Reconstructions of geometric data of pointed stable curves in positive characteristic

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July 9, 2021

In the last week, I explain some philosophy aspects of the speaker for establishing a general theory of anabelian geometry of curves over algebraically closed fields of positive characteristic. In this talk, I go to some technical aspects a little.

Notations

- k_i: a field
- $X_i^\bullet \stackrel{\text{def}}{=} (X_i, D_{X_i})$, $i \in \{1, 2\}$: a pointed stable curve of type (g_{X_i}, n_{X_i}) over k_i, where g_{X_i} denotes the genus of X_i, n_{X_i} denotes the cardinality of D_{X_i}
- $\Gamma_{X_i^\bullet}$: the dual semi-graph of $X_i^\bullet$
- Π_{X_i}: algebraic fundamental group of $X_i^\bullet$ in the sense of SGA1 (e.g. étale, tame, log étale, and so on)

Fundamental problem of anabelian geometry

Roughly speaking, the main problem of the anabelian geometry of curves is as follows:

Problem 1

How much geometric information of $X_i^\bullet$ is contained in various versions of its fundamental group?

More precisely, the ultimate goal of anabelian geometry is the following:

- Reconstruct $X_i^\bullet$ (as a scheme) group-theoretically from various versions of its fundamental group.

First step

For reconstructing $X_i^\bullet$ (as scheme, or k_i-scheme), the first step is to reconstruct the following geometric data (=Data 1) from $\Pi_{X_i^\bullet}$:

- the type (g_{X_i}, n_{X_i})
- the inertia subgroups associated to marked points

Moreover, if $X_i^\bullet$ is a singular pointed stable curve, we also want to reconstruct the following geometric data (=Data 2) from $\Pi_{X_i^\bullet}$:

- the dual semi-graph $\Gamma_{X_i^\bullet}$
- the fundamental groups of smooth pointed stable curves associated to irreducible components of $X_i^\bullet$

Note that Data2 $\Rightarrow$ Data1.

Combinatorial anabelian geometry

S. Mochizuki observed that many similar techniques (in particular, around the prime-to-p fundamental groups, for instance, the theory of weight of ℓ-adic Galois representations) about reconstructions of Data 1 were used in the proofs of H. Nakamura, A. Tamagawa, and himself concerning Grothendieck's anabelian conjecture of curves over arithmetic fields.

Moreover, he observed that there exists a general theory for reconstructing Data 1 and Data 2 associated arbitrary pointed stable curves from (prime-to-p, if $\mathrm{char}(k_i) = p > 0$) admissible fundamental groups (=geometric log étale fundamental groups) with certain outer Galois action arose from log stable curves over log points. This general theory was called the combinatorial anabelian geometry by Mochizuki which was formulated by using his theory of the semi-graphs of anabelioids), and which plays an important role in his theory of IUT.

Combinatorial anabelian geometry

Roughly speaking, the combinatorial anabelian geometry is a kind of anabelian theory of curves over algebraically closed fields which focus on reconstructions of geometric data (i.e., topological and combinatorial data) of arbitrary pointed stable curves via geometric fundamental groups with geometric (not arithmetic) Galois actions (i.e., they are not depend on the arithmetic properties of base fields).

More precisely,

- In characteristic 0, we mainly focus on "geometric fundamental groups+Galois action arose from Dehn twists (or in other world, arose from log structures induced by nodes of curves)"

Combinatorial anabelian geometry

- In the case of positive characteristic, we only use "geometric fundamental groups" and use some completely different techniques (from that of characteristic 0)
- Combinatorial anabelian geometry in positive characteristic is an important part of the theory of anabelian geometry of curves over algebraically closed fields considered by Tamagawa and the speaker

Combinatorial Grothendieck conjecture in characteristic 0

Since the theory of combinatorial anabelian geometry introduced by Mochizuki, and completely developed by Hoshi-Mochizuki is a theory for prime-to-p admissible fundamental groups over algebraically closed fields, for simplicity, we may assume that $\mathrm{char}(k_i) = 0$.

