existence of such a path is guaranteed by Lemma 4.1. We choose v_1 such that $N \cup \{(v_0, v_1)\}$ is square-free if possible. Furthermore, we assume the minimality of P, that is, any subpath $(v_0, v_1, v_2, \ldots, v_p)$ does not satisfy the above conditions for $1 \le p \le l-1$. Let $P^{(p)}$ be the subpath $(v_0, v_1, v_2, \ldots, v_p)$ of P, and define $M^{(p)} = M^\circ \Delta P^{(p)}$ and $N^{(p)} = N^\circ \Delta P^{(p)}$.

If $M^{(l)}$ and $N^{(l)}$ are square-free, then $t^\circ := d_{M^{(l)}} - d_{M^\circ} - s^\circ$ is an $(x^\circ + s^\circ, y^\circ - s^\circ)$-increment, and $M^{(l)}, N^{(l)}$, and t° are the desired outputs. Otherwise, let p be the integer such that $M^{(0)}, M^{(1)}, \ldots, M^{(p)}$ and $N^{(0)}, N^{(1)}, \ldots, N^{(p)}$ are square-free, and $M^{(p+1)}$ or $N^{(p+1)}$ contains a square.

We consider the case when p is even, that is, $M^{(p+1)}$ is square-free and $N^{(p+1)}$ has a square containing (v_p, v_{p+1}). The case when p is odd can be dealt with in the same way. Let $C_1 = (v_{p+1}, v_p, u_1, u_2)$ be the square in $N^{(p+1)}$. When $p \ge 1$, by the minimality of l, $M^{(p)}$ is not a 2-matching, that is, $d_{M^{(p)}}(v_p) = 3$. Therefore $\{(v_p, v_{p+1}), (v_p, u_1)\} \subseteq M^{(p)}$, because G° is subcubic. Furthermore, $\{(v_p, v_{p+1}), (v_p, u_1)\} \subseteq M^{(p)}$ is also true when $p = 0$ by the following claim and the definition of P.

Claim 4.3. *One of the followings holds:*

- *there exists an edge $e \in \delta v_0 \cap (M^\circ \setminus N^\circ)$ such that $N^\circ \cup \{e\}$ is square-free, or*
- *G° has a square $C = (v_0, u_1, u_2, u_3)$ such that $\{(v_0, u_1), (v_0, u_3)\} \subseteq M^\circ$ and $\{(v_0, u_1), (u_1, u_2), (u_2, u_3)\} \subseteq N^\circ$ (see Figure 2).*

Proof. It is obvious because $d_{M^\circ}(v_0) > d_{N^\circ}(v_0)$. □

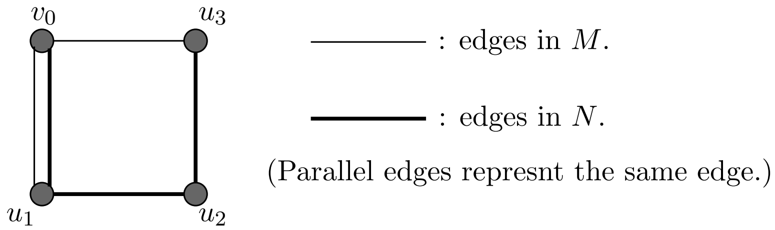


Figure 2: An illustration of Claim 4.3.

Then, by the condition (B), $(v_{p+1}, u_2), (u_1, u_2) \notin M^{(p)}$. Since the graph is subcubic and $(v_{p+1}, u_2), (u_1, u_2) \notin M^{(p)}$, we have $d_{M^{(p)}}(u_2) \leq 1$.

Now we define

$$M' = (M^{(p)} \setminus \{(v_p, v_{p+1})\}) \cup \{(v_{p+1}, u_2)\}, \qquad N' = (N^{(p)} \cup \{(v_p, v_{p+1})\}) \setminus \{(v_{p+1}, u_2)\}$$

(see Figure 3). Obviously, N' is square-free. Since $d_{M^{(p)}}(u_2) \leq 1$ and $d_{N^{(p)}}(u_2) = 2$, M' and N' are 2-matchings and $d_{M'} - d_{M^\circ} - s^\circ = \chi_{u_2}$ is a $(d_{M^\circ} + s^\circ, d_{N^\circ} - s^\circ)$-increment. Therefore, if M' is square-free, then M' and N' are the desired 2-matchings and $t^\circ = \chi_{u_2}$ is the desired unit vector.

Otherwise, M' has a square $C_2 = (v_{p+1}, u_2, u_3, u_4)$ containing (v_{p+1}, u_2). Then, the following claim holds.

Claim 4.4. $u_3 \neq v_p$.

Proof. Assume that $u_3 = v_p$. Since $(v_p, u_1) \in M'$, we have $u_1 = u_4$ and $(u_1, v_{p+1}) \in M'$. Then, $|M^\circ \cap E[C_2]| + |N^\circ \cap E[C_2]| = |M' \cap E[C_2]| + |N' \cap E[C_2]| = 7$, where $E[C_2]$ is the set of edges whose end-vertices are both in $V(C_2)$. This contradicts that M° and N° are square-free 2-matchings. □

By this claim, $\{u_3, u_4\} \cap \{v_p, v_{p+1}\} = \emptyset$. Now we define

$$M'' = M' \setminus \{(u_2, u_3)\}, \qquad N'' = N' \cup \{(u_2, u_3)\}$$

(see Figure 4). Obviously, M'' is a square-free 2-matching. Furthermore, N'' is square-free, because N'' contains (u_3, u_2), (u_2, u_1), (u_1, v_p), (v_p, v_{p+1}), which means that it has no square containing (u_2, u_3). If $d_{N'}(u_3) \leq 1$, then M'' and N'' are the desired 2-matchings and $t^\circ = -\chi_{u_3}$ is the desired unit vector, because $d_{M'}(u_3) = 2$.

Otherwise, $d_{N'}(u_3) = 2$ and $d_{N''}(u_3) = 3$. Since G° is subcubic, $(u_3, u_4) \in N'$.

Claim 4.5. $(u_4, v_{p+1}) \notin N'$.

Proof. If $(u_4, v_{p+1}) \in N'$, then $|M^\circ \cap E(C_2)| + |N^\circ \cap E(C_2)| = |M' \cap E(C_2)| + |N' \cap E(C_2)| = 6$, which contradicts the condition (B). □

We define

$$M''' = (M'' \setminus \{(u_2, v_{p+1})\}) \cup \{(u_2, u_3)\}), \qquad N''' = (N'' \setminus \{(u_3, u_4)\}) \cup \{(u_4, v_{p+1})\}$$

(see Figure 5). Then, $\delta v_{p+1} \cap M''' = \{(v_{p+1}, u_4)\}$ and $\delta v_{p+1} \cap N''' = \{(v_p, v_{p+1}), (v_{p+1}, u_4)\}$. Hence M''' and N''' are square-free 2-matchings and $t^\circ = d_{M'''} - d_{M^\circ} - s^\circ = -\chi_{v_{p+1}}$ is a $(d_{M^\circ} + s^\circ, d_{N^\circ} - s^\circ)$-increment.

