ON FINITE QUOTIENTS OF TAME FUNDAMENTAL GROUPS OF CURVES IN POSITIVE CHARACTERISTIC

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Abstract. Let (X_j, D_{X_j}), $j \in \{1, 2\}$, be a smooth pointed stable curve of type (g_{X_j}, n_{X_j}) over an algebraically closed field k_j of characteristic $p > 0$, $U_{X_j} \stackrel{\text{def}}{=} X_j \setminus D_{X_j}$, and $\pi_1^{\mathrm{t}}(U_{X_j})$ the tame fundamental group of (X_j, D_{X_j}). Suppose that $g_{X_1} = 0$, that k_1 is an algebraic closure of the finite field $\mathbb{F}_p$, and that (the minimal models of) U_{X_1} and U_{X_2} are not isomorphic as schemes. In the present paper, we give an *explicit construction of differences* between $\pi_1^{\mathrm{t}}(U_{X_1})$ and $\pi_1^{\mathrm{t}}(U_{X_2})$ via their *finite quotients*. In particular, our construction deduces a strong generalization of Tamagawa's results concerning Grothendieck's anabelian conjecture for curves over algebraically closed fields of characteristic p. This generalization shows that the anabelian phenomena for curves in positive characteristic can be understood by using not only entire tame fundamental groups but also certain *finite quotients* of them.

Keywords: smooth pointed stable curve, tame covering, tame fundamental group, anabelian geometry, positive characteristic.

Mathematics Subject Classification: Primary 14H30; Secondary 14H10, 14G32.

Contents

1. Introduction

Let k be an algebraically closed field of characteristic $p > 0$, and let (X, D_X) be a smooth pointed stable curve of type (g_X, n_X) over k, where X denotes the (smooth) underlying curve of genus g_X and D_X denotes the (finite) set of marked points with cardinality $n_X \stackrel{\text{def}}{=} \#(D_X)$ satisfying $2g_X + n_X - 2 > 0$. By choosing a base point of $x \in U_X \stackrel{\text{def}}{=} X \setminus D_X$, we have the étale fundamental group $\pi_1(U_X, x)$ of U_X and the tame fundamental group $\pi_1^{\mathrm{t}}(U_X, x)$ of (X, D_X).

1.1. **Finite quotients of fundamental groups and main problem.** We maintain the notation introduced above. For simplicity, write $\pi_1(U_X)$ and $\pi_1^{\mathrm{t}}(U_X)$ for $\pi_1(U_X, x)$ and $\pi_1^{\mathrm{t}}(U_X, x)$, respectively, and denote by

$$\pi_A^{\text{ét}}(U_X),\ \pi_A^{\mathrm{t}}(U_X)$$

the sets of finite quotients of $\pi_1(U_X)$ and $\pi_1^{\mathrm{t}}(U_X)$, respectively. Since there is a natural surjection $\pi_1(U_X) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_X)$, we have $\pi_A^{\mathrm{t}}(U_X) \subseteq \pi_A^{\text{ét}}(U_X)$.

1.1.1. Suppose that U_X is *affine* (i.e., $n_X > 0$). In 1957, S. Abhyankar ([A]) made a famous conjecture which gives a precise description for the set $\pi_A^{\text{ét}}(U_X)$. In particular, it says that $\pi_A^{\text{ét}}(U_X)$ *can be completely determined* by the type (g_X, n_X), and that $\pi_A^{\text{ét}}(U_X)$ *cannot* determine the isomorphism class of U_X as a scheme. The solvable case of Abhyankar's conjecture was solved by J-P. Serre ([Ser2]) and the full conjecture was proved by M. Raynaud ([R1]) where $U_X = \mathbb{A}_k^1$ is an affine line, and by D. Harbater ([H]) where U_X is an arbitrary affine curve over k. The next step is naturally to ask how many information about the structure of $\pi_1(U_X)$ can be carried by $\pi_A^{\text{ét}}(U_X)$. Note that since $\pi_1(U_X)$ is *not* topologically finitely generated when U_X is affine, the isomorphism class of $\pi_1(U_X)$ (as a profinite group) cannot be determined by the set $\pi_A^{\text{ét}}(U_X)$.

Furthermore, A. Tamagawa ([T1]) discovered surprisingly that there exist *anabelian phenomena* for *étale* fundamental groups of curves over algebraically closed fields of characteristic p. These kind of anabelian phenomena say that the isomorphism classes of curves as schemes can be completely determined by the isomorphism classes of their *étale* fundamental groups as profinite groups. Tamagawa's result tell us that *there are essential differences* between $\pi_1(U_X)$ and $\pi_A^{\text{ét}}(U_X)$, and that almost no information about $\pi_1(U_X)$ can be carried by $\pi_A^{\text{ét}}(U_X)$.

1.1.2. Next, we return to the the case where (X, D_X) is an arbitrary smooth pointed stable curve over k (i.e., $n_X \geq 0$). Since the tame fundamental group $\pi_1^{\mathrm{t}}(U_X)$ is topologically finitely generated, the isomorphism class of $\pi_1^{\mathrm{t}}(U_X)$ as a profinite group can be completely determined by the set of finite quotients $\pi_A^{\mathrm{t}}(U_X)$ ([FJ, Proposition 16.10.7]). So the information carried by $\pi_1^{\mathrm{t}}(U_X)$ is equivalent to the information carried by $\pi_A^{\mathrm{t}}(U_X)$.

However, unlike the case of étale fundamental groups, the situation is becoming very elusive. At the present, very little is known about $\pi_A^{\mathrm{t}}(U_X)$. For instance, we still do not know whether a finite group G is contained in $\pi_A^{\mathrm{t}}(U_X)$ or not even in the simplest case where G is an extension of an abelian group by an abelian p-group (note that the problem is trivial if G is abelian). On the other hand, if U_X is *generic* (in the sense of moduli spaces), there exist criteria to determine whether a finite group G is contained in $\pi_A^{\mathrm{t}}(U_X)$ or not, where G is an extension of an abelian group by a p-group ([B], [N], [OP], [PaSt], [Y4], [Z]). These criteria are deduced from the following geometric observation: Some evidence suggests that the *p-rank* of all abelian tame coverings (i.e., Galois tame coverings whose Galois groups are abelian) of a generic curve *can attain maximum* (e.g. all étale coverings of generic curves are *ordinary* if $n_X = 0$ ([N], [Z])). However, the method of above criteria cannot be extended to the case of arbitrary finite groups since a result of Raynaud ([R2]) says that the p-rank of the Galois tame coverings of a generic curve *cannot* attain maximum in general. On the other hand, even in the case of generic curves, we still do not know whether the p-rank of all abelian tame coverings of a generic curve can attain maximum or not if $n_X > 0$ (but see [B] for the case where $n_X \leq 4$, and see [Y4] for a criterion for ordinary abelian tame coverings of an arbitrary generic curve).

1.1.3. *Main problem.* In fact, *one cannot expect an explicit description for* $\pi_A^{\rm t}(U_X)$ since anabelian phenomena also exist for *tame* fundamental groups. Tamagawa ([T3]) generalized the main result of [T1] to the case of tame fundamental groups. In particular, it shows that the set of $\pi_A^{\rm t}(U_X)$ depends on not only the type (g_X, n_X) but also the isomorphism class of U_X. See [PoSa], [R3], [T4], [Y1], [Y3] for more results concerning these kind of anabelian phenomena.

In order to understand more precisely the relationship between the structures of tame fundamental groups and the anabelian phenomena (or equivalently, the relationship between the sets of finite quotients of tame fundamental groups and the scheme-theoretical structures of curves) in positive characteristic world, we ask a problem from a different view of point of 1.1.2:

Problem 1.1. *How does the scheme-theoretical structure of a curve affect explicitly the set of finite quotients of its tame fundamental group? Or more precisely, what exactly are the differences for the sets of finite quotients of the tame fundamental groups of non-isomorphic curves?*

1.2. Main result.

In the present paper, we solve the above problem for certain curves.

1.2.1. We fix some notation. Let (X_j, D_{X_j}), $j \in \{1, 2\}$, be a smooth pointed stable curve of type (g_{X_j}, n_{X_j}) over an algebraically closed field k_j of characteristic $p > 0$, $U_{X_j} \overset{\rm def}{=} X_j \setminus D_{X_j}$, $\pi_1^{\rm t}(U_{X_j})$ the tame fundamental group of (X_j, D_{X_j}), and $\pi_A^{\rm t}(U_{X_j})$ the set of finite quotients of $\pi_1^{\rm t}(U_{X_j})$. Let $s, b \in \mathbb{N}$ be positive natural numbers. We put $D_b^{(1)}(\pi_1^{\rm t}(U_{X_j})) \overset{\rm def}{=} [\pi_1^{\rm t}(U_{X_j}), \pi_1^{\rm t}(U_{X_j})]\big(\pi_1^{\rm t}(U_{X_j})\big)^b$ and $D_b^{(i)}(\pi_1^{\rm t}(U_{X_j})) \overset{\rm def}{=} D_b^{(1)}\big(D_b^{(i-1)}(\pi_1^{\rm t}(U_{X_j})\big)$ for $i \in \{2, \dots, s\}$, where $[\pi_1^{\rm t}(U_{X_j}), \pi_1^{\rm t}(U_{X_j})]$ denotes the commutator subgroup of $\pi_1^{\rm t}(U_{X_j})$.

We denote by $k_j^{\rm m}$ the *minimal* algebraically closed subfield of k_j over which U_{X_j} can be defined. Thus, by considering the function field of X_j, we obtain a smooth pointed stable curve $(X_j^{\rm m}, D_{X_j^{\rm m}})$ (i.e., *a minimal model of* (X_j, D_{X_j}) in the sense of [T2, Definition 1.30 and Lemma 1.31]) such that

$$U_{X_j} \overset{\sim}{\to} U_{X_j^{\rm m}} \times_{k_j^{\rm m}} k_j$$

as k_j-schemes, where $U_{X_j^{\rm m}} \overset{\rm def}{=} X_j^{\rm m} \setminus D_{X_j^{\rm m}}$. Note that the tame fundamental group $\pi_1^{\rm t}(U_{X_j^{\rm m}})$ of $(X_j^{\rm m}, D_{X_j^{\rm m}})$ is naturally isomorphic to $\pi_1^{\rm t}(U_{X_j})$.

1.2.2. The main result of the present paper is as follows (see Theorem 6.2 for a precise statement):

Theorem 1.2. *We maintain the notation introduced above. Suppose that* $k_1^{\rm m}$ *is an algebraic closure of the finite field* $\mathbb{F}_p$*, that* $g_{X_1} = 0$*, and that* $U_{X_1^{\rm m}} \not\cong U_{X_2^{\rm m}}$*. Then we can construct explicitly a finite group* G *depending on* $U_{X_1^{\rm m}}$ *and* $U_{X_2^{\rm m}}$ *such that* $G \notin \pi_A^{\rm t}(U_{X_1})$ *and* $G \in \pi_A^{\rm t}(U_{X_2})$*.*

Our construction given in Theorem 1.2 (i.e., Theorem 6.2) implies the following interesting anabelian result *without any assumptions between the full tame fundamental groups* $\pi_1^{\rm t}(U_{X_1})$ and $\pi_1^{\rm t}(U_{X_2})$ (see Theorem 6.3 for a precise statement):

Theorem 1.3. *We maintain the notation introduced above. Suppose that* k_1^{m} *is an algebraic closure of the finite field* $\mathbb{F}_p$ *and that* $g_{X_1}=0$*. Then we can construct explicitly a natural number* $c(\mathcal{I})\in\mathbb{N}$ *depending on* $U_{X_1^{\mathrm{m}}}$ *and finite groups* $G_1 \overset{\mathrm{def}}{=} \pi_1^{\mathrm{t}}(U_{X_1})/D^{(2)}_{c(\mathcal{I})}(\pi_1^{\mathrm{t}}(U_{X_1}))\in\pi_A^{\mathrm{t}}(U_{X_1})$*,* $G_2 \overset{\mathrm{def}}{=} \pi_1^{\mathrm{t}}(U_{X_2})/D^{(6)}_{c(\mathcal{I})}(\pi_1^{\mathrm{t}}(U_{X_2}))\in\pi_A^{\mathrm{t}}(U_{X_2})$ *such that*

$$U_{X_1^{\mathrm{m}}}\cong U_{X_2^{\mathrm{m}}}$$

as schemes if and only if

$$G_1, G_2\in\pi_A^{\mathrm{t}}(U_{X_1})\cap\pi_A^{\mathrm{t}}(U_{X_2}).$$

Moreover, suppose further $g_{X_1}=g_{X_2}=0$ *and* $n_{X_1}=n_{X_2}$*. Then we have that*

$$U_{X_1^{\mathrm{m}}}\cong U_{X_2^{\mathrm{m}}}$$

as schemes if and only if

$$G_2\in\pi_A^{\mathrm{t}}(U_{X_1})\cap\pi_A^{\mathrm{t}}(U_{X_2}).$$

Remark 1.3.1. Theorem 1.3 shows that the anabelian phenomena for curves over algebraically closed fields of positive characteristic can be understood by using not only entire étale or tame fundamental groups but also certain *finite quotients* of them.

In [T1] and [T3], Tamagawa proved the following results concerning Grothendieck's anabelian conjecture:

> We maintain the notation introduced above. Suppose that $g_{X_1}=0$, and that $k_1=k_2$ is an algebraic closure of the finite field $\mathbb{F}_p$. Then the following statements hold:
>
> (i) $U_{X_1}\cong U_{X_2}$ as schemes if and only if $\pi_1(U_{X_1})\cong\pi_1(U_{X_2})$ (see [T1, Theorem 0.2]), where $\pi_1(U_{X_j})$, $j\in\{1,2\}$, is the étale fundamental group of U_{X_j}.
>
> (ii) $U_{X_1}\cong U_{X_2}$ as schemes if and only if $\pi_1^{\mathrm{t}}(U_{X_1})\cong\pi_1^{\mathrm{t}}(U_{X_2})$ (see [T3, Theorem 0.2]).

(i) and (ii) are the main results of [T1] and [T3], respectively, moreover, we have that (i) can be deduced from (ii) ([T1, Corollary 1.5]), and that (ii) is much harder than (i). At the present, these results are also the only results that we know about *Grothendieck's anabelian conjecture for smooth curves over algebraically closed fields of characteristic p*.

A direct consequence of Theorem 1.3 is the following strong generalization of the above results obtained by Tamagawa which can be regarded as a "*finite version*" of Grothendieck's anabelian conjecture (see Corollary 6.4 for a precise statement):

Corollary 1.4. *We maintain the notation introduced above. Suppose that* k_1^{m} *is an algebraic closure of the finite field* $\mathbb{F}_p$ *and that* $g_{X_1}=0$*. Then we can construct explicitly a natural number* $c(\mathcal{I})\in\mathbb{N}$ *depending on* $U_{X_1^{\mathrm{m}}}$ *such that*

$$U_{X_1^{\mathrm{m}}}\cong U_{X_2^{\mathrm{m}}}$$

as schemes if and only if

$$G_1'\cong G_2',$$

where $G_j' \overset{\mathrm{def}}{=} \pi_1^{\mathrm{t}}(U_{X_j})/D^{(6)}_{c(\mathcal{I})}(\pi_1^{\mathrm{t}}(U_{X_j}))\in\pi_A^{\mathrm{t}}(U_{X_j})$*,* $j\in\{1,2\}$*.*

Remark 1.4.1. Note that we have $G_1\neq G_1'$ and $G_2=G_2'$, where G_j, G_j' are finite groups constructed in Theorem 1.3 and Corollary 1.4, respectively. Moreover, although Theorem 1.3 implies that the condition $G_1, G_2\in\pi_A^{\mathrm{t}}(U_{X_1})\cap\pi_A^{\mathrm{t}}(U_{X_2})$ mentioned in Theorem 1.3 and the condition $G_1'\cong G_2'$ mentioned in Corollary 1.4 are *equivalent*, Theorem 1.3 is *much stronger* than Corollary 1.4, and it *cannot* be deduced from Corollary 1.4. More precisely, the condition $G_1, G_2\in\pi_A^{\mathrm{t}}(U_{X_1})\cap\pi_A^{\mathrm{t}}(U_{X_2})$ only says that there exists a surjection $G_1'\twoheadrightarrow G_2'$

which is *much weaker* than the condition $G'_1 \cong G'_2$. The difficulties of anabelian geometry under the conditions $G'_1 \twoheadrightarrow G'_2$ and $G'_1 \cong G'_2$ are *essentially different*, the former is a *Hom-type* problem and the latter is an *Isom-type* problem.

On the other hand, the condition $G_1, G_2 \in \pi^{\mathrm{t}}_A(U_{X_1}) \cap \pi^{\mathrm{t}}_A(U_{X_2})$ is very important when we apply Theorem 1.3 to study the topological properties concerning the moduli spaces of fundamental groups (e.g. see §1.3 below), but the condition $G'_1 \cong G'_2$ is *far from enough*.

1.2.3. Next, we briefly explain the method of proving Theorem 1.2 which is completely different from the method used in [T1], [T3], and whose main ingredients are a formula concerning maximum of generalized Hasse-Witt invariants and the theory of combinatorial anabelian geometry in positive characteristic developed in the papers [Y2], [Y5]. For simplicity, we may assume $(g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2}) = (0, n)$ which is the most difficult part of the present paper. Moreover, under this assumption, Theorem 1.2 is equivalent to the "moreover" part of Theorem 1.3.

The "only if" part of the "moreover" part of Theorem 1.3 is trivial. In order to prove the "if" part of the "moreover" part of Theorem 1.3, we need to construct explicitly a *suitable* finite group $G \in \pi^{\mathrm{t}}_A(U_{X_2})$ such that the scheme-theoretical structures of $U_{X_1^{\mathrm{m}}}$ and $U_{X_2^{\mathrm{m}}}$ can be controlled by *an arbitrary surjection* $\pi^{\mathrm{t}}_1(U_{X_1}) \twoheadrightarrow G$. This is an extremely difficult problem in general since a finite quotient of the tame fundamental group of a curve cannot contain all information of the scheme-theoretical structure of the curve in general ([Y6, Theorem 3.6]).

To overcome the difficulty, we introduce the so-called "*quasi-anabelian pairs*" (see Definition 4.1) associated to tame fundamental groups. Roughly speaking, a quasi-anabelian pair consists of two finite quotients of a tame fundamental group which allows us to consider anabelian geometry via finite quotients. In §4, by using a formula concerning maximum of generalized Hasse-Witt invariants and the theory of combinatorial anabelian geometry in positive characteristic, we give an explicit construction for a quasi-anabelian pair associated to the tame fundamental group of an arbitrary smooth pointed stable curve (Theorem 4.6). Once a general method for constructing quasi-anabelian pairs has been established, moreover, in the particular case where X_1^{m} is a smooth pointed stable curve of type $(0, n)$ over $\overline{\mathbb{F}}_p$, we may construct a quasi-anabelian pair (Q_{N_2}, Q_{H_2}) associated to $\pi^{\mathrm{t}}_1(U_{X_2})$ depending on $U_{X_1^{\mathrm{m}}}$ and $U_{X_2^{\mathrm{m}}}$ which contains the information of scheme-theoretical structure of $U_{X_2^{\mathrm{m}}}$. Then we put $G \overset{\mathrm{def}}{=} Q_{N_2}$ and prove that the information of scheme-theoretical structure of X_2^{m} can be determined completely by the information of scheme-theoretical structure of X_1^{m} via an arbitrary surjection $\pi^{\mathrm{t}}_1(U_{X_1}) \twoheadrightarrow G$. This completes the proof of Theorem 1.3.

1.3. **A further motivation.** Let us explain a further background that motivated the theory developed in the present paper. In [Y6], the author of the present paper introduced a topological space $\Pi_{g,n}$ (or more general, $\overline{\Pi}_{g,n}$). We call $\Pi_{g,n}$ (or more general, $\overline{\Pi}_{g,n}$) *the moduli space of fundamental groups of curves of type* (g, n), whose underlying set is the sets of isomorphism classes of tame fundamental groups (or more general, admissible fundamental groups), and *whose topology is determined by the sets of finite quotients of tame fundamental groups* (or more general, the sets of finite quotients of admissible fundamental groups). Furthermore, in [Y6], we posed the so-called *homeomorphism conjecture*, roughly speaking, which says that (by quotiening a certain equivalence relation induced by Frobenius actions) the moduli spaces of curves are homeomorphic to the moduli spaces of fundamental groups. The main results of [Y6], [Y7] say that the homeomorphism conjecture holds for 1-*dimensional moduli spaces of pointed stable curves*.

The homeomorphism conjecture generalizes all of the conjectures appeared in the (tame or admissible) anabelian geometry of curves over algebraically closed fields of positive characteristic. It sheds some new light on the theory of the anabelian geometry of curves over algebraically closed fields of positive characteristic based on the following consideration:

> The *anabelian properties* of pointed stable curves of type (g, n) is equivalent to the *topological properties* of the topological space $\overline{\Pi}_{g,n}$.

Moreover, this consideration supplies a point of view to see what anabelian phenomena for curves over algebraically closed fields of positive characteristic that we can reasonably expect. Then it is important to understand the precise relationship between the open subsets of $\Pi_{g,n}$ (or more general, the open subsets of $\overline{\Pi}_{g,n}$) and the sets of finite quotients of tame fundamental groups (or more general, the sets of finite quotients of admissible fundamental groups). Theorem 1.2 implies the following result concerning the topological properties of $\Pi_{0,n}$:

> We maintain the notation introduced in 1.2. Let $q_j \in \Pi_{0,n}$, $j \in \{1, 2\}$, be the point of $\Pi_{0,n}$ corresponding to the isomorphism class of $\pi_1^{\mathrm{t}}(U_{X_j})$. Then we *can construct explicitly an open neighborhood* $\mathcal{U} \subseteq \Pi_{0,n}$ of q_2 such that $q_1 \notin \mathcal{U}$.

1.4. **Structure of the present paper.** The present paper is organized as follows. In §2, we fix some notation concerning curves, tame coverings, and tame fundamental groups. In §3, we prove that various geometric objects can be reconstructed group-theoretically from certain finite quotients of tame fundamental groups. In §4, we introduce "quasi-anabelian pairs" associated to tame fundamental groups and give an explicit construction for quasi-anabelian pairs. In §5, we prove that the field structures associated to inertia subgroups and linear structures associated to affine lines can be reconstructed group-theoretically from quasi-anabelian pairs. In §6, by applying various results obtained in previous sections, we prove our main result.

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2. Preliminaries

In this section, we fix some notation which will be used in the remainder of the present paper.

2.1. Curves and their tame fundamental groups.

2.1.1. Let k be an algebraically closed field of characteristic $p > 0$, and let

$$(X, D_X)$$

be a smooth pointed stable curve of type (g_X, n_X) over k, where X denotes the (smooth) underlying curve of genus g_X and D_X denotes the finite set of marked points with cardinality $n_X \stackrel{\text{def}}{=} \#(D_X)$ satisfying [Kn, Definition 1.1 (iv)] (i.e., $2g_X + n_X - 2 > 0$). We put $U_X \stackrel{\text{def}}{=} X \setminus D_X$. Then U_X is a hyperbolic curve over k.

Let (W, D_W) be a smooth pointed stable curve over k and $f : (W, D_W) \to (X, D_X)$ a morphism of smooth pointed stable curves over k. We shall say that f is *étale* (resp. *tame*, *Galois étale*, *Galois tame*) if the underlying morphism $W \to X$ induced by f is étale (resp. the morphism $U_W \to U_X$ induced by f is étale and is at most tamely ramified over D_X, f is a Galois covering and is étale, f is a Galois covering and is tame).

2.1.2. By choosing a base point of $x \in U_X$, we have the tame fundamental group $\pi_1^{\rm t}(U_X, x)$ of (X, D_X) and the étale fundamental group $\pi_1(X, x)$ of X. Since we only focus on the isomorphism classes of fundamental groups in the present paper, for simplicity of notation, we omit the base point, and denote by

$$\pi_1^{\rm t}(U_X)$$

the tame fundamental group $\pi_1^{\rm t}(U_X, x)$ of (X, D_X) and $\pi_1(X)$ the étale fundamental group $\pi_1(X, x)$ of X. Note that there is a natural continuous surjection $\pi_1^{\rm t}(U_X) \twoheadrightarrow \pi_1(X)$.

We shall write

$$\pi_A^{\rm t}(U_X)$$

for the *set of finite quotients* of $\pi_1^{\rm t}(U_X)$. Since $\pi_1^{\rm t}(U_X)$ is topologically finitely generated, the isomorphism class of $\pi_1^{\rm t}(U_X)$ is completely determined by the set $\pi_A^{\rm t}(U_X)$ ([FJ, Proposition 16.10.7]).

2.1.3. Let $H \subseteq \pi_1^{\rm t}(U_X)$ be an arbitrary open subgroup. We shall denote by (X_H, D_{X_H}) the smooth pointed stable curve of type (g_H, n_H) over k corresponding to H and $f_H : (X_H, D_{X_H}) \to (X, D_X)$ the tame covering of smooth pointed stable curves over k corresponding to the natural injection $H \hookrightarrow \pi_1^{\rm t}(U_X)$. Note that the tame fundamental group $\pi_1^{\rm t}(U_{X_H})$ of (X_H, D_{X_H}) is naturally isomorphic to H.

We put

$$\widehat{X} \overset{\rm def}{=} \varprojlim_{H \subseteq \pi_1^{\rm t}(U_X) \text{ open}} X_H,\ D_{\widehat{X}} \overset{\rm def}{=} \varprojlim_{H \subseteq \pi_1^{\rm t}(U_X) \text{ open}} D_{X_H},$$

and call $(\widehat{X}, D_{\widehat{X}})$ the universal tame covering of (X, D_X) corresponding to $\pi_1^{\rm t}(U_X)$ and $D_{\widehat{X}}$ the set of marked points of $(\widehat{X}, D_{\widehat{X}})$. Then there is a natural action of $\pi_1^{\rm t}(U_X)$ on $D_{\widehat{X}}$ such that $D_{\widehat{X}}/\pi_1^{\rm t}(U_X) = D_X$.