The main problem in combinatorial anabelian geometry is the so-called the combinatorial Grothendieck conjecture (or ComGC) which, roughly speaking, is the following:

Combinatorial Grothendieck conjecture in characteristic 0

Conjecture 1 (Combinatorial Grothendieck conjecture in characteristic 0)

Let k_i be an algebraically closed field, $X_i^\bullet$ a pointed stable curve over k_i, and $\Pi_{X_i^\bullet}$ the admissible fundamental group of $X_i^\bullet$. Let I_i be a profinite group and $\rho_{I_i} : I_i \to$ Out$(\Pi_{X_i^\bullet})$ an "certain" outer Galois representation. Suppose that

$$\begin{array}{ccc} I_1 & \xrightarrow{\rho_{I_1}} & \mathrm{Out}(\Pi_{X_1^\bullet}) \\ \beta \big\downarrow & & \big\downarrow \mathrm{out}(\alpha) \\ I_2 & \xrightarrow{\rho_{I_2}} & \mathrm{Out}(\Pi_{X_2^\bullet}), \end{array}$$

is commutative, where $\beta : I_1 \overset{\sim}{\to} I_2$ and $\alpha : \Pi_{X_1^\bullet} \overset{\sim}{\to} \Pi_{X_2^\bullet}$. Then α induces an "isomorphsim" between Data 1 and Data 2 associated to $X_1^\bullet$ and $X_2^\bullet$.

Combinatorial Grothendieck conjecture in characteristic 0

- the outer Galois representation ρ_{I_i} plays an essential role in the formulation of the conjecture
- the above conjecture does not hold for arbitrary outer Galois representations (i.e., we need some "non-degenerate" conditions that limits the deformations of curves)
- the above conjecture was proved by Mochizuki in the case of IPSC-type (outer Galois actions arose from curves over DVR), and by Hoshi and Mochizuki in the case of certain NN-type (outer Galois actions arose from curves over completions of local rings associated points of moduli stacks whose log structures determined by the nodes of curves) under certain assumptions about inertia subgroups of marked points.

The results of Hoshi and Mochizuki about the ComGC formulated above play a central role in combinatorial anabelian geometry in characteristic 0.

The world of positive characteristic

Around 2000, Raynaud, F. Pop, M. Saïdi, and Tamagawa showed evidence for very strong anabelian phenomena for curves over

algebraically closed fields of positive characteristic.

This kind of anabelian phenomena is quite different from that over arithmetic fields and go beyond Grothendieck's anabelian geometry. Moreover, this shows that the geometric étale (or tame) fundamental group of a smooth pointed stable curve in positive characteristic must encode "moduli" of the curve. This is the reason that we do not have an explicit description of the étale (or tame) fundamental group of any hyperbolic curve in positive characteristic.

In the remainder of this talk, I will explain the

tame (due to Tamagawa)/admissible (due to Yang) anabelian geometry

of curves over algebraically closed fields of characteristic $p > 0$ with the main focus on reconstructions of Data 1 and Data 2 associated to arbitrary pointed stable curves.

Settings

- $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$: moduli stacks of smooth pointed stable curves and pointed stable curves of type (g, n) over $\overline{\mathbb{F}}_p$
- $M_{g,n}$ and $\overline{M}_{g,n}$: coarse moduli spaces of $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$, respectively
- $q \in \overline{M}_{g,n}$: an arbitrary point
- k: an arbitrary algebraically closed field which contains the residue field $k(q)$ of q

Settings

- $X^\bullet = (X, D_X)$: pointed stable curve determined by the natural morphism $\operatorname{Spec} k \to \operatorname{Spec} k(q) \to \overline{M}_{g,n}$
- $X^{\log}$: the log stable curve whose log structure is induced by the log stack $\overline{\mathcal{M}}_{g,n}^{\log}$ (whose log structure is induced by $\overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$)
- $\Gamma_{X^\bullet}$: the dual semi-graph of $X^\bullet$
- r_X: the Betti number of $\Gamma_{X^\bullet}$

Admissible fundamental groups

Denote by

$$\Delta^{\mathrm{adm}}$$

the geometric log étale fundamental group of $X^{\log}$ (or admissible fundamental group of $X^{\bullet}$) which depends only on q (i.e., Δ^{adm} does not depend on the choices of k).