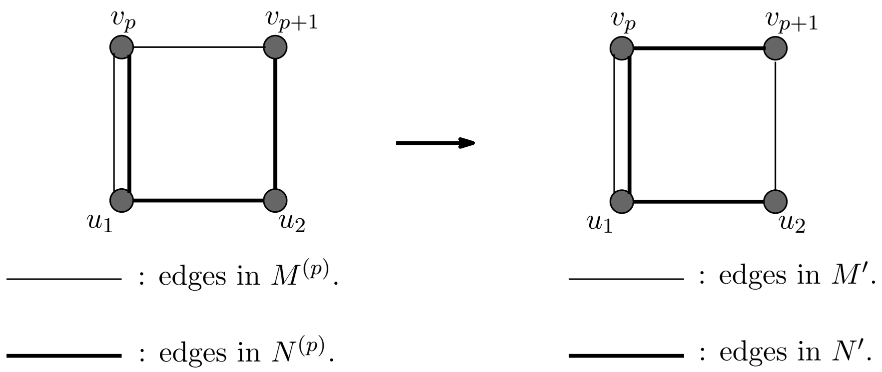


Figure 3: Definitions of M' and N'.

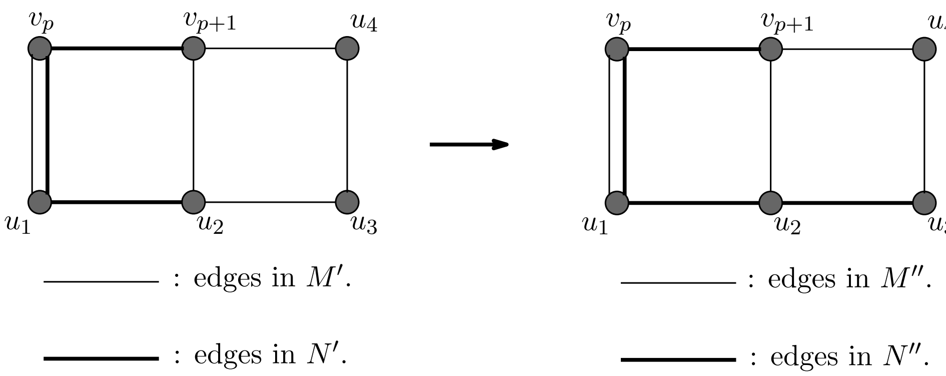


Figure 4: Definitions of M'' and N''.

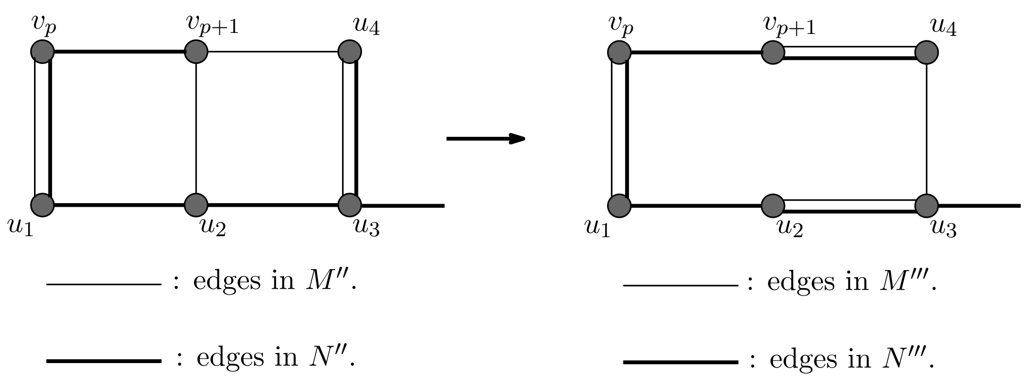


Figure 5: Definitions of M''' and N'''.

4.3 Finding an $(x+s, y)$-increment

We have already presented a procedure to find an $(x+s, y-s)$-increment. To obtain an $(x+s, y)$-increment t, we choose M and N satisfying the following assumption.

Assumption 4.6. For $x, y \in J_{\rm sq}(G)$, let M and N be square-free 2-matchings with $d_M = x$ and $d_N = y$ maximizing $|M \cap N|$.

We show that under Assumption 4.6 we can find an $(x+s, y)$-increment by the procedure in the previous subsection. It suffices to show that we can find an $(x^\circ + s^\circ, y^\circ)$-increment t° in the shrunk graph G°. Note that an $(x^\circ + s^\circ, y^\circ - s^\circ)$-increment t° is not an $(x^\circ + s^\circ, y^\circ)$-increment if and only if $t^\circ = -s^\circ$. We also note that, by Assumption 4.6, M° and N° maximize $|M^\circ \cap N^\circ|$ among all square-free 2-matchings in G° such that both of them contain all square-edges and their degree sequences are x° and y°, respectively. Clearly, the modified 2-matchings in our proof contain all square-edges in each step, since the path is alternating and we modify in squares, where a square is a cycle of length four whose vertices are not incident to a square-edge.

Suppose that the output (M^*, N^*, t°) in the previous subsection satisfies that $t^\circ = -s^\circ$, that is, $d_{M^*} = d_{M^\circ}$ and $d_{N^*} = d_{N^\circ}$. Then, a pair of square-free 2-matchings (M^*, N°) satisfies that $d_{M^*} = x^\circ$, $d_{N^\circ} = y^\circ$, and $|M^* \cap N^\circ| > |M^* \cap N^*|$. More precisely,

- if $(M^*, N^*) = (M', N')$, then $|M^* \cap N^\circ| = |M^* \cap N^*| + \frac{p+2}{2}$,
- if $(M^*, N^*) = (M'', N'')$, then $|M^* \cap N^\circ| = |M^* \cap N^*| + \frac{p+3}{2}$, and
- if $(M^*, N^*) = (M''', N''')$, then $|M^* \cap N^\circ| = |M^* \cap N^*| + \frac{p+1}{2}$.

This contradicts Assumption 4.6.

Thus the output t° is an $(x^\circ + s^\circ, y^\circ)$-increment and its corresponding unit vector $t \in \mathbf{Z}^V$ is an $(x+s, y)$-increment, which completes the proof of Theorem 3.3.

