Moduli spaces of fundamental groups of curves in positive characteristic

Yu Yang

RIMS, Kyoto University

June 30, 2021

In this talk, I will explain some philosophy aspects of the speaker's constructions of a general theory for the anabelian geometry of curves over algebraically closed fields of positive characteristic. I do not touch any technical aspects.

Grothendieck's anabelian philosophy

One of the main problems in anabelian geometry is the so-called "Grothendieck's anabelian conjectures".

In the case of curves, roughly speaking, Grothendieck's anabelian conjectures are the various formulations based on the following anabelian philosophy which was suggested in Grothendieck's letter to Faltings:

> **Hom-version:** *The set of dominate morphisms of hyperbolic curves can be determined group-theoretically by the set of open continuous homomorphisms of their algebraic fundamental groups (in the sense of SGA1).*

Grothendieck's anabelian philosophy

In particular, the Hom-version implies

Isom-version: *The sets of isomorphisms of hyperbolic curves can be determined group-theoretically from the sets of isomorphisms of their algebraic fundamental groups.*

Weak Isom-version: *The isomorphism class of a hyperbolic curve can be determined group-theoretically from the isomorphism class of its algebraic fundamental group.*

The Grothendieck's anabelian conjectures can be also formulated for general varieties.

In the case of arithmetic fields (e.g. number fields, p-adic fields, finite fields, etc.), Grothendieck's anabelian conjectures have been proven by many mathematicians.

All of the proofs require the use of the outer Galois representations induced by the homotopy exact sequence of fundamental groups.

In this talk, I will explain a new kind of anabelian phenomenon observed by the speaker that exists only in the world of positive characteristic, and that cannot be explained by using Grothendieck's original anabelian philosophy.

All of the results of the speaker mentioned in this talk can be found in the speaker's papers, see my homepage:

https://www.kurims.kyoto-u.ac.jp/~yuyang/

Fundamental groups of curves over algebraically closed fields

Suppose that the base fields are algebraically closed.

In the case of characteristic 0, the étale (or tame) fundamental groups of curves of type (g, n) are isomorphic to the profinite completion of the topological fundamental groups of Riemann surfaces of type (g, n). Then their structures depend only on (g, n).

In the case of characteristic $p > 0$, the Abhyankar conjecture (proved by M. Raynaud and D. Harbater) shows that the sets of finite quotients of the étale fundamental groups of affine curves of type (g, n) can be completely determined by (g, n). However, the global properties and the structures of étale fundamental groups are very mysterious and no longer known (even the simplest case $\mathbb{A}^1_{\overline{\mathbb{F}}_p}$).

Tamagawa's theory: anabelian geometry of curves over algebraically closed fields of positive characteristic

Around 1996, A. Tamagawa discovered that there exist very strong anabelian phenomena for curves over algebraically closed fields of positive characteristic.

This means that the geometry of curves can be possibly determined by their geometric fundamental groups. This kind of anabelian phenomena is quite different from that over arithmetic fields and go beyond Grothendieck's anabelian geometry. This is the reason that we do not have an explicit description of the étale (or tame) fundamental group of any hyperbolic curve in positive characteristic.

The techniques used in this situation are much different from that of arithmetic fields, and depend heavily on the geometry of coverings of curves in positive characteristic.

Tamagawa's theory: anabelian geometry of curves over algebraically closed fields of positive characteristic

From 1996 to 2001, as a founder, Tamagawa contributed many fundamental ideas to this theory and proved many significant results (e.g. reconstructions of inertia groups of cusps and their associated field structures, the relations between generalized Hasse-Witt invariants and liner conditions arose from curves, the theory of Raynaud-Tamagawa theta divisors, p-averages of tame fundamental groups, local Torelli for Phym varietes, etc.).

In his paper "Fundamental groups and geometry of curves in positive characteristic", he made a conjectural world concerning (étale and tame) fundamental groups of curves over algebraically closed fields of positive characteristic. All of his conjectures are still wide open. Next, let me explain one of them concerning anabelian geometry.