Note that $\Delta^{\mathrm{adm}} \cong \pi_1^{\mathrm{tame}}(X \setminus D_X)$ when X is nonsingular.

Main goal of this talk

The main goal of this talk is to explain the following results obtained by Tamagawa (smooth case) and the speaker (general case):

- There exists a group-theoretical formula for the topological type (g, n). In particular, (g, n) is a group-theoretical invariant. Moreover, there exists a group-theoretical algorithm whose input datum is Δ^{adm}, and whose output datum is Data 1 associated to $X^\bullet$.

- There exists a group-theoretical algorithm whose input datum is Δ^{adm}, and whose output datum is Data 2 associated to $X^\bullet$.

Remarks

- When k is an "arithmetic field", (g, n) can be reconstructed by applying outer Galois actions (e.g. weight-monodromy filtration). However, in the case of tame/admissible fundamental groups of curves over algebraically closed fields of positive characteristic, the reconstruction of (g, n) is very difficult.

- Suppose that $X^\bullet$ is smooth over k. Tamagawa also obtained a group-theoretical formula for (g, n) by using the étale fundamental group of $X \setminus D_X$, whose proof is much simpler (only 1 page!) than the case of tame fundamental groups. Moreover, a result of Tamagawa sayas that the tame fundamental group can be reconstructed group-theoretically from the étale fundamental group, then the tame fundamental group version is stronger than the étale fundamental group version.

Remarks

- The most important reason for using tame/admissible fundamental groups is that tame/admissible fundamental groups are "good" invariants if one considers the theory of anabelian geometry of curves in positive characteristic from the point of view of moduli spaces.

Generalized Hasse-Witt invariants of cyclic coverings

- H: an open normal subgroup such that $G \stackrel{\mathrm{def}}{=} \Delta^{\mathrm{adm}}/H$ is a cyclic group whose order is prime to p
- $Y^\bullet = (Y, D_Y)$: pointed stable curve over k corresponding to H

Then we obtain a natural representation

$$\rho_H : G \to \mathrm{Aut}_k(H^1_{et}(Y, \mathbb{F}_p) \otimes k)$$

and a decomposition

$$H^1_{et}(Y, \mathbb{F}_p) \otimes k \cong \bigoplus_{\chi : G \to k^\times} H_\chi.$$

Generalized Hasse-Witt invariants of cyclic coverings

We shall say that

$$\{\dim_k(H_\chi)\}_\chi$$

is the set of generalized Hasse-Witt invariants of cyclic covering $Y^\bullet \to X^\bullet$. Note that since $H^1_{et}(Y, \mathbb{F}_p) \cong H^{\mathrm{ab}} \otimes \mathbb{F}_p$, we have that generalized Hasse-Witt invariants are group-theoretical invariants associated to Δ^{adm}, where $(-)^{\mathrm{ab}}$ denotes the abelianization of $(-)$.

- $\{\dim_k(H_\chi)\}_{\chi,H}$ (or $\{\rho_H\}_H$) plays a role of "outer Galois representations" in the theory of anabelian geometry of curves over algebraically closed fields of characteristic $p > 0$ (i.e., a lot of geometric information concerning $X^\bullet$ can be carried out from $\{\dim_k(H_\chi)\}_{\chi,H}$).

Raynaud-Tamagawa theta divisors

The theory of Raynaud-Tamagawa theta divisors is a powerful tool to study generalized Hasse-Witt invariants of cyclic coverings. Let me explain this theory roughly in just few slices. For simplicity, we suppose that $X^\bullet$ is smooth over k.