5 NP-hardness of the weighted problem

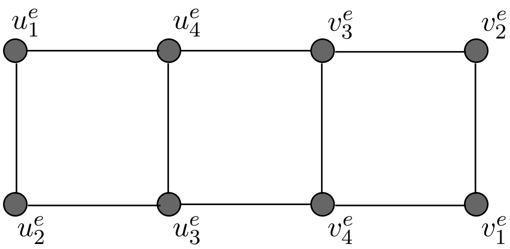
The objective of this section is to show the NP-hardness of the weighted square-free 2-matching problem in subcubic graphs. Actually, we show the following stronger result, which extends Z. Király's result for bipartite graphs.

Theorem 5.1. *The weighted square-free 2-matching problem is NP-hard even if the given graph is cubic, bipartite, and planar.*

First, we show the NP-hardness of the problem of finding a square-free 2-factor of maximum total weight, called the *weighted square-free 2-factor problem*. After that we derive Theorem 5.1 from this result.

Theorem 5.2. *The weighted square-free 2-factor problem is NP-hard even if the given graph is cubic, bipartite, and planar.*

Proof. We give a polynomial reduction from the independent set problem in planar cubic graphs to the weighted square-free 2-factor problem. For a graph $G = (V, E)$, a vertex set $I \subseteq V$ is *independent* if there exists no edge in E connecting two vertices in I. The independent set problem is to find an independent set I of maximum size, and this problem is NP-hard even if the input graph is cubic and planar [16].



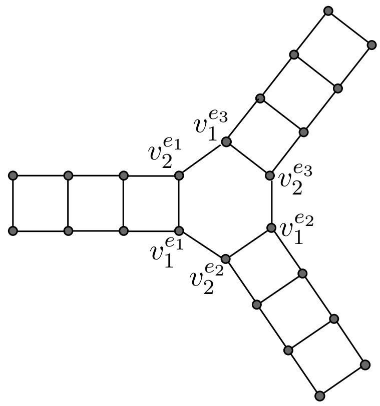


Figure 6: Definitions of V^e, E^e, and E^v.

Let $G = (V, E)$ be a cubic planar graph which is an instance of the independent set problem. We construct a new graph $G' = (V', E')$ as follows. As shown in Figure 6, define a vertex set V^e and an edge set E^e corresponding to $e = (u, v) \in E$ by

$$\begin{aligned} V^e &= \{u_1^e, u_2^e, u_3^e, u_4^e, v_1^e, v_2^e, v_3^e, v_4^e\}, \\ E^e &= \{(u_1^e, u_2^e), (u_2^e, u_3^e), (u_3^e, u_4^e), (u_4^e, u_1^e), \\ &\qquad (v_1^e, v_2^e), (v_2^e, v_3^e), (v_3^e, v_4^e), (v_4^e, v_1^e), (u_3^e, v_4^e), (v_3^e, u_4^e)\}. \end{aligned}$$

For any vertex $v \in V$ with $\delta v = \{e_1, e_2, e_3\}$, define an edge set E^v by

$$E^v = \{(v_1^{e_1}, v_2^{e_2}), (v_1^{e_2}, v_2^{e_3}), (v_1^{e_3}, v_2^{e_1})\},$$

and define

$$\begin{aligned} V' &= \bigcup_{e \in E} V^e, \\ E' &= \left(\bigcup_{e \in E} E^e \right) \cup \left(\bigcup_{v \in V} E^v \right). \end{aligned}$$

Note that E^v is depending on the ordering of e_1, e_2, and e_3, and if three edges in δv are arranged in an appropriate order for each $v \in V$, then G' is planar. It is obvious that G' is cubic and bipartite.

Set $L = 3|V| + 1$, and define the weight $w : E' \to \mathbf{R}_+$ by

$$w(e') = \begin{cases} L & \text{if } e' = (u_1^e, u_2^e), (v_1^e, v_2^e), (u_3^e, v_4^e), (v_3^e, u_4^e) \text{ for some } e = (u, v) \in E, \\ 1 & \text{if } e' \in E^v \text{ for some } v \in V, \\ 0 & \text{otherwise.} \end{cases}$$

Then the following claim holds.

Claim 5.3. *The original graph $G = (V, E)$ has an independent set of size k if and only if $G' = (V', E')$ contains a square-free 2-factor whose total weight is $4|E|L + 3k$.*

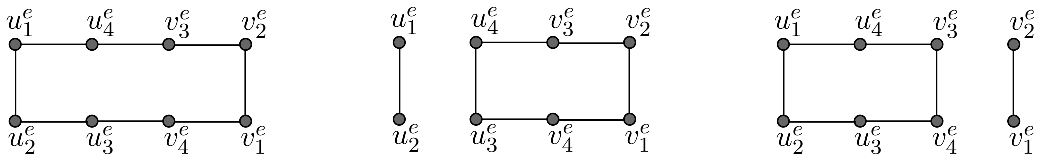


Figure 7: Three patterns of $M \cap E^e$.

Proof for the Claim 5.3. Let $M \subseteq E'$ be a square-free 2-factor in G' whose total weight is at least $4|E|L$. We show that such a square-free 2-factor in G' and an independent set of G correspond to each other. First, by the definition of L, one can see that M contains all edges of weight L. Then, since M is a square-free 2-factor, we have the following three possibilities for each $e = (u, v) \in E$ (see Figure 7);

$$M \cap E^e = \begin{cases} E^e \setminus \{(u_3^e, u_4^e), (v_3^e, v_4^e)\}, \\ E^e \setminus \{(u_1^e, u_4^e), (u_2^e, u_3^e), (v_3^e, v_4^e)\}, \\ E^e \setminus \{(v_1^e, v_4^e), (v_2^e, v_3^e), (u_3^e, u_4^e)\}. \end{cases} \tag{2}$$

Note that a 2-factor is a collection of cycles covering all vertices.

For a vertex $v \in V$ with $\delta v = \{e_1, e_2, e_3\}$, let C^v be a cycle of length six in G' through $v_1^{e_1}, v_2^{e_1}, v_1^{e_2}, v_2^{e_2}, v_1^{e_3}$, and $v_2^{e_3}$. Then, each cycle in M is contained in E^e for some $e \in E$ or coincides with C^v for some $v \in V$.

Let $V_M \subseteq V$ be a vertex set defined by $V_M = \{v \mid v \in V,\ E(C^v) \subseteq M\}$. By (2), V_M is an independent set of G. On the other hand, when we are given an independent set I of G, we can construct a square-free 2-factor M in G' such that M contains C^v for $v \in I$ and $w(M) \geq 4|E|L$ by (2). As above, an independent set I of G and a square-free 2-factor M in G' with $w(M) \geq 4|E|L$ correspond to each other.

Since M contains $3|V_M|$ edges of weight 1, $w(M) = 4|E|L + 3|V_M|$, which shows the claim. □

By this claim, the independent set problem in G is equivalent to the weighted square-free 2-factor problem in (G', w). □

Now we can easily give a proof for Theorem 5.1.