Settings

- k_i, $i \in \{1, 2\}$: an algebraically closed field of characteristic $p > 0$
- $X_i^\bullet \overset{\text{def}}{=} (X_i, D_{X_i})$: a smooth pointed stable curve of type (g_{X_i}, n_{X_i}) over k_i, where X_i denotes the underlying (projective) curve, D_{X_i} denotes the set of marked points, g_{X_i} denotes the genus of X_i, and $n_{X_i} \overset{\text{def}}{=} \#D_{X_i}$
- $U_{X_i} \overset{\text{def}}{=} X_i \setminus D_{X_i}$
- $\pi_1(U_{X_i})$: the étale fundamental group of U_{X_i} (we omit the base point)
- $\pi_1^{\mathrm{t}}(X_i^\bullet)$: the tame fundamental group of $X_i^\bullet$
- $\Pi_{X_i^\bullet}$: $\pi_1(U_{X_i})$ or $\pi_1^{\mathrm{t}}(X_i^\bullet)$

The Weak Isom-version Conjecture

One of the main conjectures in the anabelian geometry of curves over algebraically closed fields of positive characteristic is as follows:

Conjecture 1 (Weak Isom-version Conjecture)

Suppose that $\Pi_{X_i^\bullet}$*,* $i \in \{1, 2\}$*, is the étale (resp. tame) fundamental group of* $X_i^\bullet$*. Then* $X_1^\bullet$ *and* $X_2^\bullet$ *are isomorphic as schemes (resp. the minimal models of* $X_1^\bullet$ *and* $X_2^\bullet$ *are isomorphic as schemes) if and only if* $\Pi_{X_1^\bullet} \cong \Pi_{X_2^\bullet}$*.*

Results around the Weak Isom-version Conjecture

First, we have the following famous (highly non-trivial in the case of tame) result of Tamagawa:

- $\Pi_{X_1^\bullet} \cong \Pi_{X_2^\bullet} \Rightarrow (g_{X_1}, n_{X_1}) = (g_{X_2}, n_{X_2})$.

The following results concerning the Weak Isom-version Conjecture are the only cases which we known:

- Suppose that $\Pi_{X_i^\bullet}$ is the étale fundamental group of $X_i^\bullet$ and $k_1 = \overline{\mathbb{F}}_p$. Then Weak Isom-version Conjecture is true if either $g_{X_1} = 0$ (Tamagawa) or $(g_{X_1}, n_{X_1}) = (1, 1)$ holds (A. Sarashina).
- The above results also holds when $\Pi_{X_i^\bullet}$ is the tame fundamental group of $X_i^\bullet$ (Tamagawa).

The finiteness theorem

For arbitrary (g_{X_i}, n_{X_i}), nothing is known about the Weak Isom-version Conjecture. On the other hand, we have the following weak version of the Weak Isom-version Conjecture: The finiteness theorem.

This famous theorem was proved by Raynaud, F. Pop-M. Saïdi in special cases, and by Tamagawa in the general case, which says that

> *over $\overline{\mathbb{F}}_p$, only finitely many isomorphism classes of smooth pointed stable curves have the same tame fundamental group.*

Specialization homomorphisms

The finiteness theorem is a direct consequence of the following result:

> *Let $\mathcal{X}$ be a non-isotrivial smooth stable curve over $\overline{\mathbb{F}}_p[[t]]$. The specialization map of geometric étale (tame) fundamental groups between the generic fiber and the special fiber of $\mathcal{X}$ is not an isomorphism.*

- Raynaud also suggested that the above theorem can be generalized to the case where we may replace the étale (or tame) fundamental groups by their very small quotients (i.e., the "new part")
- The above suggestion was proved by J. Tong under certain assumptions concerning the Jacobian of special fibers.

Specialization homomorphisms

- Motivated by the Weak Isom-version Conjecture mentioned above, we expect that the above result concerning specialization map holds for arbitrary DVR of characteristic $p > 0$. However, nothing is known. On the other hand, recently, Saïdi-Tamagawa proved a weak version about the finiteness theorem over arbitrary algebraically closed fields of characteristic $p > 0$ which says that non-isotrivial family of smooth pointed stable curves has non-constant tame fundamental groups.

Specialization homomorphisms

- The speaker showed that the above result does not hold for singular non-isotrivial pointed stable curves over a DVR whose residue field is not $\overline{\mathbb{F}}_p$. This means that the specialization map of geometric (log) étale fundamental group of the generic fiber and the special fiber may be isomorphism. This is the motivation that the speaker introduced the so-called "Frobenius equivalence" (which I will explain later), and this phenomenon can be explained by using the clutching maps of moduli spaces of fundamental groups introduced by the speaker.