Let $N \overset{\text{def}}{=} p^f - 1$, $f \in \mathbb{N}_{>0}$, D an effective divisor on X such that $\mathrm{Supp}(D) \subseteq D_X$ and $\mathrm{ord}_Q(D) < p^f$ for each $Q \in \mathrm{Supp}(D)$, and $\mathcal{I}$ a line bundle on X such that $\mathcal{I}^{\otimes N} \cong \mathcal{O}_X(-D)$. Let F_k^f be the fth absolute Frobenius morphism of k,

$$X_f \overset{\text{def}}{=} X \times_{k, F_k^f} k$$

the fth Frobenius twist of X, $F_{X/k}^f : X \to X_1 \to \ldots \to X_f$ the fth relative Frobenius morphism of X, and $\mathcal{I}_f$ the pulling back of the line bundle $\mathcal{I}$ under the natural morphism $X_f \to X$.

Raynaud-Tamagawa theta divisors

We obtain a vector bundle $\mathcal{B}_D^f \stackrel{\text{def}}{=} (F_{X/k}^f)_*(\mathcal{O}_X(D))/\mathcal{O}_{X_f}$, and put

$$\mathcal{E}_D^f \stackrel{\text{def}}{=} \mathcal{B}_D^f \otimes \mathcal{I}_f$$

on X_f. Consider the following condition $(\star)$:

$$0 = \min\{H^0(X_f, \mathcal{E}_D^f \otimes \mathcal{L}), H^1(X_f, \mathcal{E}_D^f \otimes \mathcal{L})\}, \ [\mathcal{L}] \in J_{X_f},$$

where J_{X_f} denotes the Jacobian of X_f. We put

$$\Theta_{\mathcal{E}_D^f} \stackrel{\text{def}}{=} \{[\mathcal{L}] \in J_{X_f} \mid \mathcal{L} \text{ does not satisfy } (\star)\}.$$

In fact, $\Theta_{\mathcal{E}_D^f}$ is a closed subscheme of J_{X_f} with codimension ≤ 1. We shall say $\Theta_{\mathcal{E}_D^f}$ the Raynaud-Tamagawa theta divisor associated to D if $\Theta_{\mathcal{E}_D^f} \neq J_{X_f}$.

Raynaud-Tamagawa theta divisors

- The theory of $\Theta_{\mathcal{E}_D^f}$ was developed by Raynaud (1982) when $D = 0$, and the ramified version (i.e., $D \neq 0$) was developed by Tamagawa (2003).
- If $D = 0$ (resp. $\deg(D) = N$), the existence of $\Theta_{\mathcal{E}_D^f}$ was proved by Raynaud (resp. Tamagawa). The existence of $\Theta_{\mathcal{E}_D^f}$ is a very difficult problem, and it does not exist in general.

Raynaud-Tamagawa theta divisors

If $\Theta_{\mathcal{E}_D^f}$ exists, we may use intersection theory to estimate the cardinality of $\mathbb{Z}/N\mathbb{Z}$-tame covering of $X^\bullet$ whose ramification divisor is equal to D, and whose generalized Hasse-Witt invariant attains the maximum. This is the main idea and purpose of Raynaud and Tamagawa's theory on theta divisors.