Proof for Theorem 5.1. Let $G = (V, E)$ and w be an instance of the weighted square-free 2-factor problem. Define a new weight function $w' : E \to \mathbf{R}_+$ by $w'(e) = L + w(e)$, where $L = n(\max_{e \in E} w(e)) + 1$. We consider an instance (G, w') of the weighted square-free 2-matching problem. Then, by the definition of w', the optimal solution M of the weighted square free 2-matching problem must be a 2-factor if $w'(M) \geq nL$, and in this case M is also an optimal solution of the original problem. If $w'(M) < nL$, we can conclude that G has no 2-factors.

Therefore, we can reduce the weighted square-free 2-factor problem to the weighted square-free 2-matching problem, which means that Theorem 5.1 can be derived from Theorem 5.2. □

Since the graph G' in the proof of Theorem 5.2 contains no complete bipartite graph with five vertices (i.e. $K_{1,4}$ and $K_{2,3}$) as a subgraph, its complement graph is $(|V'| - 4)$-connected. Hence, we also obtain the following theorem.

Theorem 5.4. *The weighted $(n-3)$-connectivity augmentation problem is NP-hard.*

6 Weighted square-free 2-matchings

We have already seen in Section 5 that the weighted square-free 2-matching problem in subcubic graphs is NP-hard for general weight functions. In this section, we show that the weighted square-free 2-matching problem is polynomially solvable if the weight function is vertex-induced on every square.

Suppose that for a weighted (not necessarily simple) graph (G, w) and for a vector $x \in \{0, 1, 2\}^V$, we can find in γ_2 time an edge set $F \subseteq E$ maximizing $w(F)$ such that $d_F = x$. Note that γ_2 is bounded by $\mathrm{O}(n(m+n\log n))$ [14] and $\mathrm{O}(m\log(nw(E))\sqrt{n\alpha(m,n)\log n})$ [15], where α is the inverse of the Ackermann function.

Theorem 6.1. *In a weighted subcubic graph (G, w), if w is vertex-induced on every square in G, then the weighted square-free 2-matching problem is solvable in $\mathrm{O}(n^3\gamma_2)$ time.*

In what follows, in this section, we give a proof for Theorem 6.1. In our proof, we show the relation between the weighted square-free 2-matching problem and M-concave functions, which are a quantitative extension of jump systems.

6.1 M-concave functions

An *M-concave (M-convex) function on a constant-parity jump system* is a quantitative extension of a jump system, which is a generalization of valuated matroids [5, 7], valuated delta-matroids [6], and M-concave (M-convex) functions on base polyhedra [33, 34].

Definition 6.2 (M-concave function on a constant-parity jump system [35])**.** For $J \subseteq \mathbf{Z}^V$, we call $f : J \to \mathbf{R}$ an *M-concave function on a constant-parity jump system* if it satisfies the following exchange axiom:

(M-EXC) For any $x, y \in J$ and for any (x, y)-increment s, there exists an $(x + s, y)$-increment t such that $x + s + t \in J$, $y - s - t \in J$, and $f(x) + f(y) \leq f(x + s + t) + f(y - s - t)$.

It directly follows from (M-EXC) that J satisfies (EXC), and hence J is a constant-parity jump system. We call a function $f : J \to \mathbf{R}$ an *M-convex function* if $-f$ is an M-concave function on a constant-parity jump system. M-concave functions on constant-parity jump systems appear in many combinatorial optimization problems such as the weighted matching problem, the minsquare factor problem [1], and the weighted even factor problem in odd-cycle-symmetric digraphs [29]. Some properties of M-concave functions are investigated in [28], and efficient algorithms for maximizing an M-concave function on a constant-parity jump system are given in [36, 38].

Theorem 6.3 (Murota and Tanaka [36])**.** *Let $J \subseteq \mathbf{Z}^V$ be a finite constant-parity jump system, and $f : J \to \mathbf{Z}$ be an M-concave function on J. Suppose that a vector $x_0 \in J$ is given, and we can check whether $x \in J$ or not and evaluate $f(x)$ in γ time. Then we can find a vector $x \in J$ maximizing $f(x)$ in $\mathrm{O}(n^3\Phi(J))\gamma)$ time.*

Note that $\mathrm{O}(n^4(\log\Phi(J))^2\gamma)$ time algorithm is proposed in [38] also for this problem.

6.2 Relation with M-concave functions

In this subsection, we consider a generalization of Theorem 3.3. For a weighted subcubic graph (G, w), define a function f_{sq} on $J_{\mathrm{sq}}(G)$ by

$$f_{\mathrm{sq}}(x) = \max\left\{ \sum_{e\in M} w(e) \;\middle|\; M \text{ is a square-free 2-matching, } d_M = x \right\}.$$

Then, the following theorem holds.

Theorem 6.4. *For a weighted subcubic graph (G, w), if w is vertex-induced on every square in G, $f_{\rm sq}$ is an M-concave function on the constant-parity jump system $J_{\rm sq}(G)$.*

In what follows, in this subsection, we give a proof for this theorem. In a similar way as Theorem 3.3, we use the procedure in Section 4.2 to find an $(x + s, y)$-increment t satisfying (M-EXC) for given x, y, and s. We now consider the weight of the output. Define $E_1 \subseteq E$ as the set of all shrunk edges, that is, $E_1 = E(C_1) \cup \cdots \cup E(C_q)$, and let $E_0 = E \setminus E_1$. Define $w(F) = \sum_{e \in F} w(e)$ for $F \subseteq E$. Then the following lemma holds.

Lemma 6.5. *Let M and N be square-free 2-matchings in G, whose shrunk edge sets in G° are M° and N°, respectively. Let M^*, N^* be square-free 2-matchings in G° obtained from M and N by the procedure in Section 4.2. Then, $w(M^* \cap E_0) + w(N^* \cap E_0) = w(M^\circ \cap E_0) + w(N^\circ \cap E_0)$.*

Proof. If $(M^*, N^*) = (M^{(l)}, N^{(l)}), (M', N'), (M'', N'')$, then $M^* + N^* = M^\circ + N^\circ$, where "+" means the union when we consider the multiplicity of the edges. Hence, $w(M^* \cap E_0) + w(N^* \cap E_0) = w(M^\circ \cap E_0) + w(N^\circ \cap E_0)$. If $(M^*, N^*) = (M''', N''')$ then $M^* + N^* = M^\circ + N^\circ - \{(u_2, v_{p+1}), (u_3, u_4)\} + \{(u_2, u_3), (v_{p+1}, u_4)\}$, where "−" means the difference of sets when we consider the multiplicity of the edges. Since w is vertex-induced on (v_{p+1}, u_2, u_3, u_4), we have $w(M^* \cap E_0) + w(N^* \cap E_0) = w(M^\circ \cap E_0) + w(N^\circ \cap E_0)$. □