Let $x \in D_X$ be a marked point and $\widehat{x} \in D_{\widehat{X}}$ a point over x (i.e., the image of $\widehat{x}$ of the natural surjection $D_{\widehat{X}} \twoheadrightarrow D_X$ is x). We denote by $I_{\widehat{x}} \subseteq \pi_1^{\rm t}(U_X)$ the stabilizer subgroup of $\widehat{x}$. Let $\widehat{K}_{X,x}$ be the quotient field of $\widehat{\mathcal{O}}_{X,x}$ and $\widehat{K}^{\rm t}_{X,x}$ a maximal tamely ramified extension of $\widehat{K}_{X,x}$. Then the subgroup $I_{\widehat{x}}$ is (outer) isomorphic to $\mathrm{Gal}(\widehat{K}^{\rm t}_{X,x}/\widehat{K}_{X,x})$. Thus, we have $I_{\widehat{x}} \cong \widehat{\mathbb{Z}}(1)^{p'}$, where $(-)^{p'}$ denotes the maximal prime-to-p quotient of $(-)$. We put

$$\mathrm{Edg}^{\rm op}(\pi_1^{\rm t}(U_X)) \overset{\rm def}{=} \{I_{\widehat{x}}\}_{x \in D_{\widehat{X}}},$$

where "Edg" and "op" mean "edge" and "open edge", respectively, since the set of marked points of a pointed stable curve corresponds to the set of open edges of its dual semi-graph. The set $\mathrm{Edg}^{\rm op}(\pi_1^{\rm t}(U_X))$ admits a natural action of $\pi_1^{\rm t}(U_X)$ (i.e., the conjugacy action), and we have the following bijection

$$\mathrm{Edg}^{\rm op}(\pi_1^{\rm t}(U_X))/\pi_1^{\rm t}(U_X) \overset{\sim}{\to} D_X,\ I_{\widehat{x}} \mapsto x.$$

2.1.4. Let $a, b, s \in \mathbb{N}$ be positive natural numbers, $\mathcal{I}_0 \overset{\rm def}{=} \emptyset$, and $\mathcal{I}_i \overset{\rm def}{=} \{b_1, \dots, b_i\} \subseteq \mathbb{N}$, $i \in \{1, \dots, a\}$, a finite set of positive natural numbers. Let Δ be a profinite group. We denote by $D_b(\Delta) \subseteq \Delta$ the topological closure of $[\Delta, \Delta]\Delta^b$, where $[\Delta, \Delta]$ denotes the commutator subgroup of Δ. We define the closed normal subgroup $D_{\mathcal{I}_i}(\Delta)$ of Δ inductively by $\mathcal{D}_{\mathcal{I}_0}(\Delta) \overset{\rm def}{=} \Delta$, $D_{\mathcal{I}_1}(\Delta) \overset{\rm def}{=} D_{b_1}(\Delta)$ and $D_{\mathcal{I}_{i+1}}(\Delta) \overset{\rm def}{=} D_{b_{i+1}}(D_{\mathcal{I}_i}(\Delta))$, $i \in \{1, \dots, a-1\}$. We put $G_\Delta^{\mathcal{I}_i} \overset{\rm def}{=} \Delta/D_{\mathcal{I}_i}(\Delta)$, $i \in \{1, \dots, a\}$. Moreover, we define the closed normal subgroup $D_b^{(s)}(\Delta)$ of Δ inductively by $D_b^{(0)}(\Delta) \overset{\rm def}{=} \Delta$, $D_b^{(1)}(\Delta) \overset{\rm def}{=} D_b(\Delta)$, and $D_b^{(s)}(\Delta) \overset{\rm def}{=} D_b(D_b^{(s-1)}(\Delta))$. We put $G_\Delta^{s,b} \overset{\rm def}{=} \Delta/D_b^{(s)}(\Delta)$. Note that, if Δ is topologically finitely generated, then $D_{\mathcal{I}_i}(\Delta)$ and $D_b^{(s)}(\Delta)$ are open characteristic subgroups of Δ (in particular, we have $\#(G_\Delta^{\mathcal{I}_i}) < \infty$, $\#(G_\Delta^{s,b}) < \infty$).

2.2. **Cohomology classes and sets of marked points.**

2.2.1. *Notation and Settings.* We maintain the notation introduced in 2.1.1. Moreover, *we suppose* $g_X \geq 2$ *and* $n_X > 0$ (i.e., U_X is affine).

2.2.2. Let $h : (W, D_W) \to (X, D_X)$ be a connected Galois tame covering over k. We put

$$\mathrm{Ram}_h \stackrel{\text{def}}{=} \{x \in D_X \mid h \text{ is ramified over } x\}.$$

Let (Y, D_Y) be a smooth pointed stable curve over k. We shall say that

$$\mathfrak{T}_{U_X} \stackrel{\text{def}}{=} (\ell, d, f_X : (Y, D_Y) \to (X, D_X))$$

is *an mp-triple associated to* (X, D_X), where "mp" means "marked point", if the following conditions hold:

- (i) ℓ and d are prime numbers distinct from each other such that $(\ell, p) = (d, p) = 1$ and $\ell \equiv 1 \pmod{d}$; then all dth roots of unity are contained in $\mathbb{F}_\ell$.
- (ii) f_X is a Galois *étale* covering (2.1.1) over k whose Galois group is isomorphic to μ_d, where $\mu_d \subseteq \mathbb{F}_\ell^\times$ denotes the subgroup of dth roots of unity.

Then we have an injection $H^1_{\text{ét}}(Y, \mathbb{F}_\ell) \hookrightarrow H^1_{\text{ét}}(U_Y, \mathbb{F}_\ell)$ induced by the surjection $\pi_1^{\mathrm{t}}(U_Y) \twoheadrightarrow \pi_1(Y)$. Note that every non-zero element of $H^1_{\text{ét}}(U_Y, \mathbb{F}_\ell)$ induces a connected Galois tame covering of (Y, D_Y) of degree ℓ. Moreover, we obtain an exact sequence

$$0 \to H^1_{\text{ét}}(Y, \mathbb{F}_\ell) \to H^1_{\text{ét}}(U_Y, \mathbb{F}_\ell) \to \mathrm{Div}^0_{D_Y}(Y) \otimes \mathbb{F}_\ell \to 0$$

with a natural action of μ_d, where $\mathrm{Div}^0_{D_Y}(Y) \stackrel{\text{def}}{=} \{D \in \mathrm{Div}(Y) \mid \deg(D) = 0,\ \mathrm{Supp}(D) \subseteq D_Y\}$.

2.2.3. Let $\left(\mathrm{Div}^0_{D_Y}(Y) \otimes \mathbb{F}_\ell\right)_{\mu_d} \subseteq \mathrm{Div}^0_{D_Y}(Y) \otimes \mathbb{F}_\ell$ be the subset of elements on which μ_d acts via the character $\mu_d \hookrightarrow \mathbb{F}_\ell^\times$ and $E^*_{\mathfrak{T}_{U_X}} \subseteq H^1_{\text{ét}}(U_Y, \mathbb{F}_\ell)$ the subset of elements whose images in $\left(\mathrm{Div}^0_{D_Y}(Y) \otimes \mathbb{F}_\ell\right)_{\mu_d}$ are non-zero. Write $g_\alpha : (Y_\alpha, D_{Y_\alpha}) \to (Y, D_Y)$, $\alpha \in E^*_{\mathfrak{T}_{U_X}}$, for the Galois tame covering over k whose Galois group is isomorphic to $\mathbb{Z}/\ell\mathbb{Z}$ induced by α. We define $\epsilon : E^*_{\mathfrak{T}_{U_X}} \to \mathbb{Z},\ \alpha \mapsto \#(D_{Y_\alpha})$, and put $E^\star_{\mathfrak{T}_{U_X}} \stackrel{\text{def}}{=} \{\alpha \in E^*_{\mathfrak{T}_{U_X}} \mid \#(\mathrm{Ram}_{g_\alpha}) = d\}$. Since $d | \#(\mathrm{Ram}_{g_\alpha})$ for all $\alpha \in E^*_{\mathfrak{T}_{U_X}}$, we see

$$E^\star_{\mathfrak{T}_{U_X}} = \{\alpha \in E^*_{\mathfrak{T}_{U_X}} \mid \epsilon(\alpha) = \ell(dn_X - d) + d\}.$$

Note that $E^\star_{\mathfrak{T}_{U_X}}$ is not empty.

Let $\alpha \in E^\star_{\mathfrak{T}_{U_X}}$. Since the image of α is contained in $(\mathrm{Div}^0_{D_Y}(Y) \otimes \mathbb{F}_\ell)_{\mu_d}$, the action of μ_d on $\mathrm{Ram}_{g_\alpha} \subseteq D_Y$ is transitive. Thus, there exists a unique marked point $x_\alpha \in D_X$ such that $f_X(y) = x_\alpha$ for all $y \in \mathrm{Ram}_{g_\alpha}$. Then we may define

$$E^\star_{\mathfrak{T}_{U_X}, x} \stackrel{\text{def}}{=} \{\alpha \in E^\star_{\mathfrak{T}_{U_X}} \mid g_\alpha \text{ is ramified over } f_X^{-1}(x)\},\ x \in D_X.$$

Note that we have $E^\star_{\mathfrak{T}_{U_X}, x'} \cap E^\star_{\mathfrak{T}_{U_X}, x''} = \emptyset$ for all marked points $x', x'' \in D_X$ distinct from each other and the disjoint union

$$E^\star_{\mathfrak{T}_{U_X}} = \bigsqcup_{x \in D_X} E^\star_{\mathfrak{T}_{U_X}, x}.$$

The following result says that the set of marked points D_X can be described by using the set $E^\star_{\mathfrak{T}_{U_X}}$.

Proposition 2.1. *(i) We define a pre-equivalence relation* $\sim$ *on* $E^\star_{\mathfrak{T}_{U_X}}$ *as follows:*

Let $\alpha, \beta \in E^{\star}_{\mathfrak{T}_{U_X}}$. We have that $\alpha \sim \beta$ if, for each $\lambda, \mu \in \mathbb{F}_\ell^\times$ for which $\lambda\alpha + \mu\beta \in E^{*}_{\mathfrak{T}_{U_X}}$, we have $\lambda\alpha + \mu\beta \in E^{\star}_{\mathfrak{T}_{U_X}}$.

Then the pre-equivalence relation $\sim$ on $E^{\star}_{\mathfrak{T}_{U_X}}$ is an equivalence relation.

(ii) Denote by $E_{\mathfrak{T}_{U_X}}$ the quotient set of $E^{\star}_{\mathfrak{T}_{U_X}}$ by $\sim$ defined in (a). Then we have a natural bijection

$$\vartheta_{\mathfrak{T}_{U_X}} : E_{\mathfrak{T}_{U_X}} \xrightarrow{\sim} D_X,\ [\alpha] \mapsto x_\alpha,$$

where $[\alpha]$ denotes the equivalence class of α.

Proof. The proposition is a special case of [Y2, Proposition 2.2] (i.e., the part of the proposition concerning "$(-)^{\mathrm{op}}$"). □

Remark 2.1.1. The bijection $\vartheta_{\mathfrak{T}_{U_X}}$ does not depend on the choices of $\mathfrak{T}_{U_X}$ in the following sense: Let $\mathfrak{T}'_{U_X}$ be an arbitrary mp-triples associated to (X, D_X). Then [Y2, Remark 2.2.1] says that we have a natural bijection

$$E_{\mathfrak{T}'_{U_X}} \xrightarrow{\sim} E_{\mathfrak{T}_{U_X}}$$

which fits into the following commutative diagram:

$$\begin{array}{ccc} E_{\mathfrak{T}'_{U_X}} & \xrightarrow{\vartheta_{\mathfrak{T}'_{U_X}}} & D_X \\ \downarrow & & \| \\ E_{\mathfrak{T}_{U_X}} & \xrightarrow{\vartheta_{\mathfrak{T}_{U_X}}} & D_X. \end{array}$$

2.3. Generalized Hasse-Witt invariants.

2.3.1. *Notation and Settings.* We maintain the notation introduced in 2.1.1.

2.3.2. Let n be an arbitrary positive natural number prime to p and $\mu_n \subseteq k^\times$ the group of nth roots of unity. Fix a primitive nth root ζ, then we may identify μ_n with $\mathbb{Z}/n\mathbb{Z}$ via the isomorphism $\zeta^i \mapsto i$. Let $\alpha \in \mathrm{Hom}(\pi_1^{\mathrm{t}}(U_X), \mathbb{Z}/n\mathbb{Z})$. We denote by $f_\alpha : (X_\alpha, D_{X_\alpha}) \to (X, D_X)$ the Galois tame covering (possibly disconnected) over k with Galois group $\mathbb{Z}/n\mathbb{Z}$ corresponding to α. Write F_{X_α} for the absolute Frobenius morphism on X_α. Then there exists a decomposition ([Ser1, Section 9])

$$H^1(X_\alpha, \mathcal{O}_{X_\alpha}) = H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\mathrm{st}} \oplus H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\mathrm{ni}},$$

where F_{X_α} is a bijection on $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\mathrm{st}}$ and is nilpotent on $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\mathrm{ni}}$. Moreover, we have $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{\mathrm{st}} = H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{F_{X_\alpha}} \otimes_{\mathbb{F}_p} k$, where $H^1(X_\alpha, \mathcal{O}_{X_\alpha})^{F_{X_\alpha}}$ denotes the subspace of $H^1(X_\alpha, \mathcal{O}_{X_\alpha})$ on which F_{X_α} acts trivially. Then Artin-Schreier theory implies that we may identify $H_\alpha \overset{\mathrm{def}}{=} H^1_{\text{ét}}(X_\alpha, \mathbb{F}_p) \otimes_{\mathbb{F}_p} k$ with the largest subspace of $H^1(X_\alpha, \mathcal{O}_{X_\alpha})$ on which F_{X_α} is a bijection.

The finite dimensional k-linear space H_α is a finitely generated $k[\mu_n]$-module induced by the natural action of μ_n on X_α, moreover, it admits the following canonical decomposition

$$H_\alpha = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} H_{\alpha,i},$$

where $\zeta \in \mu_n$ acts on $H_{\alpha,i}$ as the ζ^i-multiplication. We call $\gamma_{\alpha,i} \overset{\mathrm{def}}{=} \dim_k(H_{\alpha,i}),\ i \in \mathbb{Z}/n\mathbb{Z}$, a *generalized Hasse-Witt invariant* (see [Ka], [N], [T1], [Y5]) of the cyclic tame covering f_α. In particular, we call $\gamma_{\alpha,1}$ the *first* generalized Hasse-Witt invariant of the cyclic tame covering f_α.

2.3.3. Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$, $H \subseteq \pi_1^{\mathrm{t}}(U_X)$ an open characteristic subgroup, and $Q_H \stackrel{\mathrm{def}}{=} \pi_1^{\mathrm{t}}(U_X)/H$. Let $\#(Q_H^{\mathrm{ab}}) = p^d m$ such that $m \neq 1$ and $(p, m) = 1$, where $(-)^{\mathrm{ab}}$ denotes the abelianization of $(-)$.

Let $\chi \in \mathrm{Hom}(Q_H, \overline{\mathbb{F}}_p^\times)$ and $Q_{H,\chi} \subseteq Q_H$ the kernel of χ. Then the finite group $Q_{H,\chi}$ admits a natural action of Q_H via the conjugation action. We put

$$N_\chi \stackrel{\mathrm{def}}{=} \{\pi \in H_{\chi,p} \stackrel{\mathrm{def}}{=} \mathrm{Hom}(Q_{H,\chi}, \mathbb{Z}/p\mathbb{Z}) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \mid \tau \cdot \pi = \chi(\tau)\pi \text{ for all } \tau \in Q_H\},$$

$$\gamma_{N_\chi} \stackrel{\mathrm{def}}{=} \dim_{\overline{\mathbb{F}}_p}(N_\chi),$$

where $(\tau \cdot \pi)(x) \stackrel{\mathrm{def}}{=} \pi(\tau^{-1} \cdot x)$ for all $x \in Q_{H,\chi}$. We define a group-theoretical invariant associated to the finite group Q_H as follows:

$$\gamma_{Q_H}^{\max} \stackrel{\mathrm{def}}{=} \max\{\gamma_{N_\chi} \mid \chi \in \mathrm{Hom}(Q_H, \overline{\mathbb{F}}_p^\times) \text{ and } \chi \neq 1\}.$$

Let $\mu_{m'} \stackrel{\mathrm{def}}{=} \chi(Q_H) \subseteq \overline{\mathbb{F}}_p^\times$ be the image of χ which is the group of m'th roots of unity for some natural number $m'|m$ prime to p. Write $(X_\chi, D_{X_\chi}) \to (X, D_X)$ for the Galois tame covering over k with Galois group $\mu_{m'}$ induced by the composition of surjections $\pi_1^{\mathrm{t}}(U_X) \twoheadrightarrow Q_H \stackrel{\chi}{\twoheadrightarrow} \overline{\mathbb{F}}_p^\times$. Suppose

$$H \subseteq D_p^{(1)}(D_{m'}^{(1)}(\pi_1^{\mathrm{t}}(U_X))).$$

Then $\chi : Q_H \to \overline{\mathbb{F}}_p^\times$ factors through the natural surjection $Q_H \twoheadrightarrow \pi_1^{\mathrm{t}}(U_X)/D_p^{(1)}(D_{m'}^{(1)}(\pi_1^{\mathrm{t}}(U_X)))$. Thus, we have a natural Q_H-equivariant isomorphism $H^1_{\text{ét}}(X_\chi, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \cong H_{\chi,p}$. Moreover, since the actions of Q_H on $H^1_{\text{ét}}(X_\chi, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p$ and $H_{\chi,p}$ factor through $Q_H/Q_{H,\chi} \cong \mu_{m'}$, the isomorphism $H^1_{\text{ét}}(X_\chi, \mathbb{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \cong H_{\chi,p}$ is also $\mu_{m'}$-equivariant. This means that γ_{N_χ} is a generalized Hasse-Witt invariant of the cyclic tame covering $(X_\chi, D_{X_\chi}) \to (X, D_X)$.

3. Reconstructions of marked points via finite quotients

In this section, we prove that the sets of marked points of smooth pointed stable curves can be reconstructed group-theoretically from certain finite quotients of tame fundamental groups. The main result of the present section is Proposition 3.5.

3.1. **Reconstructions of types.**

3.1.1. *Notation and Settings.* We maintain the notation introduced in 2.1.1. Moreover, *suppose* $n_X > 0$ (i.e., U_X is affine).

Let $m \in \mathbb{Z}_{\geq 0}$ be an arbitrary non-negative integer and

$$C(m) \stackrel{\mathrm{def}}{=} \begin{cases} 0, & \text{if } m = 0, \\ 3^{m-1}m!, & \text{if } m \neq 0. \end{cases}$$

Let

$$\widehat{\Gamma}_{g_X,n_X} \stackrel{\mathrm{def}}{=} \langle \alpha_1, \dots, \alpha_{g_X}, \beta_1, \dots, \beta_{g_X}, \gamma_1, \dots, \gamma_{n_X} \mid \prod_{r=1}^{g_X}[\alpha_r, \beta_r] \prod_{s=1}^{n_X} \gamma_s = 1\rangle^{\mathrm{pro}}$$

be the profinite completion of the topological fundamental group of a Riemann surface of type (g_X, n_X). Now, we fix natural numbers

$$\ell,\ d,\ \mathcal{I}_a \stackrel{\mathrm{def}}{=} \{b_1, \dots, b_a\} \subseteq \mathbb{N},\ c(\mathcal{I}_a),$$

such that the following conditions are satisfied:

- ℓ and d are prime numbers distinct from p and distinct from each other such that $\ell \equiv 1 \pmod{d}$ (then all dth roots of unity are contained in $\mathbb{F}_\ell$).

- $(\ell, \prod_{i=1}^{a} b_i) = (d, \prod_{i=1}^{a} b_i) = 1$.
- Let $e(\mathcal{I}_a)' \stackrel{\text{def}}{=} \ell d \prod_{i=1}^{a} b_i$. We put $e(\mathcal{I}_a) \stackrel{\text{def}}{=} \#\big(\widehat{\Gamma}_{g_X,n_X}/D^{(a+2)}_{e(\mathcal{I}_a)'}(\widehat{\Gamma}_{g_X,n_X})\big)$ (see 2.1.4 for $D^{(a+2)}_{e(\mathcal{I}_a)'}(-)$). Then we have $p|c(\mathcal{I}_a)$, $e(\mathcal{I}_a)|c(\mathcal{I}_a)$, and $(p^{t_{\mathcal{I}_a}}-1)|c(\mathcal{I}_a)$, where $t_{\mathcal{I}_a} \in \mathbb{N}$ is a natural number satisfying $p^{t_{\mathcal{I}_a}} - 1 > C\big(e(\mathcal{I}_a)(2g_X + 2n_X)\big)$.

On the other hand, let $s, b \in \mathbb{N}$, $\mathcal{I}_0 \stackrel{\text{def}}{=} \emptyset$, and $\mathcal{I}_i \stackrel{\text{def}}{=} \{b_1, \dots, b_i\}$, $i \in \{1, \dots, a\}$. For simplicity, we put (see 2.1.4 for $D_{\mathcal{I}_i}(-)$, $D_b^{(s)}(-)$, $G^{\mathcal{I}_i}_{(-)}$, $G^{s,b}_{(-)}$)

$$D_{\mathcal{I}_i} \stackrel{\text{def}}{=} D_{\mathcal{I}_i}(\pi_1^{\mathrm{t}}(U_X)),\ D_b^{(s)} \stackrel{\text{def}}{=} D_b^{(s)}(\pi_1^{\mathrm{t}}(U_X)),$$

$$G^{\mathcal{I}_i} \stackrel{\text{def}}{=} G^{\mathcal{I}_i}_{\pi_1^{\mathrm{t}}(U_X)} = \pi_1^{\mathrm{t}}(U_X)/D_{\mathcal{I}_i}(\pi_1^{\mathrm{t}}(U_X)),\ G^{s,b} \stackrel{\text{def}}{=} G^{s,b}_{\pi_1^{\mathrm{t}}(U_X)} = \pi_1^{\mathrm{t}}(U_X)/D_b^{(s)}(\pi_1^{\mathrm{t}}(U_X))$$

the open characteristic subgroups and the finite quotients of $\pi_1^{\mathrm{t}}(U_X)$, respectively. Note that we have $D^{(a)}_{c(\mathcal{I}_a)} \subseteq D^{(a)}_{e(\mathcal{I}_a)} \subseteq D_{\mathcal{I}_a} \subseteq \pi_1^{\mathrm{t}}(U_X)$.

3.1.2. *Anabelian reconstructions.* Let $\mathcal{F}$ be a geometric object and $\Pi_{\mathcal{F}}$ a profinite (possibly finite) group associated to $\mathcal{F}$. Suppose that we are given an invariant $\mathrm{Inv}_{\mathcal{F}}$ depending on the isomorphism class of $\mathcal{F}$ (in a certain category), and that we are given an additional structure $\mathrm{Add}_{\mathcal{F}}$ (e.g., a family of subgroups, a family of quotient groups, etc.) on the profinite group $\Pi_{\mathcal{F}}$ depending functorially on $\mathcal{F}$.

We shall say that $\mathrm{Inv}_{\mathcal{F}}$ can be *group-theoretically reconstructed from* $\Pi_{\mathcal{F}}$ if there exists a group-theoretical algorithm whose input datum is $\Pi_{\mathcal{F}}$, and whose output datum is $\mathrm{Inv}_{\mathcal{F}}$. We shall say that $\mathrm{Add}_{\mathcal{F}}$ *can be group-theoretically reconstructed from* $\Pi_{\mathcal{F}}$ if there exists a group-theoretical algorithm whose input datum is $\Pi_{\mathcal{F}}$, and whose output datum is $\mathrm{Add}_{\mathcal{F}}$.

3.1.3. Firstly, we have the following lemma:

Lemma 3.1. *Let $H \subseteq \pi_1^{\mathrm{t}}(U_X)$ be an arbitrary open subgroup, (X_H, D_{X_H}) the smooth pointed stable curve of type (g_H, n_H) over k corresponding to H, and $c \stackrel{\text{def}}{=} p(p^t - 1)$ a positive natural number satisfying $p^t - 1 \geq C\big(2g_H + n_H\big)$. Then the following statements hold (see 2.1.4 for $G_H^{1,\ell}$, $G_H^{2,c}$):*

(i) We have $2g_H + n_H \leq \#(\pi_1^{\mathrm{t}}(U_X)/H)(2g_X + 2n_X)$.

(ii) Let $\ell|c$ be a prime divisor of c distinct form p. Then the natural number $2g_H + n_H$ can be reconstructed group-theoretically from the finite group $G_H^{1,\ell} = H^{\mathrm{ab}} \otimes \mathbb{F}_\ell$.

(iii) The type (g_H, n_H) can be reconstructed group-theoretically from the finite group $G_H^{2,c}$.

Proof. (i) The Riemann–Hurwitz formula implies

$$g_H \leq \#(\pi_1^{\mathrm{t}}(U_X)/H)g_X + (\frac{n_X}{2} - 1)(\#(\pi_1^{\mathrm{t}}(U_X)/H) - 1).$$

On the other hand, we have $n_H \leq n_X \#(\pi_1^{\mathrm{t}}(U_X)/H)$. This completes the proof of (i).

(ii) Since we assume that U_X is affine, we obtain $2g_H + n_H = \dim_{\mathbb{F}_\ell}(G_H^{1,\ell}) + 1$. This completes the proof of (ii).

(iii) By [Y5, Theorem 5.4] and its proof (in particular, line 4, page 82 of [Y5]; note that the cardinality $\#\big(e(\Gamma_{(X_H, D_{X_H})})\big)$ of the set of edges of the dual semi-graph of (X_H, D_{X_H}) is equal to $n_H < C\big(2g_H + n_H\big)$ if X is non-singular), we have (see 2.3.3 for $\gamma^{\max}_{G_H^{2,c}}$)

$$\gamma^{\max}_{G_H^{2,c}} = g_H + n_H - 2.$$

In particular, $g_H + n_H - 2$ can be reconstructed group-theoretically from $G_H^{2,c}$. On the other hand, let $\ell'|c$ be a prime divisor of c distinct form p. Since $G_H^{2,c,\mathrm{ab}} \otimes \mathbb{F}_{\ell'} = G_H^{1,\ell'}$, (ii) implies that $2g_H + n_H - 1 = \dim_{\mathbb{F}_{\ell'}}(G_H^{1,\ell'})$ can be reconstructed group-theoretically from $G_H^{2,c}$. Then

$$g_H = \dim_{\mathbb{F}_{\ell'}}(G_H^{1,\ell'}) - \gamma^{\max}_{G_H^{2,c}} - 1,\ n_H = 2\gamma^{\max}_{G_H^{2,c}} - \dim_{\mathbb{F}_{\ell'}}(G_H^{1,\ell'}) + 1$$

can be reconstructed group-theoretically from $G_H^{2,c}$. We complete the proof of (iii). □

The above lemma implies the following proposition:

Proposition 3.2. *We maintain the notation and the settings introduced in 3.1.1. Then the following statements hold:*

(i) Let $H \subseteq \pi_1^{\mathrm{t}}(U_X)$ be an open subgroup such that $D_{e(\mathcal{I}_a)}^{(a+2)} \subseteq H$, Then the type (g_H, n_H) can be reconstructed group-theoretically from the finite group $\overline{H} \stackrel{\mathrm{def}}{=} H/D_{c(\mathcal{I}_a)}^{(a+4)} \subseteq G^{a+4,c(\mathcal{I}_a)}$.

(ii) Let $N, H \subseteq \pi_1^{\mathrm{t}}(U_X)$ be open subgroups such that $D_{e(\mathcal{I}_a)}^{(a+2)} \subseteq N \subseteq H$. Write $f_{N,H} : (X_N, D_{X_N}) \to (X_H, D_{X_H})$ for the tame covering over k corresponding to $N \hookrightarrow H$. Then we can detect whether $f_{N,H}$ is étale or not, group-theoretically from the finite groups $\overline{H} \stackrel{\mathrm{def}}{=} H/D_{c(\mathcal{I}_a)}^{(a+4)}, \overline{N} \stackrel{\mathrm{def}}{=} N/D_{c(\mathcal{I}_a)}^{(a+4)} \subseteq G^{a+4,c(\mathcal{I}_a)}$.

Proof. (i) We see $G_H^{2,c(\mathcal{I}_a)} = \overline{H}/D_{c(\mathcal{I}_a)}^{(2)}(\overline{H})$. Then the proposition follows immediately from Lemma 3.1 (iii).

(ii) Note that we have $\deg(f_{N,H}) = \#(H/N) = \#(\overline{H}/\overline{N})$. The Riemann-Hurwitz formula implies that $f_{H,N}$ is étale if and only if $g_N - 1 = \deg(f_{N,H})(g_H - 1)$ holds. Then (ii) follows immediately from (i). □

3.2. **Reconstructions of marked points.**

3.2.1. *Notation and Settings.* We maintain the notation and the settings introduced in 3.1.1, and put

$$\mathscr{T}(D_{\mathcal{I}_a}) \stackrel{\mathrm{def}}{=} \{H \subseteq \pi_1^{\mathrm{t}}(U_X) \mid D_{\mathcal{I}_a} \subseteq H\},$$

$$\mathscr{T}(G^{\mathcal{I}_a}) \stackrel{\mathrm{def}}{=} \{\overline{H} \stackrel{\mathrm{def}}{=} H/D_{c(\mathcal{I}_a)}^{(a+4)} \mid H \in \mathscr{T}(D_{\mathcal{I}_a})\}.$$

Moreover, in this subsection, *we suppose* $g_X \geq 2$.