By using $\Theta_{\mathcal{E}_D^f}$, Raynaud obtained the following deep theorem, which is the first result concerning the global structure of tame fundamental group of curves over algebraically closed fields of characteristic $p > 0$:

- Let $X^\bullet$ be a pointed stable curves over k. Then Δ^{adm} (i.e., the étale fundamental group of X) is not a prime-to-p profinite group. This means that, for each open subgroup $H_1 \subseteq \Delta^{\mathrm{adm}}$, there exists an open subgroup $H_2 \subseteq H_1$ such that $H_2^{\mathrm{ab}} \otimes \mathbb{F}_p \neq 0$.

p-average of admissible fundamental groups

- K_N: the kernel of the natural surjection $\Delta^{\mathrm{adm}} \twoheadrightarrow \Delta^{\mathrm{adm,ab}} \otimes \mathbb{Z}/N\mathbb{Z}$, where $N \stackrel{\mathrm{def}}{=} p^f - 1$

Tamagawa introduced an important group-theoretical invariant as following, which is called the limit of p-average of Δ^{adm}:

$$\mathsf{Avr}_p(\Delta^{\mathrm{adm}}) \stackrel{\mathrm{def}}{=} \lim_{f\to\infty} \frac{\dim_{\mathbb{F}_p}(K_N^{\mathrm{ab}} \otimes \mathbb{F}_p)}{\#(\Delta^{\mathrm{adm,ab}} \otimes \mathbb{Z}/N\mathbb{Z})}.$$

- Roughly speaking, when $N >> 0$, almost all of the generalized Hasse-Witt invariants of $\mathbb{Z}/N\mathbb{Z}$-admissible coverings are equal to $\mathsf{Avr}_p(\Delta^{\mathrm{adm}})$.

p-average theorem: smooth case

We have the following highly nontrivial theorem which was proved by Tamagawa by using Raynaud-Tamagawa theta divisors.

Theorem 1 (Tamagawa)

Suppose that $X^\bullet$ is smooth over k. Then we have

$$\mathrm{Avr}_p(\Delta^{\mathrm{adm}}) = \begin{cases} g-1, & \text{if } n \leq 1, \\ g, & \text{if } n \geq 2. \end{cases}$$

- The smooth version of p-average theorem means that $\mathrm{Avr}_p(\Delta^{\mathrm{adm}})$ contains the information concerning (g, n) when $X^\bullet$ is smooth over k.
- Tamagawa also proved p-average theorem for 2-connected pointed stable curves which is a main step in his proof of resolutions of non-singularities.

p-average theorem: general case

We have the following theorem which was proved by the speaker by using Raynaud-Tamagawa theta divisors.

Theorem 2 (Y)

(1) Suppose that $X^\bullet$ is component-generic. Then we have

$$\mathrm{Avr}_p(\Delta^{\mathrm{adm}}) = g_X - r_X - \#v(\Gamma_{X^\bullet}) + \#V_{X^\bullet}^{\mathrm{tre}, g_v=0} + \#E_{X^\bullet}^{\mathrm{tre}} + \sum_{v \in v(\Gamma_{X^\bullet})} \#E_v^{>1}.$$

(2) Suppose that $\#E_v^{>1} \leq 1$ for each vertex v of $\Gamma_{X^\bullet}$ (e.g. $\Gamma_{X^\bullet}$ is 2-connected, $X^\bullet$ is smooth, etc.). Then we have

$$\mathrm{Avr}_p(\Delta^{\mathrm{adm}}) = g_X - r_X - \#V_{X^\bullet}^{\mathrm{tre}} + \#V_{X^\bullet}^{\mathrm{tre}, g_v=0} + \#E_{X^\bullet}^{\mathrm{tre}}.$$

Remarks

- The data appeared in the above formulas depend only on the structure of dual semi-graph $\Gamma_{X^\bullet}$.
- For some technical reasons (arose from the existences of Raynaud-Tamagawa theta divisors), we do not obtain a general result for $\mathrm{Avr}_p(\Delta^{\mathrm{adm}})$ of arbitrary pointed stable curves.
- By using the methods developed by the speaker concerning combinatorial anabelian geometry in positive characteristic, the generalized version of formula for $\mathrm{Avr}_p(\Pi_{X^\bullet})$ will give us a strong anabelian result with less assumptions concerning reconstructions of Data 2 via open continuous homomorphisms. This was one of motivations for generalizing Tamagawa's p-average theorem to the case of pointed stable curves.