Lemma 6.6. *Let M^*, N^* and t° be the outputs of the procedure in Section 4.2. Suppose that M^{**} and N^{**} are square-free 2-matchings which are expanded edge sets of M^* and N^*, respectively, and t is a $(d_M + s, d_N - s)$-increment corresponding to t° such that $d_{M^{**}} = d_M + s + t$ and $d_{N^{**}} = d_N - s - t$. Then, $w(M^{**}) + w(N^{**}) = w(M) + w(N)$.*

Proof. By Lemma 6.5, it suffices to show that

$$w(M^{**} \cap E(C_i)) + w(N^{**} \cap E(C_i)) = w(M \cap E(C_i)) + w(N \cap E(C_i)) \tag{3}$$

for any shrunk cycle C_i. Since $d_{M^{**} \cap E_0} + d_{N^{**} \cap E_0} = d_{M \cap E_0} + d_{N \cap E_0}$ and $d_{M^{**}} + d_{N^{**}} = d_M + d_N$, it holds that $d_{M^{**} \cap E(C_i)} + d_{N^{**} \cap E(C_i)} = d_{M \cap E(C_i)} + d_{N \cap E(C_i)}$. Then the equation (3) holds because w is vertex-induced on C_i. □

We are now ready to show Theorem 6.4.

Proof for Theorem 6.4. For $x, y \in J_{\rm sq}(G)$ and an (x, y)-increment s, let M and N be square-free 2-matchings such that $d_M = x$, $d_N = y$, $w(M) = f_{\rm sq}(x)$, and $w(N) = f_{\rm sq}(y)$. As with Assumption 4.6, we assume that M and N maximize $|M \cap N|$ among such 2-matchings.

Let M^{**}, N^{**}, and t be as in Lemma 6.6. If t is not an $(x + s, y)$-increment, then $d_{M^{**}} = d_M$ and $d_{N^{**}} = d_N$. Since $w(M^{**}) + w(N^{**}) = w(M) + w(N)$ by Lemma 6.6, $w(M^{**}) = w(M)$ and $w(N^{**}) = w(N)$. However, $|M^{**} \cap N| > |M \cap N|$ contradicts the maximality of $|M \cap N|$. Thus, t is an $(x + s, y)$-increment.

On the other hand, by Lemma 6.6, we have

$$\begin{aligned} f_{\rm sq}(x) + f_{\rm sq}(y) &= w(M) + w(N) \\ &= w(M^{**}) + w(N^{**}) \\ &\leq f_{\rm sq}(x + s + t) + f_{\rm sq}(y - s - t). \end{aligned}$$

Hence $f_{\rm sq}$ is an M-concave function on $J_{\rm sq}$. □

6.3 Polynomial time algorithm

In this section, we give a proof for Theorem 6.1 with the aid of previous works on M-concave functions. As a generalization of Lemma 3.4, we show the following lemma.

Lemma 6.7. *Given a weighted subcubic graph* (G, w) *and a vector* $x \in J_{\mathrm{sq}}(G)$*, we can calculate* $f_{\mathrm{sq}}(x)$ *in* $\mathrm{O}(\gamma_2)$ *time if* w *is vertex-induced on every square.*

Proof. Take edge-disjoint cycles $C_1, C_2, \dots, C_q$ of length four maximally such that $x(v) = 2$ for each $v \in \bigcup V(C_i)$. Let $G^\circ = (V^\circ, E^\circ)$ denote the graph obtained from $G = (V, E)$ by shrinking $C_1, C_2, \dots, C_q$. Let u_1^i and u_2^i denote the vertices arising when shrinking $C_i = (v_1^i, v_2^i, v_3^i, v_4^i)$. Let p be a function on $\bigcup V(C_i)$ that induces w on $E(C_i)$ simultaneously. Since the cycles $C_1, \dots, C_q$ are disjoint we can find such p. Let E_0, E_1, V_0, V_1 and x° be the same as in the proof of Lemma 3.4. We define $w^\circ : E^\circ \to \mathbf{R}$ as follows (see Figure 8):

$$w^\circ(e) = \begin{cases} w(e) & \text{if } e = (u, v) \text{ where } u, v \in V_0, \\ w(e) - p(v) & \text{if } e = (u, v) \text{ where } u \in V_0 \text{ and } v \in V^\circ \setminus V_0, \\ w(e) - p(u) - p(v) & \text{if } e = (u, v) \text{ where } u, v \in V^\circ \setminus V_0, \\ p(v_1^i) + p(v_2^i) + p(v_3^i) + p(v_4^i) & \text{if } e = (u_1^i, u_2^i). \end{cases}$$

We will show that $f_{\mathrm{sq}}(x) = f(x^\circ) + p(V_1)$ where

$$f(x^\circ) = \max \left\{ \sum_{e \in M^\circ} w^\circ(e) \;\middle|\; M^\circ \text{ is a 2-matching in } G^\circ,\ d_{M^\circ} = x^\circ \right\}.$$

Clearly, that would prove the lemma since $f(x^\circ)$ can be calculated in $\mathrm{O}(\gamma_2)$ time.

For a square-free 2-matching M with $d_M = x$ we can get a 2-matching M° in G° with $d_{M^\circ} = x^\circ$, and conversely, for any 2-matching M° of G° with $d_{M^\circ} = x^\circ$ we can define a square-free 2-matching M of G with $d_M = x$ as described in Lemma 3.4. One only has to observe that for a corresponding pair M, M°, we have $w(M) = w^\circ(M^\circ) + p(V_1)$. This means that for any M with $d_M = x$ and $w(M) = f_{\mathrm{sq}}(x)$ we can find an M° with $w^\circ(M^\circ) = f_{\mathrm{sq}}(x) - p(V_1)$, and conversely, for any M° with $d_{M^\circ} = x^\circ$ and $w^\circ(M^\circ) = f(x^\circ)$ we can find an M with $w(M) = f(x^\circ) + p(V_1)$, hence we are done. $\square$

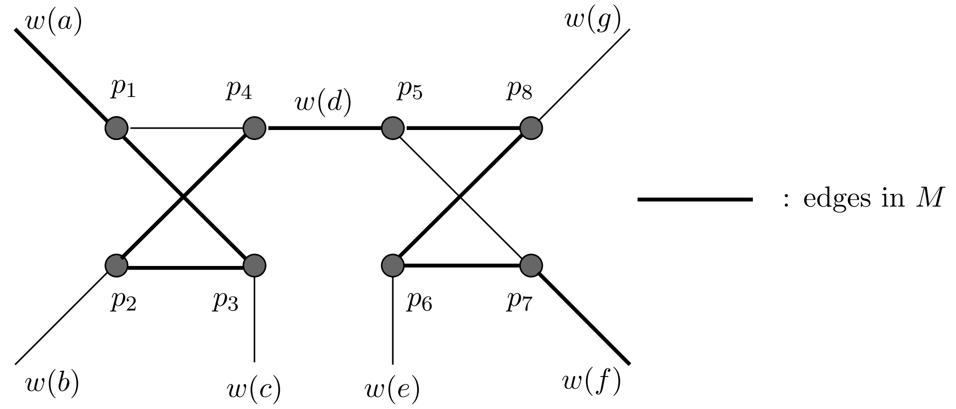
Theorem 6.1 follows from Lemma 6.7 and Theorems 6.3 and 6.4.