3.2.2. Let $H \subseteq \pi_1^{\mathrm{t}}(U_X)$ be an open subgroup such that $D_{e(\mathcal{I}_a)}^{(a+1)} \subseteq H$. Let $\overline{H} \stackrel{\mathrm{def}}{=} H/D_{c(\mathcal{I}_a)}^{(a+4)}$ and $\ell' \in \{\ell, d\}$ a prime number. Note that there exists a bijection $\mathrm{Hom}(H, \mathbb{Z}/\ell'\mathbb{Z}) \stackrel{\sim}{\to} \mathrm{Hom}(\overline{H}, \mathbb{Z}/\ell'\mathbb{Z})$, $\beta \mapsto \overline{\beta}$, induced by the natural surjection $H \twoheadrightarrow \overline{H}$. Let $\beta \in \mathrm{Hom}(H, \mathbb{Z}/\ell\mathbb{Z})$ be an element and $H_\beta \stackrel{\mathrm{def}}{=} \ker(\beta) \subseteq \pi_1^{\mathrm{t}}(U_X)$. We put

$$\mathrm{Hom}^{\text{ét}}(\overline{H}, \mathbb{Z}/\ell'\mathbb{Z}) \stackrel{\mathrm{def}}{=} \{\overline{\beta} \in \mathrm{Hom}(\overline{H}, \mathbb{Z}/\ell'\mathbb{Z}) \mid \text{the Galois tame covering}$$
$$(X_{H_\beta}, D_{X_{H_\beta}}) \to (X_H, D_{X_H}) \text{ corresponding to } H_\beta \hookrightarrow H \text{ is étale}\}.$$

Note that since we assume $g_X \geq 2$, the set $\mathrm{Hom}^{\text{ét}}(\overline{H}, \mathbb{Z}/\ell'\mathbb{Z})$ is *not* equal to 0.

Lemma 3.3. *We maintain the notation and the settings introduced above. Then* $\mathrm{Hom}^{\text{ét}}(\overline{H}, \mathbb{Z}/\ell'\mathbb{Z})$ *can be reconstructed group-theoretically from the finite groups $\overline{H}$ and $G^{a+4,c(\mathcal{I}_a)}$.*

Proof. Let $\overline{\beta} \in \mathrm{Hom}(\overline{H}, \mathbb{Z}/\ell'\mathbb{Z})$ be an arbitrary element and $\beta \in \mathrm{Hom}(H, \mathbb{Z}/\ell'\mathbb{Z})$ the element corresponding to $\overline{\beta}$. Since $D_{e(\mathcal{I}_a)}^{(a+1)} \subseteq H$, by the assumptions concerning ℓ, d, and $e(\mathcal{I}_a)$ (see 3.1.1), we see $D_{e(\mathcal{I}_a)}^{(a+2)} \subseteq H_\beta \subseteq H$ and $\overline{H}_\beta \stackrel{\mathrm{def}}{=} H_\beta/D_{c(\mathcal{I}_a)}^{(a+4)} = \ker(\overline{\beta}) \subseteq G^{a+4,c(\mathcal{I}_a)}$. Then the lemma follows immediately from Proposition 3.2 (ii). □

If $\ell' = d$ and $\beta \in \mathrm{Hom}^{\text{ét}}(\overline{H}, \mathbb{Z}/d\mathbb{Z})$ is a non-zero element, then the triple $(\ell, d, (X_{H_\beta}, D_{X_{H_\beta}}) \to (X_H, D_{X_H}))$ is an mp-triple associated to (X_H, D_{X_H}) (2.2.2). We shall call

$$\mathfrak{T}_{\overline{H}} \overset{\text{def}}{=} (\ell, d, \overline{\beta}), \ \overline{\beta} \in \mathrm{Hom}^{\text{ét}}(\overline{H}, \mathbb{Z}/d\mathbb{Z}) \setminus \{0\},$$

an mp-triple associated to $\overline{H}$. Lemma 3.3 implies immediately the following corollary:

Corollary 3.4. *We maintain the notation and the settings introduced above. Then we can construct an mp-triple associated to* $\overline{H}$ *group-theoretically from the finite groups* $\overline{H}$ *and* $G^{a+4,c(\mathcal{I}_a)}$.

3.2.3. Let $H \in \mathscr{T}(D_{\mathcal{I}_a})$ be an element and $\overline{H} \in \mathscr{T}(G^{\mathcal{I}_a})$ the finite quotient of H. In the remainder of the present section, *we fix an mp-triple*

$$\mathfrak{T}_{\overline{H}} \overset{\text{def}}{=} (\ell, d, \overline{\beta})$$

associated to $\overline{H}$. Let $\beta \in \mathrm{Hom}(H, \mathbb{Z}/d\mathbb{Z})$ be the element corresponding $\overline{\beta}$. Then we have $D^{(a+1)}_{e(\mathcal{I}_a)} \subseteq H_\beta$ and $\overline{H}_\beta \overset{\text{def}}{=} H_\beta/D^{(a+4)}_{c(\mathcal{I}_a)} = \ker(\overline{\beta}) \subseteq G^{a+4,c(\mathcal{I}_a)}$. Denote by

$$M_{\overline{H}_\beta} \overset{\text{def}}{=} \mathrm{Hom}(\overline{H}_\beta, \mathbb{Z}/\ell\mathbb{Z}), \ M^{\text{ét}}_{\overline{H}_\beta} \overset{\text{def}}{=} \mathrm{Hom}^{\text{ét}}(\overline{H}_\beta, \mathbb{Z}/\ell\mathbb{Z}).$$

Then Lemma 3.3 and Corollary 3.4 imply that the exact sequence (as $\mathbb{F}_\ell[\mu_d]$-modules)

$$0 \to M^{\text{ét}}_{\overline{H}_\beta} \to M_{\overline{H}_\beta} \to M^{\text{ra}}_{\overline{H}_\beta} \overset{\text{def}}{=} M_{\overline{H}_\beta}/M^{\text{ét}}_{\overline{H}_\beta} \to 0$$

can be reconstructed group-theoretically from the finite groups $\overline{H}$, $G^{a+4,c(\mathcal{I}_a)}$, and the mp-triple $\mathfrak{T}_{\overline{H}}$ associated to $\overline{H}$.

We denote by $M^{\text{ra}}_{\overline{H}_\beta,\mu_d} \subseteq M^{\text{ra}}_{\overline{H}_\beta}$ the subset of elements on which μ_d acts via the character $\mu_d \hookrightarrow \mathbb{F}_\ell^\times$ and $E^*_{\mathfrak{T}_{\overline{H}}} \subseteq M_{\overline{H}_\beta}$ the subset of elements whose images in $M^{\text{ra}}_{\overline{H}_\beta,\mu_d}$ are non-zero. Let $\overline{\alpha} \in E^*_{\mathfrak{T}_{\overline{H}}}$, $\alpha \in \mathrm{Hom}(H_\beta, \mathbb{Z}/\ell\mathbb{Z})$ the element corresponding to $\overline{\alpha}$, and $H_\alpha \overset{\text{def}}{=} \ker(\alpha)$. Denote by

$$E^\star_{\mathfrak{T}_{\overline{H}}} \overset{\text{def}}{=} \{\overline{\alpha} \in E^*_{\mathfrak{T}_{\overline{H}}} \mid n_{H_\alpha} = \ell(dn_H - d) + d\},$$

where n_H, n_{H_α} denote the cardinalities of the sets of marked points of smooth pointed stable curves corresponding to H, H_α, respectively. Since $D^{(a+2)}_{e(\mathcal{I}_a)} \subseteq H_\alpha \subseteq H$, Lemma 3.1 (iii) and Corollary 3.4 imply that $E^\star_{\mathfrak{T}_{\overline{H}}}$ can be reconstructed group-theoretically from the finite groups $\overline{H}$ and $G^{a+4,c(\mathcal{I}_a)}$.

Note that we have the following natural isomorphisms

$$M_{\overline{H}_\beta} \cong H^1_{\text{ét}}(U_{X_{H_\beta}}, \mathbb{F}_\ell), \ M^{\text{ét}}_{\overline{H}_\beta} \cong H^1(X_{H_\beta}, \mathbb{F}_\ell), \ M^{\text{ra}}_{\overline{H}_\beta} \cong \mathrm{Div}^0_{D_Y}(Y) \otimes \mathbb{F}_\ell$$

as $\mathbb{F}_\ell[\mu_d]$-modules. On the other hand, by replacing (X, D_X) by (X_H, D_{X_H}) and by applying the constructions obtained in 2.2.3, we obtain $E^\star_{\mathfrak{T}_{U_{X_H}}}$ determined by the mp-triple

$$\mathfrak{T}_{U_{X_H}} \overset{\text{def}}{=} (\ell, d, (X_{H_\beta}, D_{X_{H_\beta}}) \to (X_H, D_{X_H}))$$

associated to (X_H, D_{X_H}). We see immediately that there is a natural bijection $E^\star_{\mathfrak{T}_{\overline{H}}} \overset{\sim}{\to} E^\star_{\mathfrak{T}_{U_{X_H}}}$. Then we obtain a bijection

$$E_{\mathfrak{T}_{\overline{H}}} \overset{\text{def}}{=} E^\star_{\mathfrak{T}_{\overline{H}}}/\sim \overset{\sim}{\to} E_{\mathfrak{T}_{U_{X_H}}} \overset{\text{def}}{=} E^\star_{\mathfrak{T}_{U_{X_H}}}/\sim,$$

where $\sim$ is the equivalence relation defined in Proposition 2.1 (i).

Let $N \in \mathscr{T}(D_{\mathcal{I}_a})$ be an element and $\overline{N} \in \mathscr{T}(G^{\mathcal{I}_a})$ the finite quotient of N. Suppose $N \subseteq H$. Since we assume $(d, \prod_{i=1}^a b_i) = 1$ (see 3.1.1), $\overline{N} \cap \overline{H}_\beta$ is a subgroup of $\overline{N}$ such that $\overline{N}/(\overline{N} \cap \overline{H}_\beta)$ is naturally isomorphic to $\overline{H}/\overline{H}_\beta \overset{\sim}{\to} \mathbb{Z}/d\mathbb{Z}$, where $\overline{H}/\overline{H}_\beta \overset{\sim}{\to} \mathbb{Z}/d\mathbb{Z}$ is the isomorphism induced by $\overline{\beta}$. Denote by $\overline{\beta}_N : \overline{N} \twoheadrightarrow \overline{N}/(\overline{N} \cap \overline{H}_\beta) \xrightarrow{\sim} \overline{H}/\overline{H}_\beta \xrightarrow{\sim} \mathbb{Z}/d\mathbb{Z}$ the composition of homomorphisms and $\overline{N}_{\beta_N} \stackrel{\text{def}}{=} \ker(\overline{\beta}_N) = \overline{N} \cap \overline{H}_\beta$. Then the finite groups $\overline{H}$, $\overline{N}$, and the mp-triple $\mathfrak{T}_{\overline{H}}$ associated to $\overline{H}$ determine group-theoretically an mp-triple

$$\mathfrak{T}_{\overline{N}} \stackrel{\text{def}}{=} (\ell, d, \overline{\beta}_N)$$

associated to $\overline{N}$. Furthermore, by replacing $\overline{H}$ by $\overline{N}$ and by similar arguments to the arguments given above, we have that $\overline{N}$ and $\mathfrak{T}_{\overline{N}}$ determine group-theoretically the following sets

$$E^{\star}_{\mathfrak{T}_{\overline{N}}},\ E_{\mathfrak{T}_{\overline{N}}} \stackrel{\text{def}}{=} E^{\star}_{\mathfrak{T}_{\overline{N}}}/\sim .$$

Then we have the following result:

Proposition 3.5. *We maintain the notation and the settings introduced above. Then the following statements hold:*

(i) The set of marked points D_{X_H} of (X_H, D_{X_H}) can be reconstructed group-theoretically from the finite groups $\overline{H}$ and $G^{a+4,c(\mathcal{I}_a)}$. Namely, we can identify D_{X_H} with $E_{\mathfrak{T}_{\overline{H}}}$ via the composition of bijections

$$\vartheta_{\mathfrak{T}_{\overline{H}}} : E_{\mathfrak{T}_{\overline{H}}} \xrightarrow{\sim} E_{\mathfrak{T}_{U_X}} \overset{\vartheta_{\mathfrak{T}_{U_X}}}{\xrightarrow{\sim}} D_X.$$

(ii) Let $f_{N,H} : (X_N, D_{X_N}) \to (X_H, D_{X_H})$ be the tame covering of smooth pointed stable curves over k corresponding to $N \hookrightarrow H$ and $f^{\mathrm{mp}}_{N,H} : D_{X_N} \twoheadrightarrow D_{X_H}$ the surjection of sets of marked points induced by $f_{N,H}$. Then the natural injection $\overline{N} \hookrightarrow \overline{H}$ induces a surjection

$$\gamma_{\mathfrak{T}_{\overline{H}},\overline{N}} : E_{\mathfrak{T}_{\overline{N}}} \twoheadrightarrow E_{\mathfrak{T}_{\overline{H}}}$$

which fits into the following commutative diagram:

$$\begin{array}{ccc} E_{\mathfrak{T}_{\overline{N}}} & \xrightarrow{\vartheta_{\mathfrak{T}_{\overline{N}}}} & D_{X_N} \\ {\scriptstyle \gamma_{\mathfrak{T}_{\overline{H}},\overline{N}}}\downarrow & & \downarrow{\scriptstyle f^{\mathrm{mp}}_{N,H}} \\ E_{\mathfrak{T}_{\overline{H}}} & \xrightarrow{\vartheta_{\mathfrak{T}_{\overline{H}}}} & D_{X_H}. \end{array}$$

Moreover, suppose that $\overline{N} \subseteq \overline{H}$ is a normal subgroup. Then $E_{\mathfrak{T}_{\overline{N}}}$ admits an action of $\overline{H}/\overline{N}$ such that $\vartheta_{\mathfrak{T}_{\overline{N}}}$ is compatible with $\overline{H}/\overline{N}$-actions (i.e., $\vartheta_{\mathfrak{T}_{\overline{N}}}$ is $\overline{H}/\overline{N}$-equivariant).

Proof. (i) The statement (i) follows immediately from Proposition 2.1 (i), (ii).

(ii) Let $\alpha_{\overline{H}} \in E^{\star}_{\mathfrak{T}_{\overline{H}}} \subseteq \mathrm{Hom}(\overline{H}_\beta, \mathbb{Z}/\ell\mathbb{Z})$. Then $\alpha_{\overline{H}}$ induces an element $\alpha_{\overline{N},\overline{H}} \in \mathrm{Hom}(\overline{N}_{\beta_N}, \mathbb{Z}/\ell\mathbb{Z})$ via the natural homomorphism $\mathrm{Hom}(\overline{H}_\beta, \mathbb{Z}/\ell\mathbb{Z}) \to \mathrm{Hom}(\overline{N}_{\beta_N}, \mathbb{Z}/\ell\mathbb{Z})$ induced by $\overline{N}_{\beta_N} \subseteq \overline{H}_\beta$. Since we assume $(\ell, \prod_{i=1}^a b_i) = 1$ (see 3.1.1), $\alpha_{\overline{N},\overline{H}}$ is non-zero. Moreover, we have

$$\alpha_{\overline{N},\overline{H}} = \sum_{\beta \in J} c_\beta \beta,\ c_\beta \in \mathbb{F}_\ell^\times,$$

where J is a subset of $E^{\star}_{\mathfrak{T}_{\overline{N}}}$ such that, for $\beta', \beta'' \in J$ distinct from each other, the equivalence classes $[\beta'], [\beta''] \in E_{\mathfrak{T}_{\overline{N}}}$ of β', β'' are distinct from each other.

Let $[\alpha_{\overline{N}}] \in E_{\mathfrak{T}_{\overline{N}}}$. We define

$$\gamma_{\mathfrak{T}_{\overline{H}},\overline{N}}([\alpha_{\overline{N}}]) = [\alpha_{\overline{H}}]$$

if $[\beta]=[\alpha_{\overline{N}}]$ for some $\beta \in J$. It is easy to check that $\gamma_{\mathfrak{I}_{\overline{H}},\overline{N}}$ is well-defined, and that the following diagram

$$\begin{array}{ccc} E_{\mathfrak{I}_{\overline{N}}} & \xrightarrow{\vartheta_{\mathfrak{I}_{\overline{N}}}} & D_{X_N} \\ {\scriptstyle\gamma_{\mathfrak{I}_{\overline{H}},\overline{N}}}\downarrow & & {\scriptstyle f^{\mathrm{mp}}_{N,H}}\downarrow \\ E_{\mathfrak{I}_{\overline{H}}} & \xrightarrow{\vartheta_{\mathfrak{I}_{\overline{H}}}} & D_{X_H}. \end{array}$$

is commutative.

Moreover, suppose that $\overline{N}$ is a normal subgroup of $\overline{H}$. Since $\overline{N}, \overline{H} \in \mathscr{T}(D_{\mathcal{I}_a})$ (3.2.1) and $(d, \prod_{i=1}^{a} b_i)=1$, we have $\overline{H}/\overline{N}_{\beta_N} \cong \overline{H}/\overline{N} \times \mathbb{Z}/d\mathbb{Z}$. Then the natural exact sequence

$$1 \to \overline{N}_{\beta_N} \to \overline{H} \to \overline{H}/\overline{N}_{\beta_N} \to 1$$

induces an outer representation $\overline{H}/\overline{N} \hookrightarrow \overline{H}/\overline{N}_{\beta_N} \to \mathrm{Out}(\overline{N}_{\beta_N}) \overset{\mathrm{def}}{=} \mathrm{Aut}(\overline{N}_{\beta_N})/\mathrm{Inn}(\overline{N}_{\beta_N})$. Thus, we obtain an action of $\overline{H}/\overline{N}$ on $E^{\star}_{\mathfrak{I}_{\overline{N}}} \subseteq \mathrm{Hom}(\overline{N}_{\beta_N}, \mathbb{Z}/\ell\mathbb{Z})$ induced by the outer representation. Let $\sigma \in \overline{H}/\overline{N}$ and $\alpha'_{\overline{N}}, \alpha''_{\overline{N}} \in E^{\star}_{\mathfrak{I}_{\overline{N}}}$. We obverse that $\alpha'_{\overline{N}} \sim \alpha''_{\overline{N}}$ if and only if $\sigma(\alpha'_{\overline{N}}) \sim \sigma(\alpha''_{\overline{N}})$. Thus, we obtain an action of $\overline{H}/\overline{N}$ on $E_{\mathfrak{I}_{\overline{N}}}$ induced by the natural injection $\overline{N} \hookrightarrow \overline{H}$. On the other hand, it is easy to check that the above commutative diagram is compatible with the $\overline{H}/\overline{N}$-actions. This completes the proof of (ii) of the proposition. □

4. Quasi-anabelian pairs of finite groups

In this section, we introduce the so-called "*quasi-anabelian pairs*" associated to tame fundamental groups. Roughly speaking, quasi-anabelian pairs are pairs of finite quotients of tame fundamental groups which admit certain anabelian properties. The main result of the present section is Theorem 4.6.

4.1. Definition of quasi-anabelian pairs.

4.1.1. *Notation and Settings.* We maintain the notation introduced in 2.1.1. Moreover, we suppose $n_X > 0$ (i.e., U_X is affine).

4.1.2. Let $H \subseteq \pi_1^{\mathrm{t}}(U_X)$ be an open *characteristic subgroup*, $Q_H \overset{\mathrm{def}}{=} \pi_1^{\mathrm{t}}(U_X)/H$ the finite quotient, and $x_H \in D_{X_H}$ an arbitrary marked point of (X_H, D_{X_H}). Then (X_H, D_{X_H}) admits a natural action of Q_H. We denote by $I_{x_H} \subseteq Q_H$ the stabilizer subgroup of x_H, and put

$$\mathrm{Edg}^{\mathrm{op}}(Q_H) \overset{\mathrm{def}}{=} \{I_{x_H}\}_{x_H \in D_{X_H}}.$$

We introduce the quasi-anabelian pairs associated to $\pi_1^{\mathrm{t}}(U_X)$ as follows:

Definition 4.1. Let $N, H \subseteq \pi_1^{\mathrm{t}}(U_X)$ be open characteristic subgroups such that $N \subseteq H$. We put $Q_N \overset{\mathrm{def}}{=} \pi_1^{\mathrm{t}}(U_X)/N$, $Q_H \overset{\mathrm{def}}{=} \pi_1^{\mathrm{t}}(U_X)/H$ the finite quotients. Let (Y, D_Y) be an arbitrary smooth pointed stable curve of type (g_X, n_X) over an algebraically closed field l of characteristic $p > 0$ and $\pi_1^{\mathrm{t}}(U_Y)$ the tame fundamental group of (Y, D_Y).

We shall say that

$$(Q_N, Q_H)$$

is a *quasi-anabelian pair associated to* $\pi_1^{\mathrm{t}}(U_X)$ if, for any surjection $\phi: \pi_1^{\mathrm{t}}(U_Y) \twoheadrightarrow Q_N$, the following conditions are satisfied:

- Let $\psi: \pi_1^{\mathrm{t}}(U_Y) \overset{\phi}{\twoheadrightarrow} Q_N \twoheadrightarrow Q_H$ be the composition of surjections, where $Q_N \twoheadrightarrow Q_H$ is the natural surjection induced by $N \subseteq H$. Then for any $I_{\widehat{y}} \in \mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_Y))$, there exists a marked point $x_H \in D_{X_H}$ such that $\psi(I_{\widehat{y}}) = I_{x_H} \in \mathrm{Edg}^{\mathrm{op}}(Q_H)$.

- For any $I_{x'_H} \in \mathrm{Edg}^{\mathrm{op}}(Q_H)$, there exists an element $I_{\widehat{y}'} \in \mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_Y))$ such that $\psi(I_{\widehat{y}'}) = I_{x'_H}$.

Namely, (Q_N, Q_H) is a quasi-anabelian pair associated to $\pi_1^{\mathrm{t}}(U_X)$ if, for any surjection $\phi : \pi_1^{\mathrm{t}}(U_Y) \twoheadrightarrow Q_N$, the composition of surjections $\psi : \pi_1^{\mathrm{t}}(U_Y) \overset{\phi}{\twoheadrightarrow} Q_N \twoheadrightarrow Q_H$ induces a surjection

$$\psi^{\mathrm{op}} : \mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_Y)) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(Q_H),\ I_{\widehat{y}} \mapsto \psi(I_{\widehat{y}}).$$

Remark 4.1.1. Let $N, H \subseteq \pi_1^{\mathrm{t}}(U_X)$ be open characteristic subgroups. The pair (Q_N, Q_H) is not quasi-anabelian in general. For instance, we put $N = H \overset{\mathrm{def}}{=} \ker(\pi_1^{\mathrm{t}}(U_X) \to \pi_1^{\mathrm{t}}(U_X)^{\mathrm{ab}} \otimes \mathbb{Z}/n\mathbb{Z})$ for a positive natural number n prime to p. Then (Q_N, Q_H) is *not* a quasi-anabelian pair if all elements of $\mathrm{Edg}^{\mathrm{op}}(Q_H)$ are non-trivial.

4.2. **Explicit constructions of quasi-anabelian pairs.** In this subsection, we give an explicit construction of a quasi-anabelian pair associated the tame fundamental group of an arbitrary smooth pointed stable curve.

4.2.1. *Notation and Settings.* Let $j \in \{1, 2\}$, and let (X_j, D_{X_j}) be an arbitrary smooth pointed stable curve of type (g_X, n_X) over an algebraically closed field k_j of characteristic $p > 0$ and $\pi_1^{\mathrm{t}}(U_{X_j})$ the tame fundamental group of (X_j, D_{X_j}). Moreover, *suppose* $g_X \geq 2$ *and* $n_X > 0$.

We fix the natural numbers

$$\ell,\ d,\ \mathcal{I}_a \overset{\mathrm{def}}{=} \{b_1, \dots, b_a\} \subseteq \mathbb{N},\ c(\mathcal{I}_a)$$

introduced in 3.1.1, and put

$$D_{\mathcal{I}_i, j} \overset{\mathrm{def}}{=} D_{\mathcal{I}_i}(\pi_1^{\mathrm{t}}(U_{X_j})),\ G_j^{\mathcal{I}_i} \overset{\mathrm{def}}{=} \pi_1^{\mathrm{t}}(U_{X_j})/D_{\mathcal{I}_i}(\pi_1^{\mathrm{t}}(U_{X_j})),\ i \in \{1, \dots, a\},$$

$$D_{b,j}^{(s)} \overset{\mathrm{def}}{=} D_b^{(s)}(\pi_1^{\mathrm{t}}(U_{X_j})),\ G_j^{s,b} \overset{\mathrm{def}}{=} \pi_1^{\mathrm{t}}(U_{X_j})/D_b^{(s)}(\pi_1^{\mathrm{t}}(U_{X_j})),\ s, b \in \mathbb{N}.$$

Let $\phi : \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow G_2^{a+4, c(\mathcal{I}_a)}$ be an arbitrary surjection and $\psi : \pi_1^{\mathrm{t}}(U_{X_1}) \overset{\phi}{\twoheadrightarrow} G_2^{a+4, c(\mathcal{I}_a)} \twoheadrightarrow G_2^{\mathcal{I}_a}$ the composition of surjections, where $G_2^{a+4, c(\mathcal{I}_a)} \twoheadrightarrow G_2^{\mathcal{I}_a}$ is the natural surjection induced by $D_{c(\mathcal{I}_a), 2}^{(a+4)} \subseteq D_{\mathcal{I}_a, 2}$. Note that ϕ and ψ fit into the following commutative diagram

$$\begin{array}{ccc}
\pi_1^{\mathrm{t}}(U_{X_1}) & = & \pi_1^{\mathrm{t}}(U_{X_1}) \\
\downarrow & & \downarrow \phi \\
G_1^{a+4, c(\mathcal{I}_a)} & \xrightarrow{\overline{\phi}} & G_2^{a+4, c(\mathcal{I}_a)} \\
\downarrow & & \downarrow \\
G_1^{\mathcal{I}_a} & \xrightarrow{\overline{\psi}} & G_2^{\mathcal{I}_a},
\end{array}$$

where all vertical arrows are surjections, and $\overline{\phi}, \overline{\psi}$ are surjections induced by ϕ, ψ, respectively.

Let $D_{e(\mathcal{I}_a), 2}^{(a+1)} \subseteq H_2 \subseteq \pi_1^{\mathrm{t}}(U_{X_2})$ (see 3.1.1 for $e(\mathcal{I}_a)$) be an open subgroup and $\overline{H}_2 \overset{\mathrm{def}}{=} H_2/D_{c(\mathcal{I}_a), 2}^{(a+4)} \subseteq G_2^{a+4, c(\mathcal{I}_a)}$ the finite quotient of H_2. We put $\overline{H}_1 \overset{\mathrm{def}}{=} \overline{\phi}^{-1}(\overline{H}_2) \subseteq G_1^{a+4, c(\mathcal{I}_a)}$, $(D_{e(\mathcal{I}_a), 1}^{(a+1)} \subseteq) H_1 \subseteq \pi_1^{\mathrm{t}}(U_{X_1})$ the inverse image of $\overline{H}_1$ via the natural surjection $\pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow G_1^{a+4, c(\mathcal{I}_a)}$, and

$$\overline{\phi}_{H_1} \overset{\mathrm{def}}{=} \phi|_{\overline{H}_1} : \overline{H}_1 \twoheadrightarrow \overline{H}_2$$

the surjection induced by $\overline{\phi}$. Write $(X_{H_j}, D_{X_{H_j}})$ for the smooth pointed stable curve of type (g_{H_j}, n_{H_j}) over k_j corresponding to $H_j \subseteq \pi_1^{\mathrm{t}}(U_{X_j})$.