Reconstructions for Data 1: notations

- $b^i \overset{\text{def}}{=} \dim_{\mathbb{Q}_\ell}(H^1_{et}(X \setminus D_X, \mathbb{Q}_\ell))$ (i.e., the Betti number of the ith ℓ-adic étale cohomology group), $i \in \{0,1,2\}$, where ℓ is a prime number distinct from p. Moreover, we may prove that b^i, $i \in \{0,1,2\}$, is a group-theoretical invariant.

- Let $\ell' \in \mathfrak{Primes} \setminus \{p\}$ be an arbitrary prime number distinct from p. Write $\mathrm{Nom}_{\ell'}(\Delta^{\mathrm{adm}})$ for the set of normal subgroups of Δ^{adm} such that $\#(\Delta^{\mathrm{adm}}/\Delta^{\mathrm{adm}}(\ell')) = \ell'$ for each $\Delta^{\mathrm{adm}}(\ell') \in \mathrm{Nom}_{\ell'}(\Delta^{\mathrm{adm}})$. We put

$$c \overset{\text{def}}{=} \begin{cases} 1, & \text{if } b^2 = 1, \\ 1, & \text{if } b^2 = 0,\ \mathrm{Avr}_p(\Delta^{\mathrm{adm}}(\ell)) - 1 = \ell(\mathrm{Avr}_p(\Delta^{\mathrm{adm}})), \\ & \ell \in \mathfrak{Primes} \setminus \{p\},\ \Delta^{\mathrm{adm}}(\ell) \in \mathrm{Nom}_\ell(\Delta^{\mathrm{adm}}), \\ 0, & \text{otherwise.} \end{cases}$$

Reconstructions for Data 1: smooth case

By applying the p-average theorem (smooth case), Tamagawa proved the following result:

Theorem 3 (An anabelian formula for (g, n) (smooth case))

Suppose that $X^\bullet$ is smooth over k. Then we have

$$g = \mathrm{Avr}_p(\Delta^{\mathrm{adm}}) + c,\ \ n = b^1 - 2\mathrm{Avr}_p(\Delta^{\mathrm{adm}}) - 2c - b^2 + 1.$$

Moreover, there exists an group-theoretical algorithm whose input datum is Δ^{adm}, and whose output datum is Data 1.

- This result is a key step in Tamagawa's proof of the weak Isom-version of Grothendieck conjecture for smooth curves of type $(0, n)$ over $\overline{\mathbb{F}}_p$, which says that the isomorphism classes of smooth curves of type $(0, n)$ over $\overline{\mathbb{F}}_p$ can be determined group-theoretically from the isomorphism classes of their tame fundamental groups.

Remarks

The approach to finding an anabelian formula for (g, n) by applying the limit of p-averages associated to Δ^{adm} explained above cannot be generalized to the case where $X^\bullet$ is an arbitrary (possibly singular) pointed stable curve. The reason is that the singular version of p-average theorem is very complicated in general, and $\mathrm{Avr}_p(\Delta^{\mathrm{adm}})$ depends not only on (g, n) but also on the graphic structure of $\Gamma_{X^\bullet}$.

Maximum generalized Hasse-Witt invariants theorem

By proving the existence of Raynaud-Tamagawa theta divisor for certain effective divisor D on X, the speaker obtained the following result:

Theorem 4 (Y)

There exists a prime-to-p cyclic admissible covering of $X^\bullet$ such that a generalized Hasse-Witt invariant of the cyclic admissible covering attains the maximum

$$\gamma_{X^\bullet}^{\max} = \begin{cases} g-1, & \textit{if } n=0, \\ g+n-2, & \textit{if } n \neq 0. \end{cases}$$

Moreover, $\gamma_{X^\bullet}^{\max}$ is a group-theoretical invariant.