7 A min-max formula

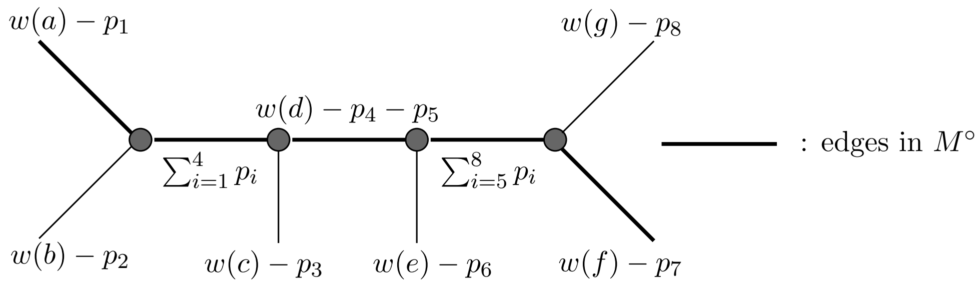
In this section we give a min-max formula that characterizes the maximum size of a square-free 2-matching in a subcubic graph. The proof is based on the connection between square-free 2-matchings in G and 2-matchings in G° that was described in Section 3.

We begin with a few definitions. For a vertex set S of $G = (V, E)$ and a subgraph T of $G - S$, let $E(T, S)$ be the set of edges of E connecting S and T. Similarly, for a vertex $v \in V$, $E(v, S)$ is the set of edges of E connecting v and S. For $S \subseteq V$ and $u \in S$, $S \setminus \{u\}$ is simply denoted by $S - u$.

The following characterization of the maximum size of a 2-matching (not necessarily square-free) can be derived from a construction of Tutte [41].



$$w(M) = w(a) + w(d) + w(f) + p_1 + p_4 + p_5 + p_7 + 2p_2 + 2p_3 + 2p_6 + 2p_8$$



$$w^\circ(M^\circ) = w(a) + w(d) + w(f) + p_2 + p_3 + p_6 + p_8$$

Figure 8: Example of $w^\circ(M^\circ)$

Theorem 7.1. *Let $G = (V, E)$ be a graph. The maximum size of a 2-matching in G is equal to the minimum value of*

$$\tau_G(U, S) = |V| + |U| - |S| + \sum_T \lfloor \tfrac{1}{2} |E(T, S)| \rfloor, \tag{4}$$

where U and S are disjoint subsets of V, S is stable, and T ranges over the components of $G - U - S$.

We drop the subscript G if it is clear from the context. Our first observation is that U can be eliminated from the formula in the subcubic case.

Theorem 7.2. *Let $G = (V, E)$ be a subcubic graph. The maximum size of a 2-matching in G is equal to the minimum value of*

$$\tau'_G(S) = |V| - |S| + \sum_T \lfloor \tfrac{1}{2} |E(T, S)| \rfloor, \tag{5}$$

where S is a stable subset of V, and T ranges over the components of $G - S$.

Proof. Let U and S be disjoint subsets of V that minimize (4). If $U = \emptyset$, then we are done, otherwise take a node $u \in U$. As G is subcubic, $d(u) \leq 3$ and so we have the following cases.

- If u has all of its neighbors in $U \cup S$, then u is a component of $G - (U - u) - S$ and $\lfloor \frac{1}{2}|E(u, S)| \rfloor \leq 1$. Hence $\tau(U - u, S) \leq \tau(U, S)$.
- If u has exactly one neighbor in $V \setminus (U \cup S)$, then let T be the component of $G - U - S$ containing the neighbor of u. Then $\lfloor \frac{1}{2}|E(T + u, S)| \rfloor \leq \lfloor \frac{1}{2}|E(T, S)| \rfloor + 1$, hence $\tau(U - u, S) \leq \tau(U, S)$.
- If u has exactly two neighbors in $V \setminus (U \cup S)$, then we have two subcases. If these neighbors are contained in the same component T of $G - U - S$ then $\lfloor \frac{1}{2}|E(T + u, S)| \rfloor \leq \lfloor \frac{1}{2}|E(T, S)| \rfloor + 1$ so $\tau(U - u, S) \leq \tau(U, S)$. If the two neighbors are contained in T_1 and T_2, then $T_1 + T_2 + u$ will form one component of $G - (U - u) - S$. It is easy to see that $\lfloor \frac{1}{2}|E(T_1 + T_2 + u, S)| \rfloor \leq \lfloor \frac{1}{2}|E(T_1, S)| \rfloor + \lfloor \frac{1}{2}|E(T_2, S)| \rfloor + 1$ which implies $\tau(U - u, S) \leq \tau(U, S)$ again.
- If u has three neighbors in $V \setminus (U \cup S)$, then, depending on the position of these neighbors in the components of $G - U - S$, we may get one from two or three components when leaving u out from U. One can easily check that the sum in (4) belonging to the components of $G - U - S$ may increase only by one in each case while the size of U always decreases by one. That means that $\tau(U - u, S) \leq \tau(U, S)$.

The observations above imply that if U and S attain the minimum in (4) and the graph is subcubic, then we can make U empty by trimming its nodes one by one so that the value $\tau(U, S)$ does not increase. At the end, we get a stable set S for which $\tau'(S) = \tau(U, S)$, and we are done. $\square$

Now we turn to the min-max formula characterizing the maximum size of a square-free 2-matching. Let G be a subcubic graph, let S be a stable subset of V, and take a set $\mathcal{C}$ of edge-disjoint cycles $C_1, \ldots, C_q$ of length four. We define the *$\mathcal{C}$-components* of $G - S$ as follows.

Definition 7.3 ($\mathcal{C}$-component)**.** We say that $u, v \in V \setminus S$ are in the same *$\mathcal{C}$-component* of $G - S$ if and only if one of the followings hold:

- u and v are in the same component of $G - S$, or
- $u \in V(T_1)$, $v \in V(T_2)$ (where T_1 and T_2 are components of $G - S$), and there is a cycle $C = (v_1, v_2, v_3, v_4) \in \mathcal{C}$ such that $v_1 \in V(T_1)$, $v_3 \in V(T_2)$, $v_2, v_4 \in S$.