If $(g_{H_1}, n_{H_1}) = (g_{H_2}, n_{H_2})$ (note that this condition holds if $H_j = \pi_1^{\mathrm{t}}(U_{X_j})$), we see that $\overline{\phi}_{H_1}$ induces an isomorphism of finite groups

$$\overline{\phi}^{p'}_{H_1} : \overline{H}_1^{p'} \xrightarrow{\sim} \overline{H}_2^{p'},$$

where $(-)^{p'}$ denotes the maximal prime-to-p quotient of $(-)$. Furthermore, let $\ell' \in \{\ell, d\}$ be a prime number. Note that we have $\ell' | \#(\overline{H}_j^{p'})$. Then the isomorphism $(\overline{\phi}^{p'}_{H_1})^{-1}$ induces a bijection

$$\overline{\varphi}^{\ell'}_{H_1} : \mathrm{Hom}(\overline{H}_1, \mathbb{Z}/\ell'\mathbb{Z}) = \mathrm{Hom}(\overline{H}_1^{p'}, \mathbb{Z}/\ell'\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}(\overline{H}_2^{p'}, \mathbb{Z}/\ell'\mathbb{Z}) = \mathrm{Hom}(\overline{H}_2, \mathbb{Z}/\ell'\mathbb{Z}).$$

4.2.2. We have the following lemma:

Lemma 4.2. *We maintain the notation and the settings introduced in 4.2.1. Suppose $(g_{H_1}, n_{H_1}) = (g_{H_2}, n_{H_2})$. Then $\overline{\varphi}^{\ell'}_{H_1}$ induces a bijection (see 3.2.2 for $\mathrm{Hom}^{\text{ét}}(\overline{H}_j, \mathbb{Z}/\ell'\mathbb{Z})$)*

$$\overline{\varphi}^{\ell', \text{ét}}_{H_1} : \mathrm{Hom}^{\text{ét}}(\overline{H}_1, \mathbb{Z}/\ell'\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}^{\text{ét}}(\overline{H}_2, \mathbb{Z}/\ell'\mathbb{Z}).$$

Proof. Let $\overline{\omega}_2 \in \mathrm{Hom}^{\text{ét}}(\overline{H}_2, \mathbb{Z}/\ell'\mathbb{Z})$ be an arbitrary element, $\overline{\omega}_1 \overset{\text{def}}{=} (\overline{\varphi}^{\ell'}_{H_1})^{-1}(\overline{\omega}_2)$, $\omega_j \in \mathrm{Hom}(H_j, \mathbb{Z}/\ell'\mathbb{Z})$ the element corresponding to $\overline{\omega}_j$ (3.2.2), and $H_{\omega_j} \overset{\text{def}}{=} \ker(\omega_j) \subseteq H_j$. Write $f_{\omega_j} : (X_{H_{\omega_j}}, D_{X_{H_{\omega_j}}}) \to (X_{H_j}, D_{X_{H_j}})$ for the Galois tame covering over k_j with Galois group $\mathbb{Z}/\ell'\mathbb{Z}$ corresponding to $H_{\omega_j} \subseteq H_j$ and $(g_{H_{\omega_j}}, n_{H_{\omega_j}})$ for the type of $(X_{H_{\omega_j}}, D_{X_{H_{\omega_j}}})$. The Riemann-Hurwitz formula implies (see 2.2.2 for $\#(\mathrm{Ram}_{f_{\omega_j}})$)

$$g_{H_{\omega_j}} = \ell'(g_{H_j} - 1) + \frac{1}{2}\#(\mathrm{Ram}_{f_{\omega_j}})(\ell' - 1) + 1, \ n_{H_{\omega_j}} = \ell'\big(n_{H_j} - \#(\mathrm{Ram}_{f_{\omega_j}})\big) + \#(\mathrm{Ram}_{f_{\omega_j}}).$$

In particular, we have $g_{H_{\omega_2}} = \ell'(g_{X_{H_2}} - 1) + 1$ and $n_{H_{\omega_2}} = \ell' n_{H_2}$ since $\overline{\omega}_2 \in \mathrm{Hom}^{\text{ét}}(\overline{H}_2, \mathbb{Z}/\ell'\mathbb{Z})$.

Let $\overline{H}_{\omega_j} \overset{\text{def}}{=} \ker(\overline{\omega}_j) = H_{\omega_j}/D^{(a+4)}_{c(\mathcal{I}_a), j}$. We see $G^{2, c(\mathcal{I}_a)}_{H_{\omega_j}} \overset{\text{def}}{=} H_{\omega_j}/D^{(2)}_{c(\mathcal{I}_a)}(H_{\omega_j}) = \overline{H}_{\omega_j}/D^{(2)}_{c(\mathcal{I}_a)}(\overline{H}_{\omega_j}) \subseteq \overline{H}_{\omega_j}$. By [Y5, Theorem 5.4] and its proof (in particular, line 4, page 82 of [Y5]), the surjection $\overline{\phi}_{H_1}|_{\overline{H}_{\omega_1}} : \overline{H}_{\omega_1} \twoheadrightarrow \overline{H}_{\omega_2}$ induces

$$\gamma^{\max}_{G^{2, c(\mathcal{I}_a)}_{H_{\omega_1}}} = g_{H_{\omega_1}} + n_{H_{\omega_1}} - 2 \geq \gamma^{\max}_{G^{2, c(\mathcal{I}_a)}_{H_{\omega_2}}} = g_{H_{\omega_2}} + n_{H_{\omega_2}} - 2.$$

We obtain $\#(\mathrm{Ram}_{f_{\omega_1}}) = 0$. This means that f_{ω_1} is étale. Namely, we have $\overline{\omega}_1 \in \mathrm{Hom}^{\text{ét}}(\overline{H}_1, \mathbb{Z}/\ell'\mathbb{Z})$. Thus, $\overline{\varphi}^{\ell'}_{H_1}$ induces an injection

$$(\overline{\varphi}^{\ell'}_{H_1})^{-1}|_{\mathrm{Hom}^{\text{ét}}(\overline{H}_2, \mathbb{Z}/\ell'\mathbb{Z})} : \mathrm{Hom}^{\text{ét}}(\overline{H}_2, \mathbb{Z}/\ell'\mathbb{Z}) \hookrightarrow \mathrm{Hom}^{\text{ét}}(\overline{H}_1, \mathbb{Z}/\ell'\mathbb{Z}).$$

Moreover, $\#(\mathrm{Hom}^{\text{ét}}(\overline{H}_1, \mathbb{Z}/\ell'\mathbb{Z})) = \#(\mathrm{Hom}^{\text{ét}}(\overline{H}_2, \mathbb{Z}/\ell'\mathbb{Z}))$ implies that $(\overline{\varphi}^{\ell'}_{H_1})^{-1}|_{\mathrm{Hom}^{\text{ét}}(\overline{H}_2, \mathbb{Z}/\ell'\mathbb{Z})}$ is a bijection. Then we complete the proof of the lemma if we put $\overline{\varphi}^{\ell', \text{ét}}_{H_1} \overset{\text{def}}{=} ((\overline{\varphi}^{\ell'}_{H_1})^{-1}|_{\mathrm{Hom}^{\text{ét}}(\overline{H}_2, \mathbb{Z}/\ell'\mathbb{Z})})^{-1}$. □

4.2.3. We maintain the notation and the settings introduced in 4.2.1. Moreover, in 4.2.3, we suppose

- $(g_{H_1}, n_{H_1}) = (g_{H_2}, n_{H_2})$.

Let $\mathfrak{T}_{\overline{H}_2} \overset{\text{def}}{=} (\ell, d, \overline{\beta}_2)$ be an mp-triple associated to $\overline{H}_2$ (i.e., $\overline{\beta}_2 \in \mathrm{Hom}^{\text{ét}}(\overline{H}_2, \mathbb{Z}/d\mathbb{Z})$, see 3.2.2). By Lemma 4.2, we see that the triple $\mathfrak{T}_{\overline{H}_1} \overset{\text{def}}{=} (\ell, d, \overline{\beta}_1 \overset{\text{def}}{=} (\overline{\varphi}^{d}_{H_1})^{-1}(\overline{\beta}_2))$ is an mp-triple associated to $\overline{H}_1$. Namely, we have $\overline{\beta}_1 \in \mathrm{Hom}^{\text{ét}}(\overline{H}_1, \mathbb{Z}/d\mathbb{Z})$. Let $\beta_j \in \mathrm{Hom}(H_j, \mathbb{Z}/d\mathbb{Z})$ be the element corresponding to $\overline{\beta}_j$, $(D^{(a+1)}_{e(\mathcal{I}_a), j} \subseteq) H_{\beta_j} \overset{\text{def}}{=} \ker(\beta_j)$, $\overline{H}_{\beta_j} \overset{\text{def}}{=}$

$H_{\beta_j}/D^{(a+4)}_{c(\mathcal{I}_a),j}$ the finite quotient of H_{β_j}. Then we obtain an exact sequence (as $\mathbb{F}_\ell[\mu_d]$-modules, see 3.2.3)

$$0 \to M^{\text{ét}}_{\overline{H}_{\beta_j}} \to M_{\overline{H}_{\beta_j}} \to M^{\text{ra}}_{\overline{H}_{\beta_j}} \to 0.$$

By replacing H_j, $\overline{H}_j$ by H_{β_j}, $\overline{H}_{\beta_j}$, respectively, and by applying Lemma 4.2, the isomorphism $\overline{\varphi}^{\ell}_{H_{\beta_1}} : M_{\overline{H}_{\beta_1}} \overset{\sim}{\to} M_{\overline{H}_{\beta_2}}$ induces the following commutative diagram of $\mathbb{F}_\ell[\mu_d]$-modules:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M^{\text{ét}}_{\overline{H}_{\beta_1}} & \longrightarrow & M_{\overline{H}_{\beta_1}} & \longrightarrow & M^{\text{ra}}_{\overline{H}_{\beta_1}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{\varphi}^{\ell,\text{ét}}_{H_{\beta_1}}} & & \downarrow{\scriptstyle \overline{\varphi}^{\ell}_{H_{\beta_1}}} & & \downarrow & & \\ 0 & \longrightarrow & M^{\text{ét}}_{\overline{H}_{\beta_2}} & \longrightarrow & M_{\overline{H}_{\beta_2}} & \longrightarrow & M^{\text{ra}}_{\overline{H}_{\beta_2}} & \longrightarrow & 0, \end{array}$$

where all vertical arrows are isomorphisms. Then the bijection $\overline{\varphi}^{\ell}_{H_{\beta_1}}$ induces a bijection

$$\overline{\phi}^{\text{mp},*}_{H_1} : E^{*}_{\mathfrak{I}_{\overline{H}_1}} \overset{\sim}{\to} E^{*}_{\mathfrak{I}_{\overline{H}_2}}.$$

Moreover, we have the following lemma:

Lemma 4.3. *We maintain the notation and the settings introduced above (in particular, we assume $(g_{H_1}, n_{H_1}) = (g_{H_2}, n_{H_2})$). Then $\overline{\phi}^{\text{mp},*}_{H_1}$ induces a bijection $\overline{\phi}^{\text{mp},\star}_{H_1} : E^{\star}_{\mathfrak{I}_{\overline{H}_1}} \overset{\sim}{\to} E^{\star}_{\mathfrak{I}_{\overline{H}_2}}$. Moreover, $\overline{\phi}_{H_1} : \overline{H}_1 \twoheadrightarrow \overline{H}_2$ induces a bijection*

$$\overline{\phi}^{\text{mp}}_{H_1} : E_{\mathfrak{I}_{\overline{H}_1}} \overset{\sim}{\to} E_{\mathfrak{I}_{\overline{H}_2}}$$

and a bijection (see Proposition 3.5 (i) for $\vartheta_{\mathfrak{I}_{\overline{H}_j}}$)

$$\overline{\phi}^{\text{gp-mp}}_{H_1} : D_{X_{H_1}} \overset{\vartheta^{-1}_{\mathfrak{I}_{\overline{H}_1}}}{\overset{\sim}{\to}} E_{\mathfrak{I}_{\overline{H}_1}} \overset{\overline{\phi}^{\text{mp}}_{H_1}}{\overset{\sim}{\to}} E_{\mathfrak{I}_{\overline{H}_2}} \overset{\vartheta_{\mathfrak{I}_{\overline{H}_2}}}{\overset{\sim}{\to}} D_{X_{H_2}}$$

of sets of marked points, where "gp" means "group-theoretically".

Proof. Let $\overline{\alpha}_2 \in E^{\star}_{\mathfrak{I}_{\overline{H}_2}}$ be an arbitrary element, $\overline{\alpha}_1 \overset{\text{def}}{=} (\overline{\phi}^{\text{mp},*}_{H_1})^{-1}(\overline{\alpha}_2) \in E^{*}_{\mathfrak{I}_{\overline{H}_1}}$, and $\alpha_j \in \text{Hom}(H_{\beta_j}, \mathbb{Z}/\ell\mathbb{Z})$ the element corresponding to $\overline{\alpha}_j$. We put $H_{\alpha_j} \overset{\text{def}}{=} \ker(\alpha_j) \subseteq H_{\beta_j}$, $g_{\alpha_j} : (X_{H_{\alpha_j}}, D_{X_{H_{\alpha_j}}}) \to (X_{H_{\beta_j}}, D_{X_{H_{\beta_j}}})$ the Galois tame covering over k_j with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ corresponding to $H_{\alpha_j} \subseteq H_{\beta_j}$, $(g_{H_{\alpha_j}}, n_{H_{\alpha_j}})$ the type of $(X_{H_{\alpha_j}}, D_{X_{H_{\alpha_j}}})$, and $(g_{H_{\beta_j}}, n_{H_{\beta_j}})$ the type of $(X_{H_{\beta_j}}, D_{X_{H_{\beta_j}}})$. Note that since $\overline{\beta}_j \in \text{Hom}^{\text{ét}}(\overline{H}_j, \mathbb{Z}/d\mathbb{Z})$ and $(g_{H_1}, n_{H_1}) = (g_{H_2}, n_{H_2})$, we see $(g_{H_{\beta_1}}, n_{H_{\beta_1}}) = (g_{H_{\beta_2}}, n_{H_{\beta_2}})$. Moreover, since $\overline{\alpha}_2 \in E^{\star}_{\mathfrak{I}_{\overline{H}_2}}$, we obtain $\#(\text{Ram}_{g_{\alpha_2}}) = d$ (2.2.3). The Riemann-Hurwitz formula implies

$$g_{H_{\alpha_j}} = \ell(g_{H_{\beta_j}} - 1) + \frac{1}{2}\#(\text{Ram}_{g_{\alpha_j}})(\ell - 1) + 1, \; n_{H_{\alpha_j}} = \ell\big(n_{H_{\beta_j}} - \#(\text{Ram}_{g_{\alpha_j}})\big) + \#(\text{Ram}_{g_{\alpha_j}}).$$

In particular, we have $g_{H_{\alpha_2}} = \ell(g_{H_{\beta_2}} - 1) + \frac{1}{2}d(\ell - 1) + 1$ and $n_{H_{\alpha_2}} = \ell(n_{H_{\beta_2}} - d) + d$.

Let $\overline{H}_{\alpha_j} \overset{\text{def}}{=} \ker(\overline{\alpha}_j) = H_{\alpha_j}/D^{(a+4)}_{c(\mathcal{I}_a),j}$. We see $G^{2,c(\mathcal{I}_a)}_{H_{\alpha_j}} \overset{\text{def}}{=} H_{\alpha_j}/D^{(2)}_{c(\mathcal{I}_a)}(H_{\alpha_j}) = \overline{H}_{\alpha_j}/D^{(2)}_{c(\mathcal{I}_a)}(\overline{H}_{\alpha_j}) \subseteq \overline{H}_{\alpha_j}$. By [Y5, Theorem 5.4] and its proof (in particular, line 4, page 82 of [Y5]), the surjection $\overline{\phi}_{H_1}|_{\overline{H}_{\alpha_1}} : \overline{H}_{\alpha_1} \twoheadrightarrow \overline{H}_{\alpha_2}$ induces

$$\gamma^{\max}_{G^{2,c(\mathcal{I}_a)}_{H_{\alpha_1}}} = g_{H_{\alpha_1}} + n_{H_{\alpha_1}} - 2 \geq \gamma^{\max}_{G^{2,c(\mathcal{I}_a)}_{H_{\alpha_2}}} = g_{H_{\alpha_2}} + n_{H_{\alpha_2}} - 2.$$

Then we obtain $\#(\text{Ram}_{g_{\alpha_1}}) \leq \#(\text{Ram}_{g_{\alpha_2}})$. This implies

$$\#(\text{Ram}_{g_{\alpha_1}}) \in \{0, d\}.$$

Suppose $\#(\mathrm{Ram}_{g_{\alpha_1}}) = 0$. This means $\overline{\alpha}_1 \in M^{\text{ét}}_{\overline{H}_{\beta_1}}$. On the other hand, Lemma 4.2 implies $\overline{\alpha}_2 \in M^{\text{ét}}_{\overline{H}_{\beta_2}}$. This contradicts the assumption $\overline{\alpha}_2 \in E^{\star}_{\mathfrak{T}_{\overline{H}_2}}$. Thus, we obtain $\#(\mathrm{Ram}_{g_{\alpha_1}}) = d$. Namely, we have $\overline{\alpha}_1 \in E^{\star}_{\mathfrak{T}_{\overline{H}_1}}$. Furthermore, since $\#(E^{\star}_{\mathfrak{T}_{\overline{H}_1}}) = \#(E^{\star}_{\mathfrak{T}_{\overline{H}_2}})$, $\overline{\phi}^{\mathrm{mp},*}_{H_1}$ induces a bijection

$$\overline{\phi}^{\mathrm{mp},\star}_{H_1} \overset{\mathrm{def}}{=} \overline{\phi}^{\mathrm{mp},*}_{H_1}|_{E^{\star}_{\mathfrak{T}_{\overline{H}_1}}} : E^{\star}_{\mathfrak{T}_{\overline{H}_1}} \overset{\sim}{\to} E^{\star}_{\mathfrak{T}_{\overline{H}_2}}.$$

Moreover, we see immediately that $\overline{\alpha}'_1 \sim \overline{\alpha}''_1$ if and only if $\overline{\phi}^{\mathrm{mp},\star}_{H_1}(\overline{\alpha}'_1) \sim \overline{\phi}^{\mathrm{mp},\star}_{H_1}(\overline{\alpha}''_1)$ for all $\overline{\alpha}'_1, \overline{\alpha}''_1 \in E^{\star}_{\mathfrak{T}_{\overline{H}_1}}$, where "$\sim$" denotes the equivalence relation defined in Proposition 2.1 (i). Then we obtain that $\overline{\phi}_{H_1}$ induces a bijection

$$\overline{\phi}^{\mathrm{mp}}_{H_1} : E_{\mathfrak{T}_{\overline{H}_1}} \overset{\sim}{\to} E_{\mathfrak{T}_{\overline{H}_2}}.$$

Thus, the lemma follows from Proposition 3.5 (i). This completes the proof of the lemma. □

4.2.4. We maintain the notation and the settings introduced in 4.2.3. Moreover, in 4.2.4, we suppose

- $H_2 \in \mathscr{T}(D_{\mathcal{I}_a,2})$ (see 3.2.1 for $\mathscr{T}(D_{\mathcal{I}_a,2})$).

Then we see immediately $H_1 \in \mathscr{T}(D_{\mathcal{I}_a,2})$. Let $N_2 \in \mathscr{T}(D_{\mathcal{I}_a,2})$ be an element such that $N_2 \subseteq H_2$, and $\overline{N}_2 \in \mathscr{T}(G_2^{\mathcal{I}_a})$ (see 3.2.1 for $\mathscr{T}(G_2^{\mathcal{I}_a})$) the finite quotient of N_2. We put $\overline{N}_1 \overset{\mathrm{def}}{=} (\overline{\phi}_{H_1})^{-1}(\overline{N}_2) \subseteq \overline{H}_1$, $(D^{(a+1)}_{e(\mathcal{I}_a),1} \subseteq) N_1 \subseteq H_1$ the inverse image of $\overline{N}_1$ via the natural surjection $H_1 \twoheadrightarrow \overline{H}_1$, and

$$\overline{\phi}_{N_1} \overset{\mathrm{def}}{=} \overline{\phi}_{H_1}|_{\overline{N}_1} : \overline{N}_1 \twoheadrightarrow \overline{N}_2$$

the surjection induced by $\overline{\phi}_{H_1}$. Note that we have $N_1 \in \mathscr{T}(D_{\mathcal{I}_a,1})$ and $\overline{N}_1 \in \mathscr{T}(G_1^{\mathcal{I}_a})$. Write $f_{N_j,H_j} : (X_{N_j}, D_{X_{N_j}}) \to (X_{H_j}, D_{X_{H_j}})$ for the tame covering of smooth pointed stable curves over k_j corresponding to $N_j \hookrightarrow H_j$, (g_{N_j}, n_{N_j}) for the type of $(X_{N_j}, D_{X_{N_j}})$, and $f^{\mathrm{mp}}_{N_j,H_j} : D_{X_{N_j}} \twoheadrightarrow D_{X_{H_j}}$ for the surjection of sets of marked points induced by f_{N_j,H_j}.

By similar arguments to the arguments given in the fourth paragraph of 3.2.3 and Proposition 3.5 (ii), we obtain the following data:

- The finite groups $\overline{N}_j$, $\overline{H}_j$, and the mp-triple $\mathfrak{T}_{\overline{H}_j} \overset{\mathrm{def}}{=} (\ell, d, \overline{\beta}_j)$ associated to $\overline{H}_j$ induce group-theoretically an mp-triple
$$\mathfrak{T}_{\overline{N}_j} \overset{\mathrm{def}}{=} (\ell, d, \overline{\beta}_{N_j})$$
associated to $\overline{N}_j$.
- The sets $E^{\star}_{\mathfrak{T}_{\overline{N}_j}}$, $E_{\mathfrak{T}_{\overline{N}_j}}$ can be reconstructed group-theoretically from $\overline{N}_j$ and $\mathfrak{T}_{\overline{N}_j}$.
- The natural injection $\overline{N}_j \hookrightarrow \overline{H}_j$ induces a surjection
$$\gamma_{\mathfrak{T}_{\overline{H}_j}, \overline{N}_j} : E_{\mathfrak{T}_{\overline{N}_j}} \twoheadrightarrow E_{\mathfrak{T}_{\overline{H}_j}}$$
which fits into the following commutative diagram:
$$\begin{array}{ccc} E_{\mathfrak{T}_{\overline{N}_j}} & \xrightarrow{\vartheta_{\mathfrak{T}_{\overline{N}_j}}} & D_{X_{N_j}} \\ {\scriptstyle \gamma_{\mathfrak{T}_{\overline{H}_j}, \overline{N}_j}} \downarrow & & {\scriptstyle f^{\mathrm{mp}}_{N_j,H_j}} \downarrow \\ E_{\mathfrak{T}_{\overline{H}_j}} & \xrightarrow{\vartheta_{\mathfrak{T}_{\overline{H}_j}}} & D_{X_{H_j}}. \end{array}$$

Then we have the following lemma.

Lemma 4.4. *We maintain the notation and the settings introduced above. Suppose* $(g_{N_1}, n_{N_1}) = (g_{N_2}, n_{N_2})$ *and* $(g_{H_1}, n_{H_1}) = (g_{H_2}, n_{H_2})$. *Then the commutative diagram of finite groups*

$$\begin{array}{ccc} \overline{N}_1 & \xrightarrow{\overline{\phi}_{N_1}} & \overline{N}_2 \\ \big\downarrow & & \big\downarrow \\ \overline{H}_1 & \xrightarrow{\overline{\phi}_{H_1}} & \overline{H}_2 \end{array}$$

induces group-theoretically a commutative diagram

$$\begin{array}{ccc} D_{X_{N_1}} & \xrightarrow{\overline{\phi}^{\text{gp-mp}}_{N_1}} & D_{X_{N_2}} \\ f^{\text{mp}}_{N_1,H_1}\big\downarrow & & f^{\text{mp}}_{N_2,H_2}\big\downarrow \\ D_{X_{H_1}} & \xrightarrow{\overline{\phi}^{\text{gp-mp}}_{H_1}} & D_{X_{H_2}}. \end{array}$$

Namely, the diagram

$$\begin{array}{ccccccc} D_{X_{N_1}} & \xrightarrow{\vartheta^{-1}_{\mathfrak{T}_{\overline{N}_1}}} & E_{\mathfrak{T}_{\overline{N}_1}} & \xrightarrow{\overline{\phi}^{\text{mp}}_{N_1}} & E_{\mathfrak{T}_{\overline{N}_2}} & \xrightarrow{\vartheta_{\mathfrak{T}_{\overline{N}_2}}} & D_{X_{N_2}} \\ f^{\text{mp}}_{N_1,H_1}\big\downarrow & & \gamma_{\mathfrak{T}_{\overline{H}_1},\overline{N}_1}\big\downarrow & & \gamma_{\mathfrak{T}_{\overline{H}_2},\overline{N}_2}\big\downarrow & & f^{\text{mp}}_{N_2,H_2}\big\downarrow \\ D_{X_{H_1}} & \xrightarrow{\vartheta^{-1}_{\mathfrak{T}_{\overline{H}_1}}} & E_{\mathfrak{T}_{\overline{H}_1}} & \xrightarrow{\overline{\phi}^{\text{mp}}_{H_1}} & E_{\mathfrak{T}_{\overline{H}_2}} & \xrightarrow{\vartheta_{\mathfrak{T}_{\overline{H}_2}}} & D_{X_{H_2}} \end{array}$$

is commutative. Moreover, suppose that $\overline{N}_j \subseteq \overline{H}_j$ *is a normal subgroup. Note that we have* $\overline{H}_1/\overline{N}_1 \overset{\sim}{\rightarrow} \overline{H}_2/\overline{N}_2$ *induced by* $\overline{\phi}_{H_1}$. *We identify* $\overline{H}_1/\overline{N}_1$ *with* $\overline{H}_2/\overline{N}_2$ *via this isomorphism. Then* $\overline{\phi}^{\text{gp-mp}}_{N_1}$ *is* $\overline{H}_j/\overline{N}_j$*-equivariant.*

Proof. Firstly, since $(g_{H_1}, n_{H_1}) = (g_{H_2}, n_{H_2})$ and $(g_{N_1}, n_{N_1}) = (g_{N_2}, n_{N_2})$, Lemma 4.3 implies that $\overline{\phi}_{H_1}$ and $\overline{\phi}_{N_1}$ induce bijections $\overline{\phi}^{\text{mp}}_{H_1} : E_{\mathfrak{T}_{\overline{H}_1}} \overset{\sim}{\rightarrow} E_{\mathfrak{T}_{\overline{H}_2}}$ and $\overline{\phi}^{\text{mp}}_{N_1} : E_{\mathfrak{T}_{\overline{N}_1}} \overset{\sim}{\rightarrow} E_{\mathfrak{T}_{\overline{N}_2}}$, respectively.