Étale vs. Tame

- (Tamagawa) $\pi_1^{\mathrm{t}}(X_i^\bullet)$ can be group-theoretically reconstructed from $\pi_1(U_{X_i})$. Thus, tame version results are stronger than étale version results.
- Tame version results are far more difficult than étale version results which was the motivation of Tamagawa's tame version of Raynaud theta divisors. The theory of "Raynaud-Tamagawa theta divisors" is one of main techniques in the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ which was developed by Raynaud in the case of étale coverings (of projective curves), and was generalized by Tamagawa to the case of tame coverings.
- Tame fundamental groups are much better than étale fundamental groups if we consider the anabelian geometry of curves from the viewpoint of moduli spaces.

Anabelian phenomena for arbitrary pointed stable curves

By applying the theory of Raynaud-Tamagawa theta divisors (in particular, the tame version), the speaker discovered that the anabelian phenomena also exist for arbitrary (possibly singular) pointed stable curves.

By replacing the tame fundamental groups by admissible fundamental groups (i.e., the geometric log étale fundamental groups which are natural generalizations of tame fundamental groups in the case of arbitrary pointed stable curves) and generalized all the results mentioned above to the case of arbitrary pointed stable curves.

What's the Hom-version?

Tamagawa also formulated the so-called Isom-version Conjecture for tame fundamental groups. But noting is known for the Isom-version Conjecture. Note that the weak Isom-version Conjecture (or the Isom-version Conjecture) shares the same anabelian philosophy as Grothendieck originally suggested (i.e., the consideration that we mentioned at the beginning).

Moreover, we have the following natural question:

> *Can we formulate a Hom-version conjecture for tame fundamental groups of arbitrary smooth pointed stable curves in the case of algebraically closed fields of characteristic p by using Grothendieck's originally suggestion?*

The answer is "no" in general since the existence of specialization maps (which is the main reason that Tamagawa cannot formulate a Hom-version conjecture in general).

Key observation

Since we cannot formulate a Hom-version conjecture in general by following Grothendieck's originally suggestion, we may ask the following:

> *What's the geometric behaviors of pointed stable curves corresponding to the sets of open homomorphisms of tame fundamental groups (or admissible fundamental group in general)?*

The observation of the speaker is as follows: The geometric behaviors are the deformation informations of pointed stable curves, and moreover, the topological structures of moduli spaces of curves can be understood from the sets of open homomorphisms of admissible fundamental groups.

This kind of new anabelian phenomenon can be precisely captured by using the formulation of moduli spaces of admissible fundamental groups and the Homeomorphism Conjecture introduced by the speaker. Let me explain this in the remainder of my talk.

Settings

- $\overline{M}_{g,n}$: the coarse moduli space of pointed stable curves of type (g,n) over $\overline{\mathbb{F}}_p$
- $M_{g,n} \subseteq \overline{M}_{g,n}$: the open subset corresponding to smooth pointed stable curves
- $q \in \overline{M}_{g,n}$: an arbitrary point
- k_q: an algebraic closure of the residue field of q
- $X_q^\bullet$: the pointed stable curve corresponding to $\operatorname{Spec} k_q \to \overline{M}_{g,n}$
- Π_q: the admissible fundamental group of $X_q^\bullet$. Note that if $q \in M_{g,n}$, Π_q is the tame fundamental group
- $\pi_A(q)$: the set of finite quotients of Π_q
- $\overline{\Pi}_{g,n} \overset{\text{def}}{=} \{$the set of isomorphism classes of $\Pi_q, q \in \overline{M}_{g,n}\}$
- $\Pi_{g,n} \subseteq \overline{\Pi}_{g,n}$: the subset such that $q \in M_{g,n}$

Frobenius equivalence

We introduce an equivalence relation $\sim_{fe}$ on $\overline{M}_{g,n}$ which is determined by the Frobenius actions, and which we call the Frobenius equivalence. In particular, if $q_1, q_2 \in M_{g,n}$, then $q_1 \sim_{fe} q_2$ if and only if the smooth pointed stable curves $X^{\bullet}_{q_1}$ and $X^{\bullet}_{q_2}$ are isomorphic as schemes (note that this does not hold in general if $q_1, q_2 \in \overline{M}_{g,n} \setminus M_{g,n}$).