We say that $C = (v_1, v_2, v_3, v_4) \in \mathcal{C}$ *fits* a $\mathcal{C}$-component T if $v_1, v_3 \in V(T)$ and $v_2, v_4 \in S$.

In other words, a $\mathcal{C}$-component is the union of some components of $G - S$ that are connected with cycles from $\mathcal{C}$ in a special configuration. Using this definition, we can formalize our result.

Theorem 7.4. *Let $G = (V, E)$ be a subcubic graph and let $\mathcal{C}$ be a maximal set of edge-disjoint cycles of length four. The maximum size of a square-free 2-matching in G is equal to the minimum value of*

$$\tau_G(S) = |V| - |S| + \sum_T \lfloor \tfrac{1}{2}(|E(T, S)| - |\mathcal{C}_T|) \rfloor - |\mathcal{K}|, \tag{6}$$

where S is a stable subset of V, T ranges over the $\mathcal{C}$-components of $G - S$, $\mathcal{C}_T \subseteq \mathcal{C}$ denotes the set of cycles fitting T, and $\mathcal{K}$ is the set of K_4's in G.

Seemingly, the minimum value of (6) also depends on the choice of $\mathcal{C}$. The theorem implies that we can anyhow take edge-disjoint cycles maximally, the minimum value of $\tau_G(S)$ will always be the same, namely, the maximum size of a square-free 2-matching.

Proof. As a K_4 forms a component of G, first we handle such a component separately. Let $T \in \mathcal{K}$ be a K_4-subgraph of G. For a stable set $S \subseteq V$, $|S \cap V(T)| = 0$ or 1 by the definition, and in both cases, $|S \cap V(T)| = \lfloor \frac{1}{2}(|E(T,S)| - |\mathcal{C}_T|) \rfloor$. Thus, a square-free 2-matching M of maximum size satisfies that

$$|M \cap E(T)| = 3 = |V(T)| - |S \cap V(T)| + \lfloor \tfrac{1}{2}(|E(T,S)| - |\mathcal{C}_T|) \rfloor - 1,$$

and hence it suffices to consider the case when G has no K_4 as a subgraph.

First we show that the maximum is not more than the minimum. Let M be a square-free 2-matching and take a stable subset S of V. We claim that for each $\mathcal{C}$-component T of $G - S$, the number of edges in M spanned by $V(T) \cup S$ is at most $|V(T)| + \lfloor \frac{1}{2}(|E(T,S)| - |\mathcal{C}_T|) \rfloor$. Indeed,

$$\begin{aligned} 2|M \cap E(T+S)| &= 2|M \cap E(T)| + 2|M \cap E(T,S)| \\ &\leq 2|M \cap E(T)| + |M \cap E(T,S)| + |E(T,S)| - |\mathcal{C}_T| \\ &\leq 2|V(T)| + |E(T,S)| - |\mathcal{C}_T|. \end{aligned}$$

Here, $T + S$ denotes the graph induced by $V(T) \cup S$. Hence we have

$$\begin{aligned} |M| &\leq \sum_T (|V(T)| + \lfloor \tfrac{1}{2}(|E(T,S)| - |\mathcal{C}_T|) \rfloor) \\ &= |V| - |S| + \sum_T \lfloor \tfrac{1}{2}(|E(T,S)| - |\mathcal{C}_T|) \rfloor. \end{aligned}$$

Now we turn to the reverse inequality. According to the above mentioned, we may assume that G does not contain a K_4. Let $\mathcal{C} = \{C_1, \ldots, C_q\}$ and let $G^\circ = (V^\circ, E^\circ)$ denote the graph obtained from $G = (V, E)$ by shrinking $C_1, C_2, \ldots, C_q$. By Theorem 7.1, the maximum size of a 2-matching in G° is equal to the minimum value of

$$\tau'_{G^\circ}(S^\circ) = |V^\circ| - |S^\circ| + \sum_{T^\circ} \lfloor \tfrac{1}{2} |E^\circ(T^\circ, S^\circ)| \rfloor. \tag{7}$$

From now let $S^\circ \subseteq V^\circ$ be a stable set attaining the minimum in (7). In Section 3, we have already shown that the maximum size of a square-free 2-matching in G is equal to $\tau'_{G^\circ}(S^\circ) + 2q$. So we only have to find a stable subset S of V such that $\tau_G(S) = \tau'_{G^\circ}(S^\circ) + 2q$.

Let S denote the set of nodes in V that corresponds to S°. Since no self-loops are incident to vertices in S° by the definition of a stable set, S is obviously stable. We claim that $\tau_G(S) = \tau'_{G^\circ}(S^\circ) + 2q$. To see this, we will blow back the cycles one by one and show that (7) increases by two at each step. Assume that some of the cycles are already blown back, and G' and S' are the actual graph and stable set, while G'' and S'' are those arising after blowing back the next square-edge. We also use the notation $\mathcal{C}'$ and $\mathcal{C}''$ for the set of cycles already blown back.

If the edge has both of its end-vertices in $V' \setminus S'$ then $|V''| = |V'| + 2$. Also $|S''| = |S'|$ and the set of edges going between S' and $V' \setminus S'$ does not change. Hence $\tau_{G''}(S'') = \tau_{G'}(S') + 2$. Now assume that the square-edge has one of its end-vertices in S' and the other in T' where T' is a $\mathcal{C}'$-component of $G' - S'$. Then we have $|V''| = |V'| + 2$, $|S''| = |S'| + 1$, and $|E(T'', S'')| - |\mathcal{C}''_{T''}| = |E(T', S')| - |\mathcal{C}'_{T'}| + 2$. Hence $\tau_{G''}(S'') = \tau_{G'}(S') + 2$ again, and we are done. □

Remark 7.5. It is easy to see that both an algorithm and a min-max theorem can be presented in the slightly more general case when a list of forbidden squares is given in the graph. That is, if we denote by $\mathcal{L}$ the list, we are looking for a maximum $\mathcal{L}$-free 2-matching M where $\mathcal{L}$-free means that M contains at most three edges from each square in $\mathcal{L}$. The only difference is that we have to take edge-disjoint cycles of length four maximally from $\mathcal{L}$ and only shrink these cycles.

By using the min-max result, we can prove a special case of a conjecture of Jordán appeared in [8]. To describe the conjecture, first we give some definitions.

We call an ordered pair $L = (Z, \mathcal{P})$ a *clump* of G if Z is a cut of size $k-1$ and $\mathcal{P}$ is a partition of $V \setminus Z$ such that no edge of G joins two distinct member of $\mathcal{P}$. A clump L *covers* a pair of nodes u, v if u and v belong to distinct members of $\mathcal{P}$. A *bush* B is a set of clumps such that each pair of nodes is covered by at most two of them. A bush B covers a pair of nodes if it contains a clump covering them. Two bushes B_1 and B_2 are *disjoint* if no pair of nodes is covered by both of them. Let

$$\sigma(B) = \lceil \tfrac{1}{2} \sum_{(Z,\mathcal{P}) \in B} (|\mathcal{P}| - 1) \rceil.$$

It is easy to see that in order to make G k-connected, one must add a set of at least $\sum_{B \in \mathcal{D}} \sigma(B)$ edges to G for any collection $\mathcal{D}$ of disjoint bushes.