Next, to verify the commutativity of the second diagram appeared in the statement of the lemma, we only need to prove the commutativity of the following diagram

$$\begin{array}{ccc} D_{X_{N_2}} & \xrightarrow{(\overline{\phi}^{\text{gp-mp}}_{N_1})^{-1}} & D_{X_{N_1}} \\ f^{\text{mp}}_{N_2,H_2}\big\downarrow & & f^{\text{mp}}_{N_1,H_1}\big\downarrow \\ D_{X_{H_2}} & \xrightarrow{(\overline{\phi}^{\text{gp-mp}}_{H_1})^{-1}} & D_{X_{H_1}}. \end{array}$$

Let $x_{N_2} \in D_{X_{N_2}}$, $x_{N_1} \overset{\text{def}}{=} (\overline{\phi}^{\text{gp-mp}}_{N_1})^{-1}(x_{N_2}) \in D_{X_{N_1}}$, $x_{H_2} \overset{\text{def}}{=} f^{\text{mp}}_{N_2,H_2}(x_{N_2}) \in D_{X_{N_2}}$, $x_{H_1} \overset{\text{def}}{=} (f^{\text{mp}}_{N_1,H_1} \circ (\overline{\phi}^{\text{gp-mp}}_{N_1})^{-1})(x_{N_2}) \in D_{X_{H_1}}$, and $x'_{H_1} \overset{\text{def}}{=} (\overline{\phi}^{\text{gp-mp}}_{H_1})^{-1}(x_{H_2}) \in D_{X_{H_1}}$. We will prove $x_{H_1} = x'_{H_1}$. Write S'_{N_1} and S_{N_2} for the sets $(f^{\text{mp}}_{N_1,H_1})^{-1}(x'_{H_1})$ and $(f^{\text{mp}}_{N_2,H_2})^{-1}(x_{H_2})$, respectively. Namely, we have

$$\begin{array}{ccc} x_{N_2} & \longrightarrow & x_{N_1} \\ & & \big\downarrow \\ & & x_{H_1}, \end{array}$$

$$\begin{array}{ccc} x_{N_2} & & \\ \downarrow & & \\ x_{H_2} & \longrightarrow & x'_{H_1}, \end{array}$$

and

$$\begin{array}{ccc} x_{N_2} \in S_{N_2} & & S'_{N_1} \\ \downarrow & & \downarrow \\ x_{H_2} & \longrightarrow & x'_{H_1}. \end{array}$$

To verify $x_{H_1} = x'_{H_1}$, it is sufficient to prove $x_{N_1} \in S'_{N_1}$.

Let $\overline{\alpha}_{H_2} \in E^{\star}_{\mathfrak{I}_{\overline{H}_2}, x_{H_2}}$, where $E^{\star}_{\mathfrak{I}_{\overline{H}_2}, x_{H_2}}$ is the subset of $E^{\star}_{\mathfrak{I}_{\overline{H}_2}}$ corresponding to the subset $E^{\star}_{\mathfrak{I}_{U_{X_{H_2}}}, x_{H_2}}$ (see 2.2.3 for $E^{\star}_{\mathfrak{I}_{U_{X_{H_2}}}, x_{H_2}}$) of $E^{\star}_{\mathfrak{I}_{U_{X_{H_2}}}}$ via the bijection $E^{\star}_{\mathfrak{I}_{\overline{H}_2}} \overset{\sim}{\to} E^{\star}_{\mathfrak{I}_{U_{X_{H_2}}}}$ obtained in Proposition 3.5 (i). Lemma 4.3 implies that $\overline{\alpha}_{H_2}$ induces an element $\overline{\alpha}_{H_1} \in E^{\star}_{\mathfrak{I}_{\overline{H}_1}, x'_{H_1}}$.

We put $\alpha_{H_j} : H_{\beta_j} \twoheadrightarrow \overline{H}_{\beta_j} \overset{\overline{\alpha}_{H_j}}{\twoheadrightarrow} \mathbb{Z}/\ell\mathbb{Z}$ and $(X_{\alpha_{H_j}}, D_{X_{\alpha_{H_j}}}) \to (X_{H_{\beta_j}}, D_{X_{H_{\beta_j}}})$ the Galois tame covering over k_j with Galois group $\mathbb{Z}/\ell\mathbb{Z}$ corresponding to α_{H_j}. We consider the Galois tame covering $(X_{\alpha_{H_2}}, D_{X_{\alpha_{H_2}}}) \times_{(X_{H_2}, D_{X_{H_2}})} (X_{N_2}, D_{X_{N_2}}) \to (X_{N_{\beta_{N_2}}}, D_{X_{N_{\beta_{N_2}}}})$ over k_2 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$, and denote by $\overline{\alpha}_{N_2}$ an element of $E^{*}_{\mathfrak{I}_{\overline{N}_2}}$ corresponding to this Galois tame covering. Then we have

$$\overline{\alpha}_{N_2} = \sum_{c_2 \in S_{N_2}} t_{c_2} \overline{\alpha}_{c_2} \in E^{*}_{\mathfrak{I}_{\overline{N}_2}},$$

where $t_{c_2} \in (\mathbb{Z}/\ell\mathbb{Z})^{\times}$ and $\overline{\alpha}_{c_2} \in E^{\star}_{\mathfrak{I}_{\overline{N}_2}, c_2}$. Note that we have $t_{x_{N_2}} \neq 0$. On the other hand, Lemma 4.3 implies that $\overline{\alpha}_{c_2}$ induces an element $\overline{\alpha}_{(\phi^{\text{gp-mp}}_{N_1})^{-1}(c_2)} \in E^{\star}_{\mathfrak{I}_{\overline{N}_1}, (\phi^{\text{gp-mp}}_{N_1})^{-1}(c_2)}$. Then $\overline{\alpha}_{N_2}$ induces an element

$$\overline{\alpha}_{N_1} \overset{\text{def}}{=} \sum_{c_2 \in S_{N_2} \setminus \{x_{N_2}\}} t_{c_2} \overline{\alpha}_{(\phi^{\text{gp-mp}}_{N_1})^{-1}(c_2)} + t_{x_{N_2}} \overline{\alpha}_{x_{N_1}} \in E^{*}_{\mathfrak{I}_{\overline{N}_1}}.$$

Note that since $\overline{\alpha}_{N_1}$ is an element corresponding to the Galois tame covering

$$(X_{\alpha_{H_1}}, D_{X_{\alpha_{H_1}}}) \times_{(X_{H_1}, D_{X_{H_1}})} (X_{N_1}, D_{X_{N_1}}) \to (X_{N_{\beta_{N_1}}}, D_{X_{N_{\beta_{N_1}}}})$$

over k_1 with Galois group $\mathbb{Z}/\ell\mathbb{Z}$, the composition of the Galois tame coverings $(X_{\alpha_{H_1}}, D_{X_{\alpha_{H_1}}}) \times_{(X_{H_1}, D_{X_{H_1}})} (X_{N_1}, D_{X_{N_1}}) \to (X_{N_{\beta_{N_1}}}, D_{X_{N_{\beta_{N_1}}}}) \to (X_{N_1}, D_{X_{N_1}})$ is tamely ramified over S'_{N_1}. This means that x_{N_1} is contained in S'_{N_1}.

On the other hand, the "moreover part" of the proposition follows from Proposition 3.5 (ii). This completes the proof of the lemma. □

4.2.5. We have the following proposition:

Proposition 4.5. *We maintain the notation and the settings introduced in 4.2.1 (in particular, we assume $g_X \geq 2$). Then the pair of finite groups*

$$(G_2^{a+4, c(\mathcal{I}_a)}, G_2^{\mathcal{I}_a})$$

is a quasi-anabelian pair (Definition 4.1) associated to $\pi_1^{\text{t}}(U_{X_2})$.

Proof. By the definition of quasi-anabelian pairs (Definition 4.1), to verify the proposition, it is sufficient to prove that the surjection (see 4.2.1 for ϕ) $\psi : \pi_1^{\text{t}}(U_{X_1}) \overset{\phi}{\twoheadrightarrow} G_2^{a+4, c(\mathcal{I}_a)} \twoheadrightarrow G_2^{\mathcal{I}_a}$ induces a surjection

$$\psi^{\text{op}} : \text{Edg}^{\text{op}}(\pi_1^{\text{t}}(U_{X_1})) \twoheadrightarrow \text{Edg}^{\text{op}}(G_2^{\mathcal{I}_a}).$$

We see immediately that the kernel $N_1 \overset{\text{def}}{=} \ker(\psi) \subseteq \pi_1^{\rm t}(U_{X_1})$ of ψ contains $D_{\mathcal{I}_a,1}$. We put $\overline{\psi}_{Q_{N_1}} : Q_{N_1} \overset{\text{def}}{=} \pi_1^{\rm t}(U_{X_1})/N_1 \overset{\sim}{\to} G_2^{\mathcal{I}_a}$ the isomorphism induced by ψ. Note that there is a surjection $\text{Edg}^{\text{op}}(\pi_1^{\rm t}(U_{X_1})) \twoheadrightarrow \text{Edg}^{\text{op}}(Q_{N_1})$ induced by the natural surjection $\pi_1^{\rm t}(U_{X_1}) \twoheadrightarrow Q_{N_1}$ (i.e., the map $I_{\widehat{y}_1} \mapsto I_{\widehat{y}_1}/(I_{\widehat{y}_1} \cap N_1) \in Q_{N_1}$). Then to verify the proposition, it is sufficient to prove that the isomorphism $\overline{\psi}_{Q_{N_1}}$ induces a bijection map

$$\overline{\psi}^{\text{op}}_{Q_{N_1}} : \text{Edg}^{\text{op}}(Q_{N_1}) \overset{\sim}{\to} \text{Edg}^{\text{op}}(G_2^{\mathcal{I}_a}).$$

For simplicity, we write D_1, D_2 for $N_1, D_{\mathcal{I}_a,2}$, respectively. Moreover, we denote by $f_{D_j} : (X_{D_j}, D_{X_{D_j}}) \to (X_j, D_{X_j})$ the Galois tame covering of smooth pointed stable curves over k_j corresponding to $D_j \subseteq \pi_1^{\rm t}(U_{X_j})$, $f_{D_j}^{\text{mp}} : D_{X_{D_j}} \twoheadrightarrow D_{X_j}$ the surjection of sets of marked points induced by f_{D_j}, and (g_{D_j}, n_{D_j}) the type of $(X_{D_j}, D_{X_{D_j}})$. We claim the following:

Claim: We have $(g_{D_1}, n_{D_1}) = (g_{D_2}, n_{D_2})$.
Let us prove the claim. Firstly, we have filtrations

$$\{e\} = D_{\mathcal{I}_a}(Q_{N_1}) \subseteq D_{\mathcal{I}_{a-1}}(Q_{N_1}) \subseteq \cdots \subseteq D_{\mathcal{I}_0}(Q_{N_1}) = Q_{N_1},$$

$$D_2 \overset{\text{def}}{=} D_{\mathcal{I}_a,2} \subseteq D_{\mathcal{I}_{a-1},2} \subseteq \cdots \subseteq D_{\mathcal{I}_0,2} \overset{\text{def}}{=} \pi_1^{\rm t}(U_{X_2}).$$

Write $D_{1,i} \subseteq \pi_1^{\rm t}(U_{X_1})$, $i \in \{0, \dots, a\}$, for the inverse image of $D_{\mathcal{I}_i}(Q_{N_1})$ of the natural surjection $\pi_1^{\rm t}(U_{X_1}) \twoheadrightarrow Q_{N_1}$ and $D_{2,i}$ for $D_{\mathcal{I}_i,2}$, $i \in \{0, \dots, a\}$. Note that we have $D_{1,a} = D_1 = N_1$, $D_{2,a} = D_2 = D_{\mathcal{I}_a,2}$, and $D_{j,0} = \pi_1^{\rm t}(U_{X_j})$. Moreover, we denote by $(g_{D_{j,i}}, n_{D_{j,i}})$ the type of smooth pointed stable curve over k_j corresponding to $D_{j,i} \subseteq \pi_1^{\rm t}(U_{X_j})$.

There is an isomorphism

$$\big(D_{\mathcal{I}_i}(Q_{N_1})/D_{\mathcal{I}_{i+1}}(Q_{N_1})\big)^{p'} \overset{\sim}{\to} \big(D_{2,i}/D_{2,i+1}\big)^{p'}, \ i \in \{0, \dots, a-1\},$$

induced by $\overline{\psi}_{Q_{N_1}}$. Moreover, denote by $m_i \overset{\text{def}}{=} \#\Big(\big(D_{\mathcal{I}_i}(Q_{N_1})/D_{\mathcal{I}_{i+1}}(Q_{N_1})\big)^{p'}\Big)$. We see

$$D_{1,i}^{\text{ab}} \otimes \mathbb{Z}/m_i\mathbb{Z} \overset{\sim}{\to} \big(D_{\mathcal{I}_i}(Q_{N_1})/D_{\mathcal{I}_{i+1}}(Q_{N_1})\big)^{p'} \overset{\sim}{\to} \big(D_{2,i}/D_{2,i+1}\big)^{p'} \overset{\sim}{\to} D_{2,i}^{\text{ab}} \otimes \mathbb{Z}/m_i\mathbb{Z}$$

for all $i \in \{0, \dots, a-1\}$. Then we obtain that $(g_{D_{1,i+1}}, n_{D_{1,i+1}}) = (g_{D_{2,i+1}}, n_{D_{2,i+1}})$ if $(g_{D_{1,i}}, n_{D_{1,i}}) = (g_{D_{2,i}}, n_{D_{2,i}})$. Thus, the claim follows immediately from $(g_{D_{1,0}}, n_{D_{1,0}}) = (g_{D_{2,0}}, n_{D_{2,0}}) = (g_X, n_X)$. This completes the proof of the claim.

On the other hand, we have the commutative diagram of finite groups (see 4.2.1 for $\overline{\phi}$)

$$\begin{array}{ccc}
\overline{D}_1 \overset{\text{def}}{=} D_1/D^{(a+4)}_{c(\mathcal{I}_a),1} & \xrightarrow{\overline{\phi}_{D_1} \overset{\text{def}}{=} \overline{\phi}|_{\overline{D}_1}} & \overline{D}_2 \overset{\text{def}}{=} D_2/D^{(a+4)}_{c(\mathcal{I}_a),2} \\
\downarrow & & \downarrow \\
G_1^{a+4,c(\mathcal{I}_a)} & \xrightarrow{\overline{\phi}} & G_2^{a+4,c(\mathcal{I}_a)},
\end{array}$$

where all vertical arrow are injections, and $G_1^{a+4,c(\mathcal{I}_a)}/\overline{D}_1 = Q_{N_1} \overset{\sim}{\to} \pi_1^{\rm t}(U_{X_2})/D_2 = G_2^{\mathcal{I}_a}$. Since $\overline{D}_j, G_j^{a+4,c(\mathcal{I}_a)} \in \mathscr{T}(G_j^{\mathcal{I}_a})$ (see 3.2.1 for $\mathscr{T}(G_j^{\mathcal{I}_a})$), by the claim, Lemma 4.4 implies that the above commutative diagram of finite groups induces group-theoretically a commutative diagram of sets of marked points

$$\begin{array}{ccc} D_{X_{D_1}} & \xrightarrow{\overline{\phi}^{\text{gp-mp}}_{D_1}} & D_{X_{D_2}} \\ {\scriptstyle f^{\text{mp}}_{D_1}}\downarrow & & {\scriptstyle f^{\text{mp}}_{D_2}}\downarrow \\ D_{X_1} & \xrightarrow{\overline{\phi}^{\text{gp-mp}}} & D_{X_2}, \end{array}$$

and that $\overline{\phi}^{\text{gp-mp}}_{D_1}$ is a $Q_{N_1}(\overset{\sim}{\to} G_2^{\mathcal{I}_a})$-equivariant. Then by the definitions of $\text{Edg}^{\text{op}}(Q_{N_1})$ and $\text{Edg}^{\text{op}}(G_2^{\mathcal{I}_a})$ (4.1.2), $\overline{\psi}_{Q_{N_1}} : Q_{N_1} \overset{\sim}{\to} G_2^{\mathcal{I}_a}$ induces a bijection map $\overline{\psi}^{\text{op}}_{Q_{N_1}} : \text{Edg}^{\text{op}}(Q_{N_1}) \overset{\sim}{\to} \text{Edg}^{\text{op}}(G_2^{\mathcal{I}_a})$. This completes the proof of the proposition. □

4.2.6. Now, we can state the main result of the present section.

Theorem 4.6. *Let $b_0 = p^u b_0' \in \mathbb{N}$ be a positive natural number such that $(p, b_0') = 1$ and $b_0' \neq 1$, and let*

$$(E_2, D_{E_2})$$

be a smooth pointed stable curve of type (g, n) over k_2 and $\pi_1^{\text{t}}(U_{E_2})$ the tame fundamental group of (E_2, D_{E_2}). Suppose $n > 0$. We shall write

$$(X_2, D_{X_2})$$

for the smooth pointed stable curve of type (g_X, n_X) over k_2 corresponding to the open subgroup $D^{(1)}_{b_0}(\pi_1^{\text{t}}(U_{E_2})) \subseteq \pi_1^{\text{t}}(U_{E_2})$. Note that we have $g_X \geq 2$ and $n_X > 0$ since $b_0 \neq 1$, and that the tame fundamental group $\pi_1^{\text{t}}(U_{X_2})$ of (X_2, D_{X_2}) is naturally isomorphic to $D^{(1)}_{b_0}(\pi_1^{\text{t}}(U_{E_2}))$. Let $c(\mathcal{I}_a)$ be the natural number defined in 3.1.1. Then the pair of finite groups

$$\Big(\pi_1^{\text{t}}(U_{E_2})/D^{(a+4)}_{c(\mathcal{I}_a)}(D^{(1)}_{b_0}(\pi_1^{\text{t}}(U_{E_2}))), \pi_1^{\text{t}}(U_{E_2})/D_{\mathcal{I}_a}(D^{(1)}_{b_0}(\pi_1^{\text{t}}(U_{E_2})))\Big)$$

is a quasi-anabelian pair associated to $\pi_1^{\text{t}}(U_{E_2})$.

Proof. Let (E_1, D_{E_1}) be an arbitrary smooth pointed stable curve of type (g, n) over k_1 and $\pi_1^{\text{t}}(U_{E_1})$ the tame fundamental group of (E_1, D_{E_1}). Let

$$\phi_{E_1} : \pi_1^{\text{t}}(U_{E_1}) \twoheadrightarrow \pi_1^{\text{t}}(U_{E_2})/D^{(a+4)}_{c(\mathcal{I}_a)}(D^{(1)}_{b_0}(\pi_1^{\text{t}}(U_{E_2})))$$

be an arbitrary surjection and

$$\psi_{E_1} : \pi_1^{\text{t}}(U_{E_1}) \overset{\phi_{E_1}}{\twoheadrightarrow} \pi_1^{\text{t}}(U_{E_2})/D^{(a+4)}_{c(\mathcal{I}_a)}(D^{(1)}_{b_0}(\pi_1^{\text{t}}(U_{E_2}))) \twoheadrightarrow \pi_1^{\text{t}}(U_{E_2})/D_{\mathcal{I}_a}(D^{(1)}_{b_0}(\pi_1^{\text{t}}(U_{E_2})))$$

the composition of surjections, where the second arrow is the natural surjection.

We put $(D^{(1)}_{b_0}(\pi_1^{\text{t}}(U_{E_1})) \subseteq) H_1 \overset{\text{def}}{=} \phi_{E_1}^{-1}\big(D^{(1)}_{b_0}(\pi_1^{\text{t}}(U_{E_2}))/D^{(a+4)}_{c(\mathcal{I}_a)}(D^{(1)}_{b_0}(\pi_1^{\text{t}}(U_{E_2})))\big) \subseteq \pi_1^{\text{t}}(U_{E_1})$ and write (X_1, D_{X_1}) for the smooth pointed stable curve of type (g_{X_1}, n_{X_1}) over k_1 corresponding to the open subgroup $H_1 \subseteq \pi_1^{\text{t}}(U_{E_1})$. Note that the tame fundamental group $\pi_1^{\text{t}}(U_{X_1})$ of (X_1, D_{X_1}) is naturally isomorphic to H_1, and that we have

$$\big(\pi_1^{\text{t}}(U_{E_1})/D^{(1)}_{b_0}(\pi_1^{\text{t}}(U_{E_1}))\big)^{p'} \overset{\sim}{\to} \big(\pi_1^{\text{t}}(U_{E_1})/H_1\big)^{p'} \overset{\sim}{\to} \big(\pi_1^{\text{t}}(U_{E_2})/D^{(1)}_{b_0}(\pi_1^{\text{t}}(U_{E_2}))\big)^{p'}.$$

Then since the types of (E_1, D_{E_1}) and (E_2, D_{E_2}) are equal to (g, n), we obtain $(g_{X_1}, n_{X_1}) = (g_X, n_X)$.

We identify $\pi_1^{\text{t}}(U_{X_1}), \pi_1^{\text{t}}(U_{X_2})$ with H_1, $D^{(1)}_{b_0}(\pi_1^{\text{t}}(U_{E_2}))$, respectively. Let

$$\phi : \pi_1^{\text{t}}(U_{X_1}) \twoheadrightarrow G_2^{a+4, c(\mathcal{I}_a)} \overset{\text{def}}{=} D^{(1)}_{b_0}(\pi_1^{\text{t}}(U_{E_2}))/D^{(a+4)}_{c(\mathcal{I}_a)}(D^{(1)}_{b_0}(\pi_1^{\text{t}}(U_{E_2}))),$$

$$\psi : \pi_1^{\text{t}}(U_{X_1}) \twoheadrightarrow G_2^{\mathcal{I}_a} \overset{\text{def}}{=} D^{(1)}_{b_0}(\pi_1^{\text{t}}(U_{E_2}))/D_{\mathcal{I}_a}(D^{(1)}_{b_0}(\pi_1^{\text{t}}(U_{E_2})))$$

be the surjections induced by ϕ_{E_1} and ψ_{E_1}, respectively. By Proposition 4.5, ψ induces group-theoretically a surjection

$$\psi^{\mathrm{op}}: \mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{X_1})) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(G_2^{\mathcal{I}_a}).$$

On the other hand, write $\mathcal{N}_{\pi_1^{\mathrm{t}}(U_{E_1})}(\mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{X_1})))$ for the set of normalizers of elements of $\mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{X_1}))$ in $\pi_1^{\mathrm{t}}(U_{E_1})$ and $\mathcal{N}_{\pi_1^{\mathrm{t}}(U_{E_2})/D_{\mathcal{I}_a}(D_{b_0}^{(1)}(\pi_1^{\mathrm{t}}(U_{E_2})))}(\mathrm{Edg}^{\mathrm{op}}(G_2^{\mathcal{I}_a}))$ for the set of normalizers of elements of $\mathrm{Edg}^{\mathrm{op}}(G_2^{\mathcal{I}_a})$ in $\pi_1^{\mathrm{t}}(U_{E_2})/D_{\mathcal{I}_a}(D_{b_0}^{(1)}(\pi_1^{\mathrm{t}}(U_{E_2})))$, respectively. Then we see immediately

$$\mathcal{N}_{\pi_1^{\mathrm{t}}(U_{E_1})}(\mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{X_1}))) = \mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{E_1})),$$

$$\mathcal{N}_{\pi_1^{\mathrm{t}}(U_{E_2})/D_{\mathcal{I}_a}(D_{b_0}^{(1)}(\pi_1^{\mathrm{t}}(U_{E_2})))}(\mathrm{Edg}^{\mathrm{op}}(G_2^{\mathcal{I}_a})) = \mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{E_2})/D_{\mathcal{I}_a}(D_{b_0}^{(1)}(\pi_1^{\mathrm{t}}(U_{E_2})))).$$

Thus, ψ^{op} induces a surjection

$$\psi_{E_1}^{\mathrm{op}}: \mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{E_1})) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{E_2})/D_{\mathcal{I}_a}(D_{b_0}^{(1)}(\pi_1^{\mathrm{t}}(U_{E_2})))).$$

This completes the proof of the theorem. □

5. Reconstructions of additive structures and linear structures via finite quotients

In this section, we prove that the field structures associated to inertia subgroups can be reconstructed group-theoretically from certain finite quotients of tame fundamental groups. Moreover, for smooth pointed stable curves of genus 0, we prove that the linear structures induced by the underlying curves can be reconstructed group-theoretically from certain finite quotients of tame fundamental groups. The main results of the present section are Proposition 5.2 and Proposition 5.3.

5.1. **Additive structures.**

5.1.1. *Notation and Settings.* Let $j \in \{1,2\}$, and let (X_j, D_{X_j}) be an arbitrary smooth pointed stable curve of type (g_X, n_X) over an algebraically closed field k_j of characteristic $p > 0$ and $\pi_1^{\mathrm{t}}(U_{X_j})$ the tame fundamental group of (X_j, D_{X_j}). Let $t \in \mathbb{N}$ be a positive natural number, and let $H_2, O_2 \overset{\mathrm{def}}{=} D_{p^t-1}^{(1)}(\pi_1^{\mathrm{t}}(U_{X_2})) \subseteq \pi_1^{\mathrm{t}}(U_{X_2})$ be open characteristic subgroups such that $H_2 \subseteq O_2$.

Let $\psi: \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow Q_{H_2} \overset{\mathrm{def}}{=} \pi_1^{\mathrm{t}}(U_{X_2})/H_2$ be a surjection such that ψ induces a surjection (e.g. there exists an open characteristic subgroup $N_2 \subseteq \pi_1^{\mathrm{t}}(U_{X_2})$ such that $(Q_{N_2} \overset{\mathrm{def}}{=} \pi_1^{\mathrm{t}}(U_{X_2})/N_2, Q_{H_2})$ is a quasi-anabelian pair associated to $\pi_1^{\mathrm{t}}(U_{X_2})$)

$$\psi^{\mathrm{op}}: \mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{X_1})) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(Q_{H_2}),\ I_{\widehat{x}_1} \mapsto \psi(I_{\widehat{x}_1}).$$

We put $H_1 \overset{\mathrm{def}}{=} \ker(\psi)$ and $O_1 \overset{\mathrm{def}}{=} \psi^{-1}(O_2/H_2) \subseteq \pi_1^{\mathrm{t}}(U_{X_1})$. Note that we have $O_1 = D_{p^t-1}^{(1)}(\pi_1^{\mathrm{t}}(U_{X_1}))$ since $p^t - 1$ is prime to p. Write $f_{H_j}: (X_{H_j}, D_{X_{H_j}}) \to (X_j, D_{X_j})$, $f_{O_j}: (X_{O_j}, D_{X_{O_j}}) \to (X_j, D_{X_j})$ for the tame coverings over k_j corresponding to $H_j, O_j \subseteq \pi_1^{\mathrm{t}}(U_{X_j})$, respectively, and (g_{H_j}, n_{H_j}), (g_{O_j}, n_{O_j}) for the types of $(X_{H_j}, D_{X_{H_j}})$, $(X_{O_j}, D_{X_{O_j}})$ respectively. Moreover, we denote by $Q_{H_1} \overset{\mathrm{def}}{=} \pi_1^{\mathrm{t}}(U_{X_1})/H_1$, $Q_{O_j} \overset{\mathrm{def}}{=} \pi_1^{\mathrm{t}}(U_{X_j})/O_j$, and $\overline{\psi}: Q_{H_1} \twoheadrightarrow Q_{H_2}$ the surjection induced by ψ. The composition of surjections $\pi_1^{\mathrm{t}}(U_{X_1}) \overset{\psi}{\twoheadrightarrow}$

$Q_{H_2} \twoheadrightarrow Q_{O_2}$ factors through the natural surjection $\pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow Q_{O_1}$. Namely, we have

$$\begin{array}{ccc} \pi_1^{\mathrm{t}}(U_{X_1}) & =\!= & \pi_1^{\mathrm{t}}(U_{X_1}) \\ \downarrow & & \downarrow{\scriptstyle \psi} \\ Q_{H_1} & \xrightarrow{\overline{\psi}} & Q_{H_2} \\ \downarrow & & \downarrow \\ Q_{O_1} & \xrightarrow{\overline{\rho}} & Q_{O_2}, \end{array}$$

where $\overline{\rho}$ is an isomorphism. Furthermore, the above commutative diagram implies the following commutative diagram

$$\begin{array}{ccc} \mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{X_1})) & \xrightarrow{\psi^{\mathrm{op}}} & \mathrm{Edg}^{\mathrm{op}}(Q_{H_2}) \\ \downarrow & & \downarrow \\ \mathrm{Edg}^{\mathrm{op}}(Q_{O_1}) & \xrightarrow{\overline{\rho}^{\mathrm{op}}} & \mathrm{Edg}^{\mathrm{op}}(Q_{O_2}), \end{array}$$

where the vertical arrows are the natural surjections induced by $\pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow Q_{O_1}$ and $Q_{H_2} \twoheadrightarrow Q_{O_2}$, respectively, and $\overline{\rho}^{\mathrm{op}}$ denotes the surjection induced by ψ^{op}.