Reconstructions for Data 1: general case

The maximum generalized Hasse-Witt invariant theorem implies the following formula immediately:

Theorem 5 (An anabelian formula for (g, n) (general case))

Let $X^\bullet$ be an arbitrary pointed stable curve of type (g, n) over k. Then we have

$$g = b^1 - \gamma^{\max}_{X^\bullet} - 1, \ n = 2\gamma^{\max}_{X^\bullet} - b^1 - b^2 + 3.$$

Moreover, there exists an group-theoretical algorithm whose input datum is Δ^{adm}, and whose output datum is Data 1.

Reconstructions for Data 2

On the other hand, $\mathrm{Avr}_p(\Delta^{\mathrm{adm}})$ contains the information concerning the Betti number of $\Gamma_{X^\bullet}$ if $\Gamma_{X^\bullet}$ is "good" enough. This means that the weight-monodromy filtration associated to the first ℓ-adic étale cohomology group of every admissible covering of $X^\bullet$ can be reconstructed group-theoretically from the corresponding open subgroup of Δ^{adm}. Note that, if k is an "arithmetic field", the weight-monodromy filtration can be reconstructed group-theoretically by using the theory of "weights".

This observation is a key in the speaker's proof of combinatorial Grothendieck conjecture in positive characteristic.

Reconstructions for Data 2

Let us show the second main result of this talk.

Theorem 6 (Combinatorial Grothendieck conjecture in positive characteristic)

Let $X^\bullet$ be an arbitrary pointed stable curve of type (g, n) over k. Then there exists a group-theoretical algorithm whose input datum is Δ^{adm}, and whose output datum is Data 2.

Remarks

- By applying combinatorial Grothendieck conjecture, all the results concerning the tame anabelian geometry of smooth curves over algebraically closed fields of characteristic $p > 0$ can be extended to the case of pointed stable curves.
- Note that we obtain two ways (i.e., via MaxGHW invariants and ComGC, respectively) for reconstructing (g, n). Since the group-theoretical algorithm appeared in combinatorial Grothendieck conjecture is not an explicit algorithm, the formula for (g, n) cannot be deduced by combinatorial Grothendieck conjecture. On the other hand, the formula for MaxGHW is not the motivation of the speaker for reconstructing (g, n).

Remarks

- The statement of Theorem 6 is mono-anabelian. Before the speaker proved Theorem 6, he also obtained a bi-anabelian version, and the mono-anabelian version is not trivial.
- The motivation of the speaker for proving a mono-anabelian version of Theorem 6 is as follows: To construct (group-theoretically) clutching maps between moduli spaces of admissible fundamental groups.

In the case of characteristic 0, the applications of combinatorial Grothendieck conjecture obtained by Mochizuki-Hoshi mainly focus on the anabelian geometry of configuration spaces and its related topics. In their cases, since the admissible fundamental groups are very "soft" (i.e., do not depend on moduli), the geometric data of fibers of $X_n^{\log} \to X_{n-1}^{\log}$ over a point of X_{n-1} can be completely determined by the log structure of the point (via ComGC).

Moreover, many geometric data associated to $X_n^{\log}$ such that $n \geq 2$ can be reconstructed group-theoretically from the geometric prime-to-p fundamental group of $X_n^{\log}$. It seems that we do not need to consider the combinatorial anabelian geometry of configuration spaces over algebraically closed fields by using (full) profinite fundamental groups when $n \geq 2$.

> *Is this means that the ComGC in positive characteristic (i.e., Theorem 6) is the ending of the combinatorial anabelian geometry in characteristic $p > 0$?*

In fact, Theorem 6 is just the first evidence discovered by the speaker about the mysteries of the fantastic object: admissible fundamental groups of pointed stable curves in positive characteristic.