- The speaker proved that the natural map $\overline{M}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n},\ q \mapsto [\Pi_q]$, factors through

$$\pi^{\mathrm{adm}}_{g,n} : \overline{\mathfrak{M}}_{g,n} \stackrel{\mathrm{def}}{=} \overline{M}_{g,n}/\sim_{fe} \twoheadrightarrow \overline{\Pi}_{g,n},\ [q] \mapsto [\Pi_q],$$

where $[q]$ denotes the equivalence class, and $[\Pi_q]$ denotes the isomorphism class.

Note that if $q_1, q_2 \in \overline{M}_{g,n} \setminus M_{g,n}$, the above result is not trivial.

The Weak Isom-version Conjecture via moduli spaces

We have the Weak Isom-version Conjecture via moduli spaces which generalizes Tamagawa's formulation to the case of arbitrary pointed stable curves.

Conjecture 2 (Weak Isom-version Conjecture)

The map $\pi^{\mathrm{adm}}_{g,n} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n}$ *is a bijection.*

The Weak Isom-version Conjecture via moduli spaces

- We put $\pi^{\mathrm{t}}_{g,n} \overset{\mathrm{def}}{=} \pi^{\mathrm{adm}}_{g,n}|_{\mathfrak{M}_{g,n}} : \mathfrak{M}_{g,n} \overset{\mathrm{def}}{=} M_{g,n}/\sim_{fe} \twoheadrightarrow \Pi_{g,n}$. We have the following commutative diagram

$$\begin{array}{ccc} \mathfrak{M}_{g,n} & \xrightarrow{\pi^{\mathrm{t}}_{g,n}} & \Pi_{g,n} \\ \downarrow & & \downarrow \\ \overline{\mathfrak{M}}_{g,n} & \xrightarrow{\pi^{\mathrm{adm}}_{g,n}} & \overline{\Pi}_{g,n} \end{array}$$

Note that the vertical maps of the above commutative diagram are injections.

- A consequence of a result of the speaker says that

$$\pi^{\mathrm{adm}}_{g,n}(\overline{\mathfrak{M}}_{g,n} \setminus \mathfrak{M}_{g,n}) = \overline{\Pi}_{g,n} \setminus \Pi_{g,n}$$

Moduli space of admissible fundamental groups

Let $\mathcal{G}$ be the category of finite groups and $G \in \mathcal{G}$. We put

$$U_G \stackrel{\text{def}}{=} \{[\Pi] \in \overline{\Pi}_{g,n} \mid G \in \pi_A(\Pi)\}.$$

We introduce a topology $\mathcal{T}_{\overline{\Pi}_{g,n}}$ on the set $\overline{\Pi}_{g,n}$ which is generated by $\{U_G\}_{G \in \mathcal{G}}$ as open sets.

We shall call $(\overline{\Pi}_{g,n}, \mathcal{T}_{\overline{\Pi}_{g,n}})$ the moduli space of admissible fundamental groups of type (g, n). For simplicity, we still use $\overline{\Pi}_{g,n}$ to denote $(\overline{\Pi}_{g,n}, \mathcal{T}_{\overline{\Pi}_{g,n}})$.

- A result of the speaker says that $\Pi_{g,n}$ is an open subset of $\overline{\Pi}_{g,n}$.

The Homeomorphism Conjecture

From now on, we will regard $\overline{\mathfrak{M}}_{g,n}$ as a topological space whose topology is induced by the Zariski topology of $\overline{M}_{g,n}$.

- The speaker proved that the map $\pi_{g,n}^{\mathrm{adm}}$ is continuous.

Moreover, the main conjecture of the theory of moduli spaces of admissible fundamental groups is as follows:

Conjecture 3 (Homeomorphism Conjecture)

The continuous map $\pi_{g,n}^{\mathrm{adm}} : \overline{\mathfrak{M}}_{g,n} \twoheadrightarrow \overline{\Pi}_{g,n}$ *is a homeomorphism.*