Let $\widehat{x}_1 \in D_{\widehat{X}_1}$ (see 2.1.3 for $D_{\widehat{X}_1}$), $I_{\widehat{x}_1} \in \mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{X_1}))$, and $I_{x_{H_2}} \stackrel{\mathrm{def}}{=} \psi^{\mathrm{op}}(I_{\widehat{x}_1}) \in \mathrm{Edg}^{\mathrm{op}}(Q_{H_2})$ for some $x_{H_2} \in D_{X_{H_2}}$. Let $x_{O_1} \in D_{X_{O_1}}$ be the image of $\widehat{x}_1$ of the natural surjection $D_{\widehat{X}_1} \twoheadrightarrow D_{X_{O_1}}$ and $\widehat{x}_{O_2}$ the image of x_{H_2} of the natural surjection $D_{X_{H_2}} \twoheadrightarrow D_{X_{O_2}}$. Then we have

$$I_{x_{O_1}} \in \mathrm{Edg}^{\mathrm{op}}(Q_{O_1}),\ I_{x_{O_2}} \in \mathrm{Edg}^{\mathrm{op}}(Q_{O_2})$$

which are equal to the images of $I_{\widehat{x}_1}$, $I_{x_{H_2}}$ of the natural surjections $\mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{X_1})) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(Q_{O_1})$, $\mathrm{Edg}^{\mathrm{op}}(Q_{H_2}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(Q_{O_2})$, respectively. The above commutative diagram implies $\overline{\rho}^{\mathrm{op}}(I_{x_{O_1}}) = I_{x_{O_2}}$. We put

$$\rho_{x_{O_1}, x_{O_2}} \stackrel{\mathrm{def}}{=} \overline{\rho}|_{I_{x_{O_1}}} : I_{x_{O_1}} \xrightarrow{\sim} I_{x_{O_2}}$$

the isomorphism. Moreover, let $\widehat{x}_2 \in D_{\widehat{X}_2}$ be a marked point of $(\widehat{X}_2, D_{\widehat{X}_2})$ over x_{H_2} (i.e., a marked point of $D_{\widehat{X}_2}$ whose image of the natural surjection $D_{\widehat{X}_2} \twoheadrightarrow D_{X_{H_2}}$ is x_{H_2}) and $I_{\widehat{x}_2} \in \mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{X_2}))$.

Additive structures associated to inertia subgroups. Write $\overline{\mathbb{F}}_{p,j}$, $j \in \{1, 2\}$, for the algebraic closure of $\mathbb{F}_p$ in k_j. We put

$$\mathbb{F}_{\widehat{x}_j} \stackrel{\mathrm{def}}{=} (I_{\widehat{x}_j} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_j^{p'}) \sqcup \{*_{\widehat{x}_j}\},$$

where $\{*_{\widehat{x}_j}\}$ is an one-point set, and $(\mathbb{Q}/\mathbb{Z})_j^{p'}$ denotes the prime-to-p part of $\mathbb{Q}/\mathbb{Z}$ which can be canonically identified with $(\mathbb{Q}/\mathbb{Z})_j^{p'}(1) \stackrel{\mathrm{def}}{=} \bigcup_{(p,m)=1} \mu_m(k_j)$. Moreover, let $a_{\widehat{x}_j}$ be a generator of $I_{\widehat{x}_j}$. Then we have a natural bijection

$$I_{\widehat{x}_j} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_j^{p'} \xrightarrow{\sim} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_j^{p'},\ a_{\widehat{x}_j} \otimes 1 \mapsto 1 \otimes 1.$$

Thus, we obtain the following bijections

$$I_{\widehat{x}_j} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_j^{p'} \xrightarrow{\sim} \mathbb{Z} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_j^{p'} \xrightarrow{\sim} (\mathbb{Q}/\mathbb{Z})_j^{p'}(1) \xrightarrow{\sim} \overline{\mathbb{F}}_{p,j}^{\times}.$$

This means that $\mathbb{F}_{\widehat{x}_j}$ can be identified with $\overline{\mathbb{F}}_{p,j}$ as sets, hence, admits a structure of field, whose multiplicative group is $I_{\widehat{x}_j} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_j^{p'}$, and whose zero element is $*_{\widehat{x}_j}$.

Suppose that $I_{x_{O_j}}$ is *non-trivial* (e.g. $n_X \geq 2$). Moreover, write $a_{x_{O_j}}$ for the image of $a_{\widehat{x}_j}$ of the natural surjection $I_{\widehat{x}_j} \twoheadrightarrow I_{x_{O_j}}$. Since there are natural homomorphisms $I_{x_{O_j}} \xrightarrow{\sim} \mathbb{Z}/(p^t-1)\mathbb{Z} \xrightarrow{\sim} \mu_{p^t-1}(k_j) \hookrightarrow \overline{\mathbb{F}}_{p,j}^{\times}$, where the first arrow is determined by $a_{x_{O_j}} \mapsto 1$, the set

$$\mathbb{F}_{x_{O_j},t} \overset{\text{def}}{=} I_{x_{O_j}} \sqcup \{*_{\widehat{x}_j}\}$$

admits a structure of field induced by $\overline{\mathbb{F}}_{p,j}$ which is isomorphic to the subfield of $\overline{\mathbb{F}}_{p,j}$ with cardinality p^t.

5.1.2. Firstly, by applying similar arguments to the arguments given in the proof of [Y5, Theorem 6.4], we have the following lemma:

Lemma 5.1. *We maintain the notation and the settings introduced in 5.1.1. Suppose*

$$n_X = 3,$$

$t > \log_p(C(g_X)+1)$ *(see 3.1.1 for* $C(g_X)$*), and* $H_2 \subseteq D_p^{(1)}(O_2) = D_p^{(1)}(D_{p^t-1}^{(1)}(\pi_1^{\mathrm{t}}(U_{X_2})))$. *Then the field* $\mathbb{F}_{x_{O_j},t}$ *can be reconstructed group-theoretically from* Q_{H_j} *and* Q_{O_j}. *Moreover, the isomorphism* $\rho_{x_{O_1},x_{O_2}} : I_{x_{O_1}} \xrightarrow{\sim} I_{x_{O_2}}$ *induces a field isomorphism*

$$\rho^{\mathrm{fd}}_{x_{O_1},x_{O_2}} : \mathbb{F}_{x_{O_1},t} \xrightarrow{\sim} \mathbb{F}_{x_{O_2},t},$$

where "fd" means "field".

Proof. Let $\overline{\mathbb{F}}_p$ be an algebraic closure of $\mathbb{F}_p$ and $\mathbb{F}_{p^t} \subseteq \overline{\mathbb{F}}_p$ the subfield with cardinality p^t. The field structure of $\mathbb{F}_{x_{O_j},t}$ is equivalent to the subset

$$\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{x_{O_j},t}, \mathbb{F}_{p^t}) \subseteq \mathrm{Hom}_{\mathrm{gp}}(\mathbb{F}^{\times}_{x_{O_j},t}, \mathbb{F}^{\times}_{p^t}),$$

where "gp" means "group". Then in order to prove the first part of the theorem, it is sufficient to prove that the set $\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{x_{O_j},t}, \mathbb{F}_{p^t})$ can be reconstructed group-theoretically from Q_{H_j} and Q_{O_j}.

Let $\chi_j \in \mathrm{Hom}_{\mathrm{gp}}(Q_{O_j}, \mathbb{F}^{\times}_{p^t})$. We put

$$H_{\chi_j} \overset{\text{def}}{=} \ker(Q_{H_j} \twoheadrightarrow Q_{O_j} \xrightarrow{\chi_j} \mathbb{F}^{\times}_{p^t}),\ M_{\chi_j} \overset{\text{def}}{=} H^{\mathrm{ab}}_{\chi_j} \otimes \mathbb{F}_p.$$

Then M_{χ_j} admits a natural action of Q_{H_j} via the conjugation action. Since we assume $H_2 \subseteq D_p^{(1)}(O_2)$, we see $M_{\chi_j} = \left(\ker(\pi_1^{\mathrm{t}}(U_{X_j}) \twoheadrightarrow Q_{O_j} \xrightarrow{\chi_j} \mathbb{F}^{\times}_{p^t})\right)^{\mathrm{ab}} \otimes \mathbb{F}_p$. Denote by

$$M_{\chi_j}[\chi_j] \overset{\text{def}}{=} \{a \in M_{\chi_j} \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \mid \sigma \cdot a = \chi_j(\sigma)a \text{ for all } \sigma \in Q_{O_j}\},$$

$$\gamma_{\chi_j}(M_{\chi_j}) \overset{\text{def}}{=} \dim_{\overline{\mathbb{F}}_p}(M_{\chi_j}[\chi_j]).$$

The integer $\gamma_{\chi_j}(M_{\chi_j})$ is a generalized Hasse-Witt invariant of the cyclic tame covering of (X_j, D_{X_j}) corresponding to $\ker(\pi_1^{\mathrm{t}}(U_{X_j}) \twoheadrightarrow Q_{O_j} \xrightarrow{\chi_j} \mathbb{F}^{\times}_{p^t}) \hookrightarrow \pi_1^{\mathrm{t}}(U_{X_j})$. Note that $n_X = 3$ implies $\gamma_{\chi_j}(M_{\chi_j}) \leq g_X + 1$ ([Y5, Lemma 2.7]). We define two maps

$$\mathrm{Res}_{j,t} : \mathrm{Hom}_{\mathrm{gp}}(Q_{O_j}, \mathbb{F}^{\times}_{p^t}) \to \mathrm{Hom}_{\mathrm{gp}}(\mathbb{F}^{\times}_{x_{O_j},t}, \mathbb{F}^{\times}_{p^t}),$$

$$\Gamma_{j,t} : \mathrm{Hom}_{\mathrm{gp}}(Q_{O_j}, \mathbb{F}^{\times}_{p^t}) \to \mathbb{Z}_{\geq 0},\ \chi_j \mapsto \gamma_{\chi_j}(M_{\chi_j}),$$

where the map $\mathrm{Res}_{j,t}$ is the restriction with respect to the natural inclusion $\mathbb{F}^{\times}_{x_{O_j},t} = I_{x_{O_j}} \hookrightarrow Q_{O_j}$. Since we assume $t > C(g_X)$, the "non-moreover" part of the lemma follows from Claim A mentioned in the proof of [Y5, Theorem 6.4] (see page 95 of [Y5]) which says

$$\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{x_{O_j},t}, \mathbb{F}_{p^t}) = \mathrm{Hom}^{\mathrm{surj}}_{\mathrm{gp}}(\mathbb{F}^{\times}_{x_{O_j},t}, \mathbb{F}^{\times}_{p^t}) \setminus \mathrm{Res}_{j,t}(\Gamma^{-1}_{j,t}(\{g_X+1\})),$$

where $\mathrm{Hom}^{\mathrm{surj}}_{\mathrm{gp}}(-,-)$ denotes the subset of $\mathrm{Hom}_{\mathrm{gp}}(-,-)$ whose elements are surjections.

Next, we prove the "moreover" part of the lemma. Let $\kappa_2 \in \mathrm{Hom}_{\mathrm{gp}}(Q_{O_2}, \mathbb{F}^{\times}_{p^t})$. Then we obtain a character

$$\kappa_1 \in \mathrm{Hom}_{\mathrm{gp}}(Q_{O_1}, \mathbb{F}^{\times}_{p^t})$$

induced by $\overline{\rho} : Q_{O_1} \xrightarrow{\sim} Q_{O_2}$. Moreover, the surjection $\overline{\phi}|_{H_{\kappa_1}} : H_{\kappa_1} \twoheadrightarrow H_{\kappa_2}$ induces a surjection $M_{\kappa_1}[\kappa_1] \twoheadrightarrow M_{\kappa_2}[\kappa_2]$. Suppose $\kappa_2 \in \Gamma^{-1}_{2,r}(\{g_X+1\})$. The surjection $M_{\kappa_1}[\kappa_1] \twoheadrightarrow M_{\kappa_2}[\kappa_2]$ implies $\gamma_{\kappa_1}(M_{\kappa_1}) = g_X + 1$. Namely, we have $\kappa_1 \in \Gamma^{-1}_{1,t}(\{g_X+1\})$. Thus, the isomorphism $\rho_{x_{O_1}, x_{O_2}} : I_{x_{O_1}} \xrightarrow{\sim} I_{x_{O_2}}$ induces an injection

$$\mathrm{Res}_{2,t}(\Gamma^{-1}_{2,t}(\{g_X+1\})) \hookrightarrow \mathrm{Res}_{1,t}(\Gamma^{-1}_{1,t}(\{g_X+1\})).$$

Since $\#\big(\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{x_{O_1},t}, \mathbb{F}_{p^t})\big) = \#\big(\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{x_{O_2},t}, \mathbb{F}_{p^t})\big)$, the isomorphism $\rho_{x_{O_1}, x_{O_2}}$ induces a bijection

$$\mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{x_{O_2},t}, \mathbb{F}_{p^t}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{fd}}(\mathbb{F}_{x_{O_1},t}, \mathbb{F}_{p^t}).$$

If we choose $\mathbb{F}_{p^t} = \mathbb{F}_{x_{O_2},t}$, then the image of $\mathrm{id}_{\mathbb{F}_{x_{O_2},t}}$ via the above bijection induces a field isomorphism

$$\rho^{\mathrm{fd}}_{x_{O_1}, x_{O_2}} : \mathbb{F}_{x_{O_1},t} \xrightarrow{\sim} \mathbb{F}_{x_{O_2},t}.$$

This completes the proof of the lemma. □

The main result of the present subsection is as follows:

Proposition 5.2. *We maintain the notation and the settings introduced in 5.1.1. Suppose*

$$n_X \geq 3,$$

$t > \log_p(C(g_X)+1)$ *(see 3.1.1 for* $C(g_X)$*), and* $H_2 \subseteq D^{(1)}_p(O_2) = D^{(1)}_p(D^{(1)}_{p^t-1}(\pi^{\mathrm{t}}_1(U_{X_2})))$*. Then the field* $\mathbb{F}_{x_{O_j},t}$ *can be reconstructed group-theoretically from* Q_{H_j} *and* Q_{O_j}*. Moreover, the isomorphism* $\rho_{x_{O_1}, x_{O_2}} : I_{x_{O_1}} \xrightarrow{\sim} I_{x_{O_2}}$ *induces a field isomorphism*

$$\rho^{\mathrm{fd}}_{x_{O_1}, x_{O_2}} : \mathbb{F}_{x_{O_1},t} \xrightarrow{\sim} \mathbb{F}_{x_{O_2},t}.$$

Proof. If $n_X = 3$, the proposition follows from Lemma 5.1. To verify the proposition, we suppose $n_X > 3$.

Write $x_{2,1} \in D_{X_2}$ for the image of $x_{O_2} \in D_{X_{O_2}}$ of the surjection $D_{X_{O_2}} \twoheadrightarrow D_{X_2}$ induced by the tame covering f_{O_2} over k_2. Let $x_{2,2}, x_{2,3} \in D_{X_2} \setminus \{x_{2,1}\}$ be marked points distinct from each other, $S_2 \overset{\mathrm{def}}{=} D_{X_2} \setminus \{x_{2,1}, x_{2,2}, x_{2,3}\}$, $S_{O_2} \overset{\mathrm{def}}{=} f^{-1}_{O_2}(S_2) \subseteq D_{X_{O_2}}$, and $S_{H_2} \overset{\mathrm{def}}{=} f^{-1}_{H_2}(S_2) \subseteq D_{X_{H_2}}$. We put

$$\mathrm{Edg}^{\mathrm{op}}_{S_2}(Q_{H_2}) \overset{\mathrm{def}}{=} \{I_x \in \mathrm{Edg}^{\mathrm{op}}(Q_{H_2}) \mid x \in S_{H_2}\},\ \mathrm{Edg}^{\mathrm{op}}_{S_2}(Q_{O_2}) \overset{\mathrm{def}}{=} \{I_x \in \mathrm{Edg}^{\mathrm{op}}(Q_{O_2}) \mid x \in S_{O_2}\}$$

and

$$\mathrm{Edg}^{\mathrm{op}}_{\widehat{S}_1}(\pi^{\mathrm{t}}_1(U_{X_1})),\ \mathrm{Edg}^{\mathrm{op}}_{\widehat{S}_2}(\pi^{\mathrm{t}}_1(U_{X_2}))$$

the inverse images of the surjections $\mathrm{Edg}^{\mathrm{op}}(\pi^{\mathrm{t}}_1(U_{X_1})) \overset{\psi^{\mathrm{op}}}{\twoheadrightarrow} \mathrm{Edg}^{\mathrm{op}}(Q_{H_2}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(Q_{O_2})$ and $\mathrm{Edg}^{\mathrm{op}}(\pi^{\mathrm{t}}_1(U_{X_2})) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(Q_{O_2})$, respectively. Write

$$I_{S_{H_2}} \subseteq Q_{H_2},\ I_{S_{O_2}} \subseteq Q_{O_2},\ I_{\widehat{S}_1} \subseteq \pi^{\mathrm{t}}_1(U_{X_1}),\ I_{\widehat{S}_2} \subseteq \pi^{\mathrm{t}}_1(U_{X_2})$$

for the closed subgroups generated by elements of

$$\mathrm{Edg}^{\mathrm{op}}_{S_2}(Q_{H_2}),\ \mathrm{Edg}^{\mathrm{op}}_{S_2}(Q_{O_2}),\ \mathrm{Edg}^{\mathrm{op}}_{\widehat{S}_1}(\pi^{\mathrm{t}}_1(U_{X_1})),\ \mathrm{Edg}^{\mathrm{op}}_{\widehat{S}_2}(\pi^{\mathrm{t}}_1(U_{X_2})),$$

respectively. Note that the images of $I_{\widehat{S}_1}$ and $I_{\widehat{S}_2}$ in Q_{H_2} of the surjection $\psi : \pi^{\mathrm{t}}_1(U_{X_1}) \twoheadrightarrow Q_{H_2}$ and the natural surjection $\pi^{\mathrm{t}}_1(U_{X_2}) \twoheadrightarrow Q_{H_2}$ are equal to $I_{S_{H_2}}$, respectively, and that the image of $I_{S_{H_2}}$ in Q_{O_2} of the natural surjection $Q_{H_2} \twoheadrightarrow Q_{O_2}$ is equal to $I_{S_{O_2}}$. Then we obtain the following diagram

$$\begin{array}{ccccc}
 & & & & \pi_1^{\mathrm{t}}(U_{X_2}) \\
 & & & & \downarrow \\
 & & & & \pi_1^{\mathrm{t}}(U_{X_2})/I_{\widehat{S}_2} \\
 & & & & \downarrow \\
\pi_1^{\mathrm{t}}(U_{X_1}) & \longrightarrow & \pi_1^{\mathrm{t}}(U_{X_1})/I_{\widehat{S}_1} & \xrightarrow{\widetilde{\psi}} & Q_{H_2}/I_{S_{H_2}} \\
 & & & & \downarrow \\
 & & & & Q_{O_2}/I_{S_{O_2}},
\end{array}$$

where $\widetilde{\psi}$ is the surjection induced by $\psi : \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow Q_{H_2}$.

The above constructions concerning $I_{\widehat{S}_j}$ imply immediately that $\pi_1^{\mathrm{t}}(U_{X_j})/I_{\widehat{S}_j}$ is naturally isomorphic to the tame fundamental group of a smooth pointed stable curve of type $(g_X, 3)$ over k_j whose underlying curve is X_j, and that ψ^{op} and $\widetilde{\psi}$ induce a surjection

$$\widetilde{\psi}^{\mathrm{op}} : \mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{X_1})/I_{\widehat{S}_1}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(Q_{H_2}/I_{S_{H_2}}).$$

Furthermore, note that we have $\widetilde{O}_2 \overset{\mathrm{def}}{=} D^{(1)}_{p^t-1}(\pi_1^{\mathrm{t}}(U_{X_2})/I_{\widehat{S}_2}) = \ker(\pi_1^{\mathrm{t}}(U_{X_2})/I_{\widehat{S}_2} \twoheadrightarrow Q_{O_2}/I_{S_{O_2}})$, and that $H_2 \subseteq D^{(1)}_p(O_2) = D^{(1)}_p(D^{(1)}_{p^t-1}(\pi_1^{\mathrm{t}}(U_{X_2})))$ implies

$$\widetilde{H}_2 \overset{\mathrm{def}}{=} \ker(\pi_1^{\mathrm{t}}(U_{X_2})/I_{\widehat{S}_2} \twoheadrightarrow Q_{H_2}/I_{S_{H_2}}) \subseteq D^{(1)}_p(D^{(1)}_{p^t-1}(\pi_1^{\mathrm{t}}(U_{X_2})/I_{\widehat{S}_2})).$$

Then by replacing $\pi_1^{\mathrm{t}}(U_{X_j})$, H_2, O_2, and ψ by $\pi_1^{\mathrm{t}}(U_{X_j})/I_{\widehat{S}_j}$, $\widetilde{H}_2$, $\widetilde{O}_2$, and $\widetilde{\psi}$, respectively, the proposition follows from Lemma 5.1. We complete the proof of the proposition. □

5.2. **Linear structures.**

5.2.1. *Notation and Settings.* We maintain the notation and the settings introduced in 5.1.1. Note that we have $D_{X_1} \overset{\sim}{\to} \mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{X_1}))/\pi_1^{\mathrm{t}}(U_{X_1})$ and $D_{X_2} \overset{\sim}{\to} \mathrm{Edg}^{\mathrm{op}}(Q_{H_2})/Q_{H_2}$. Moreover, the surjection $\psi : \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow Q_{H_2}$ and the surjection $\psi^{\mathrm{op}} : \mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{X_1})) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(Q_{H_2})$ induce a bijection

$$\psi^{\mathrm{mp}} : D_{X_1} \overset{\sim}{\to} \mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{X_1}))/\pi_1^{\mathrm{t}}(U_{X_1}) \overset{\sim}{\to} \mathrm{Edg}^{\mathrm{op}}(Q_{H_2})/Q_{H_2} \overset{\sim}{\to} D_{X_2}.$$

In the remainder of this subsection, we suppose

- $(g_X, n_X) = (0, n_X)$.

Linear structures associated to affine lines. Fix two marked points $x_{j,\infty}, x_{j,0} \in D_{X_j}$ distinct from each other. We choose any field $k'_j \cong k_j$, and choose any isomorphism $\varphi_j : X_j \overset{\sim}{\to} \mathbb{P}^1_{k'_j}$ as schemes such that $\varphi_j(x_{j,\infty}) = \infty$ and $\varphi_j(x_{j,0}) = 0$. Then the set of k_j-rational points $X_j(k_j) \setminus \{x_{j,\infty}\}$ is equipped with a structure of $\mathbb{F}_p$-module via the bijection φ_j. Note that since any k'_j-isomorphism of $\mathbb{P}^1_{k'_j}$ fixing ∞ and 0 is a scalar multiplication, the $\mathbb{F}_p$-module structure of $X_j(k_j) \setminus \{x_{j,\infty}\}$ does not depend on the choices of k'_1 and φ_1 but depends only on the choices of $x_{j,\infty}$ and $x_{j,0}$. Then we shall say that $X_j(k_j) \setminus \{x_{j,\infty}\}$ *is equipped with a structure of* $\mathbb{F}_p$*-module with respect to* $x_{j,\infty}$ *and* $x_{j,0}$.

5.2.2. We have the following proposition:

Proposition 5.3. *We maintain the notation and the settings introduced in 5.2.1. Write $x_{2,\infty}$ and $x_{2,0}$ for $\psi^{\mathrm{mp}}(x_{1,\infty})$ and $\phi^{\mathrm{mp}}(x_{1,0})$, respectively. Let*

$$\sum_{x_1\in D_{X_1}\setminus\{x_{1,\infty},x_{1,0}\}} b_{x_1}x_1 = x_{1,0}$$

be a linear condition with respect to $x_{1,\infty}$ and $x_{1,0}$ on (X_1, D_{X_1}), where $b_{x_1}\in\mathbb{F}_p$ for each $x_1\in D_{X_1}\setminus\{x_{1,\infty},x_{1,0}\}$. Suppose that there exist natural numbers $t\in\mathbb{N}$ and $b'_{x_1}\in\mathbb{Z}_{\geq 0}$, $x_1\in D_{X_1}$, such that $b'_{x_1}\equiv b_{x_1}$ (mod p) and

$$p^t-2\geq \sum_{x_1\in D_{X_1}\setminus\{x_{1,\infty},x_{1,0}\}} b'_{x_1}\geq 2.$$

Then the linear condition

$$\sum_{x_1\in D_{X_1}\setminus\{x_{1,\infty},x_{1,0}\}} b_{x_1}\psi^{\mathrm{mp}}(x_1)=\psi^{\mathrm{mp}}(x_{1,0})=x_{2,0}$$

with respect to $x_{2,\infty}$ and $x_{2,0}$ on (X_2, D_{X_2}) also holds.

Proof. For each $\widehat{x}_1\in D_{\widehat{X}_1}$ over $x_1\in D_{X_1}$, write $I_{\widehat{x}_1,\mathrm{ab}}$ for the image of the composition of homomorphisms $I_{\widehat{x}_1}\hookrightarrow\pi_1^{\mathrm{t}}(U_{X_1})\twoheadrightarrow\pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{ab}}$. Moreover, since the image of $I_{\widehat{x}_1,\mathrm{ab}}$ does not depend on the choices of $\widehat{x}_1\in D_{\widehat{X}_1}$ over x_1, we may write I_{x_1} for $I_{\widehat{x}_1,\mathrm{ab}}$. The structure of maximal prime-to-p quotient of $\pi_1^{\mathrm{t}}(U_{X_1})$ implies that $\pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{ab}}$ is generated by $\{I_{x_1}\}_{x_1\in D_{X_1}}$, and that there exists a generator a_{x_1} of I_{x_1}, $x_1\in D_{X_1}$, such that $\prod_{x_1\in D_{X_1}}a_{x_1}=1$. We put

$$I_{x_{1,\infty}}\to\mathbb{Z}/(p^t-1)\mathbb{Z},\ a_{x_{1,\infty}}\mapsto 1,$$

$$I_{x_{1,0}}\to\mathbb{Z}/(p^t-1)\mathbb{Z},\ a_{x_{1,0}}\mapsto\Big(\sum_{x_1\in D_{X_1}\setminus\{x_{1,\infty},x_{1,0}\}} b'_{x_1}\Big)-1,$$

$$I_{x_1}\to\mathbb{Z}/(p^t-1)\mathbb{Z},\ a_{x_1}\mapsto -b'_{x_1},\ x_1\in D_{X_1}\setminus\{x_{1,\infty},x_{1,0}\}.$$

Then the homomorphisms of inertia subgroups defined above induce surjections

$$\delta^{\mathrm{ab}}_{p^t-1,1}:\pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{ab}}\otimes\mathbb{Z}/(p^t-1)\mathbb{Z}\twoheadrightarrow\mathbb{Z}/(p^t-1)\mathbb{Z},$$

$$\delta_{p^t-1,1}:\pi_1^{\mathrm{t}}(U_{X_1})\twoheadrightarrow\pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{ab}}\otimes\mathbb{Z}/(p^t-1)\mathbb{Z}\overset{\delta^{\mathrm{ab}}_{p^t-1,1}}{\twoheadrightarrow}\mathbb{Z}/(p^t-1)\mathbb{Z}.$$

Note that $\ker(\delta^{\mathrm{ab}}_{p^t-1,1})$, $\ker(\delta_{p^t-1,1})$ do not depend on the choices of the generators $\{a_{x_1}\}_{x_1\in D_{X_1}}$.