Some further developments: new phenomena in char. p

The motivations of the speaker for proving the formulas for maximum and p-average of generalized Hasse-Witt invariants are as followings:

> *Let $X_i^\bullet$, $i \in \{1, 2\}$, be a pointed stable curve of type (g, n) over an algebraically closed field k_i of characteristic $p > 0$ and Δ_i^{adm} the admissible fundamental group of $X_i^\bullet$. Then we have*
>
> $$\mathrm{Hom}_{\mathrm{gp}}^{\mathrm{op}}(\Delta_1^{\mathrm{adm}}, \Delta_2^{\mathrm{adm}}) \neq \mathrm{Isom}_{\mathrm{gp}}(\Delta_1^{\mathrm{adm}}, \Delta_2^{\mathrm{adm}})$$
>
> *in general (this phenomenon does not exist in characteristic 0).*

I note that this phenomenon is an essential difference between characteristic 0 and positive characteristic that leaded me to consider whether we can reconstruct Data 1 and Data 2 from an arbitrary open continuous homeomorphism (which is one of main step in the speaker's developments of the theory of the moduli spaces of admissible fundamental groups).

The difficulties of anabelian geometry via open continuous homomorphisms

I am not sure how much a non-expert of anabelian geometry can understand that there exit big gaps between isomorphisms and open continuous homomorphisms in the theory of anabelian geometry. Let me show an example which is the first step (highly non-trivial) for considering anabelian geometry via open continuous homomorphisms (this is almost trivial for isomorphisms in some important cases of characteristic 0 (e.g. with outer Galois actions of IPSC-type)).

Let $\phi \in \mathrm{Hom}^{\mathrm{op}}_{\mathrm{gp}}(\Delta_1^{\mathrm{adm}}, \Delta_2^{\mathrm{adm}})$, H_2 an arbitrary open subgroup of Δ_2^{adm}, and $H_1 \overset{\mathrm{def}}{=} \phi^{-1}(H_2) \subseteq \Delta_1^{\mathrm{adm}}$. Write $X^{\bullet}_{H_i}$ for the pointed stable curve over k_i of type (g_{H_i}, n_{H_i}) corresponding to H_i.

$$\text{Does } (g_{H_1}, n_{H_1}) = (g_{H_2}, n_{H_2}) \text{ hold?}$$

Some further developments: reconstructions Data 1 and Data 2 via open continuous homomorphisms

The speaker obtained the following results (the formulas for maximum and p-average of generalized Hasse-Witt invariants are main ingredients in the proofs):

Theorem 7 (Y)

Let $\phi \in \mathrm{Hom}^{\mathrm{op}}_{\mathrm{gp}}(\Delta_1^{\mathrm{adm}}, \Delta_2^{\mathrm{adm}})$. Then we have the following:
(1) Data 1 can be reconstructed group-theoretically from ϕ.
(2) Data 2 can be reconstructed group-theoretically from ϕ under certain assumptions of dual semi-graphs of $X_1^\bullet$ and $X_2^\bullet$.

- The proof of the above theorem is much harder than the proof of the combinatorial Grothendieck conjecture in positive characteristic (i.e., Theorem 6).

Some further developments: The Geometric Data Conjecture

Theorem 7 and the theory of moduli spaces of admissible fundamental groups leaded the speaker to formulate a conjecture for reconstructions of geometric data which is a ultimate generalization of the combinatorial Grothendieck conjecture. Roughly speaking, we have the following

Conjecture 2 (Geometric Data Conjecture)

We put $GD_{X_i^\bullet}$, $i \in \{1, 2\}$, the set of conjugacy classes of admissible fundamental groups associated to pointed stable subcurves of $X_i^\bullet$. Let $\phi \in \mathrm{Hom}_{\mathrm{gp}}^{\mathrm{op}}(\Delta_1^{\mathrm{adm}}, \Delta_2^{\mathrm{adm}})$. Then we have $\phi(GD_{X_1^\bullet}) = GD_{X_2^\bullet}$.

- The speaker believes that the Geometric Data Conjecture is a main step to prove the Homeomorphism Conjecture (for higher dimensional moduli spaces) in the theory of the moduli spaces of admissible fundamental groups.

Thank you for the attention !