The Homeomorphism Conjecture

- The Homeomorphism Conjecture has a simpler form when we only consider smooth pointed stable curves. Let $M_{g,n,\mathbb{F}_p}$ be the coarse moduli space over $\mathbb{F}_p$. Then the Homeomorphism Conjecture says that the natural surjective map

$$M_{g,n,\mathbb{F}_p} \twoheadrightarrow \Pi_{g,n}, \ q \mapsto [\Pi_q]$$

  is a homeomorphism.
- The moduli spaces of fundamental groups and the Homeomorphism Conjecture are completely different from Grothendieck's anabelian conjecture for moduli spaces (an anabelian conjecture based on the considerations explained at the beginning)

Weak Isom-version Conjecture vs. Homeomorphism Conjecture

- The Weak Isom-version Conjecture means that the moduli spaces of curves can be reconstructed group-theoretically as sets from the isomorphism classes of admissible fundamental groups.
- The Homeomorphism Conjecture means that the moduli spaces of curves can be reconstructed group-theoretically as topological spaces from sets of open continuous homomorphisms of admissible fundamental groups.
- The Weak Isom-version Conjecture is an Isom-type problem, and the Homeomorphism Conjecture is a Hom-type problem.

Predicaments of the anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$

Since Tamagawa discovered that there also exists the anabelian geometry for certain smooth pointed stable curves over algebraically closed fields of characteristic $p > 0$, twenty-five years have passed.

However, the Weak Isom-version Conjecture is still the only anabelian phenomenon (at least in the published papers) that we know in this situation, and we *cannot* even imagine what phenomena arose from curves and their fundamental groups should be anabelian. Moreover, what "anabelian" mean in this situation?

Anabelian=Topology

The moduli spaces of admissible fundamental groups and the Homeomorphism Conjecture shed some new light on anabelian geometry based on the following new anabelian philosophy:

> *The anabelian properties of pointed stable curves over algebraically closed fields of characteristic $p > 0$ are equivalent to the topological properties of the topological space $\overline{\Pi}_{g,n}$.*

The above philosophy supplies a point of view to see what anabelian phenomena that we can reasonably expect for pointed stable curves over algebraically closed fields of characteristic $p > 0$. This means that the Homeomorphism Conjecture is a dictionary between the geometry of pointed stable curves (or moduli spaces of curves) and the anabelian properties of pointed stable curves.

Results concerning the Homeomorphism Conjecture

The following result obtained by the speaker:

Theorem 1 (Y)

The Homeomorphism Conjecture holds for 1-dimensional moduli spaces.

- In fact, the above theorem is a consequence of the following strong anabelian result:
  Let $(g, n) \in \{(0, n), (1, 1)\}$ and $q_1, q_2 \in \overline{M}_{g,n}$ arbitrary points.
  Suppose that q_1 is closed. Then

$$\mathrm{Hom}^{\mathrm{open}}_{\mathrm{pg}}(\Pi_{q_1}, \Pi_{q_2}) \neq \emptyset$$

  if and only if $q_1 \sim_{fe} q_2$.

About the proof of Theorem 1

The proof of the above theorem does not directly use any results concerning the Weak Isom-version Conjecture, and it based on many results concerning admissible fundamental groups established by the speaker. For example, the followings:

- Formulas concering maximal and averages of generalized Hasse-Witt invariants of prime-to-p admissible coverings proved by using the theory of Raynaud-Tamagawa theta divisors (the above invariants were introduced by Tamagawa for tame fundamental groups).

The above formulas play a role "Galois action" in the theory of anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$.

About the proof of Theorem 1

Let $X_i^\bullet$, $i \in \{1, 2\}$, be an arbitrary pointed stable curve of type (g, n) over an algebraically closed field k_i of characteristic $p > 0$ and Π_{X_i} the admissible fundamental group of $X_i^\bullet$. Let $\phi : \Pi_{X_1} \to \Pi_{X_2}$ be an arbitrary open continuous homomorphism.

By applying the above formulas, the speaker obtained the following strong anabelian results:

- The inertia subgroups of marked points and their associated field structures can be reconstructed group-theoretically from ϕ.

About the proof of Theorem 1

- A certain combinatorial Grothendieck conjecture in positive characteristic for ϕ under certain assumption of dual semi-graphs of $X_1^\bullet$ and $X_2^\bullet$.

The original combinatorial anabelian geometry is a theory developed only in the case of characteristic 0 by using completely different techniques (i.e., the outer Galois representations play central roles), which was introduced by Mochizuki, and which was completely developed by Mochizuki and Hoshi.

Theorem 1 can be deduced (highly non-trivial) by applying the above results and the geometry of admissible coverings of semi-stable curves.