Let $I_{x_{H_2}}\overset{\mathrm{def}}{=}\psi^{\mathrm{op}}(I_{\widehat{x}_1})\in\mathrm{Edg}^{\mathrm{op}}(Q_{H_2})$ for some $x_{H_2}\in D_{X_{H_2}}$ and $x_2\in D_{X_2}$ the image of x_{H_2} of the surjection $D_{X_{H_2}}\twoheadrightarrow D_{X_2}$ induced by f_{H_2}. Write I_{x_2} for the image of the composition of homomorphisms $I_{x_{H_2}}\hookrightarrow Q_{H_2}\twoheadrightarrow Q^{\mathrm{ab}}_{H_2}$. Note that I_{x_2} does not depend on the choices of $x_{H_2}\in f^{-1}_{H_2}(x_2)$. Since $(p,p^r-1)=1$ and $H_2\subseteq D^{(1)}_{p^t-1}(\pi_1^{\mathrm{t}}(U_{X_2}))$, ψ induces an isomorphism $\psi^{\mathrm{ab}}_{p^t-1}:\pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{ab}}\otimes\mathbb{Z}/(p^t-1)\mathbb{Z}\overset{\sim}{\to}Q^{\mathrm{ab}}_{H_2}\otimes\mathbb{Z}/(p^t-1)\mathbb{Z}=Q_{O_2}$. Note that ψ^{op} implies that $\psi^{\mathrm{ab}}_{p^t-1}$ induces a bijection

$$\psi^{\mathrm{ab,op}}_{p^t-1}:\mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{X_1})^{\mathrm{ab}}\otimes\mathbb{Z}/(p^t-1)\mathbb{Z})\overset{\sim}{\to}\mathrm{Edg}^{\mathrm{op}}(Q^{\mathrm{ab}}_{H_2}\otimes\mathbb{Z}/(p^t-1)\mathbb{Z}),$$

and that $\psi^{\mathrm{ab,op}}_{p^t-1}(I_{x_1}\otimes\mathbb{Z}/(p^t-1)\mathbb{Z})=I_{\psi^{\mathrm{mp}}(x_1)}\otimes\mathbb{Z}/(p^t-1)\mathbb{Z}$. Then the surjection $\delta^{\mathrm{ab}}_{p^t-1,1}$, the isomorphism $\psi^{\mathrm{ab}}_{p^t-1}$, and the bijection $\psi^{\mathrm{ab,op}}_{p^t-1}$ imply the following homomorphisms of inertia subgroups:

$$I_{x_{2,\infty}}\to\mathbb{Z}/(p^t-1)\mathbb{Z},\ a_{x_{2,\infty}}\mapsto 1,$$

$$I_{x_{2,0}} \to \mathbb{Z}/(p^t-1)\mathbb{Z},\ a_{x_{2,0}} \mapsto (\sum_{x_1 \in D_{X_1} \setminus \{x_{1,\infty}, x_{1,0}\}} b'_{x_1}) - 1,$$

$$I_{x_2} \to \mathbb{Z}/(p^t-1)\mathbb{Z},\ a_{x_2} \mapsto -b'_{x_1},\ x_1 \in D_{X_1} \setminus \{x_{1,\infty}, x_{1,0}\},$$

where $x_2 \stackrel{\text{def}}{=} \psi^{\text{mp}}(x_1)$, $a_{x_2} \stackrel{\text{def}}{=} \psi(a_{x_1})$, $x_1 \in D_{X_1}$. Then the homomorphisms of inertia subgroups defined above induce surjections

$$\delta^{\text{ab}}_{p^t-1,2} : Q^{\text{ab}}_{H_2} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z} \twoheadrightarrow \mathbb{Z}/(p^t-1)\mathbb{Z},$$

$$\delta_{p^t-1,2} : Q_{H_2} \twoheadrightarrow Q^{\text{ab}}_{H_2} \otimes \mathbb{Z}/(p^t-1)\mathbb{Z} \stackrel{\delta^{\text{ab}}_{p^t-1,2}}{\twoheadrightarrow} \mathbb{Z}/(p^t-1)\mathbb{Z}.$$

We put $H_{\delta_{p^t-1,j}} \stackrel{\text{def}}{=} \ker(\delta_{p^t-1,j})$, $M_j \stackrel{\text{def}}{=} H^{\text{ab}}_{\delta_{p^t-1,j}} \otimes \mathbb{F}_p$. Then we obtain the following commutative diagram:

$$\begin{array}{ccc} H_{\delta_{p^t-1,1}} & \xrightarrow{\psi|_{H_{\delta_{p^t-1,1}}}} & H_{\delta_{p^t-1,2}} \\ \downarrow & & \downarrow \\ \pi_1^{\text{t}}(U_{X_1}) & \xrightarrow{\psi} & Q_{H_2} \\ \delta_{p^t-1,1}\downarrow & & \delta_{p^t-1,2}\downarrow \\ \mathbb{Z}/(p^t-1)\mathbb{Z} & = & \mathbb{Z}/(p^t-1)\mathbb{Z}. \end{array}$$

Let $\overline{\mathbb{F}}_p$ be an algebraic closure of the finite field $\mathbb{F}_p$. We fix an injection $\mathbb{Z}/(p^t-1)\mathbb{Z} \hookrightarrow \overline{\mathbb{F}}_p^{\times}$. Note that the $\mathbb{F}_p$-vector spaces M_1, M_2 admit natural actions of $I_{\widehat{x}_{1,\infty}}$, $I_{x_{H_2,\infty}} \stackrel{\text{def}}{=} \psi(I_{\widehat{x}_{1,\infty}})$ which coincides with the action via the following character

$$\chi_{I_{\widehat{x}_{1,\infty}},t} : I_{\widehat{x}_{1,\infty}} \hookrightarrow \pi_1^{\text{t}}(U_{X_1}) \stackrel{\delta_{p^t-1,1}}{\twoheadrightarrow} \mathbb{Z}/(p^t-1)\mathbb{Z} \hookrightarrow \overline{\mathbb{F}}_p^{\times},$$

$$\chi_{I_{x_{H_2,\infty}},t} : I_{x_{H_2,\infty}} \hookrightarrow Q_{H_2} \stackrel{\delta_{p^t-1,2}}{\twoheadrightarrow} \mathbb{Z}/(p^t-1)\mathbb{Z} \hookrightarrow \overline{\mathbb{F}}_p^{\times}.$$

We put

$$M_1[\chi_{I_{\widehat{x}_{1,\infty}},t}] \stackrel{\text{def}}{=} \{a \in M_1 \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \mid \sigma(a) = \chi_{I_{\widehat{x}_{1,\infty}},t}(\sigma)a \text{ for all } \sigma \in I_{\widehat{x}_{1,\infty}}\}$$

$$M_2[\chi_{I_{x_{H_2,\infty}},t}] \stackrel{\text{def}}{=} \{a \in M_2 \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p \mid \sigma(a) = \chi_{I_{x_{H_2,\infty}},t}(\sigma)a \text{ for all } \sigma \in I_{x_{H_2,\infty}}\}$$

Note that $\dim_{\overline{\mathbb{F}}_p}(M_1[\chi_{I_{\widehat{x}_{1,\infty}},t}])$ and $\dim_{\overline{\mathbb{F}}_p}(M_2[\chi_{I_{x_{H_2,\infty}},t}])$ are the first generalized Hasse-Witt invariants (see 2.3.2) of the cyclic tame coverings of U_{X_1} and U_{X_2} corresponding to $H_{\delta_{p^t-1,1}} \subseteq \pi_1^{\text{t}}(U_{X_1})$ and the inverse image of $H_{\delta_{p^t-1,2}}$ of the natural surjection $\pi_1^{\text{t}}(U_{X_2}) \twoheadrightarrow Q_{H_2}$, respectively.

Since the actions of $I_{\widehat{x}_{1,\infty}}$, $I_{x_{H_2,\infty}}$ on $M_1 \otimes \overline{\mathbb{F}}_p$, $M_2 \otimes \overline{\mathbb{F}}_p$ are semi-simple, respectively, $\psi|_{H_{\delta_{p^t-1,1}}}$ induces a surjection

$$M_1[\chi_{I_{\widehat{x}_{1,\infty}},t}] \twoheadrightarrow M_2[\chi_{I_{x_{H_2,\infty}},t}].$$

On the other hand, the third and the final paragraphs of the proof of [T1, Lemma 3.3] imply that the linear condition

$$\sum_{x_1 \in D_{X_1} \setminus \{x_{1,\infty}, x_{1,0}\}} b_{x_1} x_1 = x_{1,0}$$

with respect to $x_{1,\infty}$ and $x_{1,0}$ on (X_1, D_{X_1}) holds if and only if $M_1[\chi_{I_{\widehat{x}_{1,\infty}}, t}] = 0$. Thus, we obtain $M_2[\chi_{I_{x_{H_{2,\infty}}}, t}] = 0$. Then the third and the final paragraphs of the proof of [T1, Lemma 3.3] imply that the linear condition

$$\sum_{x_1 \in D_{X_1} \setminus \{x_{1,\infty}, x_{1,0}\}} b_{x_1} \psi^{\mathrm{mp}}(x_1) = \psi^{\mathrm{mp}}(x_{1,0}) = x_{2,0}$$

with respect to $x_{2,\infty}$ and $x_{2,0}$ on (X_2, D_{X_2}) holds. This completes the proof of the proposition. □

6. Explicit constructions of differences of tame fundamental groups

In this section, we apply the results obtained in previous sections to construct explicitly differences of tame fundamental groups of certain non-isomorphic curves. The main result of the present section is Theorem 6.2.

6.1. Anabelian conjecture via finite quotients.

6.1.1. *Notation and Settings.* Let $j \in \{1, 2\}$, and let (X_j, D_{X_j}) be a smooth pointed stable curve of type (g_X, n_X) over an algebraically closed field k_j of characteristic $p > 0$, $\pi_1^{\mathrm{t}}(U_{X_j})$ the tame fundamental group of (X_j, D_{X_j}), and $\overline{\mathbb{F}}_{p,j}$ the algebraic closure of $\mathbb{F}_p$ in k_j. Moreover, in the present section, we suppose the following conditions hold:

- $(g_X, n_X) = (0, n)$ (note that we have $n \geq 3$);
- $k_1 \overset{\mathrm{def}}{=} \overline{\mathbb{F}}_{p,1}$.

Then, without loss of generality, we may assume

$$X_1 = \mathbb{P}^1_{k_1},\ D_{X_1} = \{x_{1,\infty} \overset{\mathrm{def}}{=} \infty, x_{1,0} \overset{\mathrm{def}}{=} 0, x_{1,1} \overset{\mathrm{def}}{=} 1, x_{1,2}, \ldots, x_{1,n-2}\},$$

where $x_{1,u} \in k_1$ for all $u \in \{2, \ldots, n-2\}$.

Minimal models of curves. Let (X, D_X) be a smooth pointed stable curve over an algebraically closed field k of characteristic $p > 0$. We denote by k^{m} the *minimal* algebraically closed subfield of k over which U_X can be defined. Thus, by considering the function field of X, we obtain a smooth pointed stable curve $(X^{\mathrm{m}}, D_{X^{\mathrm{m}}})$ (i.e., *a minimal model of* (X, D_X) in the sense of [T2, Definition 1.30 and Lemma 1.31]) such that $U_X \cong U_{X^{\mathrm{m}}} \times_{k^{\mathrm{m}}} k$ as k-schemes, where $U_{X^{\mathrm{m}}} \overset{\mathrm{def}}{=} X^{\mathrm{m}} \setminus D_{X^{\mathrm{m}}}$. Note that the tame fundamental group $\pi_1^{\mathrm{t}}(U_{X^{\mathrm{m}}})$ of $(X^{\mathrm{m}}, D_{X^{\mathrm{m}}})$ is naturally isomorphic to the tame fundamental group $\pi_1^{\mathrm{t}}(U_X)$ of (X, D_X).

6.1.2. Since $x_{1,u} \in k_1 \overset{\mathrm{def}}{=} \overline{\mathbb{F}}_{p,1}$ for all $u \in \{2, \ldots, n-2\}$, there exists a positive number

$$r \in \mathbb{N}$$

prime to p such that $\mathbb{F}_p(\zeta_r) \subseteq \overline{\mathbb{F}}_{p,1}$ is a subfield contains rth roots of $x_{1,2}, \ldots, x_{1,n-2}$, where ζ_r denotes a fixed primitive rth root of unity in $\overline{\mathbb{F}}_{p,1}$. We put

$$t \overset{\mathrm{def}}{=} [\mathbb{F}_p(\zeta_r) : \mathbb{F}_p].$$

For each $x_{1,u} \in \{x_{1,2}, \ldots, x_{1,n-2}\}$, we fix a rth root $x_{1,u}^{1/r}$ in $\mathbb{F}_p(\zeta_r)$. Then we have the following linear condition:

$$x_{1,u}^{1/r} = \sum_{v=0}^{t-1} b_{1,uv} \zeta_r^v,\ u \in \{2, \ldots, n-2\},$$

where $b_{1,uv} \in \mathbb{F}_p$ for each $u \in \{2, \ldots, n-2\}$ and each $v \in \{0, \ldots, t-1\}$.

Let $X_1 \setminus \{x_{1,\infty}\} = \mathrm{Spec}\, \overline{\mathbb{F}}_{p,1}[\mathfrak{x}_1]$, and let $f_{B_1} : (X_{B_1}, D_{X_{B_1}}) \to (X_1, D_{X_1})$ be the Galois tame covering over $\overline{\mathbb{F}}_{p,1}$ with Galois group $\mathbb{Z}/r\mathbb{Z}$ determined by the equation $\mathfrak{y}_1^r = \mathfrak{x}_1$,

(g_B, n_B) the type of $(X_{B_1}, D_{X_{B_1}})$, and B_1 the open normal subgroup of $\pi_1^{\mathrm{t}}(U_{X_1})$ corresponding to the tame covering f_{B_1}. Then f_{B_1} is totally ramified over $\{x_{1,\infty} = \infty, x_{1,0} = 0\}$ and is étale over $D_{X_1} \setminus \{x_{1,\infty}, x_{1,0}\}$. Note that $X_{B_1} = \mathbb{P}^1_{\mathbb{F}_{p,1}}$, and that the marked points of $(X_{B_1}, D_{X_{B_1}})$ over $\{x_{1,\infty}, x_{1,0}\}$ are $\{x_{B_1,\infty} \overset{\mathrm{def}}{=} \infty, x_{B_1,0} \overset{\mathrm{def}}{=} 0\}$. We put

$$x_{B_1,u} \overset{\mathrm{def}}{=} x_{1,u}^{1/r} \in D_{X_{B_1}},\ u \in \{2, \dots, n-2\},$$

$$x_{B_1,1}^v \overset{\mathrm{def}}{=} \zeta_r^v \in D_{X_{B_1}},\ v \in \{0, \dots, t-1\}.$$

Note that we have $x_{B_1,1}^0 = 1$. Thus, we obtain a linear condition

$$x_{B_1,u} = \sum_{v=0}^{t-1} b_{1,uv} x_{B_1,1}^v \text{ (or equivalently, } 0 = x_{B_1,0} = x_{B_1,u} - \Big(\sum_{v=0}^{t-1} b_{1,uv} x_{B_1,1}^v\Big))$$

with respect to $x_{B_1,\infty}$ and $x_{B_1,0}$ on $(X_{B_1}, D_{X_{B_1}})$ for each $u \in \{2, \dots, n-2\}$. Moreover, let $b'_{1,uv} \in \mathbb{Z}_{\geq 0}$, $u \in \{2, \dots, n-2\}$, $v \in \{0, \dots, s-1\}$, be a natural number such that $b'_{1,uv} \equiv b_{1,uv} \pmod p$ and $\sum_{v=0}^{t-1} b'_{1,uv} \geq 2$, $u \in \{2, \dots, n-2\}$.

Let

$$t'$$

be a positive natural number such that $p^{t'} - 2 \geq \max_{u \in \{2,\dots,n-2\}} \{\sum_{v=0}^{t-1} b'_{1,uv}\}$ holds.

6.1.3. Now, we fix natural numbers

$$\ell,\ d,\ \mathcal{I} \overset{\mathrm{def}}{=} \{b_0, b_1\} \subseteq \mathbb{N},\ c(\mathcal{I}),$$

satisfying the following conditions hold

- ℓ and d are prime numbers distinct from p and distinct from each other such that $\ell \equiv 1 \pmod d$. Note that all dth roots of unity are contained in $\mathbb{F}_\ell$.
- $r | b_0$, $(p^t - 1) | b_0$, $(p^{t'} - 1) | b_1$, $p | b_1$, and $(\ell, b_0 b_1) = (d, b_0 b_1) = 1$, where r, t, t' are the natural numbers defined in 6.1.2.
- Let $e(\mathcal{I})' \overset{\mathrm{def}}{=} \ell d b_0 b_1$. We put $e(\mathcal{I}) \overset{\mathrm{def}}{=} \#\big(\widehat{\Gamma}_{0,n} / D^{(a+2)}_{e(\mathcal{I})'}(\widehat{\Gamma}_{0,n})\big)$ (see 2.1.4 for $D^{(a+2)}_{e(\mathcal{I})'}(-)$ and 3.1.1 for $\widehat{\Gamma}_{0,n}$). Then we have $p | c(\mathcal{I})$, $e(\mathcal{I}) | c(\mathcal{I})$, $(p^t - 1) | p^{t_\mathcal{I}} - 1$, and $(p^{t'} - 1) | p^{t_\mathcal{I}} - 1$, and $(p^{t_\mathcal{I}} - 1) | c(\mathcal{I})$, where $t_\mathcal{I}$ satisfies $p^{t_\mathcal{I}} - 1 > C\big(e(\mathcal{I})(2n)\big)$ (see 3.1.1 for $C(-)$).

Note that $c(\mathcal{I})$ *depends only on the isomorphism class of* $U_{X_1^{\mathrm{m}}}$.

We put $D_\mathcal{I}(\pi_1^{\mathrm{t}}(U_{X_2})) \overset{\mathrm{def}}{=} D^{(1)}_{b_1}(D^{(1)}_{b_0}(\pi_1^{\mathrm{t}}(U_{X_2})))$ and (Y_2, D_{Y_2}) the smooth pointed stable curve over k_2 of type (g_Y, n_Y) corresponding to $D^{(1)}_{b_0}(\pi_1^{\mathrm{t}}(U_{X_2})) \subseteq \pi_1^{\mathrm{t}}(U_{X_2})$. Note that we have $g_Y \geq 2$. If we put $\mathcal{I}_1 \overset{\mathrm{def}}{=} \{b_1\}$, $e(\mathcal{I})' \overset{\mathrm{def}}{=} \ell d b_1$, and $e(\mathcal{I}_1) \overset{\mathrm{def}}{=} \#\big(\widehat{\Gamma}_{g_Y,n_Y} / D^{(a+2)}_{e(\mathcal{I}_1)'}(\widehat{\Gamma}_{g_Y,n_Y})\big)$ (see 3.1.1 for $\widehat{\Gamma}_{g_Y,n_Y}$), then we have $p^{t_\mathcal{I}} - 1 > C(e(\mathcal{I}_1)(2g_Y + 2n_Y))$. This means that $\ell, d, c(\mathcal{I})$ satisfy the conditions introduced in 3.1.1.

6.1.4. We have the following result:

Theorem 6.1. *We maintain the notation and the settings introduced in 6.1.1. Let* $c(\mathcal{I})$ *be a natural number depending on* $U_{X_1^{\mathrm{m}}}$ *constructed in 6.1.3 and* $\pi_1^{\mathrm{t}}(U_{X_2}) / D^{(6)}_{c(\mathcal{I})}(\pi_1^{\mathrm{t}}(U_{X_2}))$ *a finite group depending on* $U^{\mathrm{m}}_{X_1}$ *and* $U^{\mathrm{m}}_{X_2}$. *Then*

$$U_{X_1^{\mathrm{m}}} \cong U_{X_2^{\mathrm{m}}}$$

as schemes if and only if

$$\pi_1^{\mathrm{t}}(U_{X_2}) / D^{(6)}_{c(\mathcal{I})}(\pi_1^{\mathrm{t}}(U_{X_2})) \in \pi_A^{\mathrm{t}}(U_{X_1}) \cap \pi_A^{\mathrm{t}}(U_{X_2}).$$

Proof. The "only if" part of the theorem is trivial. We treat the"if" part of the theorem. Denote by $N_2 \overset{\text{def}}{=} D^{(6)}_{c(\mathcal{I})}(\pi_1^{\text{t}}(U_{X_2}))$, $H_2 \overset{\text{def}}{=} D_{\mathcal{I}}(\pi_1^{\text{t}}(U_{X_2}))$, $Q_{N_2} \overset{\text{def}}{=} \pi_1^{\text{t}}(U_{X_2})/D^{(6)}_{c(\mathcal{I})}(\pi_1^{\text{t}}(U_{X_2}))$, and $Q_{H_2} \overset{\text{def}}{=} \pi_1^{\text{t}}(U_{X_2})/D_{\mathcal{I}}(\pi_1^{\text{t}}(U_{X_2}))$. Since $Q_{N_2} \in \pi_A^{\text{t}}(U_{X_1}) \cap \pi_A^{\text{t}}(U_{X_2})$, we take an arbitrary surjection $\phi : \pi_1^{\text{t}}(U_{X_1}) \twoheadrightarrow Q_{N_2}$ and put $\psi : \pi_1^{\text{t}}(U_{X_1}) \overset{\phi}{\twoheadrightarrow} Q_{N_2} \twoheadrightarrow Q_{H_2}$ the composition of surjections, where $Q_{N_2} \twoheadrightarrow Q_{H_2}$ is the natural surjection induced by $N_2 \subseteq H_2$. By Theorem 4.6, we obtain that (Q_{N_2}, Q_{H_2}) is a quasi-anabelian pair associated to $\pi_1^{\text{t}}(U_{X_2})$. Then ψ induces a surjection

$$\psi^{\text{op}} : \text{Edg}^{\text{op}}(\pi_1^{\text{t}}(U_{X_1})) \twoheadrightarrow \text{Edg}^{\text{op}}(Q_{H_2}).$$

Denote by $N_1 \overset{\text{def}}{=} \ker(\phi) \subseteq H_1 \overset{\text{def}}{=} \ker(\psi) \subseteq \pi_1^{\text{t}}(U_{X_1})$, $Q_{N_1} \overset{\text{def}}{=} \pi_1^{\text{t}}(U_{X_1})/N_1$, and $Q_{H_1} \overset{\text{def}}{=} \pi_1^{\text{t}}(U_{X_1})/H_1$. Let $F_2 \subseteq \pi_1^{\text{t}}(U_{X_2})$ be an arbitrary open subgroup such that $H_2 \subseteq F_2$, and that $\#(\pi_1^{\text{t}}(U_{X_2})/F_2)$ is prime to p. Denote by $Q_{F_2} \overset{\text{def}}{=} \pi_1^{\text{t}}(U_{X_2})/F_2$, $F_1 \overset{\text{def}}{=} \psi^{-1}(Q_{F_2}) \subseteq \pi_1^{\text{t}}(U_{X_1})$, and $Q_{F_1} \overset{\text{def}}{=} \pi_1^{\text{t}}(U_{X_1})/F_1$. Then ϕ and ψ induce a commutative diagram

$$\begin{array}{ccc}
\pi_1^{\text{t}}(U_{X_1}) & = & \pi_1^{\text{t}}(U_{X_1}) \\
\downarrow & & \downarrow{\scriptstyle \psi} \\
Q_{H_1} & \xrightarrow{\overline{\psi}} & Q_{H_2} \\
\downarrow & & \downarrow \\
Q_{F_1} & \xrightarrow{\overline{\rho}_{F_1}} & Q_{F_2},
\end{array}$$

where $\overline{\psi}$, $\overline{\rho}_{F_1}$ are isomorphisms. Furthermore, the above commutative diagram implies the following commutative diagram

$$\begin{array}{ccc}
\text{Edg}^{\text{op}}(\pi_1^{\text{t}}(U_{X_1})) & \xrightarrow{\psi^{\text{op}}} & \text{Edg}^{\text{op}}(Q_{H_2}) \\
\downarrow & & \| \\
\text{Edg}^{\text{op}}(Q_{H_1}) & \xrightarrow{\overline{\psi}^{\text{op}}} & \text{Edg}^{\text{op}}(Q_{H_2}) \\
\downarrow & & \downarrow \\
\text{Edg}^{\text{op}}(Q_{F_1}) & \xrightarrow{\overline{\rho}^{\text{op}}_{F_1}} & \text{Edg}^{\text{op}}(Q_{F_2}),
\end{array}$$

where the vertical arrows are the natural surjections induced by $\pi_1^{\text{t}}(U_{X_1}) \twoheadrightarrow Q_{F_1}$ and $Q_{H_2} \twoheadrightarrow Q_{F_2}$, respectively, and $\overline{\psi}^{\text{op}}, \overline{\rho}^{\text{op}}_{F_1}$ denote the bijections induced by ψ^{op}. Note that we have the following bijections

$$\text{Edg}^{\text{op}}(\pi_1^{\text{t}}(U_{X_1}))/\pi_1^{\text{t}}(U_{X_1}) \overset{\sim}{\to} \text{Edg}^{\text{op}}(Q_{H_1})/Q_{H_1} \overset{\sim}{\to} \text{Edg}^{\text{op}}(Q_{F_1})/Q_{F_1} \overset{\sim}{\to} D_{X_1},$$

$$\text{Edg}^{\text{op}}(Q_{H_2})/Q_{H_2} \overset{\sim}{\to} \text{Edg}^{\text{op}}(Q_{F_2})/Q_{F_2} \overset{\sim}{\to} D_{X_2}.$$

Then ψ^{op} (or $\overline{\psi}^{\text{op}}$, $\overline{\rho}^{\text{op}}_{F_1}$) induces a bijection $\psi^{\text{mp}} : D_{X_1} \overset{\sim}{\to} D_{X_2}$ of sets of marked points.

Reconstructing field structures. Let $F_j = O_j \overset{\text{def}}{=} D_{p^t-1}(\pi_1^{\text{t}}(U_{X_j}))$. Note that the conditions $(p^t-1)|b_0$ and $p|b_1$ (see 6.1.3) imply $H_2 \subseteq D_p^{(1)}(O_2) = D_p^{(1)}(D^{(1)}_{p^t-1}(\pi_1^{\text{t}}(U_{X_2})))$. Let $x_{2,\infty} \overset{\text{def}}{=} \psi^{\text{mp}}(x_{1,\infty})$, $x_{2,u} \overset{\text{def}}{=} \psi^{\text{mp}}(x_{1,u})$, $u \in \{0, \dots, n-2\}$, and let $\widehat{x}_{1,\infty}, \widehat{x}_{1,u}, u \in \{0, \dots, n-2\}$, be the elements of $D_{\widehat{X}_1}$ over $x_{1,\infty}, x_{1,u}, u \in \{0, \dots, n-2\}$, respectively. We put

$$I_{x_{H_2,\infty}} \overset{\text{def}}{=} \psi^{\text{op}}(I_{\widehat{x}_{1,\infty}}),\ I_{x_{H_2,u}} \overset{\text{def}}{=} \psi^{\text{op}}(I_{\widehat{x}_{1,u}}), u \in \{0, \dots, n-2\},$$

and put

$$I_{x_{O_1,\infty}},\ I_{x_{O_1,u}}, u \in \{0,\dots,n-2\},$$

$$I_{x_{O_2,\infty}},\ I_{x_{O_2,u}}, u \in \{0,\dots,n-2\},$$

the images of $I_{\widehat{x}_{1,\infty}}, I_{\widehat{x}_{1,u}}, u \in \{0,\dots,n-2\}$ of the surjections $\mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{X_1})) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(Q_{O_1})$, $\mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{X_1})) \overset{\psi^{\mathrm{op}}}{\twoheadrightarrow} \mathrm{Edg}^{\mathrm{op}}(Q_{H_2}) \twoheadrightarrow \mathrm{Edg}^{\mathrm{op}}(Q_{O_2})$, respectively. In particular, we have $\overline{\rho}^{\mathrm{op}}_{O_1}(I_{x_{O_1,0}}) = I_{x_{O_2,0}})$.