Conjecture 7.6 (Bushy-conjecture)**.** *Let G be a $(k-1)$-connected graph. Then the minimum number of edges that must be added to G to make it k-connected is equal to the maximum value of $\sum_{B \in \mathcal{D}} \sigma(B)$, where the maximum is taken over all sets of pairwise disjoint bushes $\mathcal{D}$ of G.*

The conjecture can be easily verified for $k = n-1$ and $n-2$. Now we show how it follows from our min-max result when $k = n-3$.

Theorem 7.7. *Let G be a $(n-4)$-connected graph. Then the minimum number of edges that must be added to G to make it $(n-3)$-connected is equal to the maximum value of $\sum_{B \in \mathcal{D}} \sigma(B)$, where the maximum is taken over all sets of pairwise disjoint bushes $\mathcal{D}$ of G.*

Proof. Obviously, the maximum is at most the minimum. We prove the reverse inequality. Let $\overline{G} = (V, \overline{E})$ be the complement of the graph, which is a subcubic graph. We have already seen that a graph is $(n-3)$-connected if and only if its complement is a square-free 2-matching. Take edge-disjoint cycles $C_1, \dots, C_q$ of length four maximally in $\overline{G}$. However, we know, by the min-max result, that the minimum number of edges that must be added to G to make it $(n-3)$-connected is equal to the maximum value of

$$|\overline{E}| - (|V| - |S| + \sum_T \lfloor \tfrac{1}{2}(|\overline{E}(T,S)| - |\mathcal{C}_T|\rfloor - |\mathcal{K}|), \tag{8}$$

where S is a stable subset of V in $\overline{G}$, T ranges over the $\mathcal{C}$-components of $\overline{G} - S$, and $\mathcal{K}$ is the set of K_4's of $\overline{G}$. Assume that S attains the minimum in (8). Let $T_1, \dots, T_t$ be the $\mathcal{C}$-components of $\overline{G} - S$ intersecting no K_4. We will define a set of disjoint bushes $\mathcal{D}$ of G such that

$$\sum_{B \in \mathcal{D}} \sigma(B) \geq |\overline{E}| - (|V| - |S| + \sum_T \lfloor \tfrac{1}{2}(|\overline{E}(T,S)| - |\mathcal{C}_T|\rfloor - |\mathcal{K}|), \tag{9}$$

which would clearly prove the theorem.

For $i = 1, \dots, t$, let B_i be the set of the following clumps:

- for $v \in T_i$ with $d_{\overline{G}}(v) = 3$, let L be the star of v, namely $L = (Z, \mathcal{P})$ where $Z = V \setminus (N_{\overline{G}}(v) \cup \{v\})$ and $\mathcal{P} = \{\{v\}, N_{\overline{G}}(v)\}$;
- for a cycle $C = (v_1, v_2, v_3, v_4) \in \mathcal{C}$ fitting T_i, let $L = (Z, \mathcal{P})$ be a clump such that $Z = V \setminus V(C)$ and $\mathcal{P} = \{\{v_1, v_3\}, \{v_2, v_4\}\}$.

Here $N_G(v)$ is the set of vertices adjacent to v in G.

Clearly, these pairs are clumps in G. Moreover, each pair of nodes is covered by at most two of them. Hence the B_i's form a set $\mathcal{D}$ of pairwise disjoint bushes of G. We have

$$\begin{aligned}
\sigma(B_i) &= \lceil \tfrac{1}{2} \sum_{(Z,\mathcal{P})\in B_i} (|\mathcal{P}|-1) \rceil \\
&= \lceil \tfrac{1}{2}(|\{v \in V(T_i) \mid d_{\overline{G}}(v) = 3\}| + |\mathcal{C}_{T_i}|)\rceil \\
&\geq \lceil \tfrac{1}{2}(\sum_{v\in T_i}(d_{\overline{G}}(v)-2) + |\mathcal{C}_{T_i}|)\rceil \\
&= \lceil \tfrac{1}{2}(2|\overline{E}(T_i)| + |\overline{E}(T_i,S)| - 2|V(T_i)| + |\mathcal{C}_{T_i}|)\rceil \\
&= |\overline{E}(T_i)| - |V(T_i)| + \lceil \tfrac{1}{2}(|\overline{E}(T_i,S)| + |\mathcal{C}_{T_i}|)\rceil \\
&= |\overline{E}(T_i \cup S)| - |V(T_i)| - \lfloor \tfrac{1}{2}(|\overline{E}(T_i,S)| - |\mathcal{C}_{T_i}|)\rfloor
\end{aligned}$$

Note that for a subgraph T of $\overline{G} = (V, \overline{E})$, $\overline{E}(T)$ is the set of edges of T.

For $T \in \mathcal{K}$, the bush B_T will contain a single clump twice. Namely, if $V(T) = \{v_1, v_2, v_3, v_4\}$, then $L = (Z, \mathcal{P})$ is defined by $Z = V \setminus V(T)$ and $\mathcal{P} = \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}\}$. Clearly, $\sigma(B_T) = 3$. By summing these values over the bushes defined above we get

$$\begin{aligned}
\sum_{B\in\mathcal{D}} \sigma(B) &\geq \sum_{i=1}^{t}(|\overline{E}(T_i + S)| - |V(T_i)| - \lfloor \tfrac{1}{2}(|\overline{E}(T_i,S)| - |\mathcal{C}_{T_i}|)\rfloor) + 3|\mathcal{K}| \\
&= \sum_{T}(|\overline{E}(T + S)| - |V(T)| - \lfloor \tfrac{1}{2}(|\overline{E}(T,S)| - |\mathcal{C}_{T}|)\rfloor) + |\mathcal{K}| \\
&= |\overline{E}| - (|V| - |S| + \sum_{T}\lfloor \tfrac{1}{2}(|\overline{E}(T,S)| - |\mathcal{C}_{T}|\rfloor - |\mathcal{K}|),
\end{aligned}$$

where T ranges over the $\mathcal{C}$-components of $G - S$ and the second equality follows from $|\overline{E}(T+S)| = 6$, $|V(T)| = 4$ if $T \in \mathcal{K}$ and $|\overline{E}(T+S)| = 6$, $|V(T)| = 3$, $|\overline{E}(T,S)| = 3$ if $T + v \in \mathcal{K}$ for some $v \in S$. $\square$

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