Towards the Homeomorphism Conjecture for higher dimensional moduli spaces

The speaker believes that the Homeomorphism Conjecture can be solved by following the steps below:

- Closed points of $\mathfrak{M}_{g,n}$: This means that the images of closed points of $\mathfrak{M}_{g,n}$ are closed points of $\overline{\Pi}_{g,n}$.

If $g = 0$, this case has been completely solved by the speaker.

- Non-closed points of $\mathfrak{M}_{g,n}$: The speaker formulated a conjecture that we called the Pointed Collection Conjecture which says that the tame fundamental groups of a non-closed point $[q] \in \mathfrak{M}_{g,n}$ can be reconstructed from the tame fundamental groups of the closed points of the closure of $[q]$ in $\mathfrak{M}_{g,n}$

This is the most difficult step. We know nothing except some very special cases.

Towards the Homeomorphism Conjecture for higher dimensional moduli spaces

- From smooth to singular: The speaker formulated a conjecture that we call the Geometric Data Conjecture which says that the topological and combinatorial data associated to pointed stable curves can be reconstructed group-theoretically from open continuous homomorphisms. This conjecture is a ultimate generalization of the combinatorial Grothendieck conjecture which exists only in the world of positive characteristic.

If $g = 0$, the Geometric Data Conjecture has been proven by the speaker.

Some other open problems about $\overline{\Pi}_{g,n}$

By using the consideration "anabelian=topology" mentioned above, we have many new problems (or conjectures) which cannot be seen if we only consider the Weak Isom-version Conjecture. We select just a few of them.

- Let $[q] \in \overline{\mathfrak{M}}_{g,n}$ and $[\Pi_q] = \pi_{g,n}^{\mathrm{adm}}([q]) \in \overline{\Pi}_{g,n}$. Write V_q and V_{Π_q} for the topological closures of $[q]$ and $[\Pi_q]$. Then we have (Krull dimension)
$$\dim(V_q) = \dim(V_{\Pi_q}).$$
In particular, we have $\dim(\overline{M}_{g,n}) = \dim(\overline{\mathfrak{M}}_{g,n}) = \dim(\overline{\Pi}_{g,n})$.

Some other open problems about $\overline{\Pi}_{g,n}$

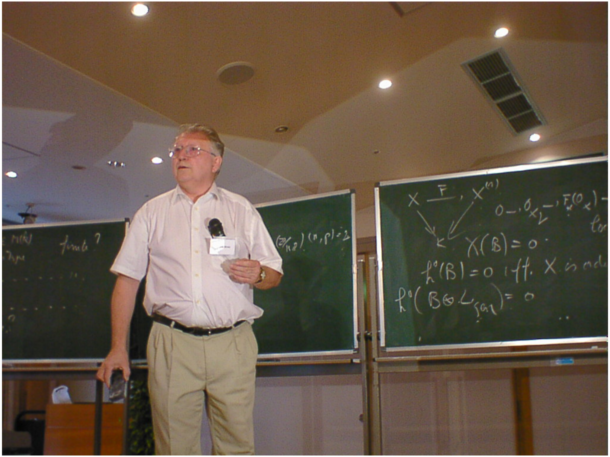
- Let $[q_1], [q_2] \in \overline{\mathfrak{M}}_{g,n}$ such that $V_{\Pi_{q_1}} \supseteq V_{\Pi_{q_2}}$. Then we have

$$\dim(V_{q_1}) \geq \dim(V_{q_2}).$$

The above problem is a generalization of Tamagawa's essential dimension conjecture which says that the "essential dimensions" can be reconstructed group-theoretically from tame fundamental groups.

- We can also define p-rank strata $\{\overline{\Pi}^f_{g,n}\}_{0\leq f\leq g}$ for $\overline{\Pi}_{g,n}$ which are locally closed. Does exist purity for $\{\overline{\Pi}^f_{g,n}\}_{0\leq f\leq g}$?

All the results explained above are based on the theories of admissible fundamental groups, geometry of semi-stable curves, and fundamental groups in positive characteristic which were established by Professors Mochizuki, Raynaud, and Tamagawa.



Prof. Michel Raynaud (Algebraic Geometry 2000, Nagano, Japan)

To the memory of Professor Michel Raynaud

Thank you for the attention!