Let $\mathbb{F}_{\widehat{x}_{j,0}} \overset{\mathrm{def}}{=} (I_{\widehat{x}_{j,0}} \otimes (\mathbb{Q}/\mathbb{Z})_j^{p'}) \sqcup \{*_{\widehat{x}_{j,0}}\}$ (5.1.1). Then $\mathbb{F}_{\widehat{x}_{j,0}}$ can be identified with $\overline{\mathbb{F}}_{p,j}$ as fields, whose multiplicative group is $I_{\widehat{x}_{j,0}} \otimes_{\mathbb{Z}} (\mathbb{Q}/\mathbb{Z})_j^{p'}$, and whose zero element is $*_{\widehat{x}_{j,0}}$. By applying Proposition 5.2, the isomorphism $\rho_{x_{O_1,0},x_{O_2,0}} \overset{\mathrm{def}}{=} \overline{\rho}_{O_1}|_{I_{x_{O_1,0}}} : I_{x_{O_1,0}} \overset{\sim}{\to} I_{x_{O_2,0}}$ induces a field isomorphism

$$\rho^{\mathrm{fd}}_{x_{O_1,0},x_{O_2,0}} : \mathbb{F}_{x_{O_1,0},t} \overset{\sim}{\to} \mathbb{F}_{x_{O_2,0},t},$$

where $\mathbb{F}_{x_{O_j,0},t} \overset{\mathrm{def}}{=} I_{x_{O_j,0}} \sqcup \{*_{\widehat{x}_{j,0}}\}$ admits a structure of field induced by $\overline{\mathbb{F}}_{p,j}$ which is isomorphic to the subfield of $\overline{\mathbb{F}}_{p,j}$ with cardinality p^t. Thus, $\mathbb{F}_{x_{O_1,0},t}$ can be regarded as the subfield $\mathbb{F}_p(\zeta_r)$ of $\overline{\mathbb{F}}_{p,1}$. Moreover, we put

$$\xi_r \overset{\mathrm{def}}{=} \rho^{\mathrm{fd}}_{x_{O_1,0},x_{O_2,0}}(\zeta_r) \in \mathbb{F}_{x_{O_2,0},t}.$$

Then $\mathbb{F}_{x_{O_2,0},t}$ can be regarded as the subfield $\mathbb{F}_p(\xi_r)$ of $\overline{\mathbb{F}}_{p,2}$.

Constructing tame covering of (X_2, D_{X_2}) *corresponding* f_{B_1}. Let $B_1 \subseteq \pi_1^{\mathrm{t}}(U_{X_1})$ be the open normal subgroup introduced in 6.1.2 and B_2 the inverse image of $\psi(B_1) \subseteq Q_{H_2}$ of the natural surjection $\pi_1^{\mathrm{t}}(U_{X_2}) \twoheadrightarrow Q_{H_2}$. Note that $r = \#(\pi_1^{\mathrm{t}}(U_{X_1})/B_1) = \#(\pi_1^{\mathrm{t}}(U_{X_2})/B_2)$ is prime to p (6.1.2). Let $F_j = B_j$. Write

$$f_{B_2} : (X_{B_2}, D_{X_{B_2}}) \to (X_2, D_{X_2})$$

for the Galois tame covering over k_2 with Galois group $\mathbb{Z}/r\mathbb{Z}$ corresponding to B_2. Then the isomorphism $\overline{\rho}_{B_1}$ and the bijection $\overline{\rho}^{\mathrm{op}}_{B_1}$ imply that f_{B_2} is totally ramified over $\{x_{2,\infty}, x_{2,0}\}$ and is étale over $D_{X_2} \setminus \{x_{2,\infty}, x_{2,0}\}$. Note that we have $X_{B_2} = \mathbb{P}^1_{k_2}$, and that the types of $(X_{B_1}, D_{X_{B_1}})$ and $(X_{B_2}, D_{X_{B_2}})$ are equal (i.e., (g_B, n_B)).

The construction concerning $c(\mathcal{I})$ (6.1.3) implies $N_j \subseteq H_j \subseteq B_j$. We put $\overline{B}_{N_j} \overset{\mathrm{def}}{=} B_j/N_j \subseteq Q_{N_j}$, $\overline{B}_{H_j} \overset{\mathrm{def}}{=} B_j/H_j \subseteq Q_{H_j}$. Then ϕ and ψ induce the following commutative diagram

$$\begin{array}{ccc}
B_1 & = & B_1 \\
\downarrow & & \downarrow{\scriptstyle \phi_{B_1}} \\
\overline{B}_{N_1} & \xrightarrow{\overline{\phi}_{B_1}} & \overline{B}_{N_2} \\
\downarrow & & \downarrow \\
\overline{B}_{H_1} & \xrightarrow{\overline{\psi}_{B_1}} & \overline{B}_{H_2}.
\end{array}$$

On the other hand, we see $\mathrm{Edg}^{\mathrm{op}}(B_j) = \{I \cap B_j \mid I \in \mathrm{Edg}^{\mathrm{op}}(\pi_1^{\mathrm{t}}(U_{X_j}))\}$, $\mathrm{Edg}^{\mathrm{op}}(B_{H_j}) = \{I \cap B_{H_j} \mid I \in \mathrm{Edg}^{\mathrm{op}}(Q_{H_j})\}$. Then ψ^{op} and $\overline{\psi}^{\mathrm{op}}$ induce the following commutative diagram

$$\begin{array}{ccc} \mathrm{Edg}^{\mathrm{op}}(B_1) & \xrightarrow{\psi^{\mathrm{op}}_{B_1}} & \mathrm{Edg}^{\mathrm{op}}(\overline{B}_{H_2}) \\ \downarrow & & \| \\ \mathrm{Edg}^{\mathrm{op}}(\overline{B}_{H_1}) & \xrightarrow{\overline{\psi}^{\mathrm{op}}_{B_1}} & \mathrm{Edg}^{\mathrm{op}}(\overline{B}_{H_2}). \end{array}$$

Note that we have

$$\mathrm{Edg}^{\mathrm{op}}(B_1)/B_1 \xrightarrow{\sim} \mathrm{Edg}^{\mathrm{op}}(\overline{B}_{H_1})/\overline{B}_{H_1} \xrightarrow{\sim} D_{X_{B_1}},$$
$$\mathrm{Edg}^{\mathrm{op}}(\overline{B}_{H_2})/\overline{B}_{H_2} \xrightarrow{\sim} D_{X_{B_2}}.$$

Then $\overline{\psi}^{\mathrm{op}}_{B_1}$ induces a bijection $\overline{\psi}^{\mathrm{mp}}_{B_1} : D_{X_{B_1}} \xrightarrow{\sim} D_{X_{B_2}}$ of sets of marked points. We put

$$x_{B_2,\infty} \overset{\mathrm{def}}{=} \overline{\psi}^{\mathrm{mp}}_{B_1}(x_{B_1,\infty}),\ x_{B_2,0} \overset{\mathrm{def}}{=} \overline{\psi}^{\mathrm{mp}}_{B_1}(x_{B_2,0}),$$
$$x_{B_2,u} \overset{\mathrm{def}}{=} \overline{\psi}^{\mathrm{mp}}_{B_1}(x_{B_2,u}),\ u \in \{0, \ldots, n-2\},\ x^v_{B_2,1} \overset{\mathrm{def}}{=} \overline{\psi}^{\mathrm{mp}}_{B_1}(x^v_{B_1,1}),\ v \in \{0, \ldots, t-1\},$$

where $x_{B_1,\infty}, x_{B_1,0}, x_{B_1,u}, x^v_{B_1,1} \in D_{X_{B_1}}$ are the marked points introduced in 6.1.2. Without loss of generality, we may put $x^0_{B_2,1} = 1$.

Constructing a linear condition on $(X_{B_2}, D_{X_{B_2}})$. By the constructions given in 6.1.2, we have a linear condition

$$x_{B_1,u} = \sum_{v=0}^{t-1} b_{1,uv} x^v_{B_1,1} = \sum_{v=0}^{t-1} b_{1,uv}(\zeta^v_r \cdot x^0_{B_1,1}) = \sum_{v=0}^{t-1} b_{1,uv}\zeta^v_r$$

with respect to $x_{B_1,\infty}$ and $x_{B_1,0}$ on $(X_{B_1}, D_{X_{B_1}})$ for each $u \in \{2, \ldots, n-2\}$. Note that the condition concerning b_1 implies $H_2 \subseteq D^{(1)}_{p^{t'}-1}(B_2)$. Then by replacing $\pi_1^{\mathrm{t}}(U_{X_j})$, H_j, O_j, Q_{H_2}, and $\psi : \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow Q_{H_2}$ in 5.1.1 by B_j, H_j, $D^{(1)}_{p^{t'}-1}(B_j)$, $\overline{B}_{H_2}$, and $\psi_{B_1} : B_1 \overset{\phi_{B_1}}{\twoheadrightarrow} \overline{B}_{N_2} \twoheadrightarrow \overline{B}_{H_2}$, respectively, and by applying Proposition 5.3, the linear condition

$$x_{B_2,u} = \sum_{v=0}^{t-1} b_{1,uv} x^v_{B_2,1}$$

with respect to $x_{B_2,\infty}$ and $x_{B_2,0}$ on $(X_{B_2}, D_{X_{B_2}})$ holds for each $u \in \{2, \ldots, n-2\}$. Since $\xi^v_r \cdot x^0_{B_2,1} = x^v_{B_2,1}$, we obtain

$$x_{B_2,u} = \sum_{v=0}^{t-1} b_{1,uv} x^v_{B_2,1} = \sum_{v=0}^{t-1} b_{1,uv}(\xi^v_r \cdot x^0_{B_2,1}) = \sum_{v=0}^{t-1} b_{1,uv}\xi^v_r.$$

Then we have

$$x_{1,u} = x^r_{B_1,u} = \left(\sum_{v=0}^{t-1} b_{1,uv}\zeta^v_r\right)^r,\ x_{2,u} = x^r_{B_2,u} = \left(\sum_{v=0}^{t-1} b_{1,uv}\xi^v_r\right)^r,\ u \in \{2, \ldots n-2\}.$$

Moreover, $\rho^{\mathrm{fd}}_{x_{O_1,0}, x_{O_2,0}}(\xi_r) = \zeta_r$ implies

$$\rho^{\mathrm{fd}}_{x_{O_1,0}, x_{O_2,0}}(x_{1,u}) = x_{2,u}.$$

Thus, we obtain

$$\mathbb{P}^1_{\mathbb{F}_{x_{O_1,0},t}} \setminus \{x_{1,\infty} = \infty, x_{1,0} = 0, x_{1,1} = 1, x_{1,2}, \ldots, x_{1,n-2}\}$$
$$\xrightarrow{\sim} \mathbb{P}^1_{\mathbb{F}_{x_{O_2,0},t}} \setminus \{x_{2,\infty} = \infty, x_{2,0} = 0, x_{2,1} = 1, \rho^{\mathrm{fd}}_{x_{O_1,0}, x_{O_2,0}}(x_{1,2}), \ldots, \rho^{\mathrm{fd}}_{x_{O_1,0}, x_{O_2,0}}(x_{1,n-2})\}.$$

Since

$$U_{X_1} = U_{X_1^{\mathrm{m}}} \xrightarrow{\sim} \left(\mathbb{P}^1_{\mathbb{F}_{x_{O_1,0,t}}} \setminus \{x_{1,\infty} = \infty, x_{1,0} = 0, x_{1,1} = 1, x_{1,2}, \dots, x_{1,n-2}\}\right) \times_{\mathbb{F}_{x_{O_1,0,t}}} k_1,$$

$$U_{X_2^{\mathrm{m}}} \xrightarrow{\sim} \left(\mathbb{P}^1_{\mathbb{F}_{x_{O_2,0,t}}} \setminus \{x_{1,\infty} = \infty, x_{2,0} = 0, x_{2,1} = 1, x_{2,2}, \dots, x_{1,n-2}\}\right) \times_{\mathbb{F}_{x_{O_2,0,t}}} \overline{\mathbb{F}}_{p,2},$$

we obtain $U_{X_1^{\mathrm{m}}} \cong U_{X_2^{\mathrm{m}}}$ as schemes. This completes the proof of the theorem. □

6.2. Main result.

6.2.1. Now, we can state our main result of the present paper:

Theorem 6.2. *Let $j \in \{1, 2\}$, and let (X_j, D_{X_j}) be a smooth pointed stable curve of type (g_{X_j}, n_{X_j}) over an algebraically closed field k_j of characteristic $p > 0$ and $\pi_1^{\mathrm{t}}(U_{X_j})$ the tame fundamental group of (X_j, D_{X_j}). Then the following statements hold (see 2.1.4 for $D^{(-)}_{(-)}(-)$):*

(i) Suppose $2g_{X_1} + n_{X_1} \neq 2g_{X_2} + n_{X_2}$. Let ℓ' be a prime number distinct from p. Then we have

$$\pi_1^{\mathrm{t}}(U_{X_2})/D^{(1)}_{\ell'}(\pi_1^{\mathrm{t}}(U_{X_2})) \notin \pi_A^{\mathrm{t}}(U_{X_1}),\ \pi_1^{\mathrm{t}}(U_{X_2})/D^{(1)}_{\ell'}(\pi_1^{\mathrm{t}}(U_{X_2})) \in \pi_A^{\mathrm{t}}(U_{X_2}).$$

(ii) Suppose $m \overset{\mathrm{def}}{=} 2g_{X_1}+n_{X_1} = 2g_{X_2}+n_{X_2}$. Let c be a positive natural number satisfying $p(p^t-1)|c$ and $p^t - 1 \geq C(m)$ (see 3.1.1 for $C(m)$). If $g_{X_1} + n_{X_1} < g_{X_2} + n_{X_2}$, we have

$$\pi_1^{\mathrm{t}}(U_{X_2})/D^{(2)}_{c}(\pi_1^{\mathrm{t}}(U_{X_2})) \notin \pi_A^{\mathrm{t}}(U_{X_1}),\ \pi_1^{\mathrm{t}}(U_{X_2})/D^{(2)}_{c}(\pi_1^{\mathrm{t}}(U_{X_2})) \in \pi_A^{\mathrm{t}}(U_{X_2}).$$

If $g_{X_1} + n_{X_1} > g_{X_2} + n_{X_2}$, we have

$$\pi_1^{\mathrm{t}}(U_{X_1})/D^{(2)}_{c}(\pi_1^{\mathrm{t}}(U_{X_1})) \in \pi_A^{\mathrm{t}}(U_{X_1}),\ \pi_1^{\mathrm{t}}(U_{X_1})/D^{(2)}_{c}(\pi_1^{\mathrm{t}}(U_{X_1}))) \notin \pi_A^{\mathrm{t}}(U_{X_2}).$$

(iii) Suppose that $2g_{X_1} + n_{X_1} = 2g_{X_2} + n_{X_2}$ and $g_{X_1} + n_{X_1} = g_{X_2} + n_{X_2}$ (or equivalently, $(g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$), that $U_{X_1^{\mathrm{m}}} \not\cong U_{X_2^{\mathrm{m}}}$ as schemes (see 6.1.1 for $U_{X_j^{\mathrm{m}}}$), that k_1^{m} (6.1.1) is an algebraic closure of the finite field $\mathbb{F}_p$, and that $g_{X_1} = 0$. Let $c(\mathcal{I})$ be a natural number depending on $U_{X_1^{\mathrm{m}}}$ constructed in 6.1.3. Then we have

$$\pi_1^{\mathrm{t}}(U_{X_2})/D^{(6)}_{c(\mathcal{I})}(\pi_1^{\mathrm{t}}(U_{X_2})) \notin \pi_A^{\mathrm{t}}(U_{X_1}),\ \pi_1^{\mathrm{t}}(U_{X_2})/D^{(6)}_{c(\mathcal{I})}(\pi_1^{\mathrm{t}}(U_{X_2})) \in \pi_A^{\mathrm{t}}(U_{X_2}).$$

Proof. (i) follows immediately from the structures of the maximal prime-to-p quotients of tame fundamental groups, and (iii) follows immediately from Theorem 6.1. We treat (ii). Suppose $g_{X_1} + n_{X_1} < g_{X_2} + n_{X_2}$. We put $G_j \overset{\mathrm{def}}{=} \pi_1^{\mathrm{t}}(U_{X_j})/D^{(2)}_{c}(\pi_1^{\mathrm{t}}(U_{X_j}))$.

Suppose $G_2 \in \pi_A^{\mathrm{t}}(U_{X_1})$. Let $\phi : \pi_1^{\mathrm{t}}(U_{X_1}) \twoheadrightarrow G_2$ be an arbitrary surjection. Then we see immediately that the surjection ϕ factors through G_1. This means that ϕ induces a surjection $\overline{\phi} : G_1 \twoheadrightarrow G_2$. By [Y5, Theorem 5.4] and its proof (in particular, line 4, page 82 of [Y5]), we have (see 2.3.3 for $\gamma^{\max}_{G_j}$)

$$\gamma^{\max}_{G_1} = g_{X_1} + n_{X_1} - 2 \geq g_{X_2} + n_{X_2} - 2 = \gamma^{\max}_{G_2}.$$

This contradicts the assumption $g_{X_1} + n_{X_1} < g_{X_2} + n_{X_2}$. Then we have $G_2 \notin \pi_A^{\mathrm{t}}(U_{X_1})$.

Similar arguments to the arguments given above imply (ii) holds if $g_{X_1}+n_{X_1} > g_{X_2}+n_{X_2}$. We complete the proof of (ii). □

6.2.2. Theorem 6.2 implies the following anabelian result:

Theorem 6.3. *Let* $j \in \{1, 2\}$*, and let* (X_j, D_{X_j}) *be a smooth pointed stable curve of type* (g_{X_j}, n_{X_j}) *over an algebraically closed field* k_j *of characteristic* $p > 0$ *and* $\pi_1^{\mathrm{t}}(U_{X_j})$ *the tame fundamental group of* (X_j, D_{X_j})*. Suppose that* k_1^{m} *(6.1.1) is an algebraic closure of the finite field* $\mathbb{F}_p$*, and that* $g_{X_1} = 0$*. Let* $c(\mathcal{I})$ *be a natural number depending on* $U_{X_1^{\mathrm{m}}}$ *constructed in 6.1.3 and (see 2.1.4 for* $D_{(-)}^{(-)}(-)$*)*

$$\mathfrak{G} \overset{\mathrm{def}}{=} \{\pi_1^{\mathrm{t}}(U_{X_1})/D_{c(\mathcal{I})}^{(2)}(\pi_1^{\mathrm{t}}(U_{X_1})),\ \pi_1^{\mathrm{t}}(U_{X_2})/D_{c(\mathcal{I})}^{(6)}(\pi_1^{\mathrm{t}}(U_{X_2}))\}$$

a set of finite groups depending on $U_{X_1^{\mathrm{m}}}$ *and* $U_{X_2^{\mathrm{m}}}$ *(see 6.1.1 for* $U_{X_j^{\mathrm{m}}}$*). Then we have that*

$$U_{X_1^{\mathrm{m}}} \cong U_{X_2^{\mathrm{m}}}$$

as schemes if and only if

$$\mathfrak{G} \subseteq \pi_A^{\mathrm{t}}(U_{X_1}) \cap \pi_A^{\mathrm{t}}(U_{X_2}).$$

Moreover, suppose further $g_{X_1} = g_{X_2} = 0$ *and* $n_{X_1} = n_{X_2}$*. Then we have that*

$$U_{X_1^{\mathrm{m}}} \cong U_{X_2^{\mathrm{m}}}$$

as schemes if and only if

$$\pi_1^{\mathrm{t}}(U_{X_2})/D_{c(\mathcal{I})}^{(6)}(\pi_1^{\mathrm{t}}(U_{X_2})) \in \pi_A^{\mathrm{t}}(U_{X_1}) \cap \pi_A^{\mathrm{t}}(U_{X_2}).$$

Proof. The "only if" part of the theorem is trivial. We only treat the "if" part of the theorem. Note that the construction of $c(\mathcal{I})$ (see 6.1.3) implies that there exists a prime number $\ell' | c(\mathcal{I})$ distinct from p, and that $c(\mathcal{I})$ satisfies the conditions concerning the natural number c mentioned in Theorem 6.2 (ii). We put

$$\mathfrak{G}' \overset{\mathrm{def}}{=} \{\pi_1^{\mathrm{t}}(U_{X_1})/D_{\ell'}^{(1)}(\pi_1^{\mathrm{t}}(U_{X_1})),\ \pi_1^{\mathrm{t}}(U_{X_1})/D_{c(\mathcal{I})}^{(2)}(\pi_1^{\mathrm{t}}(U_{X_1})),$$

$$\pi_1^{\mathrm{t}}(U_{X_2})/D_{\ell'}^{(1)}(\pi_1^{\mathrm{t}}(U_{X_2})),\ \pi_1^{\mathrm{t}}(U_{X_2})/D_{c(\mathcal{I})}^{(2)}(\pi_1^{\mathrm{t}}(U_{X_2}))\}$$

Moreover, $\mathfrak{G} \subseteq \pi_A^{\mathrm{t}}(U_{X_1}) \cap \pi_A^{\mathrm{t}}(U_{X_2})$ implies

$$\mathfrak{G}' \subseteq \pi_A^{\mathrm{t}}(U_{X_1}) \cap \pi_A^{\mathrm{t}}(U_{X_2})$$

since there are natural surjections

$$\pi_1^{\mathrm{t}}(U_{X_1})/D_{c(\mathcal{I})}^{(2)}(\pi_1^{\mathrm{t}}(U_{X_1})) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_1})/D_{\ell'}^{(1)}(\pi_1^{\mathrm{t}}(U_{X_1})),$$

$$\pi_1^{\mathrm{t}}(U_{X_2})/D_{c(\mathcal{I})}^{(6)}(\pi_1^{\mathrm{t}}(U_{X_2})) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})/D_{c(\mathcal{I})}^{(2)}(\pi_1^{\mathrm{t}}(U_{X_2})) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_2})/D_{\ell'}^{(1)}(\pi_1^{\mathrm{t}}(U_{X_2})).$$

Then by Theorem 6.2 (i), (ii), we see immediately $(g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$. Thus, the theorem follows from Theorem 6.2 (iii).

Furthermore, the "moreover" part of the theorem is Theorem 6.1. This completes the proof of the theorem. □

Remark 6.3.1. Moreover, Theorem 6.3 is the best form in the following sense:

> Theorem 6.3 *does not* hold if we replace $\mathfrak{G}$ mentioned in the statement of Theorem 6.3 by a set of finite groups depending only on one of the curves $U_{X_1^{\mathrm{m}}}, U_{X_2^{\mathrm{m}}}$.

Namely, the following statement *does not* hold:

> Suppose $(0, n) \overset{\mathrm{def}}{=} (0, n_{X_1}) = (0, n_{X_2})$. Then there exists a finite group $G' \in \pi_A^{\mathrm{t}}(U_{X_2})$ such that, for an *arbitrary* smooth pointed stable curve (X_1, D_{X_1}) of type $(0, n)$, $U_{X_1^{\mathrm{m}}} \cong U_{X_2^{\mathrm{m}}}$ if and only if $G' \in \pi_A^{\mathrm{t}}(U_{X_1}) \cap \pi_A^{\mathrm{t}}(U_{X_2})$.

In fact, [Y6, Theorem 3.6] (note that the admissible fundamental group of a smooth pointed stable curve coincides with its tame fundamental group) implies that, for any finite group $G' \in \pi_A^{\mathrm{t}}(U_{X_2})$, there exists a smooth pointed stable curve (X_1, D_{X_1}) of type $(0, n)$ such that $G' \in \pi_A^{\mathrm{t}}(U_{X_1}) \cap \pi_A^{\mathrm{t}}(U_{X_2})$ holds.

In particular, Theorem 6.3 implies directly the following "finite version" of Grothendieck's anabelian conjecture which is a strong generalization of the main results of [T1], [T3].

Corollary 6.4. *Let $j \in \{1, 2\}$, and let (X_j, D_{X_j}) be a smooth pointed stable curve of type (g_{X_j}, n_{X_j}) over an algebraically closed field k_j of characteristic $p > 0$ and $\pi_1^{\mathrm{t}}(U_{X_j})$ the tame fundamental group of (X_j, D_{X_j}). Suppose that k_1^{m} (6.1.1) is an algebraic closure of the finite field $\mathbb{F}_p$, and that $g_{X_1} = 0$. Let $c(\mathcal{I})$ be a natural number depending on $U_{X_1^{\mathrm{m}}}$ constructed in 6.1.3 and (see 2.1.4 for $D^{(-)}_{(-)}(-)$). Then we have that*

$$U_{X_1^{\mathrm{m}}} \cong U_{X_2^{\mathrm{m}}}$$

as schemes if and only if

$$\pi_1^{\mathrm{t}}(U_{X_1})/D^{(6)}_{c(\mathcal{I})}(\pi_1^{\mathrm{t}}(U_{X_1})) \cong \pi_1^{\mathrm{t}}(U_{X_2})/D^{(6)}_{c(\mathcal{I})}(\pi_1^{\mathrm{t}}(U_{X_2})).$$

Proof. Since there is a natural surjection $\pi_1^{\mathrm{t}}(U_{X_1})/D^{(6)}_{c(\mathcal{I})}(\pi_1^{\mathrm{t}}(U_{X_1})) \twoheadrightarrow \pi_1^{\mathrm{t}}(U_{X_1})/D^{(2)}_{c(\mathcal{I})}(\pi_1^{\mathrm{t}}(U_{X_1}))$, the condition $\pi_1^{\mathrm{t}}(U_{X_1})/D^{(6)}_{c(\mathcal{I})}(\pi_1^{\mathrm{t}}(U_{X_1})) \cong \pi_1^{\mathrm{t}}(U_{X_2})/D^{(6)}_{c(\mathcal{I})}(\pi_1^{\mathrm{t}}(U_{X_2}))$ implies that

$$\{\pi_1^{\mathrm{t}}(U_{X_1})/D^{(2)}_{c(\mathcal{I})}(\pi_1^{\mathrm{t}}(U_{X_1})),\ \pi_1^{\mathrm{t}}(U_{X_2})/D^{(6)}_{c(\mathcal{I})}(\pi_1^{\mathrm{t}}(U_{X_2}))\} \subseteq \pi_A^{\mathrm{t}}(U_{X_1}) \cap \pi_A^{\mathrm{t}}(U_{X_2}).$$

Then the corollary follows from Theorem 6.3. □

Remark 6.4.1. Note that the condition $\mathfrak{G} \subseteq \pi_A^{\mathrm{t}}(U_{X_1}) \cap \pi_A^{\mathrm{t}}(U_{X_2})$ mentioned in Theorem 6.3 is *much weaker* than the condition $\pi_1^{\mathrm{t}}(U_{X_1})/D^{(6)}_{c(\mathcal{I})}(\pi_1^{\mathrm{t}}(U_{X_1})) \cong \pi_1^{\mathrm{t}}(U_{X_2})/D^{(6)}_{c(\mathcal{I})}(\pi_1^{\mathrm{t}}(U_{X_2}))$ mentioned in Corollary 6.4, and that Theorem 6.3 *cannot* be deduced from Corollary 6.4.

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